Rotational symmetry vs. axisymmetry in shell theory

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Rotational symmetry vs. axisymmetry in Shell theory

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Rotational Symmetry vs. Axisymmetry in Shell Theory

*This paper is dedicated to Kumbakonam Rajagopal
on the occasion of his sixtieth birthday*

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Abstract

This paper treats rotationally symmetric motions of axisymmetric shells. It derives the governing equations in a convenient form and determines their mathematical structure. The complicated governing equations have the virtue of the far simpler equations for axisymmetric motions that there is but one independent spatial variable. Consequently the constitutive equations enjoy convenient monotonicity properties. The richness of rotationally symmetric motions is illustrated by a numerical treatment of an initial-boundary-value problem for a nonlinearly elastic cylindrical shell. This paper discusses the subtle question of nonexistence of general axisymmetric motions of axisymmetric shells. It briefly treats spatially autonomous motions, which are governed by ordinary differential equations in time.

1 Introduction

We say that a shell is *axisymmetric* if its (shape and) material properties are invariant under rotations about a fixed axis $\mathbf{k}$ and under reflections through a plane containing the axis $\mathbf{k}$. In this paper we formulate and study geometrically exact equations governing *rotationally symmetric* motions of (nonlinearly viscoelastic) axisymmetric shells. These motions are invariant merely under rotations about $\mathbf{k}$. Axisymmetric motions are subject to the further restriction that they are also invariant under reflections through a plane containing $\mathbf{k}$. This means that in an axisymmetric motion every material point with a reference position lying in a given plane $\mathcal{P}_0$ containing $\mathbf{k}$ stays

in that plane and that the motion in any other plane is just a rotation about $\mathbf{k}$ of the motion in $\mathcal{P}_0$. For a rotationally symmetric motion, the material points of $\mathcal{P}_0$ need not lie in a plane, but the deformation of any other plane is just a rotation about $\mathbf{k}$ of the motion in $\mathcal{P}_0$. The distinction between rotationally invariant motions and axisymmetric motions is illustrated in Figure 1. (Our methods are readily extended to treat rotationally symmetric motions of rotationally symmetric shells.)

Figure 1: Deformations of a typical section of an axisymmetric body perpendicular to the axis. (a) The reference configuration of this section showing radially disposed material fibers. (b) Projections onto the same plane of the deformed images of the radial fibers at some time when the body suffers an axisymmetric deformation. (c) Projections onto the same plane of the deformed images of the radial fibers at some time when the body suffers a rotationally symmetric deformation.

We study (Cosserat) shells whose configurations are determined by an image of a material surface and by a unit vector field defined at each material point of the material surface. These shells can suffer flexure, extension, and shear (but not transverse extension). The resulting theories accordingly are far more general than most of the classical theories of shells. Their governing equations of motion are obtained from the classical balances of force and torque. (Our methods handle more elaborate theories, but these require additional equations of motion, which are best motivated from 3-dimensional considerations.) Rotationally invariant motions of such shells are far richer than axisymmetric motions, but they enjoy the same virtue that the governing equations have but one independent spatial variable.

There are important mechanical problems in which an axisymmetric shell cannot execute axisymmetric motions; these include the rotation of an axisymmetric shell about its axis under frictional resistance offered by an ambient viscous fluid [5, 6, 7], and the combined breathing and spinning motions of such a shell [1]. We resolve a paradoxical subtlety by exhibiting the special class of azimuthally unshearable axisymmetric shells, which can execute such motions when axisymmetric shells cannot.

One of our aims is to exhibit reasonably simple formulations, having but one independent spatial variable, which are susceptible of analysis. We illustrate our theory with a numerical treatment of an initial-boundary-value problem for the combined breathing and twisting motions of a cylindrical shell, which exhibit a rich collection of phenomena. We briefly discuss spatially autonomous problems.

The reader interested in getting to our main results as rapidly as possible can skim over the complicated but elementary section 2 and the elementary section 3, relying on the summary of governing equations given at the end of section 3.

