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Time-dependent manipulation of ultracold ion bunches

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The combination of an ultracold ion source based on photoionization of a laser-cooled gas and time-dependent acceleration fields enables precise manipulation of ion beams. We demonstrate reduction in the longitudinal energy spread and transverse (de)focusing of the beam by applying time-dependent acceleration voltages. In addition, we show how time-dependent acceleration fields can be used to control both the sign and strength of the spherical aberrations. The experimental results are in close agreement with detailed charged particle tracking simulations and can be explained in terms of a simple analytical model. © 2011 American Institute of Physics. [doi:10.1063/1.3544009]

I. INTRODUCTION

Recently an ion source has been introduced\[12\] that is interesting for high-brightness applications. High brightness is achieved by going to very low temperatures, instead of reducing the emission area of the source. This source, the ultracold ion source (UCIS), is based on the photoionization of laser-cooled atoms trapped in a magneto-optical trap (MOT). It can have a brightness that is comparable\[3\] to the current industry standard Ga-liquid metal ion source (LMIS) (Ref. 4) but with much lower energy spread, as we showed recently.\[5\] In addition, many different elements can be used, in particular, the alkali-metals. Light species, such as Li, are interesting for scanning ion microscopy applications,\[6\] a recently developed alternative to scanning electron microscopy, while heavier elements, as, for example, Cs, are useful for sputtering applications such as secondary ion mass spectrometry (SIMS) and focused ion beams (FIB).

The low energy spread enables the creation of well-defined beams at energies as low as a few electron volts.\[5\] The combination of low beam energies and pulsed operation of the UCIS make it possible to change the acceleration field while the ions are being accelerated. The utilization of such time-dependent acceleration fields enables the control of both the longitudinal and the transverse phase–space distribution of the ion bunch. Generally the phase–space distribution of a freely propagating bunch is characterized by correlations between momentum and position, which correspond to either expansion or contraction of the bunch in various directions. Using time-dependent fields both the magnitude and the sign of linear momentum–position correlations can be changed, which is equivalent to (de)focusing the bunch transversely or (de)compressing the bunch longitudinally. In addition, higher order beam manipulations, i.e., changing nonlinear momentum–position correlations, are possible as well. This opens up new possibilities to correct for spherical and chromatic aberrations, which are presently limiting the spatial resolution in FIB applications.\[7\] Other high-brightness applications such as SIMS and scanning ion microscopy will of course also greatly benefit from the improved spatial resolution.

The idea of time-dependent manipulation has been discussed before. In (time-of-flight) SIMS,\[8\] for example, it can be used to improve the mass resolution. By using time-dependent fields, short primary ion bunches can be created,\[9\] combined with time-focusing of the secondary ions.\[10,11\] Also aberration correction with time-dependent fields has been the subject of several theoretical studies\[12\] but has not yet been demonstrated experimentally. In Ref. 13 a scheme was proposed to perform spherical and chromatic aberration compensation in an electron microscope with switched electric fields. In Ref. 14 the aberrations of a time-dependent magnetic lens were studied theoretically.

In Ref. 15 we demonstrated manipulation of ultra cold ion bunches with time-dependent fields. Here we will present additional measurements and discuss the experimental results in more detail. We start in Sec. II by deriving a simple general model to get insight into bunch manipulation with time-dependent fields in both the longitudinal and the transverse directions. Next, in Sec. III, we apply the model to describe the linear bunch manipulation for several specific pulse shapes that are also used in the experiment. The experimental setup is briefly discussed in Sec. IV. In Sec. V we present the experimental results of longitudinal phase–space manipulation. We show that the relative energy spread of the bunch can be reduced by switching the field off while the bunch is still in the accelerator. In Sec. VI we present experimental results of transverse bunch manipulation. We show that by using more complex pulse shapes, the accelerator field can be used as an adjustable lens with control of both the strength and the sign of the lens. And finally, in Sec. VII, we demonstrate that also nonlinear manipulation is possible. We present measurements that show that the strength and the sign of the spherical aberration coefficient of the lens can be controlled by only changing the time-dependent acceleration voltage.
II. MODEL OF TIME-DEPENDENT BUNCH MANIPULATION

The basic concept of time-dependent bunch manipulation is straightforward; if an ion-bunch is created inside an accelerating electric field, the field can be changed in time by applying a time-dependent voltage to the accelerating structure. The effective field the ions experience while being accelerated can thus, within some limits, be controlled without the need to change the geometry. For time-scales relevant for the ions, magnetic fields induced by the time-dependent electric fields and electromagnetic waves which might start to propagate in the structure do not play any significant role. The time-dependent electric field is thus simply the static electric field multiplied with a time-dependent scaling factor.

In this section a simple model is derived to shed light on the principle of bunch manipulation with these time varying fields. This model will also be used in the subsequent sections to explain the experimental results. In our UCIS experiment described in this paper, we create cold ion bunches from a laser-cooled atomic cloud, illustrated in Fig. 1. A part of the cold atomic cloud, located in the center of an accelerator structure, is pulse-ionized so an ion bunch is created that is accelerated in the z-direction. Both the longitudinal as well as the transverse phase-space of the bunch will be affected by the time-dependent field.

In the experiment a cylindrically symmetric accelerator structure is used. The symmetry of the system makes it possible to write a static acceleration field \( \tilde{E}(r,z)=E_z \bar{r}+E_\perp \bar{z} \) as an expansion in the \( z \)-component of the on-axis electric field \( E_{0z}(z) \):

\[
E_z(r,z) = -\frac{1}{2} E_{0z}(z) r + \frac{1}{16} E_{0z}''(z) r^3 + \cdots , \tag{1a}
\]

\[
E_\perp(r,z) = E_{0\perp}(z) - \frac{1}{4} E_{0\perp}''(z) r^2 + \cdots , \tag{1b}
\]

where a prime denotes the derivative with respect to \( z \) and \( r \) is the radial direction.

If a time-dependent anode voltage \( V_a(t) \) is applied, the electric field \( \tilde{E} \) varies in time. Then it is convenient to introduce the static normalized field distribution \( e_z(z) = E_{0z}(z) / \phi(z_0) \), with \( z_0 \) the starting position of the ions and the (static) potential given by \( \phi(z)=\int_z^\infty E_{0z}(z')dz' \). Due to this definition \( e_z(z) \) has the unit of \( 1/m \). In the experiment the potential at the anode \( \phi(z_0) \) is directly controlled but not the potential \( \phi(z_0) \) at the initial position of the ions. By defining \( \alpha=\phi(z_0)/\phi(z_0) \) for the static field, the time-dependent field on axis can now be written as \( E_z(z,t)=e_z(z)\alpha V_a(t) \).

