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Abstract

In this paper, we consider a degenerate pseudo-parabolic equation modeling two-phase flow in porous media, where dynamic effects in the difference of the phase pressures are included. Because of the special form of the capillary induced diffusion function, the equation becomes degenerate for certain values of the unknown. To overcome the difficulties due to the degeneracy, a regularization method is employed for proving the existence of a weak solution.

Keywords: Dynamic capillary pressure, two-phase flow, degenerate pseudo-parabolic equation, weak solution, existence.

1 Introduction

Pseudo-parabolic equations appear as models for many real life applications, such as lightning [2], seepage in fissured rocks [4], radiation with time delay [27] and heat conduction models [36]. Here we consider a pseudo-parabolic equation modeling two-phase flow in porous media, where dynamic effects are complementing the capillary pressure - saturation relationship. With a given maximal time $T > 0$ and for all $x \in \Omega$ a bounded domain in $\mathbb{R}^d$ ($d = 1, 2,$ or 3) having a Lipschitz continuous boundary $\partial\Omega$, we investigate the equation

$$\partial_t u + \nabla \cdot \mathbf{F}(u, x, t) = \nabla \cdot (H(u)\nabla p_c), \quad (t, x) \in Q := \Omega \times (0, T]. \tag{1.1}$$

This equation is obtained by including Darcy's law for both phases in the mass conservation laws. Here u stands for water saturation, $\mathbf{F}$ and H are the water fraction flow function and the capillary induced diffusion function, while p_c is the capillary pressure term. Such models are proposed in [19, 32]. For recent works providing experimental evidence for the dynamic effects in the phase pressure difference we refer to [12, 21, 5]. Similar models, but considering an "apparent saturation" are discussed in [3]. Here we consider a simplified situation, where

$$p_c = u + \tau\partial_t u. \tag{1.2}$$

Then (1.1) becomes

$$\partial_t u + \nabla \cdot \mathbf{F}(u, x, t) = \nabla \cdot (H(u)\nabla(u + \tau\partial_t u)). \tag{1.3}$$

The functions H and $\mathbf{F}$ depend on the specific model, in particular on the relative permeabilities. Commonly encountered in the engineering literature are relative permeabilities of power-like types, u^{p+1} and $(1-u)^{q+1}$, where p and q are positive reals. This leads to

$$(1.4)\qquad H(u) = \frac{\mathbf{K}}{\mu}\frac{u^{p+1}(1-u)^{q+1}}{u^{p+1}+M(1-u)^{q+1}}, \text{ and } \mathbf{F}(u,x,t) = \mathbf{Q}(x,t)\frac{u^{p+1}}{u^{p+1}+M(1-u)^{q+1}} + H(u)\rho\mathbf{g},$$

where $\mathbf{K}$ is the permeability tensor of the porous medium, that will be supposed, for the sake of simplicity, to be isotropic. Next, μ and $\tilde{\mu}$ are the viscosities of the two phases, whereas $M = \frac{\tilde{\mu}}{\mu} > 0$ is the viscosity ratio of the two fluids, and τ is a positive constant standing for the damping coefficient. Further, $\mathbf{Q}$ is the total flow in the porous medium, satisfying $\nabla \cdot \mathbf{Q} = 0$, whereas $\mathbf{g}$ is the gravity vector. Finally, ρ denotes the difference between the phase densities.

With the given function H, (1.3) becomes degenerate whenever $u = 0$ or $u = 1$. Note that the expression (1.4) makes sense only for $u \in [0,1]$. For completeness we extend H continuously by 0 outside this interval. Therefore the functions H is nonnegative, bounded and Lipschitz continuous on the entire $\mathbb{R}$. Similarly, the vector valued function $\mathbf{F}$ is extended by constants for all u outside $[0,1]$, leading to a bounded and Lipschitz continuous, function defined on $\mathbb{R}$. However, (1.4) is just a typical example appearing in the literature. In view of this, throughout this paper we assume

$$\begin{aligned} &H : \mathbb{R} \to \mathbb{R} \text{ is nonnegative, Lipschitz and } C^1, \text{ satisfying} \\ \text{(A1)}\qquad &H(u) > 0 \text{ if } 0 < u < 1, \text{ and } H(u) = 0 \text{ otherwise;} \\ &\mathbf{F} : \mathbb{R} \to \mathbb{R}^d \text{ is Lipschitz and } C^1. \text{ Further, for all } v \in \mathbb{R}, t > 0, \nabla \cdot \mathbf{F}(v,x,t) = 0. \end{aligned}$$

Pseudo-parabolic equations like (1.3) are investigated in the mathematical literature for decades. Short time existence of solutions with constant, compact support is obtained in [15], whereas a nonlinear parabolic-Sobolev equation is studied in [37]. The existence and uniqueness of weak solutions for some nonlinear pseudo-parabolic equations, where the degeneracy may appear in only one term, are proved in [17] and [34]. Long time existence of weak solutions to a closely related model is proved in [28, 29]. We further refer to [25] for the analysis of a non-degenerate pseudo-parabolic model that includes hysteresis.

The connection between pseudo-parabolic equations and shock solutions to hyperbolic conservation laws is investigated in [14] for the case of a constant function H. The analysis there, based on traveling waves, is continued in [13]. In both cases, undercompressive shocks are obtained for values of τ exceeding a threshold value. Nonclassical shocks are also obtained in [6], but in a heterogeneous medium, and in [23], but based on a different regularization. Traveling wave solutions for a pseudo-parabolic equation involving a convex flux function are analyzed in [9, 10, 31].

Concerning numerical methods for pseudo-parabolic equations, the superconvergence of a finite element approximation to similar equation is investigated in [1] and time-stepping Galerkin methods are analyzed in [16] and [18], where two finite difference approximation schemes are considered. Further, Fourier spectral methods are analyzed in [35]. For homogeneous media, discontinuous initial data and corresponding numerical schemes for pseudo-parabolic equations are considered in [11], whereas for heterogeneous media we refer to [20]. We also mention [33] for a review of different numerical methods for pseudo-parabolic equations.

In this paper we prove the existence of weak solutions to the degenerate pseudo-parabolic equation in (1.3). The exact definition will be given below. Existence results for similar models are proved in [28] and [29]. This work is closely related to the analysis in [29]. The results there require sufficiently large values of p and q, and requires that the initial data is neither 0, nor 1 for almost all $x \in \Omega$. Here we only assume $p, q \geq 0$. In particular, if $p, q \in [0, 1)$, the initial data may be 0 or 1 on a non-zero measure subset of Ω.

