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Rational representations and controller synthesis of $L_2$ behaviors

Mark Mutsaers $^1$, Siep Weiland

Eindhoven University of Technology, Department of Electrical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

**Abstract**

This paper considers linear dynamical systems restricted to square integrable trajectories. Following the behavioral formalism, a number of relevant classes of linear and shift-invariant $L_2$ systems are defined. It is shown that rational functions, analytic in specific half-spaces of the complex plane, prove most useful for representing such systems. For various classes of $L_2$ systems, this paper provides a complete characterization of system equivalence in terms of rational kernel representations of $L_2$ systems. In addition, a complete solution is given for the problem when selected (non-manifest) variables of an $L_2$ system can be completely eliminated from their behavior. This elimination theorem has considerable independent interest in general modeling problems. It is shown that the elimination result is key in the solution of the problem for realizing an arbitrary $L_2$ system as the interconnection of a given $L_2$ system and a to-be-synthesized $L_2$ system. In the context of control, this problem amounts to characterizing the existence and parameterization of all controllers that, after interconnection with a given plant, constitute a desired controlled system.

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1. Introduction

For years, the behavioral theory of dynamical systems has been advocated as a natural vantage point from which to address general questions on modeling, identification, model equivalence and control. Within this theory, quite some research effort has been devoted to studying equations of the form

$$d \left( \frac{d}{dr} \right) w = 0,$$

which represents a system of differential equations in the signal $w$ and where $P(\xi)$ is a polynomial in the indeterminate $\xi$, with real matrix-valued coefficients. Here, (1) is a compact notation for the general class of systems that can be represented by any finite number of linear, ordinary, constant coefficient differential equations in, say, $w$ variables that evolve over time. The interest in models of this type stems from the fact that many first-principle modeling exercises naturally lead to systems of ordinary differential equations with real coefficients. Eq. (1) is called a kernel representation of a system and its associated behavior is the set of sufficiently often differentiable functions $w : T \to \mathbb{R}^n$ (in $w$ variables and defined on some time set $T \subset \mathbb{R}$) that satisfy (1). If differentiation in (1) is not understood in a generalized sense of distributions, then there is a technical difficulty about the function space in which solutions $w$ of (1) are assumed to reside. Since many relevant linear, shift-invariant function spaces are dense in the space $C^\infty$ of infinitely differentiable functions, the restriction to this signal space resolves this complication and is the reason for interpreting the solution set of (1) in this sense.

It is the purpose of this paper to investigate model classes in which solutions of (1) belong to the Lebesgue space of square integrable functions on the time set $T = \mathbb{R}_+, T = \mathbb{R},$ or $T = \mathbb{R}$. The reason for investigating these model classes lies in the importance of square integrable trajectories in many control questions where performance and stability requirements are specified in terms of square integrable trajectories only. In addition, the study of solutions of (1) restricted to specific Hilbert spaces leads to important questions on system representation and system equivalence.

Although this work is inspired by the study of $L_2$ systems defined on different time sets, we heavily exploit the fact that the space of square Lebesgue integrable functions on $T \subset \mathbb{R}$ is isomorphic to complex valued Hilbert or Hardy spaces via the (unilateral or bilateral) Laplace transform. Hilbert spaces of complex valued functions $w : \mathbb{C} \to \mathbb{C}^m$ that are square integrable on the imaginary axis (possibly with different domains of analyticity) are closed under multiplication with rational functions $P(s)$ (also with different domains of analyticity). This observation
naturally leads one to investigate representations of the form
\[ P(s)w(s) = 0, \]
where \( w(s) \) is the Laplace transform of a solution of (1) and where
\( P(s) \) is a real rational function (i.e., every entry of \( P \) is a quotient of polynomials with real coefficients) in \( s \in \mathbb{C} \). Clearly, solutions of (1) with compact support satisfy (2) on taking Laplace transforms. Here, the system associated with (2) with \( P \) being real rational will be the collection of all \( w \in \mathcal{L}_2 \) that satisfy (2). This functional analytic interpretation of (2) proves useful for solving questions on synthesis, representation, normalization, elimination and interconnection of \( \mathcal{L}_2 \) systems. These questions will be addressed in this paper.

Models inferred from first principles generally lead to higher order differential equations and one may therefore argue that rational kernel operators of the form (2) are less interesting from a general modeling point of view. This is true. However, the functional analytic tools for rational model representations allow for possibilities such as scaling, normalization, projection and approximation that cannot be paralleled by polynomial methods. It is for this reason that a thorough understanding of system representations by rational operators prove a useful alternative to (polynomial) differential operators. Earlier investigations in, e.g., Trentelman (2010), Trentelman, Yoe, and Praagman (2007) and Willems and Yamamoto (2007) have studied interpretations of (1) with rational functions \( P \). In these papers, solutions of (1) with rational \( P \) are defined by all infinitely often differentiable functions \( w \) that satisfy the polynomial differential equation \( \frac{d}{dt}w = 0 \), where \( N \) is a (or any) factor in the left-coprime factorization \( P = D^{-1}N \) over the ring of polynomials. In this paper we take a different point of view. First, we do not consider \( C^\infty \) signals with time as the independent variable, but rather work with the Hardy spaces \( \mathcal{H}_q^+ \), \( \mathcal{H}_q^- \) or the Hilbert space \( \mathcal{L}_2 \) as signal spaces of interest. Second, we exploit the inner product structure on the signal space to infer a rich theory on rational representations of dynamical systems. This paper extends a number of results that were obtained in Weiland and Stoorvogel (1997) for a class of discrete \( \mathcal{L}_2 \) systems to continuous time systems.

The outline of the paper is as follows. First, questions of system equivalence, elimination and synthesis that will be discussed in this paper are introduced in Section 2. Section 3 introduces notation. Section 4 deals with rational representations of \( \mathcal{L}_2 \) systems. Three classes of \( \mathcal{L}_2 \) systems are introduced and we present for each model class complete results on system equivalence and for the elimination of latent variables. The roles of \( \mathcal{L}_2 \) behaviors in interconnected systems and specific controller synthesis problems are discussed in Section 5. Conclusions are presented in Section 6. All proofs are collected in the Appendix.

2. Problem formulation

Following the behavioral formalism, a dynamical system (Belur, 2003; Willems, 1989, 2007) is a triple
\[ \Sigma = (T, \mathcal{W}, \mathcal{B}), \]
where \( T \subseteq \mathbb{R} \) or \( T \subseteq \mathbb{C} \) is the time or frequency axis, \( \mathcal{W} \) is the signal space, which will be a \( w \)-dimensional vector space throughout, and \( \mathcal{B} \subseteq \mathcal{W}^T \) is the behavior, that is defined in more explicit terms in Section 4.

We consider \( \mathcal{L}_2 \) behaviors, which are closed, shift-invariant subspaces of \( \mathcal{L}_2 \). This means that, contrary to the usual behavioral models, ours does not consider function classes with time as the independent variable but uses frequency, i.e., \( T = \mathbb{C} \). More particularly, we distinguish between closed, invariant subspaces of \( \mathcal{L}_2 \) that contain the open right complex half-plane, the open left complex half-plane or the imaginary axis in their domain.
Let \( P \) be a subset of \( \mathcal{H}_r^\alpha \) as follows. \( \alpha > r \) rational functions with possible poles at infinity.

The norm of a complex valued matrix \( F \in \mathbb{C}^{n \times k} \) is defined as \( \|F\| := \sqrt{\lambda_{\max}(F^*F)} \), which is the largest singular value of \( F \). A complex valued matrix \( F(j\omega) \) belongs to the Lebesgue space \( L_{\infty} \) if its norm \( \|F(j\omega)\| \) is essentially bounded for all frequencies \( \omega \in \mathbb{R} \).

The norm \( \|F\|_{L_{\infty}} := \text{ess sup}_{\omega \in \mathbb{R}} \|F(j\omega)\| \) is analytic on \( \mathbb{C} \), and \( F(j\omega) \in \mathcal{H}_r^\alpha \) if \( \|F\| := \sup_{\omega \in \mathbb{R}} \text{ess sup}_{\sigma + j\omega \in \mathbb{C}} |F(\sigma + j\omega)| < \infty \). The space \( \mathcal{H}_r^\alpha \) is defined similarly.

The prefixes \( RH \) and \( UL \) denote rational matrices and units in the Hardy spaces \( \mathcal{H}_r^\alpha \) and \( \mathcal{H}_s^\alpha \), as \( \mathcal{R}H := \{F \in \mathcal{H}_r^\alpha : F \text{ is rational} \} \) and \( \mathcal{U}L := \{U \in \mathcal{H}_r^\alpha : U^{-1} \in \mathcal{R}H_{\alpha} \} \). Note that units are necessarily square rational matrices. Elements of \( \mathcal{R}H_{\alpha} \) and \( \mathcal{R}H_{\alpha}^{-1} \) will be referred to as stable and anti-stable functions, respectively. See Francis (1987) and Vidyasagar (1985) for more details about Hardy spaces.

The ring \( \mathcal{R}H_{\alpha} \) admits an extension that consists of stable rational functions with possible poles at infinity:

\[
\mathcal{R}H_{\alpha,\infty}^+: = \left\{ f | \exists k \geq 0, \exists \alpha < 0 \text{ s.t. } \left( \frac{1}{(s - \alpha)^k} f(s) \right) \in \mathcal{R}H_{\alpha} \right\}.
\]  

Matrix-valued functions in \( \mathcal{R}H_{\alpha,\infty}^+ \), are understood as matrices whose elements satisfy the right-hand side of (4) with \( f : C_+ \to C \).

Similarly, we can define the extension \( \mathcal{R}H_{\alpha,\infty}^+ \) (resp., \( \mathcal{R}L_{\alpha,\infty}^+ \)) as the space of complex valued functions \( f \) for which there exist \( k \geq 0 \) and \( \alpha > 0 \) such that \( \frac{1}{(s - \alpha)^k} f(s) \in \mathcal{R}H_{\alpha} \) (resp., \( \frac{1}{(s - \alpha)^k} f(s) \in \mathcal{R}L_{\alpha} \)) for some \( k \geq 0 \) and \( \alpha \neq 0 \). These extended spaces are characterized as follows.

**Lemma 3.2.** Let \( P \in \mathcal{R}H_{\alpha,\infty}^+ \) and \( \hat{P} \in \mathcal{R}H_{\alpha,\infty}^+ \) be defined multiplicative operators. \( (Pw)(s) = P(s)w(s) \) and \( (\hat{P}w)(s) = \hat{P}(s)w(s) \), with possible domains \( L_2, \mathcal{H}_2^\alpha \) and \( \mathcal{H}_2^\alpha \).

