The Zak Transform and Sampling Theorems for Wavelet Subspaces

Augustus J. E. M. Janssen, Senior Member, IEEE

Abstract—The Zak transform is used for generalizing a sampling theorem of G. Walter for wavelet subspaces. Cardinal series based on signal samples $f(a + n)$, $n \in \mathbb{Z}$ with $a$ possibly unequal to 0 (Walter's case) are considered. The condition number of the sampling operator and worst-case aliasing errors are expressed in terms of Zak transforms of scaling function and wavelet. This shows that the stability of the resulting interpolation formula depends critically on $a$.

I. INTRODUCTION

In [1] G. Walter presents a version of the classical Shannon sampling theorem for wavelet subspaces. The setting is a multiscale analysis of $f \in L^2(\mathbb{R})$, where the closed linear subspaces $V_n$ of $L^2(\mathbb{R})$, $m \in \mathbb{Z}$, and the real scaling function $\varphi$ satisfy the usual properties, so that in particular $(\varphi(t - n))_{n \in \mathbb{Z}}$ is an orthonormal basis for $V_0$. Also, as usual, $W_n$ is the orthogonal complement of $V_n$ in $V_{n+1}$, $\psi$ is the associated wavelet whose integer translates $(\psi(t - n))_{n \in \mathbb{Z}}$ span $W_0$, and $\varphi$ and $\psi$ are related according to

$$\varphi(t) = 2 \sum_n h_n \varphi(2t - n),$$

$$\psi(t) = 2 \sum_n (-1)^n h_{1-n} \varphi(2t - n)$$

where $2^{1/2} h_n$ are the expansion coefficients of $\varphi \in V_0 \subset V_1$ with respect to the orthonormal basis $(2^{1/2} \varphi(2t - n))_{n \in \mathbb{Z}}$ of $V_1$. The question that is raised in [1] is whether one can find a function $s(t) \in V_0$ such that any $f \in V_0$ can be represented as a cardinal series involving the translates of $s$ and the samples of $f$ at integer points, i.e.,

$$f(t) = \sum_n f(n) s(t - n), \quad t \in \mathbb{R}.$$  (2)

This question is dealt with by Walter under the assumption that $\varphi$ is continuous and that for some $\epsilon > 0$

$$|\varphi(t)| = O(|t|^{1-\epsilon}), \quad t \in \mathbb{R}.$$  (3)

Then, he shows that when the discrete Fourier transform

$$\Phi_d(w) = \sum_{n=\infty}^0 \varphi(n) e^{-2\pi i n w}$$

has no zeros, there is indeed such an $s$ and there is uniform convergence in (2) for all $f \in V_0$. The Fourier transform $S$ of $s$,

$$S(w) = \int_{-\infty}^{\infty} e^{-2\pi i w t} s(t) dt$$

is expressed by Walter in terms of the Fourier transform $\Phi$ of $\varphi$ and $\Phi_d$ in (4) as

$$S(w) = \Phi(w) \Phi_d(w).$$

Although it was not mentioned in [1], it can be shown that this $s$ satisfies $s(m) = \delta_{m,0}$, just as one would expect from an interpolating function. Walter then proceeds by estimating the aliasing error for functions $f = f_0 + f_1$ with $f_0 \in V_0, f_1 \in W_0$ in terms of the norm of the "out-of-space" component $f_1$ as

$$|e_f(t)|^2 = |f(t) - \sum_n f(n) s(t - n)|^2 \leq C \|f_1\|^2$$

where $C$ is a constant independent of $f$ and $t$.

The aim of this paper is to show that the Zak transform is a very appropriate tool to discuss this matter, especially when one is interested in sharp frame bounds and sharp estimates for the aliasing error $e_f$. We shall work under the slightly weaker assumption that $\varphi$ is bounded, and that

$$\sum_n |\varphi(t - n)|$$

converges uniformly in $t \in [0, 1]$.