To obtain the existence result we employ regularization and compactness arguments. The main difficulty appears in dealing with the nonlinear and degenerate term involving the third order derivative, for which we combine the *div-curl* lemma (see e.g. [30, 38]) with equi-integrability properties. A simplified approach is possible whenever the degeneracy H can be controlled by the convective term $\mathbf{F}$, specifically if the product $H(\cdot)^{-1/2}\,\mathbf{F}(\cdot)$ is a bounded function. This is obtained e.g. if $\mathbf{Q} \equiv \mathbf{0}$, as considered in [29]. In this case one can use the structure of the equation as in [8] to obtain uniform L^6 estimates for $\partial_t u$, and then apply the *div-curl* lemma directly. Here we consider a rather general convective flux $\mathbf{F}$ that make this latter strategy fail.

Below we use standard notations in the theory of partial differential equations, such as $L^2(\Omega), W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$. $W^{-1,2}(\Omega)$ denotes the dual space of $W_0^{1,2}(\Omega)$, while $L^2(0,T;W_0^{1,2}(\Omega))$ denotes the Bochner space of $W_0^{1,2}(\Omega)$ valued functions. By $(\cdot,\cdot)$ we mean the inner product in either $L^2(\Omega)$, or $(L^2(\Omega))^d$, and $\|\cdot\|$ stands for the corresponding norm. Furthermore, C denotes a generic positive real number.

The equation (1.3) is complemented by the following initial and boundary conditions

$$u(\cdot,0) = u^0, \quad \text{and} \quad u|_{\partial\Omega} = C_D. \tag{1.5}$$

The initial data is assumed in $W^{1,2}(\Omega)$. Furthermore, it satisfies $0 \leq u^0 \leq 1$ almost everywhere in Ω, while $C_D \in (0,1)$ is a constant. The extension to non-constant boundary data is possible, but requires more technical steps, detailed in [29], that we eliminate here for the sake of presentation. An important requirement here is that C_D is not a degeneracy value, 0, or 1. The reason for this will become clear in the proof of the main result.

To introduce the concept of a weak solution, we define the space

$$V := C_D + W_0^{1,2}(\Omega).$$

Then a weak solution solves
Problem P Find $u \in W^{1,2}(0,T;V)$ such that $u(\cdot,0) = u^0$, $H(u)\nabla\partial_t u \in \left(L^2(Q)\right)^d$, and such that

$$\begin{aligned}\int_0^T\int_\Omega \partial_t u\phi dxdt - \int_0^T\int_\Omega \mathbf{F}(u,x,t)\cdot\nabla\phi dxdt \\ + \int_0^T\int_\Omega H(u)\nabla u\cdot\nabla\phi dxdt + \tau\int_0^T\int_\Omega H(u)\nabla\partial_t u\cdot\nabla\phi dxdt = 0,\end{aligned} \tag{1.6}$$

for any $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$.

As $H(u)$ vanishes at $u = 0$ or 1, (1.3) becomes degenerate. We define the functions

$$G, \Gamma : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}, \qquad G(u) = \int_{C_D}^{u} \frac{1}{H(v)} dv, \text{ and } \Gamma(u) = \int_{C_D}^{u} G(v) dv. \tag{1.7}$$

Clearly, Γ is a convex function satisfying $\Gamma(C_D) = \Gamma'(C_D) = 0$, implying

$$\Gamma(u) \geq 0, \quad \text{for all} \quad u \in \mathbb{R}. \tag{1.8}$$

The existence results in the following sections are obtained under the assumption

$$\int_{\Omega} \Gamma(u^0) dx < \infty. \tag{A2}$$

Under hypothesis (1.4), this assumption is fulfilled if, for example, $0 < p, q < 1$. Whenever $p \geq 1$, (A2) requires that $\text{meas}\{u^0 = 0\} = 0$. Similarly, $q \geq 1$ requires $\text{meas}\{u^0 = 1\} = 0$. The construction of Γ is inspired by [28, 29], where a generalized Kullback entropy is defined. Since Γ is nonnegative, $\Gamma(u^0)$ is an element of $L^1(\Omega)$. As will be proved below, this implies

$$\int_{\Omega} \Gamma(u(t)) dx < C,$$

uniformly for $t \in (0, T]$.

The main result of this paper is the existence of weak solutions to Problem P. We start by studying a regularized problem in Section 2, where we replace H by the strictly positive function $H_\delta = H + \delta$. Some a priori estimates are provided in Section 2 and the existence of weak solutions for H_δ is proved. In Section 3, the existence of weak solutions to equation (1.3) is proved by compactness arguments. The major difficulty is to handle the nonlinear and degenerate term including the mixed, third order derivative. To identify the limit in this case we combine the *div-curl* lemma (see e.g. [30, 38]) with equi-integrability properties.

Remark 1.1 *Equation* (1.3) *is a simplified model for two-phase flow in porous media, where dynamic effects are taken into account in the capillary pressure. However, this model contains the main mathematical difficulties related to such models: a degenerate nonlinearity in the terms involving the higher order derivatives. More realistic models are proposed in [19, 32]. With minor modifications, the present analysis can be extended for dealing with the cases considered e.g. in [9, 10, 31]. For instance, a capillary pressure of the form*

$$p_c = p(u) + \tau \partial_t u$$

may be treated following the ideas presented below, provided that p *is increasing, with* $\sqrt{p'} \in L^1(0,1)$ *and* $H(\cdot)p'(\cdot) \in L^\infty(0,1)$*. In particular the degeneracy* $p'(u) = 0$ *for some* u *is allowed, as well as* $\lim_{s \to \{0,1\}} p'(s) = +\infty$*. Note that under these finer assumptions, the definition of the solution to the* **Problem P** *has to be modified slightly (see [7]).*

2 The regularized problem

To overcome the problems that are due to the degeneracy, we regularize **Problem P** by perturbing $H(u)$:

$$H_\delta(u) = H(u) + \delta, \tag{2.1}$$

where δ is a small positive number. Then we consider the equation:

$$\partial_t u + \nabla \cdot \mathbf{F}(u,x,t) = \nabla \cdot \left(H_\delta(u)\nabla(u + \tau\partial_t u)\right), \tag{2.2}$$

and investigate the limit case as $\delta \to 0$. In particular, we seek a solution to the following **Problem P$_\delta$** Find $u \in W^{1,2}(0,T;V)$ such that $u(\cdot,0) = u^0$, $\nabla\partial_t u \in \left(L^2(Q)\right)^d$ and

$$\begin{aligned}\int_0^T\int_\Omega \partial_t u\phi dxdt - \int_0^T\int_\Omega \mathbf{F}(u,x,t)\cdot\nabla\phi dxdt \\ + \int_0^T\int_\Omega H_\delta(u)\nabla u\cdot\nabla\phi dxdt + \tau\int_0^T\int_\Omega H_\delta(u)\nabla\partial_t u\cdot\nabla\phi dxdt = 0,\end{aligned} \tag{2.3}$$

for any $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$.

Clearly, any solution to **Problem P$_\delta$** depends on δ. However within Section 2, δ will be fixed. For the ease of reading, the δ-dependence of the solution will be self-understood, without involving any δ index for the solution u. We start by showing that P_δ has a solution. To do so, we use the Rothe method [22] and investigate firstly a sequence of time discrete problems.