Notation. We employ Gibbs notation for vectors and tensors: Vectors, which are elements of Euclidean 3-space $\mathbb{E}^3$, and vector-valued functions are denoted by lower-case, italic, bold-face symbols. The dot product and cross product of (vectors) $\mathbf{u}$ and $\mathbf{v}$ are denoted $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$. The value of tensor $\mathbf{A}$ at

vector $\mathbf{v}$ is denoted $\mathbf{A} \cdot \mathbf{v}$ (in place of the more usual $\mathbf{A}\mathbf{v}$). The transpose of $\mathbf{A}$ is denoted $\mathbf{A}^*$. We write $\mathbf{v} \cdot \mathbf{A} = \mathbf{A}^* \cdot \mathbf{v}$. The dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$ (used in Section 7) is the tensor denoted $\mathbf{ab}$ (in place of the more usual $\mathbf{a} \otimes \mathbf{b}$), which is defined by $(\mathbf{ab}) \cdot \mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$ for all $\mathbf{v}$.

The (Gâteaux) differential of the function $\mathbf{u} \mapsto \mathbf{f}(\mathbf{u})$ at $\mathbf{v}$ in the direction $\mathbf{h}$ is $\frac{d}{d\tau}\mathbf{f}(\mathbf{v} + \tau\mathbf{h})|_{\tau=0}$. When it is linear in $\mathbf{h}$, we denote this differential by $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{v}) \cdot \mathbf{h}$ or $\mathbf{f}_u(\mathbf{v}) \cdot \mathbf{h}$. We occasionally denote the function $\mathbf{u} \mapsto \mathbf{f}(\mathbf{u})$ by $\mathbf{f}(\cdot)$.

2 Deformation

The configuration at time t of a special Cosserat shell (with an inextensible director) is specified by a surface (given by a function) $(s, \phi) \mapsto \mathbf{r}(s, \phi, t)$ and a unit vector field $(s, \phi) \mapsto \mathbf{d}(s, \phi, t)$, the *director* field, such that $\mathbf{d}(s, \phi, t)$ is not tangent to the surface $\mathbf{r}(\cdot, \cdot, t)$ at $\mathbf{r}(s, \phi, t)$. In the reference configuration, the surface $\mathbf{r}$ is denoted $\mathbf{r}^\circ$ and is called the *base surface*. We identify the values of all kinematic variables in the reference configuration by superposed circles. The director $\mathbf{d}(s, \phi, t)$ is interpreted as characterizing the orientation of a material fiber whose reference configuration $\mathbf{d}^\circ(s, \phi)$ is normal to the base surface $\mathbf{r}^\circ$ at $\mathbf{r}^\circ(s, \phi)$.

Let $\{\mathbf{i}_1 \equiv \mathbf{k}, \mathbf{i}_2, \mathbf{i}_3\}$ be a fixed right-handed orthonormal basis for Euclidean 3-space, and set

$$(2.1) \quad \mathbf{j}_1 := \mathbf{i}_1 \equiv \mathbf{k}, \quad \mathbf{j}_2(\phi) := \cos \phi \mathbf{i}_2 + \sin \phi \mathbf{i}_3, \quad \mathbf{j}_3(\phi) := -\sin \phi \mathbf{i}_2 + \cos \phi \mathbf{i}_3.$$

(We take the axis of rotation of the body to be the $\mathbf{i}_1 \equiv \mathbf{k}$ axis. We number this axis with a 1, rather than the more usual 3, because in the problems we treat, all the deformations can be obtained by those suffered by the section of the body in its reference configuration with the $\{\mathbf{i}_1, \mathbf{i}_2\}$ -plane.)

The reference configuration $\{\mathbf{r}^\circ, \mathbf{d}^\circ\}$ of an axisymmetric shell is defined by

$$(2.2) \quad \mathbf{r}^\circ(s, \phi) = z^\circ(s)\mathbf{j}_1 + r^\circ(s)\mathbf{j}_2(\phi), \quad |\mathbf{r}_s^\circ| = 1, \quad \mathbf{d}^\circ(s, \phi) = \mathbf{j}_3(\phi) \times \mathbf{r}_s^\circ(s, \phi), \\ s \in [s_1, s_2], \quad \phi \in [0, 2\pi]$$

where $r^\circ(s) > 0$ for $s \in (s_1, s_2)$. We define the right-handed orthonormal basis