We do not require continuity of $\varphi$ until Section III. As a consequence of these assumptions we have that $\varphi \in L^1(\mathbb{R})$. Under assumption (8) we have also that, [see (1)] $|h_n| < \infty$, and that $\sum_n |\varphi(t - n)|$ converges uniformly in $t \in [0, 1]$.

The Zak transform of an $f \in L^2(\mathbb{R})$ is defined as

$$\mathbb{Z}f(t, w) = \sum_{k=-\infty}^{\infty} e^{-2\pi i k w} f(t + k), \quad t, w \in \mathbb{R}.$$  (9)

Actually, (9) should be interpreted in an $L^2(\mathbb{R}^2)$-sense, but for the functions $f = \varphi, \psi, s$, it turns out that we can also interpret the right-hand side of (9) pointwise as an absolutely and locally uniformly convergent series. We refer to [2], [3] for the various properties and names of the mapping $\mathbb{Z}$. See Section III, in particular the quasi-periodicity relations (43).

We shall, also more generally, consider cardinal series

$$f(t) = \sum_n f(a + n) s_a(t - n), \quad t \in \mathbb{R}.$$  (10)
with \( a \in \mathbb{R} \) and \( f, s_a \in V_0 \). This extension is significant since, contrary to the classical Shannon case, the function \( s \) in (2) cannot be expected to be symmetric about the point 0. Also, it turns out that the condition number of the sampling operator \( f \in V_0 \rightarrow (f(n + a))_{n \in \mathbb{Z}} \in L^2 \) as well as the aliasing errors depend critically on the choice of \( a \).

The Fourier transform \( S_f(w) \) of \( s_f(t) \) in (10) turns out to be given by

\[
S_f(w) = \Phi(w)/(Z \Phi)(a, w)
\]

provided that \( (Z \Phi)(a, w) \neq 0, w \in \mathbb{R} \). Now it is an interesting property of Zak transforms that they have zeros in any unit square provided that they are continuous. (Under assumption (8) we have that \( Z \Phi \) is continuous under the weak condition that \( \varphi \) is continuous.) Hence one should choose \( a \) such that the set \( \{a\} \times \mathbb{R} \) avoids the zeros of \( Z \Phi \).

The above observation is made more specific as follows. We shall show that

\[
\sup \inf \sum_{a=-\infty}^{\infty} |f(a + n)|^2 = \max \min_w \frac{(Z \Phi)(a, w)}{(Z \Phi)(a, w)}
\]

(12)

As to the aliasing errors, we show the following results. When \( f \in V_1 \), we define the aliasing error \( e_{f,a}(t) \) by

\[
e_{f,a}(t) = f(t) - \sum_a f(a + n) s_a(t - n).
\]

Then, we have

\[
\sup \inf \left\| e_{f,a}(t) \right\|^2 = 1 + \max \min_w \frac{(Z \Phi)(a, w)}{(Z \Phi)(a, w)}
\]

(14)

while for any \( t \in \mathbb{R} \),

\[
\sup \inf \left\| e_{f,a}(t) \right\|^2 = \int_{-1/2}^{1/2} \frac{|(Z \Phi)(t, w) - (Z \Phi)(a, w) (Z \Phi)(t, w)|^2}{(Z \Phi)(a, w)} dw,
\]

(15)

\[
\min \sup \left\| e_{f,a}(t) \right\|^2 = 0.
\]

(16)

For (13)–(16) it is assumed that \( (Z \Phi)(a, w) \neq 0, w \in \mathbb{R} \) and \( a \neq 0 \). Note that, even in the case \( a = 0 \), the Zak transform arises naturally in the expression (15) for the worst-case aliasing error.

The explicit expressions for the bounds in (12), (14), (15), should be used to guide the choice of \( a \). For instance, one could look for that particular \( a \) for which

\[
\frac{\max_w |(Z \Phi)(a, w)|}{\min_w |(Z \Phi)(a, w)|} \quad \text{or} \quad \frac{\max_w |(Z \Phi)(a, w)|}{\min_w |(Z \Phi)(a, w)|}
\]

is as low as possible so as to obtain the best bound for the condition number of the sampling operator or the lowest value for the worst-case \( L^2 \)-aliasing error.