The acceleration field is in general not homogeneous, so particles at different positions inside the bunch experience different forces. To describe these effects, we consider the difference in electric field between a test particle and the center particle, as is illustrated in Fig. 2. We consider two kinds of test particles: one that is displaced radially (white circle), and one displaced longitudinally in the \( z \)-direction (gray circle). The characteristic behavior of the bunch can be obtained by placing the test particles at an initial displacement equal to the rms bunch sizes, respectively, \( \sigma_{r0} \) and \( \sigma_z \).

By writing the field difference as a series expansion in the displacement, the characteristic relative momenta \( \Delta p_r \) and \( \Delta p_z \) of the test particles after acceleration with respect to the central particle can be calculated as follows:

\[
\Delta p_r = q \alpha \int_{0}^{\infty} V_a(t) \left[ -\frac{1}{2} e_z'(z(t))\sigma_r(t) + \frac{1}{16} e_z''(z(t))\sigma_r(t)^3 + \cdots \right] dt , \tag{2a}
\]

\[
\Delta p_z = q \alpha \int_{0}^{\infty} V_a(t) \left[ e_z'(z(t))\sigma_r(t) + \frac{1}{2} e_z''(z(t))\sigma_r(t)^2 + \cdots \right] dt , \tag{2b}
\]

where \( q \) is the ion charge and \( z(t) \) describes the position in the field of the center particle; \( \sigma_r(t) \) and \( \sigma_z(t) \) describe the bunch size as function of time, and thus the relative position of the test particles.

The model can be simplified by assuming that the relative positions of the test particles with respect to the center particle do not change during acceleration, so \( \sigma_r(t)=\sigma_{r0} \) and \( \sigma_z(t)=\sigma_{z0} \). Furthermore, an anode voltage switch function \( \tilde{V}_a(z) \) can be defined as function of the center position of the bunch instead of time. This results in

\[
\Delta p_r = q \alpha \int_{z_0}^{\infty} \tilde{V}_a(z) \left[ -\frac{1}{2} e_z'(z)\sigma_{r0} + \frac{1}{16} e_z''(z)\sigma_{r0}^3 + \cdots \right] dz , \tag{3a}
\]
A. Transverse-static field

The linear transverse focusing or defocusing behavior is described by the first order term in \( \sigma_0 \) in Eq. (3a). In the limit of a thin lens, the transverse focal length \( f_t \) is defined by

\[
\frac{1}{f_t} = - \frac{1}{\sigma_0} \frac{\Delta \rho_t}{p_z}
\]

(4)

with \( p_z \) the average ion momentum. If we assume that the change in \( v(z) \) in the fringe fields, i.e., in the region of appreciable radial field components, is negligible then we find for a static acceleration field;

\[
\frac{1}{f_t} = - \frac{1}{4} e_z(z_0).
\]

(5)

This shows that the divergent field at the exit of the accelerator works as a negative lens. To illustrate this, let us consider the accelerator used in our experiments (see Fig. 1 and for more details Ref. 17), which has a voltage \( V \) applied across a gap \( d=20 \text{ mm} \). To simplify the calculation for this example, let us assume that the acceleration field is uniform. Since the ions start at a position halfway the gap we then have \( \phi(z_0) = V/2 \), so \( e_z(z_0) = 2/d \). We thus find \( f_t = -40 \text{ mm} \). As we will see later, this is in reasonable agreement with particle tracking simulations using the exact acceleration fields.

B. Transverse-switched field

For a time-dependent acceleration field, the focal strength can be calculated by combining Eqs. (4) and (3a), resulting in

\[
\frac{1}{f_t} = \frac{1}{4} \int_{z_0}^{\infty} \frac{\bar{V}_a(z)}{\bar{V}_a(z_0)} e'_z(z)dz.
\]

(6)

Here we assumed that the change in final beam energy \( U \) is negligible in comparison to the static case; \( U \approx U_z \).

From Eq. (6) it is clear that the focal length \( f_t \) can be modified by time-dependent manipulation by choosing some specific anode voltage function \( V_a(t) \), and thus \( \bar{V}_a(z) \). We start with a simple unipolar voltage pulse of duration \( \tau \) and amplitude \( V\prime \), as shown in Fig. 3(b). The accelerating field can be turned off while the ion bunch is still being accelerated in the field. By introducing \( z_{11} \) as the position of the center of the bunch at the moment the field is turned off \( (t = \tau) \), the focal strength given by Eq. (6) can be written as

\[
\frac{1}{f_t} = - \frac{1}{4}[e_z(z_0) - e_z(z_{11})].
\]

(7)

If the field is switched off after the ions have left the accelerator field \( [e_z(z_{11})=0] \), Eq. (7) reduces to the expression derived for the static case. If the field is switched off earlier, the \( e_z(z_{11}) \) term reduces the focal strength.

To put this into experimental perspective, let us consider the case of Rb\(^+\) ions accelerated in our accelerator, with a typical voltage of \( V=1 \text{ kV} \) applied across the \( d=20 \text{ mm} \) gap. The Rb\(^+\) ions then take approximately 150 ns to reach their maximum energy of 500 V. In order to change the focal

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III. THEORY LINEAR MANIPULATION

In this section the effects of the linear (first order) terms in the general Eqs. (3a) and (3b) are studied in more detail. Expressions are derived for the focal length, both transversely and longitudinally, for specific cases of the anode voltage function \( V_a(t) \).

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FIG. 3. (Color online) Illustration of the field terms in Eqs. (3a) and (3b) for various anode voltage functions \( V_a(t) \). (a) Electric field and derivative without time-dependent switching. (b) Example of the effective electric fields when a unipolar pulse is applied and when (c) a tripolar pulse is applied.
strength by switching the field, pulse durations of, typically, \( \tau = 100 \) ns are required. For accurate control the pulse duration should therefore preferably be adjustable with an accuracy of the order of 10 ns.

Instead of simply turning the field off with a unipolar pulse, it is also possible to change the sign of the radial electric field with a bipolar or multipolar pulse, as is illustrated in Fig. 3. By choosing a suitable multipolar pulse, the radial momentum an ion receives can even be inverted, so the diverging accelerator field can now be used as a positive lens as well.