Then \( P : L_2 \to L_2, \quad \hat{P} : L_2 \to L_2 \), \( \hat{P} : L_2 \to \mathcal{H}_2^\alpha \), \( \hat{P} : L_2 \to \mathcal{H}_2^\alpha \), \( \hat{P} : L_2 \to \mathcal{H}_2^\alpha \), \( \hat{P} : L_2 \to \mathcal{H}_2^\alpha \).

The kernel (or null space) of a rational multiplication operator \( P \) defined on \( L_2, \mathcal{H}_2^\alpha \) or \( \mathcal{H}_2^\alpha \) is denoted by ker \( P \), ker \( P \), and ker \( P \), respectively. Thus, ker \( P = \{w \in L_2 : \|Pw\| = 0\} \).

Let \( P \in \mathcal{R}H_{\alpha,\infty}^+ \) and consider the corresponding multiplication operators as in Lemma 3.2. P is called \( L_2, \mathcal{H}_2^\alpha \) or \( \mathcal{H}_2^\alpha \) inner if \( \|Pw\|_2 = \|w\|_2 \) for all \( w \in L_2, \mathcal{H}_2^\alpha \), or \( \mathcal{H}_2^\alpha \), respectively. We call \( P \) co-inner if its Hermitian transpose is inner. A matrix \( P \in \mathcal{R}H_{\alpha,\infty}^+ \) (or \( P \in \mathcal{R}H_{\alpha,\infty}^\alpha \)) is called outer if for every \( \lambda \in \mathbb{C}_- \) \( \lambda P(\lambda) \) has full row rank. If \( P \) is outer, then \( P \) has a right inverse which is analytic in \( \mathbb{C}_- \). It is easily seen that all elements in \( \mathcal{U}L_{\alpha,\infty}^+ \) and \( \mathcal{U}L_{\alpha,\infty}^- \) are outer. Outer functions are necessarily square or wide while inner functions are square or tall. Similar definitions apply to \( \mathcal{R}H_{\alpha,\infty}^+ \). For further properties of inner and outer functions, we refer the reader to Francis (1987), Kailath (1980) and Vidyasagar (1985).

The \( \tau \)-shift operator \( \sigma, \omega \) on a signal \( \hat{w} : \mathbb{R} \to \mathbb{R} \) is defined as

\[
(\hat{\sigma}, \omega)(t) = \hat{w}(t - \tau).
\]  

We call \( \sigma, \omega \) a right (left) shift whenever \( \tau > 0 \) (\( \tau < 0 \)). Let \( \mathcal{L}, \mathcal{L}_2, \mathcal{L}_2 \) denote the usual bilateral and unilateral Laplace transforms defined on square integrable functions on \( \mathbb{R}, \mathbb{R}_+, \mathbb{R}_- \), respectively. We will be interested in operators \( \sigma, \omega : \mathcal{L}_2 \to \mathcal{L}_2, \sigma_t^\alpha : \mathcal{H}_2^\alpha \to \mathcal{H}_2^\alpha \) and \( \sigma_t^- : \mathcal{H}_2^\alpha \to \mathcal{H}_2^\alpha \), with \( \tau \in \mathbb{R} \), that commute with the Laplace transform according to \( \mathcal{L}_2 = \sigma, \mathcal{L}_2 = \sigma_t^\alpha \), and \( \mathcal{L}_2 = \sigma_t^- \). These operators are defined by setting:

\[
(\sigma, \omega)(s) = e^{-\tau t} w(s), \quad [\tau > 0],
\]

\[
(\sigma_t^\alpha, \omega)(s) = \left\{ \begin{array}{ll}
\frac{1}{\tau - s} w(s) - \int_0^\tau \hat{w}(t)e^{-\tau t} dt, & [\tau > 0] \\
\frac{1}{\tau + s} w(s) - \int_{-\tau}^0 \hat{w}(t)e^{\tau t} dt, & [\tau < 0]
\end{array} \right.
\]

\[
(\sigma_t^-, \omega)(s) = \left\{ \begin{array}{ll}
e^{-\tau t} w(s), & [\tau > 0] \\
e^{\tau t} w(s), & [\tau < 0]
\end{array} \right.
\]

Here, \( \hat{w} := \mathcal{L}_2^\tau w \) for \( w \in \mathcal{H}_2^\alpha \) and \( \hat{w} := \mathcal{L}_2^{-\tau} w \) for \( w \in \mathcal{H}_2^\alpha \). Obviously, \( \sigma_0 \) is the identity map. Note that \( \sigma, \omega : \mathcal{L}_2 \to \mathcal{L}_2 \) defines an isometry (for all \( \tau < 0 \)) and that \( \sigma_t^\alpha : \mathcal{H}_2^\alpha \to \mathcal{H}_2^\alpha \) and \( \sigma_t^- : \mathcal{H}_2^\alpha \to \mathcal{H}_2^\alpha \) define isometries only if \( \tau \geq 0 \) and \( \tau < 0 \), respectively. We will drop the superscript \( + \) and \( - \) in \( \sigma_t^\alpha, \sigma_t^- \) whenever the domain of the operators is clear from the context.

**Definition 3.3.** A subset \( \mathcal{B} \) of \( \mathcal{L}_2 \) (or \( \mathcal{H}_2^\alpha \) or \( \mathcal{H}_2^\alpha \)) is said to be left invariant if \( \sigma, \mathcal{B} \subseteq \mathcal{B} \) for all \( \tau < 0 \). It is said to be right invariant if \( \sigma, \mathcal{B} \subseteq \mathcal{B} \) for all \( \tau > 0 \).
4. Equivalence and elimination for rational representations

In this section, behaviors of dynamical systems are defined as closed subspaces of \( L_2, H_2^+ \) and \( H_2^- \) represented by the null spaces of rational operators (Mutsaers & Weiland, 2008). Behavioral inclusion, equivalence and elimination of variables will be discussed in terms of rational operators. The results will be compared with earlier research on infinitely smooth behaviors represented by rational differential operators (Trentelman, 2010; Willems & Yamamoto, 2006, 2008). Throughout this section, we will use the variables \( w \) and \( \ell \), which are elements of \( L_2, H_2^+ \) or \( H_2^- \).

4.1. Anti-stable rational operators

Let \( P \in \mathcal{RH}_\infty \) be a rational operator with \( w \) columns. We associate three dynamical systems with \( P \) by setting

\[
\Sigma := (C, C^w, B), \\
\Sigma_+ := (C_+, C^w, B_+), \\
\Sigma_- := (C_-, C^w, B_-)
\]

where

\[
B := \{ w \in L_2 \mid \| w \| = 0 \} = \ker P, \\
B_+ := \{ w \in H_2^+ \mid \| w \|_{H_2^+} = 1 \} = \ker \Pi_+, P, \\
B_- := \{ w \in H_2^- \mid \| w \| = 0 \} = \ker P.
\]

Here \( \Pi_+ \) denotes the canonical projection \( \Pi_+ : L_2 \to H_2^+ \). The subsets \( B \subset L_2, B_+ \subset H_2^+, B_- \subset H_2^- \) define behaviors of dynamical systems \( \Sigma, \Sigma_+ \), respectively, in the frequency domain, i.e., as subsets of complex valued functions. We refer to \( P \) as a rational kernel representation of these systems. The corresponding time domain models of (6a) are inferred via the inverse Laplace transform according to

\[
\Sigma := (\mathbb{R}, \mathbb{R}^w, L^{-1}B), \quad \Sigma_+ := (\mathbb{R}, \mathbb{R}^w, L^{-1}B_+) \quad \text{and} \quad \Sigma_- := (\mathbb{R}, \mathbb{R}^w, L^{-1}B_-).
\]

**Lemma 4.1.** For \( P \in \mathcal{RH}_\infty \) the behaviors \( B, B_+ \) and \( B_- \) in (6b) are closed left invariant subspaces of \( L_2, H_2^+ \) and \( H_2^- \), respectively.

The proof of this lemma can be found in the Appendix. Systems of the form (6) will generally be referred to as left invariant \( L_2 \) systems.

**Definition 4.2.** The classes of all linear and left invariant systems in \( L_2, H_2^+ \) and \( H_2^- \) that admit representations by anti-stable rational operators as in (6) are denoted by \( \mathcal{L}_-, \mathcal{L}_+ \) and \( \mathcal{L}_- \), respectively.

We call a rational kernel representation \( P \) minimal if any other rational kernel representation of the system has at least as many \( w \) as \( P \) has full row rank. For a dynamical system \( \Sigma \) in the class \( \mathcal{L}_- \), the output cardinality of its behavior \( B \) is defined as \( p(B) = \text{rowrank}(P), \) where \( P \in \mathcal{RH}_\infty \) represents \( B \) as in (6b). The output cardinality therefore reflects the number of independent restrictions that are imposed on the system. It is easily shown that \( p(B) \) is, in fact, independent of the representation \( P \) and that \( p(B) \) can be interpreted as the dimension of the output variable in one (or any) input-output representation of \( \Sigma \). Similarly, the input cardinality of \( B \) is the number \( m(B) = w - p(B), \) which represents the degree of under-determination of the restrictions that the system imposes on its \( w \) variables. For systems in the model classes \( \mathcal{L}_+ \) and \( \mathcal{L}_- \), the input and output cardinality are defined in a similar manner.

A complete characterization of inclusions and equivalence of systems in the model classes \( \mathcal{L}_-, \mathcal{L}_+ \) and \( \mathcal{L}_- \) is given in the following result.

**Theorem 4.3 (Inclusion and Equivalence).** Let two systems in the class \( \mathcal{L}_- \) (or \( \mathcal{L}_+ \) or \( \mathcal{L}_- \)) with behaviors \( B_1, B_2 \) (or \( B_1, B_2 \) or \( B_1, B_2 \)) be represented by full rank \( P, Q \) in \( \mathcal{R}_\infty \) respectively, as in (6). We then have:

1. **Inclusions of behaviors:**
   i. \( B_2 \subset B_1 \iff 3F \in \mathcal{R}_\infty \) s.t. \( P = FQ \),
   ii. \( B_2 \subset B_1 \iff 3F \in \mathcal{R}_\infty \) s.t. \( P = FQ \),
   iii. \( B_2 \subset B_1 \iff 3F \in \mathcal{R}_\infty \) s.t. \( P = FQ \);

2. **Equivalence of behaviors:**
   i. \( B_1 = B_2 \iff 3U \in \mathcal{U} \), s.t. \( P = UQ \),
   ii. \( B_1 = B_2 \iff 3U \in \mathcal{U} \), s.t. \( P = UQ \),
   iii. \( B_1 = B_2 \iff 3U \in \mathcal{U} \), s.t. \( P = UQ \);

3. If, in addition, \( Q \) is co-inner, then the statements in item 1 are equivalent to the existence of \( F \in \mathcal{R}_\infty, F \in \mathcal{R}_\infty \) and \( F \in \mathcal{R}_\infty \) in i-iii respectively, such that \( P = F \); if also \( P \) is co-inner, then the statements in item 2 are equivalent to the existence of \( U \in \mathcal{U} \), \( U \in \mathcal{U} \) and \( U \in \mathcal{U} \), in i-iii respectively, such that \( P = UQ \).

