Self-synchronization and controlled synchronization

Citation for published version (APA):

DOI:
10.1109/COC.1997.633464

Document status and date:
Published: 01/01/1997

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
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Download date: 09. Dec. 2019
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Abstract
An attempt is made to give a general formalism for
synchronization in dynamical systems encompassing
most of the known definitions and applications. The
proposed set-up describes synchronization of intercon-
nected systems with respect to
a
set of functionals and
captures peculiarities of both self-synchronization and
controlled synchronization. Various illustrative exam-
pies are given.

1 Introduction
Starting with the work of C. Huygens [13] synchro-
nization phenomena attracted attention of many re-
searchers. The development of small parameter and
averaging methods by H. Poincaré [23], B. Van der
Pol [25], N.N. Bogolyubov [7] in the first half of the
20th century allowed for a better understanding
and theoretical explanation of the mechanism of self-
synchronization [2, 3, 12]. Incomplete information
about the system parameters has been taken into account
(adaptive and robust synchn
ronization [9, 10, 18]) as
well as incomplete information of
the state of the system (observer-based synchronization
[19, 20]). The problems of how to cope with uncertain-
ties and incomplete measurements are traditional in
control theory. However, experts from different fields
understand synchronization in different ways which in
turn requires additional efforts to apply conventional
control methods. Therefore, there is a need for unified
definitions of synchronization which capture peculiar-
ities of both self-synchronization and controlled syn-
chronization and which allow to pose and solve various
synchronization problems. In this paper general defini-
tions of synchronization are introduced and discussed
in Section 2. Two large classes of synchronization prob-
lems are extracted, namely frequency synchronization
(Section 3) and coordinate synchronization (Section 4).
A number of examples demonstrating the potential of
the introduced definitions is given.

2 Definitions of synchronization
Synchronization in its most general interpretation
means correlated or corresponding in time behavior of
two or more processes. According to [17]: “to synchro-
nize” means to concur or agree in time, to proceed or
to operate at exactly the same rate, to happen at the
same time. Below we formalize the above description and also formulate a "controlled" version. To this end consider $k$ dynamical systems

$$\Sigma_i = \{T, U_i, X_i, Y_i, \phi_i, h_i\}, \quad i = 1, \ldots, k$$

where $T$ is the common set of time instances, $U_i, X_i, Y_i$ are sets of inputs, states and outputs, respectively; $\phi_i : T \times X_i \times U_i \to X_i$ are transition maps; $h_i : T \times X_i \times U_i \to Y_i$ are output maps. (We use here one of the standard definition of dynamical system, see e.g. [21, 14]). First consider the case when all $U_i$ are just singletons, i.e. inputs are not present and may be omitted in the formulation. Suppose $l$ functionals $g_j : Y_1 \times Y_2 \times \ldots \times Y_k \times T \to \mathbb{R}^l, j = 1, \ldots, l$, are given. Here $Y_i$ are the sets of all functions from $T$ into $Y_i$, i.e. $Y_i = \{y : T \to Y_i\}$. In the sequel, we take as time set $T$ either $T = \mathbb{R}_{\geq 0}$ (continuous time) or $T = \mathbb{Z}_{\geq 0}$ (discrete time). For any $\tau \in T$ we then define $\sigma_\tau$ as the shift operator, i.e. $\sigma_\tau : Y_i \to Y_i$ is given as $\sigma_\tau y(t) = y(t + \tau)$ for all $y \in Y_i$ and all $t \in T$.

**Definition 2.1** We call the solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ synchronized with respect to the functionals $g_1, \ldots, g_l$ if

$$g_j(\sigma_\tau y_1(\cdot), \ldots, \sigma_\tau y_k(\cdot), t) \equiv 0, \quad j = 1, \ldots, l \quad (1)$$

is valid for all $t \in T$ and some $\tau_1, \ldots, \tau_k \in T$, where $y_i(\cdot)$ denotes the output function of the system $\Sigma_i$: $y_i(t) = h_i(x_i(t), t), t \in T, i = 1, \ldots, k$. We say that the solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are asymptotically synchronized with respect to the functionals $g_1, \ldots, g_l$, if there are an $\epsilon > 0$ and $\tau_1, \ldots, \tau_k \in T$ such that

$$|g_j(\sigma_\tau y_1(\cdot), \ldots, \sigma_\tau y_k(\cdot), t)| \leq \epsilon, \quad j = 1, \ldots, l \quad (2)$$

for all $t \in T$. The solutions $x_1(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ with initial conditions $x_1(0), \ldots, x_k(0)$ are asymptotically synchronized with respect to the functionals $g_1, \ldots, g_l$, if for some $\tau_1, \ldots, \tau_k \in T$

$$\lim_{t \to \infty} g_j(\sigma_\tau y_1(\cdot), \ldots, \sigma_\tau y_k(\cdot), t) = 0, \quad j = 1, \ldots, l \quad (3)$$