We now consider the effect of a tripolar pulse, as defined in Fig. 3(c); the pulse starts with a positive amplitude \( V_p \) and duration \( \tau \), is subsequently switched to the negative voltage \( V_n \) for a duration of \( \tau_n \), and finally switched back again to the positive voltage \( V_p \). The first switch occurs at position \( z_{s1} \), the second switch occurs at position \( z_{s2} \). For this pulse the focal strength can be approximated with Eq. (6) as

\[
\frac{1}{f_l} = -\frac{1}{4} \varepsilon(z_0) - \frac{V_p - V_n}{V_p} [\varepsilon(z_{s1}) - \varepsilon(z_{s2})].
\]

The focal strength can be controlled by either changing the negative voltage \( V_n \) or by changing the pulse durations \( \tau \) and \( \tau_n \), which result in different \( z_{s1} \) and \( z_{s2} \). Note that Eq. (8) also describes bipolar pulses \( \varepsilon(z_{s2}) = 0 \) and unipolar pulses \( V_n = 0 \) and \( \varepsilon(z_{s2}) = 0 \).

C. Longitudinal-static field

In the longitudinal direction a similar result can be obtained; a longitudinal focal length can be defined as

\[
\frac{1}{f_l} = \frac{1}{\alpha_0} \frac{\Delta p_z}{p_z}.
\]

Under the same assumptions as in the transverse direction, i.e., \( \varepsilon(z) \) changes negligibly in the fringe field region, this results for the static case in

\[
\frac{1}{f_l} = \frac{1}{2} \varepsilon(z_0) = -\frac{1}{f_i},
\]

and is thus positive. This can be understood as follows: an ion created in the back of the bunch travels a larger distance in the field than an ion created in the front. The particle in the back therefore acquires a higher velocity than a front particle. Because of this correlated velocity difference in the bunch, the front particles are overtaken by the back particles. The bunch is maximally compressed at the focal point. For our accelerator the point of maximal longitudinal compression, the temporal focus, lies at a distance \( f_i = 25 \) mm from the exit of the accelerator.

Equation (10) shows that the longitudinal and transverse focal strength differ by a factor \(-2\), which is due to the zero divergence of the electric field, \( \partial E_z/\partial z = -2 \partial E_z/\partial \sigma \). The longitudinal and transverse focal lengths are thus always coupled and cannot be set independently.

This longitudinal focus determines both the bunch length and the energy spread. The contribution of the initial velocities of the cold ions to the energy spread is generally very small, and can therefore be neglected in comparison to the spread induced by the acceleration field. This will be discussed in more detail in Sec. III E. The relative energy spread \( \sigma_v/U \) due to the acceleration can be approximated by

\[
\frac{\sigma_v}{U} = 2 \frac{\sigma_{z_i}}{f_l}.
\]

D. Longitudinal-switched field

Similar to the transverse case, the longitudinal focal length for a time-dependent field is given by

\[
\frac{1}{f_l} = -\frac{1}{2} \int_{z_0}^{\infty} \frac{V(z)}{V(z_0)} \varepsilon(z) dz,
\]

resulting in

\[
\frac{\sigma_v}{U} = \alpha_0 \varepsilon(z_0).
\]

with \( \varepsilon(z_0) \) the normalized electric field at the initial position of the bunch (\( z = z_0 \)). We thus find that the relative energy spread is also independent of the beam energy. In static fields, a small relative energy spread can therefore only be realized by choosing a small longitudinal size \( \sigma_{z_i} \) of the ionization volume.

If the field is switched off after the ions have left the accelerator field, Eq. (15) reduces to the expression derived for the static case. In the idealized case of a perfectly homogeneous electric field inside the accelerator, i.e., \( \varepsilon(z_{s1}) = \varepsilon(z_0) \), the right hand side of the first term exactly cancels the second term in Eq. (15), which correspond to a focal
length $f_l = \pm \infty$. If the field is not homogeneous, the first term in the right hand side still partially reduces the energy spread.

**E. Longitudinal-thermal limitations**

So far, we neglected the initial velocities of the cold ions. These thermal velocities correspond to an initial energy spread of only $kT = 20$ neV, where $k$ is Boltzmann’s constant and $T$ the initial temperature of the ions. In this section we will show they still can play a role in the longitudinal phase–space manipulation. They set a clear limit on the reduction in the energy spread that can be achieved by turning the field off with a unipolar pulse, as discussed in the previous section. We will start by discussing the acceleration process in the longitudinal phase–space ($z-p_z$), as depicted in Fig. 4.

For clarity a perfectly homogeneous acceleration field with a hard edge is assumed and space charge effects are neglected. In Fig. 4(a) the phase–space evolution of an ion bunch with a relatively large initial length ($\sigma_{p_z0}$) is shown. Several snapshots of the bunch at different moment in time are sketched. The bunch starts in the left bottom corner with an initial momentum spread $\sigma_{p_z0}$ due to the initial temperature and an initial length $\sigma_{z0}$ due the size of the ionization laser waist. The bunch immediately starts to accelerate in the field. Because the field is homogeneous, all ions experience the same force and thus gain the same momentum. At the same time the shape undergoes a linear transformation in the $z$-direction due to the finite initial momentum spread.

In the static case (green) this continues until the bunch reaches the edge of the accelerator. Not all ions reach the edge at the same moment. Particles in the back of the bunch, that are still inside, keep on being accelerated in contrast to the particles in front, which are already outside the accelerator. This transforms the bunch in a nonlinear way, as illustrated in the figure. The momentum spread is now clearly increased compared to the initial spread. This corresponds to the (correlated) energy spread due to the initial bunch length as discussed before, see Eq. (12). If the field is turned off before the bunch reaches the edge (red), this transformation does not happen. The momentum spread stays the same and the bunch only drifts further in the $z$-direction. The momentum spread is smaller than in the static case, so by switching off the accelerating field, the energy spread can indeed be reduced.

The final momentum spread $\sigma_{p_z}$ in the static case can be reduced by decreasing the initial size as illustrated in Fig. 4(b) where the time evolution of an initially shorter bunch is shown. This is in contrast to the switched case where the spread is independent of the initial size $\sigma_{z0}$. This may even result in a larger momentum spread than in the static case.