**Example 4.4.** Let \( P(s) = \frac{s+1}{2} \) and \( Q(s) = \frac{1}{s} \). Then \( P, Q \in \mathcal{R}_\infty \) and \( P = FQ \) with \( F(s) = \frac{s+1+1}{s} \). Since \( F \) is analytic in \( C \) and \( F(s) = \infty \) for any \( \alpha > 0 \), it follows that \( F \in \mathcal{R}_\infty \), Statement ii of Theorem 4.3 thus promises that \( B_2 \subset \mathcal{B}_1, B_3 \subset \mathcal{B}_1 \) where \( B_1 = \ker \Pi_+, P \) and \( B_3 = \ker \Pi_+, Q \). Indeed, \( B_2 = \{ 0 \} \) and \( B_3 = \{ \frac{1}{s} \mid c \in C \} \subset \mathcal{H}_\infty \). Since \( B_2 \) is also represented by the (inner) and co-inner function \( Q(s) = 1 \), the same conclusion follows from statement 3 of Theorem 4.3 as \( P = FQ \) with \( F(s) = \frac{s+1}{s} \), which belongs to \( \mathcal{R}_\infty \).

**Example 4.5.** Let \( P(s) = \left[ \begin{array}{cc} 1 & -T(s) \end{array} \right] \) with \( T(s) = \frac{s+1}{s} \). Then \( P \) defines a system in the model class \( L \) whose behavior is \( \ker P = L_2 \) graph associated with the transfer function \( T, \) i.e., \( B_1 = \{ w = (y, u) \in L_2 \mid y = Tu \} \). If \( T = D^{-1}N \) is a normalized left-coprime factorization of \( T \) over \( \mathcal{R}_\infty \) then \( P = UQ \) with \( Q = \left[ \begin{array}{cc} D & -N \end{array} \right] \) and \( U = D^{-1} \). Since \( U \in \mathcal{U} \), statement 2 of Theorem 4.3 claims that \( B_1 = B_2 = \ker \). Since \( QQ^* = I \), it follows that every system in \( L \) admits a co-inner kernel representation.

**Remark 4.6.** Theorem 4.3 substantially differs from the equivalence results in Gottimukkala, Fiaz, and Trentelman (2011), Trentelman (2010) and Willems and Yamamoto (2007, 2008) where \( C^\infty \) behaviors are defined as kernels of rational differential operators \( P \). In Gottimukkala et al. (2011), it is shown that the controllable parts of the \( C^\infty \) kernels of rational operators \( P \) and \( Q \) coincide if and only if there exists a unitary matrix \( U \in \mathcal{U} \), such that \( P = UQ \).

**Remark 4.7.** The explicit construction of the operators \( F \) and \( U \) in Theorem 4.3 is an application of the Beurling–Lax theorem (Rosenblum & Rovnyak, 1997). We refer the reader to the proof of Theorem 4.3 for details.

Next, we consider latent variable systems for the three model classes \( \mathcal{L}_-, \mathcal{L}_+ \) and \( \mathcal{L}_- \). Let \( \Sigma_\ell = (C, C^w \times C, B_{\text{latent}}) \in \mathcal{L}_\ell \) be a system in which variables are decomposed into a manifest variable \( w \) and a latent variable \( \ell \). Let \( \Sigma_{\ell,+} \in \mathcal{L}_+ \) and \( \Sigma_{\ell,-} \in \mathcal{L}_- \) denote latent variable systems with behaviors \( B_{\text{latent,+}} \) and \( B_{\text{latent,-}} \) with a
similar variable decomposition. This means that there exists $P = [P_1, P_2] \in \mathcal{RH}_\infty$ such that

$$B_{\text{full}} := \{ \omega, \ell \in \mathcal{L}_2 \mid (w, \ell) = 0 \} = \ker P,$$

$$B_{\text{full},+} := \{ \omega, \ell \in \mathcal{H}_2^+ \mid (w, \ell) \in \mathcal{H}_2^+ \} = \ker_+, \Pi_+ P,$$

$$B_{\text{full},-} := \{ \omega, \ell \in \mathcal{H}_2^- \mid (w, \ell) = 0 \} = \ker_-, P,$$

where $P$ is decomposed according to the variables $(w, \ell)$. Associate with (7) the manifest behaviors

$$B_{\text{manifest}} := \{ w \in \mathcal{L}_2 \mid \exists \ell \in \mathcal{L}_2 \text{ s.t. } (w, \ell) \in B_{\text{full}}, \}$$

$$B_{\text{manifest},+} := \{ w \in \mathcal{H}_2^+ \mid \exists \ell \in \mathcal{H}_2^+ \text{ s.t. } (w, \ell) \in B_{\text{full},+}, \}$$

$$B_{\text{manifest},-} := \{ w \in \mathcal{H}_2^- \mid \exists \ell \in \mathcal{H}_2^- \text{ s.t. } (w, \ell) \in B_{\text{full},-}, \}.$$

That is, the manifest behaviors consist of the projection of the full behaviors on the manifest variable $w$. From a general modeling point of view, the modeler is interested in the manifest behavior only, but the representation of this system is typically implicitly described by means of auxiliary or latent variables. We therefore address the question of when the manifest behaviors define systems in $L_L, L_+, \text{ and } L_-$, respectively, and whether one can find explicit representations for the manifest system. This is formalized as follows.

**Definition 4.8.** The full behaviors in (7) are said to be $\ell$-eliminable if there exists a $P' \in \mathcal{RH}_\infty$ such that

$$B_{\text{manifest}} := \{ w \in \mathcal{L}_2 \mid P' w = 0 \} = \ker P'$$

$$B_{\text{manifest},+} := \{ w \in \mathcal{H}_2^+ \mid P' w \in \mathcal{H}_2^+ \} = \ker_+, \Pi_+ P'$$

$$B_{\text{manifest},-} := \{ w \in \mathcal{H}_2^- \mid P' w = 0 \} = \ker_-, P'.$$

Thus, in an $\ell$-eliminable system, one can find a kernel representation for its induced manifest behavior. The following elimination theorem is the main result of this section.

**Theorem 4.9 (Elimination).** Let $P = [P_1, P_2] \in \mathcal{RH}_\infty$ be full row rank and define the full system behaviors as in (7) and consider the equation

$$Q = P_1 + P_2 X.$$

We have, with respect to (8), that

$$B_{\text{full}} \text{ is } \ell \text{-eliminable } \iff \exists X \in \mathcal{RH}_\infty \text{ s.t. } Q \in \mathcal{RH}_\infty$$

and rowrank$(Q) = p(B_{\text{full}}) - \text{rowrank}(P_2)$,

$$B_{\text{full},+} \text{ is } \ell \text{-eliminable } \iff \exists X \in \mathcal{RH}_\infty \text{ s.t. } Q \in \mathcal{RH}_\infty$$

and rowrank$(Q) = p(B_{\text{full},+}) - \text{rowrank}(P_2)$,

$$B_{\text{full},-} \text{ is } \ell \text{-eliminable } \iff \exists X \in \mathcal{RH}_\infty \text{ s.t. } Q \in \mathcal{RH}_\infty$$

and rowrank$(Q) = p(B_{\text{full},-}) - \text{rowrank}(P_2)$.

Moreover, in each of these cases, the corresponding manifest behavior of Definition 4.8 is represented by the rational operator $P' = Q$.

The elimination problem has been investigated earlier. For polynomial representations of $C^\infty$ systems, it has been shown in Polderman and Willems (1998) that elimination of latent variables is always possible. The same result has been obtained for discrete time systems. The elimination problem for $C^\infty$ solutions of rational differential operators has been mentioned in Willems and Yamamoto (2008); however no concrete solution was presented in that paper. Theorem 4.9 shows that in the context of the Hardy and Lebesgue spaces, that we introduced here, elimination of latent variables from systems in the model classes $L_L, L_+,$ and $L_-$ is only possible under the stated conditions. For results of eliminability in terms of conditions from geometric control theory, we refer to Mutsaers and Weiland (2010, 2011).

**Example 4.10.** Consider the latent variable system with behavior given by

$$B_t = \{ (w, \ell) \in \mathcal{H}_2^+ \mid \begin{bmatrix} 2(s-2)(s-3) & s - \alpha \\ (s-7)(s-8) & s - \alpha \end{bmatrix} \begin{bmatrix} w \\ s + 4 \\ s - 7 \\ 0 \end{bmatrix} \in \mathcal{H}_2^+ \}.$$

Here, $\alpha$ is a non-zero real constant. By Theorem 4.9 this system is $\ell$-eliminable if there exists $X \in \mathcal{RH}_\infty^+$ such that $Q$ in (8) belongs to $\mathcal{RH}_\infty^+$ and satisfies the proper rank conditions. This implies that

$$\frac{2(s-2)(s-3)}{(s-7)(s-8)} - \frac{s - \alpha}{s - 7} X(s) \in \mathcal{RH}_\infty^+,$$

and the rank condition implies that

$$\frac{2(s-2)}{(s-7)(s-8)} - \frac{s - \alpha}{s - 7} X(s) = U(s) \frac{s + 4}{s - 8} = \frac{s - 3}{s - 7} \left( \frac{s + 4}{s - 8} \right),$$

which fulfills the rank condition. Moreover, also (9) holds with $X \in \mathcal{RH}_\infty^+$ if and only if $\alpha < 0$.

4.2 Stable rational operators

So far, we have considered anti-stable rational operators for defining $L_L$ systems. This subsection defines model classes of $L_2$ systems through stable rational operators. The material in this subsection is analogous to that in the previous subsection and will therefore be stated without further discussion or proof. Let $\hat{P} \in \mathcal{RH}_\infty^+$ and consider the following three dynamical systems:

$$\Sigma := (C, C^\infty, B),$$

$$\Sigma_+ := (C_+, C^\infty, B_+),$$

$$\Sigma_- := (C_-, C^\infty, B_-),$$

where

$$B := \{ w \in \mathcal{L}_2 : \hat{P} w = 0 \} = \ker \hat{P},$$

$$B_+ := \{ w \in \mathcal{H}_2^+ : \hat{P} w = 0 \} = \ker_+ \hat{P},$$

$$B_- := \{ w \in \mathcal{H}_2^- : \hat{P} w = 0 \} = \ker_- \hat{P}.$$

Here, $\Pi_\Sigma$ is the canonical projection from $\mathcal{L}_2$ onto $\mathcal{H}_2^\Sigma$.