If the synchronization phenomenon is achieved for all initial conditions $x_1(0), \ldots, x_k(0)$ it is possible to say that the systems $\Sigma_1, \ldots, \Sigma_k$ are synchronized (in the appropriate sense with respect to the given functionals). In the case of asymptotic synchronization it is also possible to define the basins of the initial conditions which yield synchronization. In the sequel, we will only consider the case when synchronization is achieved for all initial conditions. Although this definition is rather general, it can be further generalized. For example in many practical problems the time shifts $\tau_i, i = 1, \ldots, k$ are not constant but tend to constant values, so called "asymptotic phases". In this case, instead of the shift operator for each output function $y_i(\cdot)$ it is convenient to consider the time varying shift operator defined as follows

$$(\sigma_{\tau_i})y(t) = y(t_{\tau_i}(t))$$

where $t_{\tau_i}(t) = t \iff t_{\tau_i}(t) \leq t$ is proposed which, however allows for infinitely large phase shifts. In many practical synchronization problems the spaces $Y_i = \mathcal{Y}$ and the functionals $\{g_{j_{rs}}\}$ are chosen to compare similar characteristics of different systems, e.g.

$$g_{j_{rs}}(y_r(\cdot), y_s(\cdot)) = \text{dist}(J_j(\sigma_{\tau_r} y_r(\cdot)), J_j(\sigma_{\tau_r} y_s(\cdot)))$$

where $r, s = 1, \ldots, k, \quad j = 1, \ldots, l$ and $J_j : \mathcal{Y} \times \mathcal{Y} \to \mathcal{J}$, are some mappings (synchronization indices) which map the (output) trajectory $y_j(\cdot)$ of each system $\Sigma_1, \ldots, \Sigma_k$, into some metric space $\mathcal{J}_j$. This will be referred as synchronization with respect to the indices $\{J_j\}$. The specific choice of the synchronization indices depends on the essence of the mathematical, physical or engineering problem. The same is valid for the phase shifts $\tau_i$ which may be fixed in some problems and may be arbitrary in others. Naturally, the possibility of efficient solutions of the synchronization problems crucially depends on the chosen functionals and/or indices.

**Remark 1.** Note that instead of a set of the functionals it is always possible to take one functional which expresses the same synchronization phenomenon. For example one can take the functional $G$ as follows

$$G(y_1(\cdot), \ldots, y_k(\cdot), t) = \sum_{j=1}^{l} g_j(y_1(\cdot), \ldots, y_k(\cdot), t). \quad (5)$$

**Remark 2.** In applications of synchronization it is important to require that the conditions (1)-(3) are not violated (or not significantly violated) when some parameters of the systems are varied in some range. In other words the properties the (1)-(3) should be robust but in this case the phase shifts may be not constant and even not tend to constant values; however, the following condition

$$\lim_{t \to \infty} |t_{\tau_i}(t) - t| \leq \tau_i$$

may be imposed instead of (4).

In many cases the sets $U_i, X_i, Y_i$ are finite-dimensional vector spaces and the systems $\Sigma_i$ can be described by ordinary differential equations. First consider the simplest case of disconnected systems without inputs:

$$\Sigma_i : \quad dx_i \over dt = F_i(x_i, t), \quad (6)$$
where $F_i$, $i = 1, \ldots, k$ are some time-dependent vector fields. Sometimes synchronization may occur in disconnected systems (6) (e.g. all precise clocks are synchronized in the frequency sense). This case will be referred to as natural synchronization. A more interesting and important case, however, seems synchronization of interconnected systems. In this case the system models are augmented with interconnections and look as follows

$$\begin{align*}
\frac{dx_i}{dt} &= F_i(x_i, t) + \tilde{F}_i(x_0, x_1, \ldots, x_k, t), \quad i = 1, \ldots, k \\
\frac{dx_0}{dt} &= F_0(x_0, x_1, \ldots, x_k, t)
\end{align*}$$