In the switched field case $\sigma_{p_z}$ is conserved, which limits the lowest energy-spread that can be attained. Note that although the momentum spread $\sigma_{p_z}$ is conserved, the energy spread $\sigma_U$ is not. In fact, the final energy spread $\sigma_U$ is given by

$$\sigma_U = \frac{\sigma_{p_z} p_z}{m}$$

with $p_z$ the final average momentum of the bunch. This can also be written in terms of energy and temperature as

$$\sigma_U = \sqrt{2kTU}.$$  \hspace{1cm} (17)

This equation shows that the lowest reachable energy spread, if it is not limited by space charge or inhomogeneous fields, is limited by a cross-term between the initial thermal energy and the final beam energy. In Fig. 5 this contribution to the energy spread is plotted as function of $U$ for different initial temperatures. Fortunately the initial temperature is low (around 200 $\mu$K) so the resulting energy spread is well below 10 meV for beam energies up to several kilovolts.

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**FIG. 4.** (Color online) Illustration of the phase–space of a bunch in a hard-edged homogeneous acceleration field for (a) a long initial bunch length; and (b) a short initial bunch length. Two situations are shown, the static case (green) where the bunch transforms at the hard-edge of the field and the unipolar pulse case (red) where the field is switched off before it reaches the edge.

**FIG. 5.** Plot of the energy spread contribution of the cross term Eq. (17) as function of the beam energy at which the field is turned off for several initial temperatures.
To understand the physical mechanism behind Eq. (17), it is advantageous to think in terms of momentum gained during acceleration in switched fields, instead of kinetic energy. In the static case all ions with the same initial position receive the same amount of energy, independent of their initial velocity, simply because they travel the same distance. A particle with a positive initial velocity $v_0$ will exit the field at an earlier time, so less momentum is transferred. In the switched case however, such a particle will be accelerated over an extra distance of $v_0 \tau$ before the field is turned off in comparison with a particle with zero initial velocity. The extra energy received from the field, the energy spread, can thus [alternative to Eq. (17)] also be written as

$$\sigma_v = \sigma_{v_0} \tau E_0 \gamma$$

(18)

with $\sigma_{v_0}$ the initial thermal velocity spread and $E_0$ the field strength of the homogeneous accelerator field.

**IV. EXPERIMENTAL SETUP**

To perform these ultracold ion beam experiments with time varying fields, several different ingredients are required. First of all an atomic gas has to be laser-cooled and trapped, which is the basis of the UCIS. Second it has to be possible to photo-ionize a part of the cold atom cloud. This should all be done at an earlier time, so less momentum is transferred. In the transverse focusing experiments, where no time-of-flight data is required, a commercial frequency doubled diode laser system (Toptica TA-SHG 110) was used. The laser beam was then switched between three states: zero, a positive high voltage level ($V_p$), and a negative high voltage level ($V_n$) with a switch time of 50 ns and a maximum pulse repetition rate of 30 kHz. If a time-dependent voltage is used, all ions should start accelerating at the same moment in time. If the ionization is performed when the accelerating field is already present, ions created in the beginning of the ionization laser pulse already start to accelerate while others have not yet been ionized. Therefore we ionize the trapped atoms while the acceleration field is still turned off; the field is turned on several nanoseconds after the last ions have been created.

The ion bunches are detected by a 40 mm diameter multichannel plate detector (MCP) with phosphor screen, which is mounted at a distance $L=0.63 \text{ m}$ from the center of the accelerator. The MCP-phosphor-screen assembly is indicated in Fig. 1 by the vertical plate at (d). To increase the detection efficiency at low beam energies, a voltage of $-900 \text{ V}$ is applied to the front plate of the MCP. A metal grid is placed 10 mm in front of the detector to shield the electric field. A charged coupled device (CCD) camera (Apogee U9000), indicated at (e) in Fig. 1, is used to capture images of the phosphor screen that contain the spatial information of the ion bunches. Simultaneously the temporal distribution of the ion bunch is recorded on an oscilloscope by using a transimpedance amplifier ($=10 \text{ ns resolution}$) connected to the phosphor screen, which is indicated at (f) in Fig. 1.

From the recorded temporal signal of the amplifier the total charge $Q$ of the bunch, the time-of-flight $T$ to the detector and its rms spread $\sigma_T$ are extracted by fitting the signal with a Gaussian distribution function. It is important to know how this time-of-flight data can be translated to the average beam energy $U$ and the energy spread $\sigma_U$. If all ions would have been created at exactly the same position and thus all with exactly the same drift length $L$ to the detector, this would result in simple expressions. In that case the beam
energy $U$ can be calculated using $U=(1/2)m(L/T)^2$, the longitudinal bunch length follows from $a_z=L/a_t/T$, and the relative energy spread $\sigma_U/U$ is given by the relation $\sigma_U/U = 2\sigma_t/T$.

If one looks closer into the details, complications arise. The ions start at different positions in the accelerator, the acceleration field is not homogeneous and some post-acceleration occurs close to the detector. If a static acceleration field is used, as in Ref. 5, then an extra constant factor can be introduced to compensate for these effects. Namely, due to the longitudinal focusing effect of the accelerator, as discussed in Sec. III C, the bunch compresses in longitudinal direction due to the correlated velocity difference at a fixed position just outside the accelerator. This position with minimal bunch length can be used as a virtual anode from where the simple formulas are valid again. For time-dependent acceleration fields such a simple compensation factor does not work because the longitudinal focal strength changes, and therefore also the position of the minimal bunch length, the virtual anode. Fortunately, the distance to the detector is long compared to the accelerator length and the initial bunch size. Numerical simulations show that the simple relations are accurate to within 15% for the relevant cases.

V. LONGITUDINAL MANIPULATION

A. Experimental results

In Fig. 6 we present results of the longitudinal phase-space manipulation using a unipolar voltage pulse. The relative energy spread $\sigma_U/U$ obtained from the measurements is plotted as function of the pulse duration $\tau$ in Fig. 6(a) and as function of the final beam energy $U$ in Fig. 6(b).

First we discuss the static-measurements, equivalent to using a voltage pulse with $\tau=+\infty$. In this measurement series, indicated by the purple circles in Fig. 6(b) the anode voltage $V_a$ is varied from 400 to 2500 V. As discussed, the (correlated) energy spread of the bunch and the longitudinal focal length are related by Eq. (11). The focal length $f_z$ is independent of the acceleration voltage and thus also the relative energy spread. The measured data in Fig. 6(b) indeed shows that it is nearly independent of the beam energy. Lowering this energy spread is not possible in a static field without changing the accelerator field shape or ionization volume geometry. But as discussed in Sec. III D, if a time-dependent acceleration field is used, this can be done.