**Lemma 4.11.** For $\hat{P} \in \mathcal{RH}_\infty^+$, the behaviors $B, B_+, \text{ and } B_-$ in (10b) are closed, right invariant subspaces of $\mathcal{L}_2, \mathcal{H}_2^+, \text{ and } \mathcal{H}_2^-$, respectively.

Hence, kernels of anti-stable rational operators define left invariant subspaces, and kernels of stable rational operators are right invariant.

**Definition 4.12.** The classes of all linear and right invariant systems in $\mathcal{L}_2, \mathcal{H}_2^+$ and $\mathcal{H}_2^-$ that admit representations by stable rational operators as in (10) are denoted by $M, M_+$ and $M_-$, respectively.
Theorem 4.13 (Inclusion and Equivalence). Let two systems in the class \( M \) (or \( M_+ \), \( M_- \)) with behaviors \( B_1, B_2 \) (or \( B_{1,+}, B_{2,+} \) or \( B_{1,-}, B_{2,-} \)) be represented by full rank \( P, Q \in \mathcal{RH}_+^\infty \), respectively, as in (10). We then have:

1. Inclusions of behaviors:
   i. \( B_2 \subset B_1 \iff \exists \bar{F} \in \mathcal{RL}_+^\infty, s.t. \bar{P} = \bar{F} \bar{Q} \).
   ii. \( B_{2,+} \subset B_{1,+} \iff \exists \bar{F} \in \mathcal{RL}_+^\infty, s.t. \bar{P} = \bar{F} \bar{Q} \).
   iii. \( B_{2,-} \subset B_{1,-} \iff \exists \bar{F} \in \mathcal{RH}_-^\infty, s.t. \bar{P} = \bar{F} \bar{Q} \).

2. Equivalence of behaviors:
   i. \( B_1 = B_2 \iff \exists \bar{U} \in \mathcal{UL}_+^\infty, s.t. \bar{P} = \bar{U} \bar{Q} \).
   ii. \( B_{1,+} = B_{2,+} \iff \exists \bar{U} \in \mathcal{RH}_+^\infty, s.t. \bar{P} = \bar{U} \bar{Q} \).
   iii. \( B_{1,-} = B_{2,-} \iff \exists \bar{U} \in \mathcal{RH}_-^\infty, s.t. \bar{P} = \bar{U} \bar{Q} \).

3. If in addition \( \bar{Q} \) is co-inner, then the statements in item 1 are equivalent to the existence of \( \bar{F} \in \mathcal{RL}_\infty, \bar{F} \in \mathcal{RL}_\infty \) and \( \bar{F} \in \mathcal{RH}_+^\infty \) in i–iii respectively, such that \( \bar{P} = \bar{F} \bar{Q} \); if also \( \bar{P} \) is co-inner, then the statements in item 2 are equivalent to the existence of \( \bar{U} \in \mathcal{UL}_\infty, \bar{U} \in \mathcal{UL}_\infty \) and \( \bar{U} \in \mathcal{RH}_+^\infty \) in i–iii respectively, such that \( \bar{P} = \bar{U} \bar{Q} \).

Next, consider the elimination problem for latent variable systems in the model classes \( M, M_+ \) and \( M_- \). Let \( \bar{P} = [\bar{P}_1 \bar{P}_2] \in \mathcal{RH}_+^\infty \) be decomposed according to the partition of the variable column \( \ell \) and consider \( B_{\text{full}} = \ker \bar{P}, B_{\text{full},+} = \ker \bar{P}_1 \) and \( B_{\text{full},-} = \ker \bar{P}_2 \), as defined in a similar manner as in (7). In the following result we provide necessary and sufficient conditions for the complete elimination of the variable \( \ell \) and an explicit representation of the corresponding manifest behaviors as kernels of stable rational operators:

Theorem 4.14 (Elimination). Let \( \bar{P} = [\bar{P}_1 \bar{P}_2] \in \mathcal{RH}_+^\infty \) be full rank and define full system behaviors as in (10) and consider the equation

\[ \bar{Q} = \bar{P}_1 + \bar{P}_2 \bar{X}. \]  

We have, with respect to (11), that

\[ B_{\text{full}} \text{ is } \ell \text{-eliminable } \iff \exists \bar{X} \in \mathcal{RL}_\infty, s.t. \bar{Q} \in \mathcal{RH}_+^\infty \]

and rowrank\( (\bar{Q}) \) = \( \text{rowrank}(\bar{P}_1) \).

\[ B_{\text{full},+} \text{ is } \ell \text{-eliminable } \iff \exists \bar{X} \in \mathcal{RH}_+^\infty, s.t. \bar{Q} \in \mathcal{RH}_+^\infty \]

and rowrank\( (\bar{Q}) \) = \( \text{rowrank}(\bar{P}_1) \).

\[ B_{\text{full},-} \text{ is } \ell \text{-eliminable } \iff \exists \bar{X} \in \mathcal{RH}_-^\infty, s.t. \bar{Q} \in \mathcal{RH}_-^\infty \]

and rowrank\( (\bar{Q}) \) = \( \text{rowrank}(\bar{P}_2) \).

Moreover, in each of these cases, the corresponding manifest behavior is represented as the kernel of the stable rational operator \( \bar{Q} \).

The proofs of Theorems 4.13 and 4.14 are similar to the proofs of Theorems 4.3 and 4.9 and are not included in this paper.

5. Controller synthesis

This section answers the third question posed in Section 2, namely the controller synthesis problem. Given are two systems \( \Sigma_p \) and \( \Sigma_x \), both represented by means of rational kernel representations. We address the question of how to synthesize a third system \( \Sigma_c \), belonging to the same model class as \( \Sigma_p \) and \( \Sigma_x \), such that the interconnection of \( \Sigma_p \) and \( \Sigma_c \) coincides with \( \Sigma_x \). Because this question is of evident interest in control, we will refer to \( \Sigma_p \) as the plant, to \( \Sigma_c \) as the controller and to \( \Sigma_x \) as the controlled system. The problem then amounts to synthesizing a controller for a given plant that yields a given controlled system after interconnecting plant and controller. Here, we distinguish between full and partial interconnections as explained in Section 2. For the latter case, we will illustrate the results obtained by giving an example.

In this section, we focus on the system class \( L_+ \). However, all results extend to the system classes \( L_- \) and \( M_{(+)} \) without additional technical problems. For simplicity of notation, throughout this section we omit the subscript \( + \) in the definitions of systems \( (\Sigma) \) and their corresponding behaviors.

5.1. The full interconnection problem

For systems in the class \( L_+ \), the synthesis problem by full interconnection is formalized as follows.

Problem 5.1. Let two systems \( \Sigma_p = (C_+, C^w, P) \in L_+ \) and \( \Sigma_x = (C_+, C^w, K) \in L_+ \) be given.

i. Verify whether there exists \( \Sigma_c = (C_+, C^w, \mathcal{E}) \in L_+ \) such that \( P \cap \mathcal{E} = K \). Any such system is said to implement \( K \) for \( P \) by full interconnection through \( \mathcal{E} \).

ii. If such a controller exists, find a representation \( C_0 \in \mathcal{RH}^\infty_+ \) for the system \( \Sigma_c \), in the sense that its behavior \( \mathcal{E} = \ker \Pi_c C_0 \) implements \( K \) for \( P \).

iii. Characterize the set \( \mathcal{C}_{\mathcal{E}} \) of all \( C \in \mathcal{RH}^\infty_+ \) for which the behavior \( \mathcal{E} = \ker \Pi_c C \) implements \( K \) for \( P \).

The synthesis algorithm that will be derived in this section is inspired by the polynomial analog that has been treated in Polderman and Willems (1998) and Trentelman et al. (2007). Specifically, we provide an explicit algorithm that leads to the set of all rational representations of behaviors \( C \) that implement \( K \) for \( P \). The main result is stated as follows.

Theorem 5.2. Let the systems \( \Sigma_p = (C_+, C^w, P) \in L_+ \) and \( \Sigma_x = (C_+, C^w, K) \in L_+ \) be represented by the rational operators \( P, K \in \mathcal{RH}^\infty_+ \) respectively.

i. There exists a controller \( \Sigma_c = (C_+, C^w, \mathcal{E}) \in L_+ \) that implements \( K \) for \( P \) by full interconnection if and only if there exists an outer function \( X \in \mathcal{RH}^\infty_+ \) such that \( P = X K \).

ii. The set \( \mathcal{C}_{\mathcal{E}} \) of all possible kernel representations of controllers that implement \( K \) for \( P \) by full interconnection is given as the output of Algorithm 5.4.

By Theorem 4.3, the condition in item i of Theorem 5.2 implies that \( K \subseteq \mathcal{P} \). Hence, the inclusion \( K \subseteq \mathcal{P} \) is a necessary condition for the existence of a controller that implements \( K \) for \( P \) by full interconnection. This condition is, however, not sufficient. This is unlike the situation for \( \mathcal{E}^\infty \) behaviors discussed in Polderman and Willems (1998) and Trentelman et al. (2007) where the inclusion \( K \subseteq \mathcal{P} \) is a necessary and sufficient condition for guaranteeing the existence of a \( \mathcal{E}^\infty \) controller that implements \( K \) for \( P \). The fact that we consider systems in \( L_+ \) over the function space \( \mathcal{H}_+^\infty \) therefore makes an important difference in synthesis questions when compared to \( \mathcal{E}^\infty \) systems. We illustrate this in the following example.

Example 5.3. Given is the plant behavior \( \mathcal{P} = \ker \Pi_c P \) with \( P = 1/(s^2 + 3s + 1) \in \mathcal{RH}^\infty_+ \) and \( \alpha \) a non-zero real constant. The desired controlled behavior \( \mathcal{K} = \ker \Pi_c K \) is represented by \( K = \text{diag} \left( \frac{1}{\alpha}, \frac{1}{\alpha} \right) \in \mathcal{RH}^\infty_+ \). By Theorem 5.2, there exists a controller that implements \( K \) for \( P \) if and only if there exists an outer \( X \in \mathcal{RH}^\infty_+ \) such that \( P = X K \). Such an \( X \) exists and is given
by $X = \left[ \begin{array}{c} s^{-2}w_0 - s^{-3}w_1 + s^{-4}w_2 + s^{-5}w_3 \end{array} \right] \in \mathcal{RH}^c_\infty$, which is outer if and only if $\alpha > 0$. For $\alpha < 0$, we do not fulfill the condition of Theorem 5.2.

In that case $K \subset \mathcal{P}$ and the transient $w(s) = \frac{1}{s^\alpha} w_0 s^\alpha$, with $w_0 \in \mathcal{C}$ an arbitrary vector, belongs to $\mathcal{P}$ but not to $\mathcal{K}$. Now note that for any controller $E = \ker_+ \Pi_* C$, with $C = [C_1, C_2] \in \mathcal{RH}^c_\infty$, we have that $\det(P'[C]) = C_2(s^{k_2} + (s^{-k_2} + s^{-k_1}) - C_1(s^{k_2} + (s^{-k_2} + s^{-k_1})$. This implies that $w(s)$ belongs to the full interconnection of $\mathcal{P}$ and $E$. Conclude that for $\alpha < 0$, we have that $K \subset \mathcal{P}$ but $K$ cannot be implemented for $\mathcal{P}$.