In Section III we shall show that

\[
\begin{bmatrix}
(Z \Phi)(t, w) \\
(Z \Phi)(t, w)
\end{bmatrix} = U_h(w)
\begin{bmatrix}
(Z \Phi)(2t, \frac{1}{2} w) \\
(Z \Phi)(2t, \frac{1}{2} (w + 1))
\end{bmatrix}
\]

(18)

where \( U_h(w) \) is the unitary matrix

\[
U_h(w) =
\begin{bmatrix}
\frac{1}{2} w & \frac{1}{2} (w + 1) \\
- \exp(-\pi i w) h^*(\frac{1}{2} w) & \exp(-\pi i w) h^*(\frac{1}{2} w)
\end{bmatrix}
\]

(19)

and with \( h(w) \) defined as, see (1),

\[
h(w) = \sum_n h_n e^{-\pi i n w}.
\]

(20)

Since, [see (43)],

\[
(Z \Phi)(2t, \frac{1}{2} (w + 1)) = - \exp(\pi i w) (Z \Phi)
\]

\[
\cdot (2t - 1, \frac{1}{2} (w + 1))
\]

(21)

we thus see that there is a remarkable connection between Zak transforms of scaling functions and wavelets on one hand and the well-known baker's transformation

\[
(t, w) \in [0, 1] \rightarrow \begin{cases}
(2t, \frac{1}{2} w), & 2t \leq 1 \\
(2t - 1, \frac{1}{2} (w + 1)), & 2t > 1
\end{cases}
\]

(22)

on the other.

Formula (18) shall be used to show that, when \( Z \Phi \) is continuous, there are points where \( Z \Phi \) vanishes while \( Z \Psi \) does not. This implies that the second quantity in (17) is unbounded, so that the choice of \( a \) really matters.

II. Derivations

In this section we present the proofs of the results just announced. We start with the proof of (12). Let \( f \in V_0 \) and write

\[
f(t) = \sum_k \alpha_k \varphi(t - k), \quad \sum_k |\alpha_k|^2 = \|f\|^2.
\]

(23)

Since \( \varphi(t - k)_{k \in \mathbb{Z}} \in l^1(\mathbb{Z}) \) for all \( t \), we can interpret (23) pointwise, and we have

\[
\sum_n |f(a + n)|^2 = \sum_{k,l} \alpha_k \alpha_l^* \sum_n \varphi(a + n - k) \varphi^*(a + n - l)
\]

(24)

with an absolutely convergent triple series at the right-hand side. (That this latter series converges absolutely follows from the fact that \( \sum_n |\alpha_k| |\varphi(a + n - k)|_{k \in \mathbb{Z}} \in l^1(\mathbb{Z}) \) as a convolution of an \( l^1(\mathbb{Z}) \)-sequence and an \( l^1(\mathbb{Z}) \)-sequence.) Hence the required sup and inf can be expressed as the maximum and minimum of the spectrum of the infinite Toeplitz matrix

\[
(b_k)_{k \in \mathbb{Z}} \quad \text{where} \quad b_k = \sum_n \varphi(a + n - k) \varphi^*(a + n).
\]

(25)
It is well known, and easy to prove, that this maximum and minimum are the ess sup and ess inf over [0, 1] of the function

$$\sum_k b_k e^{2\pi ikw} = |(Z\varphi)(a, w)|^2.$$  \hfill (26)

The proof of (12) is now completed by noting that (Z\varphi)(a, w) is periodic in w by (43) and continuous in w by (8).

Before starting the proofs of (14)-(16) we collect some useful facts about the Zak transform and the expansion coefficients of $s(t)$.