In Figs. 6(a) and 6(b) measurements are shown performed with unipolar pulses, indicated by black and red squares. The relative energy spread $\sigma_U/U$ was measured while the pulse duration $\tau$ was varied from 100 ns up to 1500 ns. The measurements have been done for two different voltages; $V_a=2000$ V (black squares) and 1000 V (red squares); at a bunch charge of $\approx 0.5$ fC. For pulse durations above 1000 ns the ions have already left the accelerator, and are therefore not influenced by the field switching. Both curves at that duration in Fig. 6(a) level off and have the same energy spread as the static experiments, indicated by the purple arrow. For shorter $\tau$, $\sigma_U/U$ is reduced. For even shorter $\tau$, the time-of-flight $T$ becomes so long that space charge effects start to increase $\sigma_U$ again. This effect was also observed in Ref. 5.

To make a more direct comparison with the static case, the same measurements are plotted as function of $U$ in Fig. 6(b). $U$ has been calculated using the measured average time-of-flight $T$. The encircled points correspond to the cases where $\tau$ is so long that the ions already left the accelerator and thus have gained the full energy $U$. It can now be observed that if $\tau$ is decreased, not only the relative energy spread but also $U$ decreases. The relative energy spread reached with the pulsed measurement is clearly below the measurements with a static field. We conclude that the relative energy spread can be reduced by a factor three by switching the field off before the ion bunch has left the accelerator.

B. Simulations

To quantitatively understand the measurements, particle tracking simulations were performed with the GPT (Ref. 18) code. The electric field inside the accelerator was calculated with the SUPERFISH poisson solver. All the ions in the bunch are tracked individually with all mutual Coulomb in-
interactions included. The initial conditions are the same for all simulations. The ions are started with a Gaussian density distribution with experimentally determined values for the position and width. The simulation results are depicted as dashed (static case) and solid (unipolar switch) curves in Fig. 6. Good agreement with the data is obtained. Small deviations from the simulations occur in the low-energy region, where space-charge effects become significant, mostly due to experimental uncertainties in the exact initial ion distribution.

In the simulations the space-charge forces between the ions can be turned off, without changing any other parameter. The resulting curves are plotted as the dashed lines. For these curves only the interaction with the accelerator field is important, so they can be directly compared to the results of the analytical model.

C. Analytical energy-spread models

We will now make a more detailed comparison between the simulations and the model presented in Sec. III D. The calculated longitudinal focal length for the static case of \( f_l = 26 \text{ mm} \) using Eq. (10) matches within 1% with the simulation results. To compare the model for the time-dependent acceleration fields, the relative energy spread resulting from GPT simulations without space-charge effects (black solid curve), is shown in Fig. 7. It is plotted as function of the switch position \( z_{s1} \) instead of pulse duration. Plotted in this way, both simulation curves (dashed lines) of Fig. 6 fall on top each other and form the single black solid curve in Fig. 7. Small \( z_{s1} \) correspond to short durations, and large \( z_{s1} \) to long durations. If the field is switched off at large \( z_{s1} \), i.e., at a position where the field has almost reduced to zero, \( \sigma_U/U \) does not change anymore, so the curve levels off for large \( z_{s1} \).

The relative energy spread calculated with the model of Eq. (15) is depicted by the blue dashed line. First of all, it can be improved (dashed green curve) by correcting for the changing beam energy \( U \) by multiplying the focal strength with the factor \( \Gamma \) defined in Appendix A. This curve shows the same overall behavior as the simulation curve but there is a clear difference for small \( z_{s1} \).

The largest part of the remaining discrepancy can be attributed to the assumption that the bunch length \( \sigma_z(z) \) does not change in the acceleration field, as used in Eq. (3b). Without this assumption we get

\[
\frac{\sigma_U}{U} = \Gamma[\sigma_z(z_1)e_z(z_1) - \sigma_z(z_0)e_z(z_0)],
\]

with \( \sigma_z(z) \) the bunch length as function of its center position. This bunch length can be approximated with a second order model in Eq. (19), this results in an energy spread curve (red dashed line) which agrees within 10% with the simulated curve.

From the comparison between model and simulations we learn that the energy spread can be understood quantitatively only by taking higher-order derivatives of the field into account at the initial position \( z_0 \) and at the position \( z_{s1} \), where the field is switched off. The accelerator field can be designed in such a way that it is more uniform than in the present setup. A much larger energy spread reduction should then be obtainable.

VI. TRANSVERSE FOCUSING

So far we looked only at the longitudinal beam behavior. In this section we will discuss experiments where the transverse beam behavior is studied for different time-dependent voltage pulses. We will start by presenting and discussing the transverse beam results of the unipolar pulse experiments that have been described in the previous section. Subsequently we will continue with more complex pulse shapes.

A. Results unipolar pulse

The accelerator field also influences the ions in transverse direction, as discussed in Sec. III. To study this, the spatial ion bunch distribution on the detector was captured by the CCD camera, simultaneously with the time-of-flight signals that were used already in the previous section. The transverse sizes \( \sigma_x \) and \( \sigma_y \) of the ion bunches were extracted from the images by performing a fit with a two-dimensional Gaussian distribution. In Fig. 8 we present these transverse results of the same set of experiments that was used for the longitudinal measurements of Fig. 6. Both \( \sigma_x \) and \( \sigma_y \) are shown as function of the beam energy \( U \).

We will first focus on the static case, the purple circles in the figure, where the divergent field at the exit hole of the accelerator structure acts as a negative lens for the ions. The focal strength of the lens can be approximated by Eq. (6), which results in \( f_l = -51 \text{ mm} \) for our accelerator. Due to this lens, the bunches will be transversely magnified when they arrive at the detector position. The focal length and therefore also the magnification is, according to the model, independent of the beam energy \( U \). In the measurements we indeed observe a constant bunch size on the detector, except again at low beam energies where the spot blows up due to space charge forces.

The results of the unipolar experiments are also shown in Fig. 8 with again the two series \( V_p = 2000 \text{ V} \) (black squares) and 1000 V (red squares). There is a clear resem-
curves and simulations without Coulombic interactions as dashed curves. The data points are thus effectively just the lower $U$ plotted vs $t$, as well as changing the pulse duration $\tau$ from 100 ns up to 1500 ns while keeping $V_p$ constant. Two data sets are shown, obtained with $V_p=2000$ (black squares) and 1000 V (red circles). In (a) the transverse size $\sigma_x$ on the detector is plotted vs $U$ and in (b) the transverse size $\sigma_y$. Also static measurements ($\tau = \infty$, purple circles) are presented, where $U$ is changed by varying $V_p$. Particle tracking simulations with Coulombic interactions are depicted as solid curves and simulations without Coulombic interactions as dashed curves.