The algorithm

The following algorithm yields an explicit construction of all controllers $\Sigma_C$ that solve Problem 5.1 for the class $\mathcal{L}_c$ of $\mathcal{L}_c$ systems.

Algorithm 5.4. Let $P, K \in \mathcal{RH}^c_\infty$ define the behaviors $\mathcal{P}$ and $\mathcal{K}$ corresponding to the systems $\Sigma_P \in \mathcal{L}_s$ and $\Sigma_K \in \mathcal{L}_s$, respectively. Aim: Find all $C \in \mathcal{RH}^c_\infty$ that define systems $\Sigma_C \in \mathcal{L}_s$ with behavior $E = \ker_+ \Pi_* C$ that implements $\mathcal{P}$ for the system $\mathcal{P} \cap E = \mathcal{P}$ full by $K$. Full interconnection.

Step 1: Find an outer rational function $X \in \mathcal{RH}^c_\infty$, such that $P = X K$. If no such $X$ exists, the algorithm ends and no controller exists that implements $\mathcal{K}$ for $\mathcal{P}$. In this case, set $C_{\text{par}} = 0$.

Step 2: Determine a unitary function $U \in \mathcal{U} \mathcal{RH}^c_\infty$, which brings $X$ into the form: $X = XU = (X_1, 0)$, where $X_1 \in \mathcal{U} \mathcal{RH}^c_\infty$.

Step 3: Define $W := [0, 1]U^{-1} \in \mathcal{RH}^c_\infty$, where the dimension of the identity matrix equals the number of zeros-columns in $X_1$.

Step 4: Set $C_0 := W K \in \mathcal{RH}^c_\infty$. Define $\alpha > 0$ and $k \geq 0$ such that $C := \frac{1}{(s^\alpha)^k} C_0 \in \mathcal{RH}^c_\infty$. Then, let $C = \ker_+ \Pi_* C$ be such that $\Sigma_C := (C_+, C^w, C) \in \mathcal{L}_s$, implements $\mathcal{K}$ for $\mathcal{P}$.

Step 5: Set $C_{\text{par}} = \left\{ \frac{1}{(s-\alpha^k)} (Q_1 P + Q_2 W K) \in \mathcal{RH}^c_\infty \right\}$

$Q_1 \in \mathcal{RH}^c_\infty$, $Q_2 \in \mathcal{U} \mathcal{RH}^c_\infty$, $\alpha > 0$, $k \geq 0$.

Output: $C_{\text{par}}$ is a parameterization of all controllers $\Sigma_C$ that implement $\mathcal{K}$ for $\mathcal{P}$ by ranging over all kernel representations $E = \ker_+ \Pi_* C$ with $C \in C_{\text{par}}$.

This explicit construction results in full plant–controller interconnections with the property that $p(\mathcal{P}) + p(E) = p(\mathcal{K})$. In the terminology used in Polderman and Willems (1998) and Trentelman et al. (2007), these are referred to as regular interconnections and they realize the idea that controllers do not duplicate laws that are already present in the plant to establish the controlled system.

5.2. The partial interconnection problem

In this subsection we consider the more general synthesis problem with partial interconnections of dynamical systems $\Sigma_P = (C_+, C^w, C^c, \mathcal{P}_\text{full})$ and $\Sigma_K = (C_+, C^w, K) \in \mathcal{L}_s$ in the model class $\mathcal{L}_s$, represented by the rational operators $P, K \in \mathcal{RH}^c_\infty$, respectively. Here, $\Sigma_P$ is a latent variable system as introduced in Section 4, so $P = \{P_1, P_2\}$ is decomposed according to the manifest and latent variables $w$ and $c$ of dimensions $w$ and $c$, respectively.

Problem 5.5. Let two linear left invariant systems $\Sigma_P = (C_+, C^w \times C^c, \mathcal{P}_\text{full}) \in \mathcal{L}_s$, and $\Sigma_K = (C_+, C^w, K) \in \mathcal{L}_s$, be given.

i. Verify whether there exists a linear left invariant system $\Sigma_C = (C_+, C^w, C) \in \mathcal{L}_s$ such that $K = \{w \in \mathcal{H}_2^c \mid \exists c \in \mathcal{H}_2^c \text{ s.t. } (w, c) \in \mathcal{P}_\text{full} \text{ and } c \in C\}$.

Any such system is said to implement $\mathcal{K}$ for $\mathcal{P}_\text{full}$ by partial interconnection.

ii. If such a controller exists, find a representation $C \in \mathcal{RH}^c_\infty$ for the system $\Sigma_C$ in the sense that its behavior $E = \ker_+ \Pi_* C$ implements $\mathcal{K}$ for $\mathcal{P}_\text{full}$.

To solve this problem, we associate with the system $\Sigma_P$ a set $\mathcal{N}$ that we refer to as the hidden behavior. For the model class $\mathcal{L}_s$, it is defined as $\mathcal{N} := \{w \in \mathcal{H}_2^c \mid \text{col}(w, 0) \in \mathcal{P}_\text{full}\}$

$= \{w \in \mathcal{H}_2^c \mid \text{col}(w, c) \in \mathcal{P}_\text{full}\}$

$= \ker_+ \Pi_* P_1$.

According to the decomposition made between manifest and latent variables. The hidden behavior is illustrated in Fig. 2 and is named hidden since it is not possible to estimate trajectories in $\mathcal{N}$ by observing the latent variable $c$ only. Problem 5.5 can be solved, under suitable conditions as is shown in the following theorem. This result is inspired by the controller implementation theorem introduced in Willems and Trentelman (2002).

Theorem 5.6. Let the systems $\Sigma_P = (C_+, C^w \times C^c, \mathcal{P}_\text{full}) \in \mathcal{L}_s$, and $\Sigma_K = (C_+, C^w, K) \in \mathcal{L}_s$ be represented by $P, K \in \mathcal{RH}^c_\infty$, respectively. Let $P = \{P_1, P_2\}$ be decomposed according to $w$ and $c$. Suppose that $\mathcal{P}_\text{full}$ is c-eliminable. Then $\mathcal{N} = \ker_+ \Pi_* P_1$ and, by Theorem 4.9, there exists $\Pi_{\text{man}} \in \mathcal{RH}^c_\infty$ such that $\mathcal{P}_\text{manifest} = \ker_+ \Pi_* \Pi_{\text{man}}$. Moreover, there exists a controller $\Sigma_C = (C_+, C^w, C) \in \mathcal{L}_s$, with $C_{\text{par}} \subset \mathcal{L}_s$ that implements $\mathcal{K}$ for $\mathcal{P}_\text{full}$ if and only if there exist outer functions $X, Y \in \mathcal{RH}^c_\infty$, such that $\mathcal{P}_\text{man} = X K$ and $K = Y P_1$.

The proof of this theorem is also constructive and is given in the Appendix. The conditions in (13) imply that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}_\text{manifest}$, which are necessary and sufficient conditions for the existence of a $\mathcal{E}^\infty$ controller $\Sigma_C$ that implements $\mathcal{K}$ for $\mathcal{P}_\text{full}$ as discussed in Polderman and Willems (1998) and Trentelman et al. (2007). However, these conditions are not sufficient for the partial interconnection problem for systems in the model class $\mathcal{L}_s$, as in the full interconnection case.

The algorithm

An explicit construction of a controller $\Sigma_C \in \mathcal{L}_s$ that implements $\mathcal{K}$ for $\mathcal{P}_\text{full}$ by partial interconnection is given by the following algorithm.

Algorithm 5.7. Let $P, K \in \mathcal{RH}^c_\infty$ define the behaviors $\mathcal{P}_\text{full}$ and $\mathcal{K}$ corresponding to the systems $\Sigma_P \in \mathcal{L}_s$, and $\Sigma_K \in \mathcal{L}_s$, respectively.

Assumption: $\mathcal{P}_\text{full}$ is c-eliminable.
Aim: Find $C \in \mathcal{RH}_\infty$ that defines the behavior $\mathcal{C}$ of system $\Sigma_C \in L_+$ as
$$\mathcal{C} = \{ c \in \mathcal{H}_2^+ \mid Cc \in \mathcal{H}_2^+ \} = \ker_+ \Pi_{\mathcal{C}} C,$$
such that $\mathcal{C}$ implements $\mathcal{K}$ for $\mathcal{P}_{\text{null}}$ by partial interconnection through $c$.

Step 1: Use Theorem 4.9 to obtain $P_{\text{man}} \in \mathcal{RH}_\infty$ such that
$$P_{\text{manifest}} = \{ w \in \mathcal{H}_2^+ \mid P_{\text{man}}w \in \mathcal{H}_2^+ \} = \ker_+ \Pi_{\mathcal{P}} P_{\text{man}}.$$

Step 2: Find an outer rational function $X \in \mathcal{RH}_{\mathcal{C}_\infty}$ such that $K = XP_t$. If no such $X$ exists, the algorithm stops and no controller can be found.

Step 3: Find an outer rational $Y \in \mathcal{RH}_{\mathcal{C}_\infty}$, such that $P_{\text{man}} = YK$. If no such $Y$ exists, the algorithm stops here.

Step 4: Determine a unitary function $U \in \mathcal{UH}_{\mathcal{C}_\infty}$, which brings $Y$ into the form: $Y = YU$, with $Y_1 \in \mathcal{UH}_{\mathcal{C}_\infty}$.

Step 5: Define $W := [0 1]U^{-1} \in \mathcal{RH}_{\mathcal{C}_\infty}$, where the dimension of the identity matrix equals the number of zero-columns in $Y$.

Step 6: The controller $\Sigma_C$ with behavior $\mathcal{C} = \ker_+ \Pi_{\mathcal{C}} C$ is given by
$$C = \frac{1}{(s - \alpha)^k} WXP_2,$$
where $\alpha > 0$ and $k \geq 0$ are such that $C \in \mathcal{RH}_{\mathcal{C}_\infty}$.