**Lemma 1:** Suppose that $a \in \mathbb{R}$ is such that (Z\varphi)(a, w) $\neq 0$, $w \in \mathbb{R}$. Then we have

$$s_n(t) = \sum_n \sigma_{n,a} \varphi(t - n)$$  \hfill (27)

where

$$\sigma_{n,a} = \int_0^1 e^{2\pi i nw} d\nu.$$  \hfill (28)

Furthermore, $\Sigma_a |\sigma_{n,a}| < \infty$, the series $\Sigma_n |s_n(t + n)|$ converges uniformly, and

$$(Zs)(t, w) = (Z\varphi)(t, w)/(Z\varphi)(a, w), \quad t, w \in \mathbb{R}.$$  \hfill (29)

**Proof:** We have as in the proof of [1, Section III, Theorem] that

$$S_n(w) = \Phi(w)/(Z\varphi)(a, w) = \Phi(w) \sum_n \sigma_{n,a} e^{2\pi i nw}.$$  \hfill (30)

Here we have $\Sigma_a |\sigma_{n,a}| < \infty$ by Wiener's theorem since (Z\varphi)(a, w) $\neq 0$, $w \in \mathbb{R}$, has an absolutely convergent Fourier series (8). This implies (27) and the uniform convergence of the series $\Sigma_n |s_n(t + n)|$. Also formula (29) follows easily.

Consequences:

1) For any $f \in V_0$ the cardinal series at the right-hand side of (10) is uniformly convergent.
2) By taking $f(t) = \varphi(t - m)$ in (10) one obtains

$$s_n(a + m) = \delta_{n,m}, \quad m \in \mathbb{Z}.$$  \hfill (31)

We now show (14)-(16). When $f \in W_0$ we have in a similar fashion as in (12) that

$$\sum_n |f(n + a)|^2 \leq \|f\|^2 \max_w |(Z\varphi)(a, w)|^2 < \infty.$$  \hfill (32)

Hence $e_{f,a}(t)$ is pointwise well-defined as an absolutely convergent series. For the proof of (14) we restrict ourselves first to

$$f(t) = \sum_k \beta_k \psi(t - k)$$  \hfill (33)

where $\beta_k \neq 0$ for only finitely many $k$, so that $\Sigma_k |f(a + n)| < \infty$. Since $f$ and $s(t - n)$ are orthogonal for all $n \in \mathbb{Z}$ we have

$$\|e_{f,a}\|^2 = \|f\|^2 + \sum_{n,k} f(n + a) f^*(m + a) r_{nm}$$  \hfill (34)

where

$$r_{nm} = \int_{-\infty}^{\infty} s(t - n) s(t - m) dt = \sum_k \sigma_{k,n,a} \sigma_{k,m,a}^*.$$  \hfill (35)

In (35) the orthonormality of the $\varphi(t - n)$'s and (27) have been used. It readily follows from Parseval's formula for Fourier series and (27) and (28) that

$$\|e_{f,a}\|^2 \leq \|f\|^2 + \sum_k \left( \frac{(Z\varphi)(a, w)}{(Z\varphi)(a, w)} \right)^2$$  \hfill (36)

with pointwise defined Zf. Also, from (33)

$$(Z\varphi)(a, w) = (Z\psi)(a, w) \sum_k \beta_k e^{-2\pi ikw}.$$  \hfill (37)

The set of all functions

$$\left| \sum_k \beta_k e^{-2\pi ikw} \right|^2$$  \hfill (38)

with $\Sigma_k |\beta_k|^2 = 1$, $\beta_k \neq 0$ for only finitely many $k$, is dense in the set of all nonnegative $L^1$-functions with unit $L^1$-norm. It then follows from continuity of $(Z\psi)(a, w), (Z\varphi)(a, w)$ as a function of $w$ that

$$\|f\|^2 \left( 1 + \max_w |(Z\varphi)(a, w)|^2 \right) \leq \|e_{f,a}\|^2$$  \hfill (39)

for all $f$ of the considered type. It is now elementary Hilbert space theory to conclude that (39) holds for all $f \in W_0$. This completes the proof of (14).