The results of the particle tracking simulations, that were described in Sec. V B, are also depicted in Fig. 8 (solid curves). The simulation of the static field case (purple) is used to obtain the transverse focal length. A smaller focal length $f_t = -35$ mm has been found than predicted with the simple analytical model ($f_t = -51$ mm). This relatively large deviation is due to the fact that for our accelerator field, with an acceleration length comparable to the exit hole size, the assumptions made in the derivation of the model are partially violated. The beam expands already inside the field, and the velocity is not constant while passing through the divergent area. Both these effects increase the focal strength.

The overall behavior of the simulation curves in the pulsed case are comparable to the experimental data. The spot size of the pulsed measurements starts at the value of the static simulation, then decreases and eventually blows up because of space charge. The agreement between the simulation and the measured data is least satisfactory for the curve with the lowest pulse amplitude. The deviations are therefore most likely caused by the combination of space charge and uncertainties in the initial ion distribution. Without space-charge (dashed curves) a minimal bunch size would have been observed corresponding to a nearly parallel beam.

Another interesting point are the striking similarities between the transverse measurements in Fig. 8 and the longitudinal measurements in Fig. 6(b). The longitudinal bunch length $\sigma_z$ is in first order proportional to the relative energy spread and is indicated on the right $\gamma$-axis of Fig. 6. Not only are the shapes of the curves for the transverse and the longitudinal measurements very similar, the produced bunches are also comparable in size in all three dimensions. This is due to the fact that the absolute values of the longitudinal and the transverse focal length are nearly equal; $|f_t| = 26$ mm and $|f_t| = 35$ mm; and that the detector is at a distance from the accelerator much larger than the focal lengths. At distances $\varepsilon \gg |f_t|$, the sign of the focal length is not important anymore for the divergence of the bunch. So if one starts with a spherical ionization volume, as is approximately the case in this experiment, then the ion bunch at the detector will be approximately spherical as well.

B. Results multipolar pulses

By using more complex pulses it is possible to change the negative lens into a positive focusing lens, as discussed in Sec. III B. In Fig. 9 this is demonstrated with a bipolar pulse, which can be interpreted as a tripolar pulse with $\tau_e = \infty$. The transverse size $\sigma_x$ has been plotted as function of

![FIG. 8. (Color online) Measurements of transverse bunch sizes as a function of energy $U$ using a unipolar voltage pulse. The beam energy $U$ is varied by changing the pulse duration $\tau$ from 100 ns up to 1500 ns while keeping $V_p$ constant. Two data sets are shown, obtained with $V_p=2000$ (black squares) and 1000 V (red circles). In (a) the transverse size $\sigma_x$ on the detector is plotted vs $U$ and in (b) the transverse size $\sigma_y$. Also static measurements ($\tau = \infty$, purple circles) are presented, where $U$ is changed by varying $V_p$. Particle tracking simulations with Coulombic interactions are depicted as solid curves and simulations without Coulombic interactions as dashed curves.

![FIG. 9. (Color online) Focusing using bipolar voltage pulses with $V_p = 1000$ V and $r=633 \text{ ns}$. The transverse size $\sigma_x$ on the detector is measured as function of $V_n$. Phosphor images of the beam are shown at the bottom. The solid curve corresponds to particle tracking simulations.](image)
the negative voltage \( V_n \), ranging from 0 to \(-1000\) V. The positive part of the pulse is kept constant, \( V_p = 1000\) V and \( \tau = 633\) ns. In the figure it can be clearly observed that when \( V_n \) gets more negative the bunch starts to focus, so the spot size decreases at the detector. The minimal spot size is observed at \( V_n = -540\) V, when the focus lies on the detector. In that case the bunch is accelerated to \( U = 280\) eV with the positive pulse and decelerated to \( U = 180\) eV by the negative pulse. At even lower \( V_n \) the beam over-focuses so the spot size increases again at the detector. Phosphor screen images have been added in the bottom of the image to illustrate the effect. Particle tracking simulations (solid curve) agree well with the measurements.

To enable direct measurements of the focal length of the time-dependent lens, independent of aberrations, another set of experiments has been performed; instead of relating the focal length to the measured spot size, the transverse position \( x \) of the center of the spot on the detector was measured as function of the position \( x_0 \) of the initial ionization volume. When the lens is ideal, the relation between \( x_0 \) and \( x \) is simply linear: \( x = Ax_0 \). From the linear coefficient \( A \) the focal length can be obtained.

The position \( x_0 \) was changed by moving the ionization laser focus. In the experiments described in the previous sections we used two focused laser beams, an excitation and an ionization laser beam, to ionize only a fraction of the cold atoms in the overlap region. For experimental convenience we now keep the MOT cooling beams on, instead of the separate focused excitation beam, thus exciting all the atoms in the cloud to the required intermediate level. Precise alignment is then not required to overlap both lasers at the right position. In this way a vertical cylinder (in the \( y \)-direction) of ions is produced coinciding with the waist of the ionization laser.

The focal behavior resulting from the use of bipolar voltage pulses, similar to those shown in Fig. 9, has been studied with this method. In Fig. 10 the transverse ion bunch position \( x \) at the detector is plotted as function of the initial position \( x_0 \). The bipolar pulse has a fixed positive part with \( V_p = 1000\) V and \( \tau = 633\) ns, while \( V_n \) was varied from \(-50\) to \(-1000\) V. At all these different voltage pulses, the ionization laser position \( x_0 \) was scanned from \(-2\) mm to \(+4\) mm, limited by the size of the atomic cloud. Every dot in the figure corresponds to an analyzed phosphor screen image from which the center cylinder position \( x \) was determined. In this set of measurements, the focal strength of the lens is now derived from the slope \( A \) of the curves, instead of the bunch size as in Fig. 9.

The relation between the slope \( A \) and the focal length is \( f = -L/(A-1) \). For small values of \( V_n \) the lens is still negative so \( A > 1 \); as the amplitude of \( V_n \) increases a parallel beam is created which corresponds to \( A = 1 \). At further increased amplitude the beam starts to focus \( (A < 1) \), up to the point where the focus lies at the detector position \( (A = 0) \). This happens at \( V_n = -550\) V, which is consistent with the position of the smallest waist in Fig. 9. For even larger values the focus lies in front of the detector \( (A < 0) \).