2.1 Time discretization

Setting $\Delta t = T/N (N \in \mathbb{N})$, we consider the Euler-implicit discretization of **Problem P$_\delta$** which leads to a sequence of time discretized problems. Specifically, we consider
Problem P$_\delta^{n+1}$ Given $u^n \in V, n \in \{0,1,2,...,N-1\}$. Find $u^{n+1} \in V$ such that

$$\begin{aligned}(u^{n+1} - u^n, \phi) + \Delta t(\nabla\cdot\mathbf{F}(u^{n+1},x,t),\phi) + \Delta t(H_\delta(u^{n+1})\nabla u^{n+1},\nabla\phi) \\ +\tau(H_\delta(u^{n+1})\nabla(u^{n+1}-u^n),\nabla\phi) \quad = 0,\end{aligned} \tag{2.4}$$

for any $\phi \in W_0^{1,2}(\Omega)$.
For obtaining estimates we will use the elementary Young inequality

$$ab \le \frac{1}{2\delta}a^2 + \frac{\delta}{2}b^2, \quad \text{for any} \quad a,b \in \mathbb{R} \quad \text{and} \quad \delta > 0. \tag{2.5}$$

We prove the following result.

Proposition 2.1 Problem P$_\delta^{n+1}$ *has a solution.*

Proof . Formally, (2.4) can be written as

$$(\tau+\Delta t)\nabla\cdot(H_\delta(X)\nabla X)-\tau\nabla\cdot(H_\delta(X)\nabla u^n)-\Delta t\nabla\cdot\mathbf{F}(X,x,t)-X+u^n=0, \tag{2.6}$$

with X standing for the unknown function. If $u^n \in C_D + C_0^\infty(\Omega)$, the existence of a solution to (2.6) is provided by Theorem 8.2, Chapter 4 in [24]. To extend the existence result to $u^n \in V$ we make use of density arguments. Specifically, along a sequence $\varepsilon \to 0$ we consider a sequence $\{u_\varepsilon^n\}_{\varepsilon>0} \subseteq C_D + C_0^\infty(\Omega)$ that converges to u^n in $W^{1,2}(\Omega)$. For each u_ε^n there exists a solution X_ε of (2.6), where u_ε^n replaces u^n. This defines a sequence $\{X_\varepsilon\}_{\varepsilon>0} \subseteq C_D + C_0^\infty(\Omega)$. As will be seen below, this sequence is uniformly bounded in $W^{1,2}(\Omega)$, and therefore contains a weakly convergent subsequence. We will show that the limit X of this subsequence solves **Problem P**.

The weak form of (2.6) reads

$$\begin{aligned}(\tau+\Delta t)(H_\delta(X_\varepsilon)\nabla X_\varepsilon,\nabla\phi)-\tau(H_\delta(X_\varepsilon)\nabla u_\varepsilon^n,\nabla\phi)&\\ -\Delta t(\mathbf{F}(X_\varepsilon,x,t),\nabla\phi)+(X_\varepsilon,\phi)&=(u_\varepsilon^n,\phi),\end{aligned} \tag{2.7}$$

for any $\phi \in W_0^{1,2}(\Omega)$. We define the vector valued function $\mathcal{F}(X_\varepsilon) := \int_{C_D}^{X_\varepsilon}\mathbf{F}(v,x,t)dv$ and note that this is a $\mathbf{0}$-vector on $\partial\Omega$. Since $\mathbf{F}$ is divergence free in x, for $\phi = X_\varepsilon - C_D \in W_0^{1,2}(\Omega)$ one gets

$$(\mathbf{F}(X_\varepsilon,x,t),\nabla(X_\varepsilon-C_D))=\int_\Omega\nabla\mathcal{F}(X_\varepsilon,x,t)\cdot\nabla X_\varepsilon dx=\int_{\partial\Omega}\gamma\cdot\mathcal{F}(C_D)dx=0, \tag{2.8}$$

the outer normal vector to $\partial\Omega$ being here denoted by γ. In this case, (2.7) yields

$$\begin{aligned}&(\tau+\Delta t)\int_\Omega H_\delta(X_\varepsilon)|\nabla X_\varepsilon|^2dx-\tau\int_\Omega|H_\delta(X_\varepsilon)\nabla X_\varepsilon\cdot\nabla u_\varepsilon^n|dx+\int_\Omega|X_\varepsilon|^2dx\\ &\le\int_\Omega(C_D+u_\varepsilon^n)X_\varepsilon dx-C_D\int_\Omega u_\varepsilon^n dx.\end{aligned} \tag{2.9}$$

By (2.5),

$$\tau\int_\Omega|H_\delta(X_\varepsilon)\nabla X_\varepsilon\cdot\nabla u_\varepsilon^n|dx\le\frac{\tau+\Delta t}{2}\left\|\sqrt{H_\delta(X_\varepsilon)}\nabla X_\varepsilon\right\|^2+\frac{\tau^2}{2(\tau+\Delta t)}\left\|\sqrt{H_\delta(X_\varepsilon)}\nabla u_\varepsilon^n\right\|^2,$$

and from (2.9)

$$\begin{aligned}&\frac{\tau+\Delta t}{2}\left\|\sqrt{H_\delta(X_\varepsilon)}\nabla X_\varepsilon\right\|^2-\frac{\tau^2}{2(\tau+\Delta t)}\left\|\sqrt{H_\delta(X_\varepsilon)}\nabla u_\varepsilon^n\right\|^2+\|X_\varepsilon\|^2\\ &\le\frac{1}{2}\|C_D+u_\varepsilon^n\|^2+\frac{1}{2}\|X_\varepsilon\|^2-C_D\int_\Omega u_\varepsilon^n dx.\end{aligned} \tag{2.10}$$

Since u_ε^n is bounded in $W^{1,2}(\Omega)$ and $\delta \le H_\delta(X_\varepsilon) \le C$, we obtain

$$(\tau+\Delta t)\left\|\sqrt{H_\delta(X_\varepsilon)}\nabla X_\varepsilon\right\|^2+\|X_\varepsilon\|^2\le C. \tag{2.11}$$

Therefore we conclude that X_ε is uniformly bounded in $W^{1,2}(\Omega)$, so it contains a subsequence, still denoted by X_ε for convenience, converging weakly in $W^{1,2}(\Omega)$. We denote this limit by X. From the compact imbedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$, we obtain

$\mathbf{F}(X_\varepsilon, x, t) \to \mathbf{F}(X, x, t)$ strongly in $(L^2(\Omega))^d$ and $H_\delta(X_\varepsilon) \to H_\delta(X)$ strongly in $L^2(\Omega)$. Hence, for any $\phi \in W_0^{1,2}(\Omega)$, we have

$$(X_\varepsilon, \phi) \to (X, \phi), \tag{2.12}$$

$$(\mathbf{F}(X_\varepsilon, x, t), \nabla\phi) \to (\mathbf{F}(X, x, t), \nabla\phi). \tag{2.13}$$