$$(2.3) \quad \mathbf{b}_1^\circ(s, \phi) := \mathbf{r}_s^\circ(s, \phi) =: \cos \theta^\circ(s)\mathbf{j}_1 + \sin \theta^\circ(s)\mathbf{j}_2(\phi), \\ \mathbf{b}_2^\circ(s, \phi) := \mathbf{d}^\circ(s, \phi) =: -\sin \theta^\circ(s)\mathbf{j}_1 + \cos \theta^\circ(s)\mathbf{j}_2(\phi), \\ \mathbf{b}_3^\circ(\phi) := \frac{\mathbf{r}_\phi^\circ(s, \phi)}{r^\circ(s)} \equiv \mathbf{j}_3(\phi) \equiv \mathbf{j}_1 \times \mathbf{j}_2(\phi).$$

See Figure 2.

A vector-valued function $\mathbf{v}$ of s, ϕ is invariant under rotations about $\mathbf{k}$ if its components with respect to the basis $\{\mathbf{j}_k(\phi)\}$ are independent of ϕ . This means that

$$(2.4) \quad \mathbf{v}_\phi = \mathbf{k} \times \mathbf{v}.$$

The function $\mathbf{v}$ would be axisymmetric if furthermore $\mathbf{v}(s, \phi) \cdot \mathbf{j}_3(\phi) = 0$ for all s, ϕ .

We study problems in which $\mathbf{r}, \mathbf{d}$, together with the stress resultants, loads, and boundary and initial data, are invariant under rotations about $\mathbf{k}$. We term such problems *rotationally symmetric*. We shall characterize such motions by representing $\mathbf{r}, \mathbf{d}$ and their derivatives as appropriate linear combinations of orthonormal vectors with scalar coefficient functions independent of ϕ . Here we encounter a difficulty: There are several choices of bases, each with both advantages and disadvantages for describing deformations, accelerations, and constitutive equations. We shall exhibit the most promising alternatives.

Figure 2: Section of the reference configuration at a constant value of ϕ .

We introduce the function χ that accounts for a nonuniform rotation in the azimuthal direction and we introduce rotated versions of the pair $\mathbf{j}_2(\phi)$ and $\mathbf{j}_3(\phi)$ by

$$\begin{aligned}
 \mathbf{a}_1 &:= \mathbf{j}_1 \equiv \mathbf{i}_1 \equiv \mathbf{k}, \\
 \mathbf{a}_2(s, \phi, t) &:= \cos \chi(s, t) \mathbf{j}_2(\phi) + \sin \chi(s, t) \mathbf{j}_3(\phi), \\
 \mathbf{a}_3(s, \phi, t) &:= -\sin \chi(s, t) \mathbf{j}_2(\phi) + \cos \chi(s, t) \mathbf{j}_3(\phi).
 \end{aligned}
 \tag{2.5}$$

We require the image $\mathbf{r}$ of $\mathbf{r}^\circ$ at time t to have the rotationally symmetric form

$$\mathbf{r}(s, \phi, t) = z(s, t) \mathbf{k} + r(s, t) \mathbf{a}_2(s, \phi, t).
 \tag{2.6}$$

Before giving rotationally symmetric representations of $\mathbf{d}$ and before introducing other bases, it is instructive to give a 3-dimensional interpretation of the deformation: Set $\mathbf{x} := (s, \phi, \zeta)$. Consider a 3-dimensional axisymmetric shell with a reference configuration consisting of all the material points of the form

$$\tilde{\mathbf{z}}(\mathbf{x}) = \mathbf{r}^\circ(s, \phi) + \zeta \mathbf{d}^\circ(s, \phi), \quad \zeta \in [\zeta_1(s), \zeta_2(s)] \text{ with } \zeta_1(s) \leq 0 \leq \zeta_2(s),
 \tag{2.7}$$