5.3. The example

To illustrate the algorithm for controller synthesis by partial interconnection, consider the following input–state–output system:
$$\Sigma_P : \begin{cases} \dot{x} = Ax + B_1s + B_2u, \\ z = Csx + D_{11}u + D_{12}u, \\ y = Csx + D_{21}u + D_{22}u, \end{cases}$$
with
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D_{12} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad D_{21} = D_{22} = 0.$$}

In this example, $w := \text{col}(z, d)$ is the manifest variable and $c := \text{col}(y, u)$ denotes the variable that is available for (partial) interconnection with a controller. The controlled system $\Sigma_C$ is defined by the state space equations
$$\Sigma_C : \begin{cases} \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -14 \end{bmatrix} x + \begin{bmatrix} -1.4615 \\ -1.4545 \\ -4.3597 \end{bmatrix} d, \\ z = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2.8583 \\ 3.0027 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d \end{cases},$$
which were obtained by substitution of the static output feedback law $u = -3y$ in (14). The $L_2$ behaviors of the plant and the controlled system are viewed as elements in the model class $L_+$ and represented by anti-stable rational operators in $\mathcal{RH}_{\mathcal{C}_\infty}$. In this case, $\mathcal{P}_{\text{null}} = \ker_+ \Pi_{\mathcal{P}}$ and $\mathcal{K} = \ker_+ \Pi_{\mathcal{K}}$, where $P(s) = \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} \in \mathcal{RH}_{\mathcal{C}_\infty}$ is decomposed accordingly with $\text{col}(w, c)$, with
$$P_1(s) = \begin{bmatrix} s + 1 & 0 \\ s - 5 & s - 5 \end{bmatrix}, \quad P_2(s) = \begin{bmatrix} 0 & 2 \\ s + 3 & s + 5 \\ s - 3 & s - 3 \\ 0 & 3 \end{bmatrix},$$
and $K(s)$ given in Box I. Given $P$ and $K$, we apply Algorithm 5.7 to find a controller that implements $\mathcal{K}$ for $\mathcal{P}$ by partial interconnection.

Step 1: To obtain a representation of the manifest behavior $\mathcal{P}_{\text{manifest}}$, we first eliminate the latent variable $c$ in the full plant behavior. For this, we start by creating zero-rows in $P_2$, as discussed in the proof of Theorem 4.9, by pre-multiplying $P$ with $U$ defined by
$$U(s) = \begin{bmatrix} 2(s + 6) & s - 4 & 0 \\ s - 3 & s - 5 & 0 \\ s - 2 & s - 3 & 0 \end{bmatrix},$$
with
$$U(s)^{-1} = \begin{bmatrix} 0 & 2(s + 6)(s - 5) & 0 \\ s - 5 & (s - 2)(s - 4) & 0 \\ 0 & 0 & s - 2 \end{bmatrix}.$$ 

Since $U$ and $U^{-1}$ belong to $\mathcal{RH}_{\mathcal{C}_\infty}$, we infer that $U \in \mathcal{UH}_{\mathcal{C}_\infty}$ and we have that $U \in \mathcal{UH}_{\mathcal{C}_\infty}$. This results in
$$U(s)P_1(s) = \begin{bmatrix} 2(s + 1)(s + 6) & (s + 3)(s - 4) & (s + 1)(s + 4) \\ (s - 3)(s - 5) & (s - 3)(s - 5) & (s - 3)(s - 5) \\ (s - 3)(s - 2) & (s - 3)(s - 2) & (s - 3)(s - 3) \end{bmatrix},$$
and $U(s)P_2(s) = \begin{bmatrix} 0 & 0 \\ 0 & (s - 3)(s - 5) & 0 \\ (s - 1)(s - 3) & (s - 1)(s - 3) \end{bmatrix} = \begin{bmatrix} P_{11} \\ P_{12} \end{bmatrix}.$

It is now easily seen that the conditions for eliminability of $c$ in Theorem 4.9 are satisfied since there exists an $X \in \mathcal{RH}_{\mathcal{C}_\infty}$ such that $P_{12} + P_{22}X \in \mathcal{RH}_{\mathcal{C}_\infty}$ and that rowrank$(P_{11}) = \text{p}(\mathcal{P}_{\text{null}}) - \text{rowrank}(P_{12})$ (note that this operator $X$ differs from the one used in Step 2). Hence, by the elimination theorem, $\mathcal{P}_{\text{manifest}} = \ker_+ \Pi_{\mathcal{P}} P_{\text{man}}$ with
$$P_{\text{man}}(s) = \begin{bmatrix} 2(s + 1)(s + 6) & (s + 3)(s - 4) & (s + 1)(s + 4) \\ (s - 3)(s - 5) & (s - 3)(s - 5) & (s - 3)(s - 5) \end{bmatrix} \in \mathcal{RH}_{\mathcal{C}_\infty}.$$
Step 2: We need to verify the existence of an outer function \( X \in \mathcal{RH}_\infty^- \) such that \( K = X P \). The rational operator given in Box II fulfills this requirement because \( \mathcal{RH}_\infty^- \subset \mathcal{RH}_\infty^- \).

Step 3: We need to verify the existence of an outer function \( Y \in \mathcal{RH}_\infty^- \) such that \( P_{\text{min}} = Y K \). The rational operator

\[
Y(s) = \begin{bmatrix}
\frac{2(s - 10.49)}{(s - 3)(s - 5)} & \frac{s - 1.95}{(s - 3)(s - 5)}
\end{bmatrix}
\in \mathcal{RH}_\infty^-,
\]

fulfills this requirement.

Step 4: We need to post-multiply \( Y \) with a unitary operator \( U \) such that \( YU = [Y_1 \ 0] \), with \( Y_1 \) a unit. The matrix function

\[
U(s) = \begin{bmatrix}
\frac{s - 1}{s - 5} & \frac{s - 3}{s - 2} \\
0 & \frac{2(s - 1.231)(s - 3)(s - 10.49)}{(s - 1.95)(s - 2)(s - 5.286)}
\end{bmatrix},
\]

with inverse \( U(s)^{-1} = \begin{bmatrix}
\frac{s - 5}{s - 1} & \frac{0.5(s - 1.95)(s - 5)(s - 5.286)}{(s - 1)(s - 1.231)(s - 10.49)} \\
\frac{0.5(s - 1.95)(s - 5)(s - 5.286)}{(s - 1)(s - 1.231)(s - 3)(s - 10.49)} & \frac{s - 1}{s - 5}
\end{bmatrix},
\]
does indeed belong to \( \mathcal{RH}_\infty^- \). Moreover,

\[
Y_1(s) = \frac{2(s - 1)(s - 1.231)(s - 10.49)}{(s - 3)(s - 5)^2} \in \mathcal{RH}_\infty^-,
\]
yields that \( Y_1 \) is a unitary function. This meets the conditions on \( U \).

Step 5: The function \( \mathcal{W} := [0 \ 1] U^{-1} \) reads

\[
\mathcal{W}(s) = \begin{bmatrix}
0 & \frac{2(s - 1.231)(s - 3)(s - 10.49)}{(s - 1.95)(s - 2)(s - 5.286)}
\end{bmatrix}.
\]

Step 6: The controller \( \Sigma \) with behavior \( e := \ker_{\Sigma_1} \Pi_1 \), \( C \) is given by the equation in Box III. There, \( \nu(s) = \frac{12(s + 5)}{s - 1} \in \mathcal{RH}_\infty^- \). By Theorem 4.3, \( C = \ker_{\Sigma_1} \Pi_1 C_0 \) with the equivalent kernel representation \( C_0(s) = \frac{-12(s + 5)}{s - 1} - \frac{4(s + 5)}{s - 1} \in \mathcal{RH}_\infty^- \).

Note that this controller does indeed implement \( \mathcal{K} \) for \( P \), since substitution of the law \( u = 3y \) yields

\[
- \frac{12(s + 5)}{s - 1} - \frac{4(s + 5)}{s - 1} u = - \frac{12(s + 5)}{s - 1} y - \frac{4(s + 5)}{s - 1} 3y = 0.
\]

6. Conclusions

In this paper, systems are viewed as collections of functions that are square integrable on the imaginary axis. More specifically, we distinguish three classes of closed, left invariant systems that can be represented as kernels of rational operators in the class \( \mathcal{RH}_\infty^- \) of stable rational functions, and three classes of closed right invariant systems that can be modeled as the null spaces of operators in \( \mathcal{RH}_\infty^- \), the class of anti-stable rational functions. This defines six model classes of \( L_2 \) systems. For each of these model classes we addressed the question of system equivalence. Necessary and sufficient conditions on rational functions have been derived that guarantee the equivalence of systems. We have presented necessary and sufficient conditions for the complete elimination of latent variables from an \( L_2 \) latent variable system. More specifically, we presented conditions under which the induced manifest behavior of a latent variable system, represented as the kernel of a rational operator, can again be represented as the kernel of a rational operator. The results presented on equivalence and elimination of \( L_2 \) systems that are represented by rational operators substantially differ from results on the elimination and equivalence of infinitely smooth solutions systems that are represented by polynomial differential equations.

We have applied the results to solve the controller synthesis problem in an analogous approach, as described in Treutelman et al. (2007). Explicit algorithms have been presented that synthesize a controller \( e \) that after interconnection with an \( L_2 \) plant \( P \) gives a desired controlled behavior \( K \). In fact, we characterized all controllers (as \( L_2 \) systems) that after interconnection with a given plant result in the desired controlled behavior. Two possible interconnection structures, namely full and partial interconnections, are distinguished for this controller synthesis problem.

Appendix. Proofs

We start this section with a lemma that proves useful in various proofs.

**Lemma A.1.** Let \( P \in \mathcal{RH}_\infty^- \), \( k \geq 0 \) and \( \alpha > 0 \). Then,

\[
\{ w \in \mathcal{H}_2^+ \mid Pw \in \mathcal{H}_2^+ \} = \left\{ w \in \mathcal{H}_2^+ \mid \frac{1}{(s - \alpha)^k} Pw \in \mathcal{H}_2^+ \right\}.
\]

Moreover, let \( z \in \mathcal{L}_2 \). Then \( \frac{1}{(s - \alpha)^k} z \in \mathcal{H}_2^+ \) if and only if \( z \in \mathcal{H}_2^+ \).