To show that X solves Problem P_δ^{n+1}, we need to prove that

$$(H_\delta(X_\varepsilon)\nabla X_\varepsilon, \nabla\phi) \to (H_\delta(X)\nabla X, \nabla\phi). \tag{2.14}$$

The idea involved in proving this last step will be used later again. We start by observing that $H_\delta(X_\varepsilon)\nabla X_\varepsilon$ is bounded in $(L^2(\Omega))^d$, therefore it has a weak limit χ. To identify this limit, we take $\phi \in C_0^\infty(\Omega)$ as test function in (2.7). Since $H_\delta(X_\varepsilon) \to H_\delta(X)$ strongly in $L^2(\Omega)$ and $\nabla X_\varepsilon \to \nabla X$ weakly in $(L^2(\Omega))^d$, we have

$$(H_\delta(X_\varepsilon)\nabla X_\varepsilon, \nabla\phi) \to (H_\delta(X)\nabla X, \nabla\phi). \tag{2.15}$$

This implies that $H_\delta(X_\varepsilon)\nabla X_\varepsilon \rightharpoonup H_\delta(X)\nabla X$ in distributional sense. By the uniqueness of the limit, we have $\chi = H_\delta(X)\nabla X$.

Finally, since $\{u_\varepsilon^n\}_{\varepsilon>0} \subseteq V$ converges weakly to u^n in $W^{1,2}(\Omega)$, we have

$$(u_\varepsilon^n, \phi) \to (u^n, \phi), \tag{2.16}$$

$$(H_\delta(X_\varepsilon)\nabla u_\varepsilon^n, \nabla\phi) \to (H_\delta(X)\nabla u^n, \nabla\phi). \tag{2.17}$$

Combining (2.12), (2.13), (2.14), (2.16), (2.17) and (2.7), we conclude that X is a solution to **Problem $\mathbf{P}_\delta^{n+1}$**. □

In proving the existence of a solution to **Problem $\mathbf{P}_\delta$**, we use the following elementary results

Proposition 2.2 *Let $k \in \{0, 1, ..., N\}, m \geq 1$. For any set of m-dimensional real vectors $\boldsymbol{a}^k, \boldsymbol{b}^k \in \mathbb{R}^m$, we have the following identities:*

$$\sum_{k=1}^{N} < \boldsymbol{a}^k - \boldsymbol{a}^{k-1}, \sum_{n=k}^{N} \boldsymbol{b}^n >= \sum_{k=1}^{N} < \boldsymbol{a}^k, \boldsymbol{b}^k > - < \boldsymbol{a}^0, \sum_{k=1}^{N} \boldsymbol{b}^k >, \tag{2.18}$$

$$\sum_{k=1}^{N} < \boldsymbol{a}^k - \boldsymbol{a}^{k-1}, \boldsymbol{a}^k >= \frac{1}{2}(|\boldsymbol{a}^N|^2 - |\boldsymbol{a}^0|^2 + \sum_{k=1}^{N} |\boldsymbol{a}^k - \boldsymbol{a}^{k-1}|^2), \tag{2.19}$$

$$\sum_{k=1}^{N} < \sum_{k=n}^{N} \boldsymbol{a}^k, \boldsymbol{a}^n >= \frac{1}{2}|\sum_{k=1}^{N} \boldsymbol{a}^k|^2 + \frac{1}{2}\sum_{k=1}^{N} |\boldsymbol{a}^k|^2. \tag{2.20}$$

2.2 A priori estimates

For the existence of a solution to **Problem $\mathbf{P}_\delta$**, we apply compactness arguments based on the following a priori estimates.

Proposition 2.3 *For any $n \geq 1$, we have the following:*

$$
\begin{aligned}
(2.21)\qquad & ||\nabla u^n||_{L^2(\Omega)} &\leq& \; C,\\
(2.22)\qquad & \int_\Omega \Gamma_\delta(u^n)dx &\leq& \; C,\\
(2.23)\qquad & ||u^n - u^{n-1}||^2_{L^2(\Omega)} &+& \; \tau||\sqrt{H_\delta(u^n)}\nabla(u^n - u^{n-1})||^2_{L^2(\Omega)} \leq C(\Delta t)^2,\\
(2.24)\qquad & ||u^n||_{L^2(\Omega)} &\leq& \; C.
\end{aligned}
$$

Here C does not depend on δ.

Proof . **1.** Taking $\phi = G_\delta(u^{n+1}) = \int_{C_D}^{u^{n+1}} \frac{1}{H_\delta(v)} dv \in W_0^{1,2}(\Omega)$ in (2.4) gives

$$
\begin{aligned}
(2.25)\qquad & (u^{n+1} - u^n, G_\delta(u^{n+1})) + (\tau + \Delta t)||\nabla u^{n+1}||^2_{L^2(\Omega)}\\
& -\tau(\nabla u^n, \nabla u^{n+1}) + \Delta t(\nabla \cdot \mathbf{F}(u^{n+1}, x, t), G_\delta(u^{n+1})) = 0.
\end{aligned}
$$

Define $\mathcal{G}(u^{n+1}, x, t) := \int_{C_D}^{u^{n+1}} G_\delta(v)\partial_v \mathbf{F}(v, x, t)dv$. By (A1) we have

$$
(\nabla \cdot \mathbf{F}(u^{n+1}, x, t), G_\delta(u^{n+1})) = \int_\Omega \nabla \cdot \mathcal{G}(u^{n+1}, x, t)dx = \int_{\partial\Omega} \gamma \cdot \mathcal{G}(C_D)dx = 0.
$$

Here γ denotes the outer normal vector to $\partial\Omega$. Further, as in (1.7) we define $\Gamma_\delta(u) := \int_{C_D}^u G_\delta(v)dv$ and note that $\Gamma''_\delta(u) = \frac{1}{H_\delta(u)} > 0$, thus

$$
(2.26)\qquad (u^{n+1} - u^n)G_\delta(u^{n+1}) \geq \Gamma_\delta(u^{n+1}) - \Gamma_\delta(u^n).
$$

Summing (2.26) in (2.25) up from 0 to $n-1$ gives

$$
(2.27)\quad 0 \geq \int_\Omega \Gamma_\delta(u^n)dx - \int_\Omega \Gamma_\delta(u^0)dx + (\Delta t + \tau)\sum_{k=1}^n ||\nabla u^k||^2_{L^2(\Omega)} - \tau \sum_{k=1}^n (\nabla u^k, \nabla u^{k-1}).
$$