where $\mathbf{r}^\circ$ and $\mathbf{d}^\circ$ are given by (2.2), where ζ_1 and ζ_2 are prescribed (thickness) functions on $[s_1, s_2]$ with $\zeta_1(s) < \zeta_2(s)$ for $s \in (s_1, s_2)$, and where $r^\circ(s) + \zeta \mathbf{d}^\circ(s, \phi) \cdot \mathbf{j}_2(\phi) > 0$ for $s \in (s_1, s_2)$. An arbitrary rotationally symmetric motion of such a shell is determined from an arbitrary one-to-one motion of the planar $\{\mathbf{k}, \mathbf{j}_2(0)\}$ -section of the shell in this reference configuration: The motion of the $\{\mathbf{k}, \mathbf{j}_2(\phi)\}$ -section is just the rotation about $\mathbf{k}$ through the angle ϕ of that for the $\{\mathbf{k}, \mathbf{j}_2(0)\}$ -section. In the special Cosserat shell theory we use, the $\{\mathbf{k}, \mathbf{j}_2(0)\}$ -section is constrained to suffer only motions taking it into a ruled surface with generators $\mathbf{d}$ passing through the space curve $\mathbf{r}$ and with the length along each generator unchanged in the motion. In particular, we may regard the shell theory we use as being generated by constraining the position of material point $\mathbf{x}$ at time t to have the form

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{r}(s, \phi, t) + \zeta \mathbf{d}(s, \phi, t),
 \tag{2.8}$$

and we study problems in which $\mathbf{r}$ and $\mathbf{d}$ are rotationally symmetric.

In view of this interpretation of $\mathbf{d}$, we regard $s = \bar{s}$ as characterizing a surface in the reference configuration lying on the frustum of a cone with its tangent plane having the normal $\mathbf{r}_s^\circ(\bar{s}, \phi)$ at $(\bar{s}, \phi)$, and having an image at time t that is also on a frustum of a cone with a tangent plane at $\mathbf{r}(\bar{s}, \phi, t)$ spanned by $\mathbf{j}_3(\phi + \chi(\bar{s}, t))$ and $\mathbf{d}(\phi + \chi(\bar{s}, t))$. We regard $\phi = \bar{\phi}$ as characterizing

a surface in the reference configuration lying on the plane defined by $\bar{\phi}$, and having an image at time t (which need not be planar) that has a tangent plane at $\mathbf{r}(s, \bar{\phi}, t)$ spanned by $\mathbf{r}_s(s, \bar{\phi}, t)$ and $\mathbf{d}(s, \bar{\phi}, t)$.

The deformed configuration is most naturally described by vector fields $\mathbf{r}_s$ and $\mathbf{r}_\phi$, which are tangent to the deformed surface $\mathbf{r}$, and the director field $\mathbf{d}$. No two of these independent fields are constrained to be orthogonal. The acceleration $\mathbf{r}_{tt}$ is most simply expressed in the basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. The director acceleration $\mathbf{d}_{tt}$ is most simply described in a basis containing $\mathbf{d}$. We now construct alternative orthonormal bases suitable for describing the geometry, mechanics, and material response of shells undergoing rotationally symmetric motions:

Figure 3: Disposition of base vectors. The axis of symmetry is $\mathbf{a}_1 = \mathbf{j}_1 = \mathbf{i}_1 = \mathbf{k}$. The shaded vertical plane containing it also contains $\mathbf{a}_2$, $\mathbf{b}_1 = \mathbf{c}_1$, $\mathbf{b}_2$. The horizontal plane spans $\mathbf{a}_2$ and $\mathbf{a}_3 = \mathbf{b}_3$. The slanted plane, containing $\mathbf{b}_2$, $\mathbf{b}_3$, the director $\mathbf{c}_2 = \mathbf{d}$, and $\mathbf{c}_3$, is the tangent plane to a deformed edge $s = \text{constant}$ (interpreted as a surface), which lies on a frustum of a cone.