(7)

where the vector field $F_0$ describes the dynamics of the interconnection system, $\tilde{F}_i$ are vector fields describing the interconnections. For example in the synchronization of generators of a power station this interconnection is caused by a common electrical load. The model (7) can formally not be considered within the given definition. To include the case of synchronization of interconnected systems we should introduce a dynamical systems which describes interconnections between the systems. Recall that in the previous definition we supposed that the inputs of each system $\Sigma_i$, $i = 1, \ldots, k$ are just singletons. To describe the possible interconnections we suppose now that the input of each system $\Sigma_i$, $i = 1, \ldots, k$ can be composed from the output of the interconnected system $\Sigma_0 = \{T, U_0, X_0, Y_0, \phi_0, h_0\}$ where the transition and output maps are given by $\phi_0: T \times X_0 \times U_0 \rightarrow X_0$ and $h_0: T \times X_0 \times U_0 \rightarrow Y_0$ with $U_0 = Y_1 \times Y_2 \times \ldots \times Y_k$ and $Y_0 = U_1 \times U_2 \times \ldots \times U_k$. Now it is possible to define synchronization of interconnected systems.

Definition 2.2 We call the solutions $x_0(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ and interconnection system $\Sigma_0$ with initial conditions $x_0(0), \ldots, x_k(0)$ synchronized with respect to the functionals $g_1, \ldots, g_l$ if

$g_j(\sigma_{\tau_0} y_0(\cdot), \ldots, \sigma_\tau y_k(\cdot), t) = 0, j = 1, \ldots, l$ \hspace{1cm} (8)

is valid for all $t \in T$ and some $\tau_0, \ldots, \tau_k \in T$, where $y_i(\cdot)$ denotes the output function of the system $\Sigma_i$: $y_i(t) = h_i(x_i(t), t), t \in T, i = 0, \ldots, k$. We say that solutions $x_0(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ and interconnection system $\Sigma_0$ with initial conditions $x_0(0), \ldots, x_k(0)$ are approximately synchronized with respect to the functionals $g_1, \ldots, g_l$, if there are an $\varepsilon > 0$ and $\tau_0, \ldots, \tau_k \in T$ such that

$|g_j(\sigma_{\tau_0} y_0(\cdot), \ldots, \sigma_\tau y_k(\cdot), t)| \leq \varepsilon, j = 1, \ldots, l$ \hspace{1cm} (9)

for all $t \in T$. The solutions $x_0(\cdot), \ldots, x_k(\cdot)$ of the systems $\Sigma_1, \ldots, \Sigma_k$ and interconnection system $\Sigma_0$ with initial conditions $x_0(0), \ldots, x_k(0)$ are asymptotically synchronized with respect to the functionals $g_1, \ldots, g_l$, if for some $\tau_0, \ldots, \tau_k \in T$

$$\lim_{t \to \infty} g_j(\sigma_{\tau_0} y_0(\cdot), \ldots, \sigma_\tau y_k(\cdot), t) = 0, j = 1, \ldots, l$$

(10)

A remarkable and widely used observation is that the synchronization may exist, i.e. identity (8) may be valid in the interconnected system without any artificially introduced external action, i.e. when the interconnection system $\Sigma_0$ is given. In this case the system (7) can be called self-synchronized with respect to the functionals $g_1, \ldots, g_l$ or with respect to the indices $J_1, \ldots, J_k$. Similar definitions can be introduced for approximate and asymptotic self-synchronization. In cases important for applications the interconnections between the systems $\Sigma_1, \ldots, \Sigma_k$ are weak, for instance when (7) can be represented as follows

$$\begin{align*}
\frac{dx_i}{dt} &= F_i(x_i, t) + \mu \tilde{F}_i(x_0, x_1, \ldots, x_k, t), \quad i = 1, \ldots, k \\
\frac{dx_0}{dt} &= F_0(x_0, x_1, \ldots, x_k, t)
\end{align*}$$

(11)

where $\mu$ is a small parameter. Therefore finding conditions for self-synchronization in systems with small interactions is of special interest. Such conditions were found for a large class of dynamical systems (11) in particular with time-periodic vector fields $F_i$, [2, 3]. However, in many cases self-synchronization is not observed and the question arises: is it possible to affect, i.e. to control the system in such a way that the goal (2) or (3) can be achieved? The above definitions do not yet include the possibility of controlling the system. Assume for simplicity that all $\Sigma_i$, $i = 0, \ldots, k$ are smooth finite dimensional systems, described by differential equations with a finite-dimensional input, i.e.

$$\begin{align*}
\frac{dx_i}{dt} &= F_i(x_i, t) + g_j(x_0(t), \ldots, x_k(t), u(t)), \quad i = 1, \ldots, k \\
\frac{dx_0}{dt} &= F_0(x_0, x_1, \ldots, x_k, u, t)
\end{align*}$$

(12)

where $u = u(t) \in \mathbb{R}^m$ is the input or control variable.