By (2.19) we have

$$
\begin{aligned}
(2.28)\qquad 0 \;\geq\; & \int_\Omega \Gamma_\delta(u^n)dx - \int_\Omega \Gamma_\delta(u^0)dx + \Delta t \sum_{k=1}^n ||\nabla u^k||^2_{L^2(\Omega)} +\\
& \frac{\tau}{2}||\nabla u^n||^2_{L^2(\Omega)} - \frac{\tau}{2}||\nabla u^0||^2_{L^2(\Omega)} + \frac{\tau}{2}\sum_{k=1}^n ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)},
\end{aligned}
$$

implying

$$
\begin{aligned}
(2.29)\quad & \int_\Omega \Gamma_\delta(u^n)dx + \Delta t \sum_{k=1}^n ||\nabla u^k||^2_{L^2(\Omega)} + \frac{\tau}{2}||\nabla u^n||^2_{L^2(\Omega)} + \frac{\tau}{2}\sum_{k=1}^n ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)}\\
& \leq \int_\Omega \Gamma_\delta(u^0)dx + \frac{\tau}{2}||\nabla u^0||^2_{L^2(\Omega)}.
\end{aligned}
$$

Recalling (1.8), as H_δ is bounded and $u^0 \in W^{1,2}(\Omega)$, we have

$$\int_\Omega \Gamma_\delta(u^0)dx = \int_\Omega \int_{C_D}^{u^0} \int_{C_D}^{u} \frac{1}{H_\delta(v)} dvdudx \leq \int_\Omega \int_{C_D}^{u^0} \int_{C_D}^{u} \frac{1}{H(v)} dvdudx \leq C,$$

where C does not depend on δ. Therefore,

$$\int_\Omega \Gamma_\delta(u^n)dx \leq C, \tag{2.30}$$

$$||\nabla u^n||_{L^2(\Omega)} \leq C, \quad \sum_{k=1}^{n} ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} \leq C. \tag{2.31}$$

2. Taking $\phi = u^n - u^{n-1} \in W_0^{1,2}(\Omega)$ in (2.4) written at time $t_n = n\Delta t$, we have

$$\begin{aligned} &||u^n - u^{n-1}||^2_{L^2(\Omega)} + \Delta t(\nabla \cdot \mathbf{F}(u^n, x, t), u^n - u^{n-1}) + \\ &\Delta t(H_\delta(u^n)\nabla u^n, \nabla(u^n - u^{n-1})) + \tau||\sqrt{H_\delta(u^n)}\nabla(u^n - u^{n-1})||^2_{L^2(\Omega)} = 0. \end{aligned} \tag{2.32}$$

By (2.5) and (A1),

$$\begin{aligned} &||u^n - u^{n-1}||^2_{L^2(\Omega)} - \frac{1}{2}||u^n - u^{n-1}||^2_{L^2(\Omega)} - \frac{(C\Delta t)^2}{2}||\nabla u^n||^2_{L^2(\Omega)} \\ &-\frac{(\Delta t)^2}{2\tau}||\sqrt{H_\delta(u^n)}\nabla u^n||^2_{L^2(\Omega)} - \frac{\tau}{2}||\sqrt{H_\delta(u^n)}\nabla(u^n - u^{n-1})||^2_{L^2(\Omega)} \\ &+\tau||\sqrt{H_\delta(u^n)}\nabla(u^n - u^{n-1})||^2_{L^2(\Omega)} \leq 0. \end{aligned} \tag{2.33}$$

According to (2.31), since H_δ is bounded, we obtain

$$||u^n - u^{n-1}||^2_{L^2(\Omega)} + \tau||\sqrt{H_\delta(u^n)}\nabla(u^n - u^{n-1})||^2_{L^2(\Omega)} \leq C(\Delta t)^2. \tag{2.34}$$

As $H_\delta \geq \delta$, we also derive

$$||u^n - u^{n-1}||_{L^2(\Omega)} \leq C\Delta t \quad \text{and} \quad ||\nabla(u^n - u^{n-1})||_{L^2(\Omega)} \leq \frac{C\Delta t}{\sqrt{\delta}}. \tag{2.35}$$

3. Finally, since $u^n - C_D \in W_0^{1,2}(\Omega)$,

$$||u^n||_{L^2(\Omega)} \leq ||u^n - C_D||_{L^2(\Omega)} + ||C_D||_{L^2(\Omega)} \leq C(\Omega)||\nabla(u^n - C_D)||_{L^2(\Omega)} + C \leq C. \quad \square \tag{2.36}$$

2.3 Existence for Problem $\mathbf{P}_\delta$

Using Proposition 2.3, we now prove the existence of a solution to the regularized **Problem $\mathbf{P}_\delta$**.

Theorem 2.1 Problem $\mathbf{P}_\delta$ *has a solution.*

Proof . We start by defining

$$U_N(t) = u^{k-1} + \frac{t - t^{k-1}}{\Delta t}(u^k - u^{k-1}), \tag{2.37}$$

for $\quad t^{k-1} = (k-1)\Delta t \le t < t^k = k\Delta t, k = 1, 2...N$. Clearly, $U_N|_{\partial\Omega} = C_D$. Then we have

$$
\begin{aligned}
(2.38) \qquad \int_0^T ||U_N(t)||^2_{L^2(\Omega)}dt &= \sum_{k=1}^N \int_{t^{k-1}}^{t^k} ||u^{k-1} + \frac{t - t^{k-1}}{\Delta t}(u^k - u^{k-1})||^2_{L^2(\Omega)}dt \\
&\le 2\sum_{k=1}^N \int_{t^{k-1}}^{t^k} (||u^{k-1}||^2_{L^2(\Omega)} + ||u^k - u^{k-1}||^2_{L^2(\Omega)})dt \\
&= 2\Delta t \sum_{k=1}^N (||u^{k-1}||^2_{L^2(\Omega)} + ||u^k - u^{k-1}||^2_{L^2(\Omega)}) \\
&\le C,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.39) \int_0^T ||\nabla U_N(t)||^2_{L^2(\Omega)}dt &= \sum_{k=1}^N \int_{t^{k-1}}^{t^k} ||\nabla u^{k-1} + \frac{t - t^{k-1}}{\Delta t}\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)}dt \\
&\le 2\sum_{k=1}^N \int_{t^{k-1}}^{t^k} (||\nabla u^{k-1}||^2_{L^2(\Omega)} + ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)})dt \\
&= 2\Delta t \sum_{k=1}^N (||\nabla u^{k-1}||^2_{L^2(\Omega)} + ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)}) \\
&\le C.
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
(2.40) \qquad \int_0^T ||\partial_t U_N||^2_{L^2(\Omega)}dt &= \sum_{k=1}^N \int_{t^{k-1}}^{t^k} ||\frac{1}{\Delta t}(u^k - u^{k-1})||^2_{L^2(\Omega)}dt \\
&= \frac{1}{\Delta t}\sum_{k=1}^N ||u^k - u^{k-1}||^2_{L^2(\Omega)} \le C
\end{aligned}
$$

and, by (2.35),

$$
\begin{aligned}
(2.41) \qquad \int_0^T ||\partial_t \nabla U_N||^2_{L^2(\Omega)}dt &= \sum_{k=1}^N \int_{t^{k-1}}^{t^k} ||\frac{1}{\Delta t}\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)}dt, \\
&= \frac{1}{\Delta t}\sum_{k=1}^N ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} \le \frac{C}{\delta}.
\end{aligned}
$$