**Proof.** For the first claim, we first verify the inclusion \((\subseteq)\). Let \( w \in \mathcal{H}_2^+ \) be such that \( z := Pw \in \mathcal{H}_2^+ \). Since \( k \geq 0 \) and \( \alpha > 0 \), we have that \( \frac{1}{(s - \alpha)^k} \in \mathcal{RH}_\infty^- \). Hence, by Lemma 3.2,
Lemma 3.2 Let $\lambda \in \mathcal{H}_K^m$, which yields that $\frac{1}{1-2\alpha^2}Pw \in \mathcal{H}_K^m$. To verify (3), take $w \in \mathcal{H}_K^m$ such that $\frac{1}{1-2\alpha^2}Pw \in \mathcal{H}_K^m$. By Lemma 3.2, we have that $z := Pw \in \mathcal{L}_z$. Decompose $z$ as $z = z_+ + z_-$ with $z_+ = \Pi z \in \mathcal{H}_K^m$ and $z_- = \Pi_\perp z \in \mathcal{H}_K^m$. Substitution in the expression for $\tilde{z}$ in (3) we get

$$\frac{1}{(s - \alpha)^k}z_+ = \frac{1}{(s - \alpha)^k}z_+ \in \mathcal{H}_K^m.$$  (15)

We claim that $z_+$ is analytic in $\mathbb{C}$; To show this, first note that $z_+$ is analytic in $\mathbb{C}$, since $z_+ \in \mathcal{H}_K^m$. Also, $z_-$ is analytic in $\mathbb{C}$, since $z_+ = z - z_+ \in \mathcal{L}_z$. Now, suppose that $z_+$ is not analytic at a point $s_0 \in \mathbb{C}$. Then, $lim_{s \to s_0} z_+ = \infty$ and there exists $m > 0$ such that $z_+ = \frac{1}{(s - s_0)^m}z_+(s)$ analytic in $s_0$. Therefore, for $k \geq 0$ and $\alpha > 0$, we have

$$\lim_{s \to s_0} \frac{1}{(s - s_0)^m}z_+(s) = \frac{1}{(s - s_0)^m}z_+(s) = \infty,$$

which shows that $\frac{1}{1-2\alpha^2}Pw_0$ is not analytic in $s_0 \in \mathbb{C}$. This contradicts (15). Therefore, $z_+$ is analytic in $\mathbb{C}$; Since $z_+$ is bounded ($z_+ \in \mathcal{H}_K^m$) and analytic in $\mathbb{C}$, application of Liouville’s boundedness theorem proves that $z_+$ is a constant function. Since $z_+ = \Pi z \in \mathcal{H}_K^m$, and $z_+ \in \mathcal{H}_K^m$, we have $z_+ = 0$. Consequently, $Pw = z = z_+ + z_-$, which proves (3). The completeness of the second claim is immediate from the (3)-part of this proof.

**Proof of Lemma 3.1.** To prove that $\mathcal{R}_\mathcal{H}_K^m = \mathcal{R}_\mathcal{H}_K^m + \Re[s]$, we first show that $\mathcal{R}_\mathcal{H}_K^m \supseteq \mathcal{R}_\mathcal{H}_K^m + \Re[s]$. Take arbitrary $f_1 \in \mathcal{R}_\mathcal{H}_K^m$ and $f_2 \in \Re[s]$. Let $k \geq \text{degree}(f_2)$ and $\alpha < 0$. Then $\frac{1}{(s - \alpha)^k}f_2 \in \mathcal{R}_\mathcal{H}_K^m$ and

$$\frac{1}{(s - \alpha)^k}f_1 + \frac{1}{(s - \alpha)^k}f_2 = \left(\frac{1}{(s - \alpha)^k}f_1 + \frac{1}{(s - \alpha)^k}f_2\right) \in \mathcal{R}_\mathcal{H}_K^m,$$

which, by (5), shows that $(f_1 + f_2) \in \mathcal{R}_\mathcal{H}_K^m$.

To verify the converse inclusion, let $f \in \mathcal{R}_\mathcal{H}_K^m$, Following (5), $f$ is a rational function that is analytic in $\mathbb{C}$ with possible poles at infinity. Let $N = N(s)D(s)^{-1}$, with $N, D \in \Re[s]$, be a right-coprime polynomial factorization of $f$. By the analyticity of $f$, det$(D(s)) \neq 0$. Now, $\Pi \in \mathcal{H}_K^m$. Moreover, there exist polynomials $Q, R \in \Re[s]$ such that $N(s) = Q(s)D(s) + R(s)$ and $R(s)D(s)^{-1}$ is strictly proper (Vidyasagar, 1985). Hence, $f = N(s)D(s)^{-1} = Q(s) + R(s)D(s)^{-1}$ is a sum of a polynomial and a strictly proper rational function with poles in $\mathbb{C}$, i.e., $f = f_1 + f_2$ with $f_1 \in \mathcal{H}_K^m$ and $f_2 \in \Re[s]$. This completes the proof.

**Proof of Lemma 4.1.** To prove linearity, let $w_1, w_2 \in \mathcal{B}_i$. For $\lambda_1, \lambda_2 \in \Re$, we have to verify whether $w := \lambda_1 w_1 + \lambda_2 w_2 \in \mathcal{B}_i$. This is indeed the case because $Pw = \lambda_1 Pw_1 + \lambda_2 Pw_2 \in \mathcal{H}_K^m$. To prove left invariance of $\mathcal{B}_i$, we need to show that for all $\tau \leq 0$ and $w \in \mathcal{B}_i$, $\sigma_t w \in \mathcal{B}_i$ holds. For all $\tau \leq 0$ we have that

$$P(\sigma_t w)(s) = e^{-\tau s}P(s)w(s) - P(s)e^{-\tau s} \int_0^{-\tau} \hat{w}(t)e^{t\tau}dt.$$  

Since $w \in \mathcal{B}_i$, we have $P(s)w(s) \in \mathcal{H}_K^m$ and therefore also $e^{-\tau s}P(s)w(s) \in \mathcal{H}_K^m$ for $\tau \leq 0$. Moreover, with a change of variables $u := \tau + \tau$, we infer

$$e^{-\tau s} \int_0^{-\tau} \hat{w}(t)e^{t\tau}dt = e^{-\tau s} \int_0^{-\tau} \hat{w}(u)e^{u\tau}du = \int_{-\tau}^0 \hat{w}(u)e^{u\tau}du \in \mathcal{H}_K^m,$$

as $\hat{w}(\cdot - \tau) \in \mathcal{L}_z$, for $\tau \leq 0$. Hence, $P(s)e^{-\tau s} \int_0^{-\tau} \hat{w}(t)e^{t\tau}dt \in \mathcal{H}_K^m$. Consequently, $P(s)(\sigma_t w)(s) \in \mathcal{H}_K^m$ for $\tau \leq 0$. The proofs for $\mathcal{B}$ and $\mathcal{B}_i$ are similar and are omitted in this paper. □
Similarly, without using the factorization, we obtain that $B_1 = (P^*L_2)^\perp$.
If $B_2 \subset B_1$, then also $B_1^\perp \subset B_2^\perp$, and so
\[(P^*L_2)^\perp \subset (Q_2^*L_2)^\perp.\]
Equivalently, with over-bars denoting closures,
\[P^*L_2 \subset Q_2L_2. \tag{16}\]

The Beurling–Lax theorem (see the proof of Theorem 12.6 in Fuhrmann (1981, Chapter 2)) states that, if $M = QH$ for some inner function $Q$ and Hilbert space $H$, then $M$ is a closed invariant subspace of $H$. Applying this to (16) gives
\[P^*L_2 \subset Q_2^*L_2 \subset Q_2L_2 = Q_2^*L_2. \tag{17}\]

Now, we use a more general result for bounded operators $A$ and $B$ in Hilbert spaces (Theorem 7.1 in Fuhrmann (1981)), which states that if $A \subset \subset B$ and only if $A = BC$ for some bounded operator $C$.

More explicitly, as in the proof of Theorem 7.1, define $B_0 := Q_0^*(ker Q_{\infty}^*).$ Then $B_0$ is an injective mapping from $(ker Q_0^*)^\perp \to Q_2^*L_2.$ Moreover, $B_0^\perp$ exists as a closed operator mapping $Q_2^*L_2$ into $(ker Q_0^*)^\perp.$ Since $im P^* \subset im Q_*^*,$ the operator $C := B_0^\perp$ is a closed mapping from $L_2$ to $(ker Q_0^*)^\perp$ and belongs to $RL_{\infty}.$

Now,
\[Q_0^*C = Q_0^*B_0^\perp = B_0B_0^\perp = P^*. \]

Consequently, by taking adjoints it follows that $P = C^*Q_0.$ Let $F := C^*Q_{\infty}.$ Since $C^* \in RL_{\infty}$ and $Q_{\infty}^* \in RH_{\infty},$ we have that $F \in RL_{\infty}.$ Moreover, $Q_0^*C^*Q_{\infty} = Q_0^*Q_{\infty}Q_{\infty} = P,$ which completes the proof.

Let $B_2 :\subset B_1.$ Then $P \in RH_{\infty},$ such that $P = FQ.$

This proof goes in a similar manner to the one in the previous item. However, we will make use of Lemma A.1 and claim that there exist $k \geq 0$ and $\alpha > 0$ such that
\[B_2 :\subset \{ w \in H_2^1 \mid Qw = z \in H_2^1 \}
\]\
\[= \begin{cases} w \in H_2^1, & (1 - (s - \alpha)^2)Qw = 1/(s - \alpha)^2z \in H_2^1 \\ w \in H_2^1, & (1 - (s - \alpha)^2)Qz = 1/(s - \alpha)^2w \in H_2^1 \\ w \in H_2^1, & (1 - (s - \alpha)^2)Qw = 1/(s - \alpha)^2z \in H_2^1 \end{cases}. \]

Indeed, since $Q_{\infty}^{-1} \in RH_{\infty},$ the definition in (5) implies that there exist $k \geq 0$ and $\alpha > 0$ such that $1/(s - \alpha)^2Q_{\infty}^{-1} \in RH_{\infty}.$ For this choice of $k$ and $\alpha$ it follows that $\hat{z} := 1/(s - \alpha)^2Q_{\infty}^{-1}z \in H_2^1.$

Using this, and applying Lemma A.1 again, we obtain
\[B_2 :\subset \begin{cases} w \in H_2^1, & (1 - (s - \alpha)^2)Qw = 1/(s - \alpha)^2z \in H_2^1 \\ w \in H_2^1, & (1 - (s - \alpha)^2)Qz = 1/(s - \alpha)^2w \in H_2^1 \\ w \in H_2^1, & (1 - (s - \alpha)^2)Qw = 1/(s - \alpha)^2z \in H_2^1 \end{cases}.
\]

This implies $F = C^*Q_{\infty}^{-1} \in RH_{\infty},$ and satisfies $FQ = P$ as in the previous item.

Equality of behaviors:

We only show the proof for the equivalence $B_{2, +} = B_{2, +}$, which will be used in Section 5. With this proof, one can easily verify the other two equivalence conditions.

Let $B_{1, +}$ and $B_{2, +}$ be represented by full row rank operators $P, Q \in RH_{\infty}.$ Using the previous inclusion relations, we have that $B_{1, +} = B_{2, +}$ if and only if there exist $F_1 \in RH_{\infty}$ and $F_2 \in RH_{\infty},$ such that $P = F_1Q$ and $Q = F_2P.$ A direct substitution then gives that $P = F_1F_2Q$ and $Q = F_2F_1P.$ If $P$ and $Q$ have full row rank, it follows that $F_1 = F_2^{-1},$ which shows that both $F_1$ and $F_2$ belong to $UH_{\infty}$. This completes the proof.