Definition 2.3 The problem of controlled synchronization with respect to the functionals $g_j, j = 1, \ldots, l$ (respectively, controlled asymptotic synchronization with respect to the functionals $g_j, j = 1, \ldots, l$) is to find a control $u$ as a feedback function of the states $x_0, x_1, \ldots, x_k$ and time providing that (1) (respectively, (2), (9)) holds for the closed loop system.

The problem of controlled synchronization with respect to indices $J_1, \ldots, J_k$ is formulated similarly. Sometimes the goal can be ensured without measuring any variables of the systems, for instance by a time-periodic forcing. In this case the control function $u$ does not depend on system states and the problem of finding such a control is called an open loop controlled (asymptotic) synchronization problem. However, a more powerful approach assumes the possibility of measuring the states or some function of the system variables. Finding a control function in this case is called a closed loop or (asymptotic) feedback synchronization problem. The simplest form of feedback is static state feedback where the controller equation is as follows

$$u(t) = U(x_0, x_1, \ldots, x_k, t)$$

(13)
for some function \( U : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k} \times T \to \mathbb{R}^m \)

A more general form is *dynamic state feedback*

\[
\frac{d\mathbf{x}}{dt} = W(x_0, x_1, \ldots, x_k, w, t) \\
u(t) = U(x_0, x_1, \ldots, x_k, w, t)
\]

with \( w \in \mathbb{R}^p, W : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k} \times \mathbb{R}^p \times T \to \mathbb{R}^r, U : \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k} \times \mathbb{R}^p \times T \to \mathbb{R}^m \).

Now the problem of *state feedback synchronization* consists of finding a control law (13), (or (14), (15)) ensuring the asymptotic synchronization (3) in the closed loop system (12), (13) (or respectively, (12), (14), (15)).

**Remark 3.** Controlled synchronization becomes relevant only in cases when self-synchronization (1) does not occur and the inclusion of a static or dynamic state feedback (13) or (14), (15) will only lead to (1) after some transient behavior. Therefore we will only be concerned with asymptotic feedback synchronization (3).

In a variety of problems complete information about the states of the systems \( \Sigma_0, \Sigma_1, \ldots, \Sigma_k \) is not available and only some output variables \( \bar{y}_s, s = 1, \ldots, r \), with \( \bar{y}_s \) output functions of the interconnected system, so \( \bar{y}_s = \tilde{h}_s(x_0, x_1, \ldots, x_k, t) \), are available for use in the control law. The problem of *output feedback synchronization* can be posed as follows: find controller equations in the form of static output feedback

\[
u(t) = U(\bar{y}_1, \ldots, \bar{y}_r, t)
\]

or in the form of dynamic output feedback

\[
\frac{d\mathbf{y}}{dt} = W(\bar{y}_1, \ldots, \bar{y}_r, w, t) \\
u(t) = U(\bar{y}_1, \ldots, \bar{y}_r, w, t)
\]

with \( w \in \mathbb{R}^p, y_s \in \mathbb{R}^p, W : \mathbb{R}^p \times \ldots \times \mathbb{R}^p \times \mathbb{R}^p \times T \to \mathbb{R}^r, U : \mathbb{R}^p \times \ldots \times \mathbb{R}^p \times \mathbb{R}^p \times T \to \mathbb{R}^m \), are smooth parametrized vectorfields resp. functions, such that the goal (3) in system (12), (16) (or (12), (17), (18)) is achieved. To illustrate the given definitions we will discuss some special cases.