By (2.38), (2.39), (2.40), (2.41), there exists a subsequence of $\{U_N\}$ (still denoted as $\{U_N\}$) such that, as $N \to \infty$,

$$
\begin{aligned}
(2.42) \qquad & U_N \to U \quad \text{strongly in} \quad L^2(Q), \\
(2.43) \qquad & \partial_t U_N \rightharpoonup \partial_t U \quad \text{weakly in} \quad L^2(Q), \\
(2.44) \qquad & \nabla U_N \rightharpoonup \nabla U \quad \text{weakly in} \quad L^2(Q), \\
(2.45) \qquad & \nabla\partial_t U_N \rightharpoonup \nabla\partial_t U \quad \text{weakly in} \quad L^2(Q).
\end{aligned}
$$

Now we prove that U solves **Problem $\mathbf{P}_\delta$**. Firstly, for any $\phi \in L^2(0,T;W_0^{1,2}(\Omega))$, (2.4) implies

$$(2.46)\quad (\frac{u^k-u^{k-1}}{\Delta t}, \int_{t^{k-1}}^{t^k} \phi dt) + (\nabla\cdot \mathbf{F}(u^k,x,t), \int_{t^{k-1}}^{t^k} \phi dt) + (H_\delta(u^k)\nabla u^k, \int_{t^{k-1}}^{t^k} \nabla\phi dt) + \tau(H_\delta(u^k)\nabla \frac{u^k-u^{k-1}}{\Delta t} dt, \int_{t^{k-1}}^{t^k} \nabla\phi dt) = 0,$$

for $k = 1,2,..N$. Define

$$(2.47)\qquad \overline{U}_N(t) = u^k,$$

for $\quad t^{k-1} = (k-1)\Delta t \le t < t^k = k\Delta t, k = 1,2...N$. Then $\overline{U}_N|_{\partial\Omega} = C_D$ and

$$(2.48)\qquad \int_0^T\int_\Omega \partial_t U_N \phi dxdt - \int_0^T\int_\Omega \mathbf{F}(\overline{U}_N,x,t)\cdot\nabla\phi dxdt + \int_0^T\int_\Omega H_\delta(\overline{U}_N)\nabla\overline{U}_N\cdot\nabla\phi dxdt + \tau\int_0^T\int_\Omega H_\delta(\overline{U}_N)\nabla\partial_t U_N\cdot\nabla\phi dxdt = 0.$$

We now exploit a general principle that relates the piecewise linear and the piecewise constant interpolation (see e.g. [26] for a proof of the corresponding lemma): if one interpolation converges strongly in $L^2(Q)$, then the other interpolation also converges strongly in $L^2(Q)$. From the convergence of U_N, we conclude that $\overline{U}_N$ also converges strongly in $L^2(Q)$. Then we obtain $F(\overline{U}_N) \to F(U)$ strongly in $(L^2(Q))^d$ and $H_\delta(\overline{U}_N) \to H_\delta(U)$ strongly in $L^2(Q)$. Employing the same idea as in the proof of Lemma 2.1, we have

$$(2.49)\qquad H_\delta(\overline{U}_N)\nabla\overline{U}_N \rightharpoonup H_\delta(U)\nabla U \quad \text{weakly in } (L^2(Q))^d,$$

$$(2.50)\qquad H_\delta(\overline{U}_N)\nabla\partial_t U_N \rightharpoonup H_\delta(U)\nabla\partial_t U \quad \text{weakly in } (L^2(Q))^d.$$

Combining the latter results with (2.48), we obtain that U is a solution to **Problem $\mathbf{P}_\delta$**. □

3 Existence for Problem P

For any $\delta > 0$, Section 2 provides a solution u_δ to the regularized **Problem $\mathbf{P}_\delta$**. In this section, we identify a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ tending to 0, providing the limit u of the sequence $\{u_{\delta_n}\}_{n\in\mathbb{N}}$, which solves **Problem P**. This involves compactness argument, and therefore convergence should always be understood along a subsequence. From Assumption (A.2), Proposition 2.3 and Theorem 2.1, we have the following

Proposition 3.1 *We have the following estimates:*

$$(3.1)\qquad ||u_\delta||_{L^2(Q)} \le C,$$

$$(3.2)\qquad ||\partial_t u_\delta||_{L^2(Q)} \le C,$$

$$(3.3)\qquad ||\sqrt{H_\delta(u_\delta)}\nabla\partial_t u_\delta||_{L^2(0,T;(L^2(\Omega))^d)} \le C,$$

$$(3.4)\qquad ||\nabla u_\delta||_{L^\infty(0,T;(L^2(\Omega))^d)} \le C,$$

$$(3.5)\qquad \int_\Omega \Gamma_\delta(u_\delta(t))dx \le C, \quad \textit{for a.e.} \quad t > 0,$$

where C does not depend on δ.

By Proposition 3.1, there exists a $u \in H^1(Q)$ such that,

$$u_{\delta_n} \to u \quad \text{strongly in } L^2(Q), \quad \text{and a.e. on Q,} \tag{3.6}$$

$$\partial_t u_{\delta_n} \rightharpoonup \partial_t u \quad \text{weakly in } L^2(Q), \tag{3.7}$$

as well as

$$\nabla u_{\delta_n} \rightharpoonup \nabla u \quad \text{in } L^\infty(0,T;(L^2(\Omega))^d) \quad \text{in the weak-}\star \text{ sense.} \tag{3.8}$$

Further, from (3.3) there exists a $\zeta = (\zeta_1, ..., \zeta_d) \in \left(L^2(Q)\right)^d$ such that,

$$\sqrt{H_{\delta_n}(u_{\delta_n})}\partial_t \nabla u_{\delta_n} \rightharpoonup \zeta \quad \text{weakly in } \left(L^2(Q)\right)^d. \tag{3.9}$$

Let $\psi \in C_0^\infty(Q)$, then for all n, u_{δ_n} satisfies

$$A_n + B_n + C_n + D_n = 0, \tag{3.10}$$

where

$$\begin{aligned}
A_n &= \iint_Q \partial_t u_{\delta_n} \psi dxdt, \\
B_n &= -\iint_Q \mathbf{F}(u_{\delta_n}, x, t) \cdot \nabla \psi dxdt, \\
C_n &= \iint_Q H_{\delta_n}(u_{\delta_n}) \nabla u_{\delta_n} \cdot \nabla \psi dxdt, \\
D_n &= \iint_Q H_{\delta_n}(u_{\delta_n}) \partial_t \nabla u_{\delta_n} \cdot \nabla \psi dxdt.
\end{aligned}$$