Using co-inner operators $Q$ and $P$:

One can observe in the proof of the inclusions that when $Q$ is co-inner, no outer/co-inner factorization has to be applied. In this case, we can verify whether in $P^* \subset im \phi$ directly (since the closure of $Q^*L_2 = Q^*L_2$), and we obtain $F := C \in RL_{\infty}$ as a bounded operator. For the case where also $P$ is co-inner, equivalence of $B_{1, +}$ and $B_{2, +}$ holds when there exist $F_1, F_2 \in RL_{\infty}.$ Since we have shown that $F_1 = F_2^{-1},$ we know that $F_1, F_2 \in UL_{\infty}.$ Similar results can be obtained for the $H_2^1$ and $H_2$ behaviors.

Proof of Theorem 4.9.

We only show the second equivalence for systems in $L_{\infty},$ as the proofs in the other cases are similar. To show this, let $U \in UH_{\infty}$, be such that
\[U \begin{bmatrix} P_{12} \\ 1 \end{bmatrix},
\]
where $P_{12}$ has full row rank. Define the decomposition
\[\tilde{P} := U[P_{11} P_{12}]
\]
\[= \begin{bmatrix} P_{11} & P_{12} \\ 0 & 1 \end{bmatrix}.
\]

Then, by Theorem 4.3,
\[B_{full, +} = \{ (w, \ell) \in H_2^1 \mid P_{11}w + P_{12} \ell \in H_2^1 \}
\]
and $P_{12}w \in H_2^1$.

It follows that $B_{full, +} = B_{full, +} \cap B_{full, +}^2$, where
\[B_{full, +} = \{ (w, \ell) \in H_2^1 \mid P_{11}w + P_{12} \ell \in H_2^1 \}
\]
\[B_{full, +}^2 = \{ (w, \ell) \in H_2^1 \mid P_{11}w + P_{12} \ell \in H_2^1 \}
\]

Let $B_{1, manifest, +}$ be the manifest behavior associated with $B_{1, +}$, and let $B_{1, manifest, +}$ denote the manifest behavior associated with $B_{1, +}$.

Then $\Rightarrow$: Suppose that the system is $\ell$-eliminable. First consider $B_{full, +}$. We first prove that $B_{1, manifest, +} \subset H_2^1$. To see this, let $p_1 = p(ker \gamma \gamma [P_{11} P_{12}])$ be the output cardinality of $B_{full, +}$, and denote by $m_1 = m(ker \gamma \gamma [P_{11} P_{12}])$ the input cardinality of $B_{full, +}$. Since both $P_{12}$ and $|P_{11} P_{12}|$ have full row rank, it follows that $p_1 = rowrank(P_{12})$. This implies that the variables $(w, \ell)$ in $B_{1, manifest, +}$ admit a partitioning as
\[\begin{bmatrix} w \\ \ell \end{bmatrix} = \begin{bmatrix} w' \\ \ell' \end{bmatrix},
\]
where $w = col(w', \ell')$ is an input variable (i.e., an unconstrained variable in $H_2^1$) and $y = \ell'$ is an output variable. In particular,
Lemma A.1

Second, we construct the mapping X in (8). Define, for any \( w \in \mathcal{H}_2^* \), the set of latent functions that are compatible with \( w \) as \( \mathcal{L}(w) := \{ \ell \in \mathcal{H}_2^* \mid (w, \ell) \in \mathcal{B}_{\text{full}^+} \} \). Clearly, \( \mathcal{L}(w) \) is non-empty and it is easily seen that \( \mathcal{L}(w) \) is an affine set for any \( w \in \mathcal{H}_2^* \).

Indeed, if \( \ell_1, \ell_2 \in \mathcal{L}(w) \) and \( \alpha \in \mathbb{R} \), then \((w, \ell_1) \in \mathcal{B}_{\text{full}^+}\) and \((w, \ell_2) \in \mathcal{B}_{\text{full}^+}\). This shows that \( \alpha \ell_1 + (1 - \alpha)\ell_2 \in \mathcal{B}_{\text{full}^+}\). Any affine set can be written as \( \mathcal{L}(w) = \mathcal{L}_0 + X(w) \).

\[
\mathcal{L}(w) = \mathcal{L}_0 + X(w), \tag{20}
\]

where \( \mathcal{L}_0 \subseteq \mathcal{H}_2^* \) and \( X : \mathcal{H}_2^* \rightarrow \mathcal{H}_2^* \) is linear. Here, \( \mathcal{L}_0 \) does not depend on \( w \) and it follows that \( \mathcal{L}_0 = \mathcal{L}(0) \). This implies that \( \mathcal{L}_0 \) is ker \( \Pi \). Without loss of generality, define \( X : \mathcal{H}_2^* \rightarrow \mathcal{H}_2^* \) in such a manner that (20) holds where \( X(w) \) is orthogonal to \( \mathcal{L}_0 \), i.e., \((X(w), \mathcal{L}_0) = 0 \). Suppose this is the case. Then we claim that \( X \) is unique, linear and shift invariant. Linearity has already been shown.

- Uniqueness follows from the observation that whenever \( X_1 \) and \( X_2 \) satisfy \((X_i(w), \mathcal{L}_0) = 0 \) and \((X_i(w), \mathcal{L}_0) = 0 \) for all \( w \in \mathcal{B}_{\text{manifest}} \) then \((X_1(w) - X_2(w), \mathcal{L}_0) = 0 \). On the other hand, (20) implies that \((X_1(w) - X_2(w), \mathcal{L}_0) = 0 \). But then \( X_1(w) = X_2(w) \) for all \( w \in \mathcal{B}_{\text{manifest}} \).

- Shift invariance follows in a similar manner. Let \( \ell \in \mathcal{L}(w), \tau \leq 0 \). Then \( \ell = \ell + \tau w \) with \( \ell + \tau w \in \mathcal{L}_0 \) and consequently, \( \sigma \ell = \sigma \ell + \sigma w \). Since \( \mathcal{B}_{\text{full}} \) is left invariant we infer that \((\sigma w, \sigma \ell) \in \mathcal{B}_{\text{full}} \) and therefore \( \sigma \ell \in \mathcal{L}(\sigma w) = \mathcal{L}_0 + \sigma \mathcal{L}(w) \). It follows that \( \sigma \ell = \sigma \ell + \tau (\sigma w) \), and, using the uniqueness of \( X \), we have that \( X \) commutes with \( \sigma \) for any \( \tau \leq 0 \).

Since \( X : \mathcal{H}_2^* \rightarrow \mathcal{H}_2^* \) is linear and shift invariant, it admits a representation as a multiplicative operator \( X(w)(s) = X(s)w(s) \) where \( X \in \mathcal{H}_2^* \) is uniquely defined. See Theorem 1.3 in Weiss (1991). It follows that, for any \( w \in \mathcal{H}_2^* \), the latent variable \( \ell := X(w) \) is compatible with \( w \) in the sense that \((w, X(w)) \in \mathcal{B}_{\text{full}^+}^* \). In particular, \( \mathcal{B}_{\text{full}^+} = \mathcal{B}_{\text{manifest}^+} = \mathcal{B}_{\text{manifest}^+}^* \) implies that \( R \mathcal{H}_2^* = \mathcal{B}_{\text{full}^+} \).

Proceeding as above, \( \mathcal{L}(w) \) is defined and \( \mathcal{L}(w) \) is non-empty. Since \( \mathcal{L}(w) \) is orthogonal to \( \mathcal{L}_0 \), \( \mathcal{L}(w) \) is defined and \( \mathcal{L}(w) = R \mathcal{L}(w) \).

Finally, we prove that \( Q \in \mathcal{H}_2^* \) satisfies the rank conditions in Theorem 4.9. Since

\[
R \approx \bar{P} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},
\]

with \( R_1 = 0 \) and \( R_2 = P_{22} \), it is immediate that \( R \in \mathcal{H}_2^* \). Moreover, this satisfies \( \text{rowrank}(R) = \text{rowrank}(R_2) = \text{rowrank}(P_{22}) \). In (18), we have \( \bar{P} = UP \), and hence \( P = U^{-1} \bar{P} \in \mathcal{H}_2^* \), which implies that \( Q = U^{-1} R \in \mathcal{H}_2^* \). This also does not change the rank conditions; hence \( \text{rowrank}(Q) = \text{rowrank}(P_{\text{full}^+}) \), which completes the proof.

\( \Longleftarrow \): Suppose there exists \( X \in \mathcal{H}_2^* \) such that \( Q \in \mathcal{H}_2^* \) and that the given row rank condition is fulfilled. We will show that the manifest behavior is given by \( \mathcal{B}_{\text{manifest}^+} = \ker \Pi \).
where we define $C^+ \in \mathcal{RH}_{\mathcal{K}_C}$ such that $\{c_{C^+}\}$ has full rank. When $Cc \in \mathcal{H}_{\mathcal{C}_C}^+$ and also $c^+C \in \mathcal{H}_{\mathcal{C}_C}^+$, we do indeed have that $c = 0$, which should be the case for the hidden behavior. Since there exists a rational representation for $\mathcal{N}$, we know that we can eliminate $c$ in (24) and so by Theorem 4.9 $\exists \mathcal{N} \in \mathcal{RH}_{\mathcal{K}_C}$ such that $\mathcal{N} = ker_+ \Pi_+Q'$ with

$$Q' := \begin{pmatrix} p_1 & p_2 \\ 0 & C \\ 0 & C^+ \end{pmatrix} \begin{bmatrix} X_0 \\ 0 \end{bmatrix} = \begin{pmatrix} p_1 + p_2X_0 \\ CX_0 \\ C^+X_0 \end{pmatrix} \in \mathcal{RH}_{\mathcal{K}_C}.$$
where $\alpha > 0$ and $k \geq 0$ such that $\tilde{W} \in RH_{\infty}^0$. This implies that $C \in RH_{\infty}^0$, which completes the proof. □

References


Mark Mutsaers (born in Rijen, The Netherlands, 1983) received his M.Sc. degree in Control Engineering from the Eindhoven University of Technology, in 2008. He is currently working towards the degree of Ph.D. in the Control Systems Group of the Department of Electrical Engineering at the same institution. His research interests include model reduction of large-scale dynamical systems, general systems theory and model predictive control.

Siep Weiland is Professor at the Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology. He received both his M.Sc. (1986) and Ph.D. degrees in Mathematics from the University of Groningen in the Netherlands. He was a postdoctoral research associate at the Department of Electrical Engineering and Computer Engineering, Rice University, Houston, USA, from 1991 to 1992. Since 1992 he has been affiliated to Eindhoven University of Technology. His research interests are the general theory of systems and control, robust control, model approximation, modeling and control of spatial-temporal systems, identification, and model predictive control. He was Associate Editor of the IEEE Transactions on Automatic Control from 1995 to 1999, of the European Journal of Control from 1999 to 2003, of the International Journal of Robust and Nonlinear Control from 2001 to 2004 and Associate Editor for Automatica from 2003 until 2006.