### 3 Frequency (Huygens) synchronization

A frequency synchronization property, or Huygens synchronization property may be defined for periodic (oscillatory or rotational) motions with frequencies \( \omega_1, \ldots, \omega_k \). The frequency synchronization is understood (see [2, 3, 4, 5, 6]) as a coincidence or, more generally, *commensurability* of \( \omega_i \), i.e. the following relations should be fulfilled

\[
\omega_i = n_i \omega_s, \quad i = 1, \ldots, k
\]

for some integers \( n_i \) where \( \omega_s > 0 \) is the so called synchronous frequency. A typical and characteristic situation where this type of asymptotic frequency synchronization occurs, may be described as follows. Consider a one-dimensional movable platform with two pendulum clocks mounted on the platform. Of course, one could replace the clocks by other devices like unbalanced rotors ([(2)] but the clock example was already observed by Huygens ([13]), long before a detailed mathematical model for the system was at hand! Assume the horizontal motion of the platform is constrained by a spring and linear damper. Then, a displacement of the platform in horizontal direction, creates an oscillatory horizontal motion for the platform, and this in turn forces the two pendulum clocks to swing. Clearly, within the given configuration, there is no direct kinematic linking between the two clocks. Nevertheless, it turns out that the two pendula may rapidly exhibit frequency synchronization, i.e. (19) holds for the two clocks, with \( n_i, i = 1, 2, \) equal +1 or −1, see [4] for an explanation, and further generalizations. In case (19) holds, a single synchronization index is introduced as

\[
J(y_s(t)) = \omega_i
\]

while the functionals can be chosen as

\[
g_{ww}(y_s(t), y_r(t)) = \frac{\omega_s - \omega_r}{n_s - n_r}
\]

This version of synchronization can be extended to non-periodic motions if some kind of average frequencies \( \omega_i \) can be defined. Note that the case when the relations (19) hold, is usually referred in the celestial mechanics to as the *resonance*, or *commensurability* case when speaking about orbital or rotational motions of celestial bodies. Also the "piecewise-periodic" case can be considered. In this case the set of all time instances is split into disjoint intervals \( \Delta q = [t_p, t_{p+1}], q = 1, 2, \ldots \) such that all motions \( y_s(t) \) are periodic on each interval \( \Delta q \) with frequencies \( \omega_i(t) \) which are piecewise constant functions. An extended version of Huygens synchronization arises if we replace the requirement of coincidence of the average frequencies by that of agreement of spectra in the following sense. Introduce positive spectra scaling functions \( \alpha_i(\omega), \beta_i(\omega) \) for each system \( \Sigma_i, i = 1, \ldots, k \), and define the family of synchronization indices \( J_\omega \) as follows

\[
J_\omega(y_s(t)) = \alpha_i(\omega)S_i(\beta_i(\omega)\omega)
\]

where \( S_i \) is the spectral density of the output signal \( y_s(t) \) which is supposed to be well defined. The agreement of spectra can be understood as synchronization with respect to the family of functionals

\[
g_{ww}(y_s(t), y_r(t)) = ||J_\omega(y_s(t)) - J_\omega(y_r(t))||
\]

for some appropriate norm \( || \cdot || \), e.g. \( L_2 \)-norm.

A good example of such kind of synchronization is provided by a color music system. A possible description is as follows. The color system device modulates the light sources by sound signal. Human acoustic and optical analyzers evaluate the power spectra \( S_{\text{sound}}(\omega) \)
and \( S_{\text{color}}(\omega) \). Then the human brain evaluates some measure of difference
\[
g = ||S_{\text{sound}}(\omega) - \alpha(\omega)S_{\text{color}}(\beta(\omega)\omega)||
\]
where \( \alpha(\omega), \beta(\omega) \) are the scaling multipliers. The feeling of synchronization is determined by a weighted average of \( g(\omega) \) over the audio frequency band. Note that the spectra of real (e.g. audio) signals change with time. Therefore in practice the spectra of the systems:

\[
\text{average of synchronization is determined by}
\]
and may introduce more indices

\[
\text{corresponding coordinates of the other systems for}
\]
and the sections at the \( q \)-th time. If for any given \( q \), \( t_q \)

occurs if
\[
\text{coordinates of the ith system intersects the curve of the}
\]
and may be asynchronous from another.

Sometimes we need to deal with discrete coordinate synchronization when the coincidence of the outputs (or whole state vectors) only takes place at some discrete set of time instances \( \{t_q\} \). In this case, the index \( J(y_1(\cdot)) \) may be defined as the sequence
\[
J(y_1(\cdot)) = \{y_1(t_1), y_1(t_2), \ldots \}
\]
while the functionals \( g_q \) are chosen using some metric in the space of sequences, e.g. uniform or \( \ell^p \)-metric.