In view of the above, A_n, B_n and C_n converge to the desired limit as $n \to \infty$. We thus focus on the limit of D_n. To this end, let $j \in \{1, \dots, d\}$ be fixed and decompose the variable $x \in \mathbb{R}^d$ into $(x_j, \tilde{x}_j) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Define

$$\Omega_j(\tilde{x}_j) := \{x_j \in \mathbb{R} \mid (x_j, \tilde{x}_j) \in \Omega\}, \quad \text{and} \quad Q_j(\tilde{x}_j) := \Omega_j(\tilde{x}_j) \times (0,T).$$

We note that

$$D_n = \sum_{j=1}^{d} \int_{\mathbb{R}^{d-1}} D_{j,n}(\tilde{x}_j) d\tilde{x}_j, \tag{3.11}$$

where, for a.e. $\tilde{x}_j \in \mathbb{R}^{d-1}$,

$$D_{j,n}(\tilde{x}_j) = \iint_{Q_j(\tilde{x}_j)} H_{\delta_n}(u_{\delta_n}) \partial_t \partial_{x_j} u_{\delta_n} \partial_{x_j} \psi dx_j dt.$$

We have the following

Lemma 3.1 *For almost every* $\tilde{x}_j \in \mathbb{R}^{d-1}$,

$$\lim_{n\to\infty} D_{j,n}(\tilde{x}_j) = D_j(\tilde{x}_j) := \iint_{Q_j(\tilde{x}_j)} H(u)\partial_t\partial_{x_j}u\partial_{x_j}\psi dx_j dt.$$

Proof We deduce from Proposition 3.1 that, for almost every $\tilde{x}_j$,

$$\|\partial_{x_j}u_{\delta_n}(\cdot,\tilde{x}_j)\|_{L^2(Q_j(\tilde{x}_j))} \le C(\tilde{x}_j), \tag{3.12}$$

$$\|\sqrt{H(u_{\delta_n}(\cdot,\tilde{x}_j))}\partial_t\partial_{x_j}u_{\delta_n}(\cdot,\tilde{x}_j)\|_{L^2(Q_j(\tilde{x}_j))} \le C(\tilde{x}_j), \tag{3.13}$$

$$\|\partial_t u_{\delta_n}(\cdot,\tilde{x}_j)\|_{L^2(Q_j(\tilde{x}_j))} \le C(\tilde{x}_j), \tag{3.14}$$

where $C(\tilde{x}_j) \in L^2(\mathbb{R}^{d-1})$. From (3.13) and in view of (3.9), we deduce

$$\sqrt{H(u_{\delta_n}(\cdot,\tilde{x}_j))}\partial_t\partial_{x_j}u_{\delta_n}(\cdot,\tilde{x}_j) \rightharpoonup \zeta_j(\tilde{x}_j) \quad \text{weakly in } L^2(Q_j(\tilde{x}_j)). \tag{3.15}$$

We define an auxiliary C^2 function $A : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{gathered}\frac{A}{\sqrt{H}} \in L^\infty(0,1), \quad \frac{A'}{\sqrt{H}} \in L^\infty(0,1), \quad A'' \in L^\infty(0,1), \\ \text{and} \quad A(s) > 0 \text{ if } s \in (0,1).\end{gathered} \tag{3.16}$$

For instance, if $H(u) \sim u^{p+1}$ in the vicinity of 0 (as in encountered e.g. in (1.4)), one can consider $A(u) \sim u^{max(1,(p+3)/2)}$. The construction in the vicinity of 1 is similar. Note that (3.16) implies that $A(\cdot)$ is 0 outside $(0,1)$. Furthermore, the fractions in (3.16) are extended by 0 outside $(0,1)$.

Define the differential operator $\tilde{\nabla} := (\partial_{x_j}, \partial_t)^T$, and, for fixed $\tilde{x}_j$ in a full measure subset of $\mathbb{R}^{d-1}$, the two vector-valued functions

$$V_n(\tilde{x}_j) = (A'(u_{\delta_n}(\cdot,\tilde{x}_j))\partial_t u_{\delta_n}(\cdot,\tilde{x}_j), 0), \qquad W_n(\tilde{x}_j) = (\partial_{x_j}u_{\delta_n}(\cdot,\tilde{x}_j), \partial_t u_{\delta_n}(\cdot,\tilde{x}_j)). \tag{3.17}$$

For reader's convenience, we remove the parameter $\tilde{x}_j$ in the sequel. By (3.12)–(3.14) and the properties of A, we obtain that V_n and W_n are uniformly bounded in $(L^2(Q_j))^2$. Since $\tilde{\nabla} \times W_n = \tilde{\nabla} \times (\tilde{\nabla} u_{\delta_n}) = 0$, so $\{\tilde{\nabla} \times W_n, n \in \mathbb{N}\}$ is a compact subset of $W^{-1,2}(Q_j)$. Moreover, the sequence $\{\nabla \cdot V_n, n \in \mathbb{N}\}$ is uniformly bounded in $L^2(0,T;L^1(\Omega_j))$, as

$$\partial_{x_j}(A'(u_{\delta_n})\partial_t u_{\delta_n}) = A''(u_{\delta_n})\partial_t u_{\delta_n}\partial_{x_j}u_{\delta_n} + \frac{A'(u_{\delta_n})}{\sqrt{H(u_{\delta_n})}}\sqrt{H(u_{\delta_n})}\partial_t\partial_{x_j}u_{\delta_n}, \tag{3.18}$$

a.e. in $\omega_j(\tilde{x}_j) = \{(x_j,t) \in Q_j \mid \ u(x_j,t,\tilde{x}_j) \in (0,1)\}$ and in fact in the entire Q_j in view of the extension of the fractions in (3.16). The embedding $L^2(0,T;L^1(\Omega_j)) \hookrightarrow W^{-1,2}(Q_j)$ being compact (note that $\Omega_j \subset \mathbb{R}$), then, applying the *div-curl lemma* [30, 38], we get

$$V_n \cdot W_n = A'(u_{\delta_n})\partial_t u_{\delta_n}\partial_{x_j}u_{\delta_n} \rightharpoonup A'(u)\partial_t u\partial_{x_j}u \quad \text{weakly in} \quad \mathcal{D}'(Q_j). \tag{3.19}$$

Finally, let $\mathcal{A}$ be a primitive form of A. As before, the equality

$$\partial_t\partial_{x_j}\mathcal{A}(u_{\delta_n}) = A'(u_{\delta_n})\partial_t u_{\delta_n}\partial_{x_j}u_{\delta_n} + \frac{A(u_{\delta_n})}{\sqrt{H(u_{\delta_n})}}\sqrt{H(u_{\delta_n})}\partial_t\partial_{x_j}u_{\delta_n}, \tag{3.20}$$

holding a.e. in ω_j can be extended to Q_j. Since $\frac{A(u_{\delta_n})}{\sqrt{H(u_{\delta_n})}}$ converges a.e. in Q_j to $\frac{A(u)}{\sqrt{H(u)}}$ and is essentially bounded uniformly w.r.t. n, we obtain the strong convergence in $L^2(Q_j)$. Together with the weak convergence in (3.9), we pass to the limit ($n \to \infty$) in (3.20) and obtain

$$\partial_t \partial_{x_j} \mathcal{A}(u) = A'(u)\partial_t u \partial_{x_j} u + \frac{A(u)}{\sqrt{H(u)}}\zeta_j. \tag{3.21}$$