A more extensive measurement series has been performed to study the focal behavior of the tripolar pulses. The parameters \( V_n \) and \( \tau_n \) were varied, while \( V_p \) was kept constant at \( 1000\) V and \( \tau = 633\) ns. Again the initial position \( x_0 \) was varied for each of these different voltage pulses. By fitting the curves, the \( A \) parameter was determined. The resulting focal strengths \( 1/f \) of this parameter scan are presented in Fig. 11(b).

The horizontal axis represents the negative pulse duration \( \tau_n \) and the vertical axis the negative voltage \( V_n \). Every colored dot corresponds to a measurement with the color scale indicating the focal strength. Several lines of constant \( f \) are added as a guide to the eye. Two regions are visible, a blue area that represents a negative focal strength and a red area that represents a positive focal strength. For small values of \( \tau_n \) nothing is changed in comparison with the static case and the values are thus close to the expected \( 1/f = -28\) m\(^{-1}\). When \( \tau_n \) is increased, the influence of the negative voltage becomes stronger, up to the point where \( \tau_n \) is so long that the tripolar voltage pulse is effectively the same as a bipolar pulse, and therefore the focal strength \( 1/f \) becomes independent of \( \tau_n \).

By increasing the magnitude of the negative voltage \( V_n \) the focal strength increases. The beam starts at \( V_n = 0\) V as a divergent beam \( (1/f < 0) \), and it gets less divergent up to the point where a parallel beam is produced \( (1/f = 0) \). Subsequently, for even more negative \( V_n \) the beam starts to focus \( (1/f > 0) \).

### C. Simulations

For better understanding, again particle tracking simulations have been performed with use of GPT. In the simulations the same procedure is followed as in the experiment; for every voltage pulse shape a simulation is performed with particles starting from different initial \( x_0 \) positions while recording the positions \( x \) where the trajectories intersect the detector. Again the linear coefficient \( A \) is determined for ev-
the initial bunch position. We attribute the discrepancies to quantitatively, there are discrepancies; both the vertical and horizontal axes need to be scaled by, respectively, approximately 30% and 10% to have the measured and the simulated curves more or less overlap. It turns out that the outcome of the simulations is particularly sensitive to the exact electric field profile in the accelerator and the initial bunch position. We attribute the discrepancies to experimental uncertainty of these parameters.

D. Analytical model

The analytical model described in Sec. III B can also be used to describe the focusing. To compare this with the results of simulations, the pulse durations (τ and τp) have to be converted to switch positions (z_a1 and z_a2). This is done by using a numerically calculated trajectory of a particle started in the center of the bunch. Equation (8) is corrected for the change in final beam energy, due to switching, by multiplying with the factor Γ defined in Appendix A.

The results, 1/ft versus τn and Vn, are plotted in Fig. 12. The overall shape and behavior is very similar, showing that focusing with a time-dependent voltage pulse can be approximated with this simple model. Again, there is a difference with the simulation curves, specially in the region where the lens has a negative focal length, as we discussed for the static case. This is mainly caused by the assumption that v_s(z) is constant while passing through the field and the assumption of a constant bunch size.

VII. SPHERICAL aberrations

A. Experimental results

Up to now, we have shown that we can control the linear term in Eqs. (3a) and (3b) with the time-dependent fields but control of higher orders is also possible. To demonstrate this, we change the spherical aberration due to the exit fields of the accelerator. Importantly, we circumvent Scherzer’s theorem, a major restriction in conventional static cylindrical systems, which states that spherical aberration coefficients are always positive. This makes it impossible to simply cancel this aberration by combining lenses with both positive and negative coefficients. With time-dependent fields there is no such restriction.

To determine the amount of spherical aberration we use the same method as used in Sec. VI B to determine the focal length, namely, measuring the transverse position x on the detector as a function of the initial position x_0. When the exit lens is aberration free, as illustrated in Fig. 13(a), the relation between x_0 and x is strictly linear (x=A_1x_0). When spherical aberrations are present, shown in Figs. 13(b) and 13(c), an additional third order term appears (x=A_2x_0−C_3x_0^3). The aberration is the result of the e_x^2 term in Eq. (3a). By changing

![FIG. 11. (Color online) Measurements and simulations of the focal strength 1/ft as function of two parameters of a tripo pulse. The parameters V_p and τ_p are varied, while V_x=1000 V and τ=633 ns. (a) Schematic of the field and the voltage pulse, (b) experimental data, (c) crrr particle tracking simulation.](image)

![FIG. 12. (Color online) Plot of the focal strength 1/ft (solid lines) as calculated with Eq. (8) multiplied by the factor Γ defined in Appendix A. The result differs slightly from the particle tracking simulations (dotted lines).](image)
V_a(t), and thus Ṽ_a(z), the integral of this term can be controlled as illustrated in the diagrams of Fig. 14.

A deviation from the linear behavior can already be observed in the measurements in Fig. 10. By fitting the data with x=A x_0−C x_0^3 (solid curves), the parameters A and C are obtained. If the bunch is focused at the position of the detector (f_i=L), the third-order coefficient C is related to the C_s coefficient by C_s=f_i^3 C. The spot size δ_x due to spherical aberrations in a focusing system is given in general by δ_x=C_s x_0^2 α^3, with α the lens acceptance angle.\textsuperscript{12}

To make the deviations from linear behavior more clearly visible, an enlarged plot of the center curves (f_i⇒L) is presented in Fig. 14(a). Again the position at the detector x is plotted versus the initial position x_0. The data fits well with the third order fitting function (solid curves). This bipolar pulse results in C_s=−1.1±0.1×10^4 m. By changing only to a tripolar pulse, as is shown in Fig. 14(b), the sign of the aberration coefficient is reversed to C_s=+1.1±0.1×10^4 m. This shows that manipulation with time-dependent fields can be used to achieve aberration corrected systems.

To study the dependence of the spherical aberrations on the voltage pulse shape in more detail, the C coefficients have been determined for all the measurements performed with the tripolar pulses, described in Sec. VI, in Fig. 11(a). Points in parameter space which did not allow a reliable fit of the third order coefficient were left out. The results are depicted in Fig. 15(a). Two distinct regions are visible: on the left (small τ_n) a region of positive C and on the right (large τ_n) a region of negative C. In between the regions the C coefficient goes through zero. The positions in parameter space corresponding to the measurements shown in Fig. 14 are indicated by an asterisk (*). b) Results of particle tracking simulations. The black solid curves are lines of equal C.