A version of discrete-time coordinate synchronization occurs if \( t_q \) is the time instance when some coordinates or outputs \( y_1(t) \) approach some prespecified point or cross some given surface or level. Also \( t_q \) may be defined as the time of achieving the \( q \)-th local extremum of the signal. This kind of coordinate synchronization can be described similarly to the definition of the Poincaré map. Assume that at some time instances \( t_{q,i} \) for \( i = 1, \ldots, k \); \( q = 1, 2, \ldots \) solutions of each system satisfy the condition \( \varphi_i(y_i(t_{q,i})) = 0 \) (i.e. the phase curve of the \( i \)-th system intersects the Poincaré cross section at the \( q \)-th time). If for any given \( q \) and for all \( 1 \leq i \leq k \) the time instances \( t_{q,i} \) coincide then we may say that the systems \( \Sigma_i \) synchronize. In this case we may introduce infinite number of indices \( J_q(y_i(\cdot)) = t_{q,i} \), \( q = 1, 2, \ldots \), i.e. \( J_q \) is the time of \( q \)-th crossing of the surface. However it will require an infinite number of functionals \( g_q \). Alternatively, we define the index \( J(y_i(\cdot)) \) as the infinite sequence \( J(y_i(\cdot)) = \{t_q\}_{q=1}^{\infty} \) and use some norm in the space of sequences as a single functional (as in the previous case). We have demonstrated how to formalise problems of synchronization in sense of closeness of either values of signals in some specific time instances or those time instances themselves. However, the above formulations are not suitable to express the phenomenon of asymptotic synchronization because the introduced functionals do not depend explicitly on time. To capture asymptotic synchronization we may introduce the "current" indices
\[
J_i(y_i(\cdot), t) = \inf_{v \geq t} \{v : \varphi_i(y_i(v)) = 0\}
\]
or (a "causal" version)
\[
J_i(y_i(\cdot), t) = \sup_{0 \leq v \leq t} \{v : \varphi_i(y_i(v)) = 0\}.
\]
and "current" functionals as before
\[
g_{\sigma r}(y_{\sigma r}(\cdot), y_{\sigma r}(\cdot)) = ||J_{\sigma r}(y_r(\cdot), t) - J_{\sigma r}(y_r(\cdot), t)||
\]
Of course additional conditions should be imposed guaranteeing that all the introduced quantities are well defined, e.g. each trajectory crosses the section for arbitrarily large \( t \geq 0 \).

5 Conclusion

An attempt is made to give a fairly general definition of synchronization corresponding to intuition, encompassing most of the known definitions and applications, and capturing peculiarities of both self-synchronization and controlled synchronization. Synchronization in a general sense can be defined as the coincidence of any scalar characteristics of the subsystems: amplitudes, frequencies, energies, powers, fractal dimensions [4], etc. Naturally we may use several synchronization functionals and/or indices of different kind and thus obtain a great variety of combined synchronization problems. The general definition was illustrated by a number of examples. The key point of our approach is the understanding that the synchronization as a phenomenon should be understood with respect to some condition which defines the presence of synchronism in the particular problem, i.e. the systems may be in synchronous motion from one point of view and may be asynchronous from another. To give a general definition we introduced the concept of synchronization with respect to the given functional (or to a set of functionals). To capture the physical peculiarities when dealing with the problem it is also convenient to use the concept of synchronization index which allows to understand better the physical meaning of the problem. The concept of frequency synchronization extending the classical definition of Huygens is introduced and discussed as well as that of coordinate
synchronization. Based on the introduced definitions a practical problem of synchronization of vibrating actuators is described in [5, 6]. The formulation of the controlled synchronization problem allows to address and to solve the control design problem for various dynamical systems. We hope to provide a basis for the development of new controlled synchronization methods. For example, taking the combined functional $G$ in (5) as a goal functional we may apply the speed-gradient method for a synchronization algorithm design (see [9, 10, 11]). In cases when the system states are not available for measurement the observer-based methods [19, 20] may be applied. A robustness analysis in the spirit of [18] is also possible. However many problems of controlled synchronization remain unsolved. For example, still no general methods are available for controlled frequency synchronization, for synchronization with respect to times of crossings of some surfaces and for chaotic systems in the sense of fractal dimensions. The last problem seems especially challenging because it corresponds to a nonlocal in time-property and even involves noncausal functionals.

References