In the distributional sense, this implies

$$A'(u)\partial_t u \partial_{x_j} u + A(u)\partial_t \partial_{x_j} u = A'(u)\partial_t u \partial_{x_j} u + \frac{A(u)}{\sqrt{H(u)}}\zeta_j. \tag{3.22}$$

As a consequence, for almost every $\tilde{x}_j \in \mathbb{R}^{d-1}$,

$$\zeta_j(\tilde{x}_j) = \sqrt{H(u(\cdot, \tilde{x}_j))}\partial_t \partial_{x_j} u(\cdot, \tilde{x}_j). \tag{3.23}$$

Because of (3.15) and the strong $L^2(Q_j)$ convergence of $\sqrt{H_{\delta_n}(u_\delta(\cdot, \tilde{x}_j))}$ to $\sqrt{H(u(\cdot, \tilde{x}_j))}$, one has for almost every $\tilde{x}_j$ in $\mathbb{R}^{d-1}$,

$$\lim_{n\to\infty} D_{j,n}(\tilde{x}_j) = \iint_{Q_j(\tilde{x}_j)} \sqrt{H(u(\cdot, \tilde{x}_j))}\zeta_j(\tilde{x}_j)\partial_{x_j}\psi dx_j dt = D_j(\tilde{x}_j).$$

Proposition 3.2 *Let u be the limit in* (3.6)–(3.8). *Then, for all $\psi \in C_0^\infty(Q)$,*

$$\lim_{n\to\infty} \iint_Q H_{\delta_n}(u_{\delta_n})\partial_t \nabla u_{\delta_n} \cdot \nabla\psi dx dt = \iint_Q H(u)\partial_t \nabla u \cdot \nabla \psi dx dt. \tag{3.24}$$

Proof Note that, thanks to (3.11), for proving Proposition 3.2, it is sufficient to show that, for any $j \in \{1, \dots, d\}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d-1}} D_{j,n}(\tilde{x}_j) d\tilde{x}_j = \int_{\mathbb{R}^{d-1}} D_j(\tilde{x}_j) d\tilde{x}_j.$$

Since Ω is bounded, the functions $D_{j,n}$ are compactly supported. Further, the Cauchy-Schwarz inequality gives

$$(D_{j,n}(\tilde{x}_j))^2 \le C \iint_{Q_j(\tilde{x}_j)} H_{\delta_n}(u_{\delta_n}) \left(\partial_t \partial_{x_j} u_{\delta_n}\right)^2 dx_j dt,$$

and therefore

$$\int_{\mathbb{R}^{d-1}} (D_{j,n}(\tilde{x}_j))^2 d\tilde{x}_j \le C \iint_Q H_{\delta_n}(u_{\delta_n}) \left(\partial_t \partial_{x_j} u_{\delta_n}\right)^2 dx dt.$$

By (3.3), $D_{j,n}$ is uniformly bounded in $L^2(\mathbb{R}^{d-1})$. Hence the sequence $\{D_{j,n}\}_n$ is equi-integrable. Now (3.24) follows by Lemma 3.1 and Vitali's theorem.

Theorem 3.1 Problem P *has a solution u. Furthermore, this solution is essentially bounded by 0 and 1 in Q.*

Proof Let u be the limit in (3.6)–(3.8). To show that u is a weak solution of **Problem P**, it is sufficient to show that

$$\lim_{n\to\infty} A_n = \iint_Q \partial_t u\psi dxdt, \tag{3.25}$$

$$\lim_{n\to\infty} B_n = -\iint_Q \mathbf{F}(u,x,t)\cdot\nabla\psi dxdt, \tag{3.26}$$

$$\lim_{n\to\infty} C_n = \iint_Q H(u)\nabla u\cdot\nabla\psi dxdt, \tag{3.27}$$

$$\lim_{n\to\infty} D_n = \iint_Q H(u)\partial_t\nabla u\cdot\nabla\psi dxdt. \tag{3.28}$$

While (3.28) has been established in Proposition 3.2, the limit identification (3.25)–(3.27) follows straightforwardly from (3.6)–(3.8) and the strong L^2 convergence of $H_{\delta_n}(u_{\delta_n})$ to $H(u)$.

It remains to prove that $0 \le u \le 1$ a.e. in Q. To this end we consider $\epsilon > 0$ arbitrary, take $t \in (0,T)$, and define $\Omega^-_{\epsilon,n}(t) := \{x \in \Omega \mid u_{\delta_n}(x,t) < -\epsilon\}$. Then

$$\Gamma_{\delta_n}(u_{\delta_n}) = \int_{C_D}^{u_{\delta_n}}\int_{C_D}^{w} \frac{1}{H_{\delta_n}(v)} dvdw = \frac{(C_D - u_{\delta_n})^2}{2\delta_n}, \tag{3.29}$$

a.e. in $\Omega^-_{\epsilon,n}(t)$. Recalling (3.5), for all $\delta_n > 0$ and a.e. t, we write

$$C \ge \int_\Omega \Gamma_{\delta_n}(u_{\delta_n}(x,t))dx \ge \int_{\Omega^-_\epsilon(t)} \Gamma_{\delta_n}(u_{\delta_n}(x,t))dx = \frac{(C_D+\epsilon)^2}{2\delta_n}\ \mathrm{meas}(\Omega^-_{\epsilon,n}(t)). \tag{3.30}$$

Letting $\delta_n \to 0$, we obtain

$$\lim_{n\to\infty} \mathrm{meas}(\Omega^-_{\epsilon,n}(t)) = 0, \tag{3.31}$$

for a.e. $t \in (0,T]$. However, by (3.13) and (3.14), $u_{\delta_n} \to u$ in $C([0,T];L^2(\Omega))$, thus $u_{\delta_n}(\cdot,t) \to u(\cdot,t)$ a.e in Ω, for all t. Passing to the limit $\epsilon \to 0$ gives the lower bound for u. Similarly, we have $u \le 1$ a.e., and the theorem is proved. □

4 Conclusion

We consider a degenerate pseudo-parabolic equation modeling two-phase flow in porous media, which includes dynamic effects in the capillary pressure. We prove the existence of weak solutions. The major difficulty is due to the degeneracy in the higher order term, a mixed (space-time) derivative of third order. To overcome this we employ regularization techniques, and prove the existence for the regularized problem, as well as a-priori estimates that are uniform w.r.t. the regularization parameter. Then we use compactness arguments to show the existence of a solution to the original problem. For identifying the limit of the third order term we combine compensated compactness and equi-integrability arguments.

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