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The black solid curve in Fig. 15(a) corresponds to combinations of V_n and τ_n which give rise to a focal length of f_i=0.63 m, i.e., a focus on the detector surface. The curve is extracted from the measured data in Fig. 11(b). By moving over this curve of constant focal length f_i, however, the C coefficients can be changed. For small τ_n the C value is
positive. When \( \tau_n \) is increased, the \( C \) value goes through zero and eventually becomes negative. This demonstrates that the spherical aberrations can be adjusted without changing the focal strength.

**B. Simulations**

From the particle tracking simulations performed in Fig. 11(c), also the third order coefficient \( C \) can be obtained by fitting the \( x \) versus \( x_0 \) curves with an additional third order term. These results are presented in Fig. 15(b). The overall behavior of \( C \) in the parameter space in the simulations matches with the measurements. Both have a region of positive \( C \) and a region of negative \( C \) with comparable magnitudes and the regions are separated at about \( \tau_n=700 \) ns by a region of small \(|C|\). Furthermore the \( C \) coefficient increases for large \( \tau_n \) and negative \( V_n \).

Again, there are also differences in the exact shape of the distribution, such as the increase in magnitude in the top left corner. The simulations are quite sensitive to the initial positions and the exact shape of the field used. This is likely the cause of the deviations between the simulations and the measurements.

Describing the spherical aberrations reliably with a simple analytical model, as was done in Sec. VI for the focal strength is difficult. Calculating the \( C \) coefficient by only using the third order term of Eq. (3b) does not work because in practice the change in \( \sigma(z) \) cannot be neglected. Moreover, cross terms between the linear and third order field terms start to get important. In addition, in many situations different contributions nearly cancel each other making precise modeling harder. Developing an analytical model that can accurately predict the spherical aberrations is outside the scope of this paper.

**VIII. CONCLUSION**

In conclusion, we have experimentally demonstrated that manipulation of ion bunches extracted from a laser-cooled gas with use of time-dependent fields is possible. More specifically we have shown that by using a unipolar pulse the relative energy spread in the longitudinal direction can be reduced. In the transverse direction we demonstrated that by using more complex pulses, such as tripolar pulses, the negative lens effect of an accelerator structure can be converted into a versatile adjustable lens. The sign and strength of this lens, as well as the sign and strength of the spherical aberrations coefficient can be adjusted by only changing the applied time-dependent voltage.

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**APPENDIX A: BEAM ENERGY**

As a side effect of a time-dependent acceleration pulse, the ions are only accelerated over a smaller distance, or with multipolar pulses, decelerated because the field in \( z \)-direction is also reversed. This is illustrated in Fig. 16, where the final beam energy \( U \), following from the particle tracking simulation performed for Fig. 11(b), are shown. For small \( \tau_n \) the beam energy is close to the beam energy \( U_s \) of the static case. The beam energy \( U \) decreases for increasing \( \tau_n \) and more negative \( V_n \).

The final beam energy resulting from a time-dependent voltage can be analytically written as

\[
U = \int_{z_0}^{\infty} a \tilde{V}_p(z) e(z) dz, \tag{A1}
\]

which results for a tripolar pulse, such as in the simulation, in an energy of

\[
U = q V_p \varphi(z_0) + q (V_p - V_n) [\varphi(z_{r2}) - \varphi(z_{r1})] \tag{A2}
\]

with \( \varphi(z) \) the normalized static potential defined as \( \varphi(z) = \phi(z)/\phi(z_0) \). In the static case only the first term remains, so the energy is then

\[
U_s = q V_p \varphi(z_0). \tag{A3}
\]

In Eqs. (6) and (9), which are used to derive the equations in Sec. III, the final beam energy \( U \) is assumed to be equal to \( U_s \). In some situations \( U \) can be much lower than \( U_s \) so the focal strength will be enhanced in comparison with the assumed situation. To correct for this effect, we introduce the energy correction factor \( \Gamma \) as follows:

\[
\frac{1}{\Gamma} = \frac{U}{U_s} = 1 + \left(1 - \frac{V_p}{V_n}\right) \left(\frac{\varphi(z_{r2}) - \varphi(z_{r1})}{\varphi(z_0)}\right). \tag{A4}
\]

The focal strength can be corrected in first order for the energy change by multiplying with \( \Gamma \).
To model the relative energy spread in the bunch, the bunch length as function of bunch position is needed. In this section a simple approximate model is derived.

The length $\sigma_z$ immediately starts to change due to non-homogeneous fields present at the initial position $z_0$. To approximate the bunch length change we Taylor expand the field at the initial position: $e_z(z) = e_z(z_0) + e_z'(z_0)(z-z_0) + 1/2 e_z''(z_0)(z-z_0)^2 + \cdots$. The difference in field between a particle in the center and at position $z$, is then $\Delta e_z(z) = \sigma_z e_z'(z_0) + \sigma_z e_z''(z_0)(z-z_0) + 1/2 \sigma_z e_z'''(z_0)(z-z_0)^2$ to first order in $\sigma_z$. The difference in momentum can then be calculated, similar to Eq. (3b) as follows:

$$\Delta \eta(z) = q \int_{z_0}^{z} \frac{\Delta e_z(z')}{v_z(z')} \, dz'.$$

(8)

Keeping track of the length change due to this momentum difference in the bunch while accelerating results in

$$\Delta z(z) = \frac{1}{m} \int_{z_0}^{z} \Delta \eta(z') \, v_z(z') \, dz'.$$

(9)

If only the constant term of the field is used to calculate the center velocity $v_z(z)$ this results in a polynomial approximation of the bunch length given by

$$\sigma_z(z) = \sigma_z(0) + \frac{e_z'(z_0)}{e_z(z_0)} + \frac{1}{6} \frac{e_z''(z_0)}{e_z(z_0)} (z-z_0)^2 + \cdots$$

(10)

In Fig. 17 $\sigma_z(z)$ is plotted as function of the bunch position $z$. The particle tracking simulation (black solid curve) shows the pulse is already significantly compressed while still traveling inside the field ($z_1 < 20$ mm). In our specific acceleration field the ions start close to a field maximum so the linear field term is small, which results in only a minor correction by the first order term in Eq. (10), depicted as the gray dashed curve. However, if also the second order term is used (red dashed curve), the analytical bunch length expression inside the accelerator agrees within 5% with the simulations.

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