We finally prove (15) and (16). We have for $f = \sum_k \beta_k \psi(t - k) \in W_0$ where $\beta \in l^1$ by the Cauchy–Schwarz inequality

$$\|e_{f,a}\|^2 = \left( \sum_k \beta_k \right)^2 \left( \sum_n |s_n(t - k)|^2 \right)$$  \hfill (40)

with equality when

$$(\beta_k)_{k \in \mathbb{Z}} \quad \text{and} \quad \left( \gamma_k : = \psi(t - k) - \sum_n \psi(n + a - k) s_n(t - n) \right)_{k \in \mathbb{Z}}.$$  \hfill (41)
are linearly dependent. We next calculate
\[
\sum_{k} |\gamma_k|^2 = \int_{0}^{1} |(Z\psi)(t, w) - (Z\psi)(a, w)(Z\delta)(t, w)|^2 \, dw
\]
\[
= \int_{0}^{1} |(Z\psi)(t, w) - (Z\psi)(a, w)(Z\delta)(t, w)|^2 \, dw
\]
where we have used (29). This proves (15). Finally (16)
follows by taking
\[
\int_{0}^{1} |(Z\psi)(t, w) - (Z\psi)(a, w)|^2 \, dw
\]
under the additional assumption that
\[
\int_{0}^{1} |(Z\psi)(t, w)|^2 \, dw
\]
has been used that \( h(0) = 1, h(1/2) = 0 \). Consequently, by continuity of \( (Z\psi)(t, 0) \) at \( t = 0 \), we have that \( (Z\psi)(t, 0) \) is constant. This constant must be equal to one since \( 1 = \Phi(t) = \int_{0}^{1} (Z\psi)(t, 0) \, dt \). Hence,
\[
(Z\psi)(t, 0) = 1, \quad t \in \mathbb{R}. \quad (49)
\]
In a similar fashion it follows that
\[
(Z\psi)(t, o) = (Z\psi)(2t, \frac{1}{2}) \quad (50)
\]
where \( h(t) = 1 \) and \( h(0) = 0 \) for at least two \( t \in [0, 1] \), (40), (41).

We next show from (18) that there is an \( (a, w) \) such that \( Z\phi \) vanishes at \( (a, w) \) while \( Z\psi \) does not. To that end we suppose that \( Z\psi \) vanishes everywhere where \( Z\phi \) does, and derive a contradiction. When \( (a, w) \) is such that
\[
(Z\psi)(a, w) = (Z\psi)(a, 0) = 0, \quad \text{then}
\]
\[
(Z\phi)(2a, \frac{1}{2} w) = (Z\phi)(2a, \frac{1}{2} (w + 1)) = 0 \quad (51)
\]
by unitarity of \( U_{\phi}(w) \) in (19). Repeating this argument, we can find sequences \( (a_n, w_n) \in \mathbb{Z}^2 \) such that \( (Z\phi)(a_n, w_n) = 0 \), while \( w_n \to 0, n \to \infty \). This, however, contradicts (49) and the uniform continuity of \( Z\phi \).

IV. Examples

A. Example 1
Consider [1. Section IV, Example 3] with \( n = 2 \). Hence the Fourier transform \( \Phi \) of the scaling function \( \phi \) is given by
\[
\Phi(w) = \frac{\Theta_{1}(w)}{\Sigma_{1}^{1/2} (w)} \quad (52)
\]
where \( \Theta_{1} \) is the Fourier transform of the second order basic spline \( v_{2} \), and the orthogonalizing function \( \Sigma_{1} \) is given by
\[
\Sigma_{1}(w) = \sum_{k=-\infty}^{\infty} |\Theta_{1}(w + k)|^2. \quad (53)
\]
We have explicitly,
\[
\Theta_{1}(w) = \frac{1 - e^{-2\pi w}}{2\pi w}, \quad (46)
\]
and, using the method explained in [4. Section 5.4],
\[
\Sigma_{1}(w) = (\sin \pi w)^6 \sum_{k=-\infty}^{\infty} \left( \frac{1}{\pi (w + k)} \right)^6 \quad (54)
\]
and
\[
\delta_{2}(t) = \begin{cases} \frac{1}{2} t^2, & 0 \leq t \leq 1 \\ \frac{1}{2} (t - \frac{1}{2})^2, & 1 \leq t \leq 2 \\ \frac{1}{2} (3 - t)^2, & 2 \leq t \leq 3 \end{cases} \quad (54)
\]
and
\[
\Sigma_{2}(w) = (\sin \pi w)^6 \sum_{k=-\infty}^{\infty} \left( \frac{1}{\pi (w + k)} \right)^6 \quad (55)
\]
By periodicity of \( \Sigma_{1} \) we have
\[
(Z\phi)(a, w) = (Z\phi)(a, 0)/\Sigma_{1}^{1/2}(w) \quad (56)
\]
Fig. 1. The quantity \( \max_a \frac{|(Z\varphi)(a, w)|^2}{\min_a |(Z\varphi)(a, w)|^2} \) as a function of \( a \in [0, 1] \) for \( \varphi(t) = \varphi(3 - t) \) (Daubechies scaling function) so that, see (11),

\[
S_\psi(w) = \frac{\Phi(w)}{(Z\varphi)(a, w)} = \frac{\Theta_2(w)}{(Z\partial_2)(a, w)}.
\]

Furthermore, by (54), we have for \( a \in [0, 1] \)

\[
(Z\partial_2)(a, w) = \frac{1}{2} a^2 + \left( \frac{3}{4} - (a - \frac{1}{2})^2 \right) e^{-2iw} + \frac{i}{2} (1 - a)^3 e^{-4iw}.
\]

Hence when \( a = 0 \) we see that

\[
(Z\varphi)(0, \frac{1}{2}) = (Z\partial_2)(0, \frac{1}{2})/\sqrt{1/2} = 0.
\]

We shall show that

\[
(Z\varphi)(0, \frac{1}{2}) \neq 0.
\]

Indeed, when \( (Z\varphi)(0, 1/2) = 0 \), we would find, see (51), that \( (Z\varphi)(0, 1/4) = 0 \) as well. However, from (58) we have

\[
(Z\partial_2)(0, \frac{1}{2}) = -\frac{1}{2} - \frac{1}{2} i \neq 0.
\]

(In fact it can be shown that \( (Z\varphi)(0, 1/2) = -\sqrt{15}/8 \)). When \( a = 1/2 \), we obtain

\[
(Z\partial_2)(1/2, w) = e^{-2iw} \left( \frac{3}{4} + \frac{1}{2} \cos 2w \right)
\]

which vanishes for no real value of \( w \). Hence, we can base a stable interpolation formula on the sample values \( f(1/2 + n) \) of signals \( f \in V_0 \) using the interpolating function \( s_{1/2}(t) \) whose Fourier transform \( S_{1/2}(w) \) is given by (57). Such a thing is not possible when \( a = 0 \): the interpolating function \( s_0(t) \) has a singular Fourier transform \( S_0(w) \), and the quantities in (17) are infinite.

**B. Example 2**

Consider [1, Section IV, Example 5] with \( v = +1/\sqrt{3} \). In Figs. 1 and 2 we have plotted the quantities (17) for the condition number of the sampling operator and the worst-case \( L^2 \)-aliasing error as a function of \( a \in [0, 1] \). Clearly, the first quantity cannot be less than unity and the second one cannot be less than zero; for \( a = 0.37 \) we see that both quantities come very close to their lower bounds. Hence, \( a \approx 0.37 \) would give very good interpolation results. On the other hand, \( a \approx 0.8 \) gives very large values for these quantities, and thus bad interpolation results.

**Acknowledgments**

The author wishes to thank M. Klompstra for producing the figures. Also, the author has profited from a discussion with G. G. Walter, who observed that the first quantity in (17) for Example 2 has a minimum value (as a function of \( a \)) slightly larger than unity.

**References**


A. J. E. M. Janssen (SM'88) received the Eng. degree and Ph. D. degree in mathematics from the Eindhoven University of Technology, Eindhoven, The Netherlands in 1976 and 1979, respectively. From 1979 to 1981, he was a Bateman Research Instructor at the Mathematics Department of California Institute of Technology, Pasadena and joined the Philips Research Laboratories, Eindhoven, in 1981. His research interest is in mathematical analysis, in particular Fourier analysis and time-frequency analysis.