We introduce the right-handed orthonormal basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ by

$$(2.9) \quad \mathbf{b}_1 := \frac{\mathbf{d} \times \mathbf{a}_3}{|\mathbf{d} \times \mathbf{a}_3|}, \quad \mathbf{b}_2 := \mathbf{a}_3 \times \mathbf{b}_1, \quad \mathbf{b}_3 := \mathbf{a}_3.$$

Since $\mathbf{d}$ can be nowhere tangent to the surface $\mathbf{r}$, and thus nowhere tangent to $\mathbf{r}_\phi = \mathbf{r}\mathbf{b}_3$, the denominator of the right-hand side of (2.9)₁ cannot vanish. Since $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal to $\mathbf{a}_3 \equiv \mathbf{b}_3$, they have the form

$$(2.10) \quad \begin{aligned} \mathbf{b}_1(s, \phi, t) &= \cos \theta(s, t) \mathbf{a}_1(s, \phi, t) + \sin \theta(s, t) \mathbf{a}_2(s, \phi, t), \\ \mathbf{b}_2(s, \phi, t) &= -\sin \theta(s, t) \mathbf{a}_1(s, \phi, t) + \cos \theta(s, t) \mathbf{a}_2(s, \phi, t), \\ \mathbf{b}_3(s, \phi, t) &:= \mathbf{a}_3(s, \phi, t) \end{aligned}$$

where θ is independent of ϕ . (See Figure 3.) The unit vector $\mathbf{b}_2$ lies along the projection of $\mathbf{d}$ onto the $\{\mathbf{a}_1, \mathbf{a}_2\}$ -plane. The vector $\mathbf{b}_1(\bar{s}, \phi, t)$ is normal to the image of the edge surface characterized by $s = \bar{s}$ at $\mathbf{r}(\bar{s}, \phi, t)$, and the vectors $\mathbf{b}_2(\bar{s}, \phi, t)$ and $\mathbf{b}_3(\bar{s}, \phi, t)$ are tangent to this surface at $\mathbf{r}(\bar{s}, \phi, t)$. Since $\mathbf{d}$ is orthogonal to $\mathbf{b}_1$ and nowhere tangent to $\mathbf{r}$, it has the form

$$(2.11) \quad \mathbf{d}(s, \phi, t) = \cos \psi(s, t) \mathbf{b}_2(s, \phi, t) + \sin \psi(s, t) \mathbf{b}_3(s, \phi, t) \quad \text{with} \quad |\psi| < \frac{\pi}{2}$$

and with ψ independent of ϕ . Thus

$$(2.12) \quad \begin{aligned} \partial_s \mathbf{b}_1 &= \theta_s \mathbf{b}_2 + \chi_s \sin \theta \mathbf{b}_3, & \partial_\phi \mathbf{b}_1 &= \sin \theta \mathbf{b}_3, \\ \partial_s \mathbf{b}_2 &= -\theta_s \mathbf{b}_1 + \chi_s \cos \theta \mathbf{b}_3, & \partial_\phi \mathbf{b}_2 &= \cos \theta \mathbf{b}_3, \\ \partial_s \mathbf{b}_3 &= -\chi_s \mathbf{a}_2, & \partial_\phi \mathbf{b}_3 &= -\mathbf{a}_2, \\ \mathbf{d}_s &= -(\chi_s \sin \psi \sin \theta + \theta_s \cos \psi) \mathbf{b}_1 + (\psi_s + \chi_s \cos \theta) [-\sin \psi \mathbf{b}_2 + \cos \psi \mathbf{b}_3], \\ \mathbf{d}_\phi &= -\sin \psi (\sin \theta \mathbf{b}_1 + \cos \theta \mathbf{b}_2) + \cos \psi \cos \theta \mathbf{b}_3. \end{aligned}$$

The t -derivatives of $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{d}$ are obtained from the s -derivatives by replacing the subscript s with t .

In many cases, it seems more advantageous to have a basis including $\mathbf{b}_1$, which is perpendicular to the deformed edge, and the director $\mathbf{d}$. We accordingly define

$$(2.13) \quad \begin{aligned} \mathbf{c}_1(s, \phi, t) &:= \mathbf{b}_1(s, \phi, t), \\ \mathbf{c}_2(s, \phi, t) &:= \mathbf{d}(s, \phi, t) \equiv \cos \psi(s, t) \mathbf{b}_2(s, \phi, t) + \sin \psi(s, t) \mathbf{b}_3(s, \phi, t), \\ \mathbf{c}_3(s, \phi, t) &:= \mathbf{b}_1(s, \phi, t) \times \mathbf{d}(s, \phi, t) \\ &\equiv -\sin \psi(s, t) \mathbf{b}_2(s, \phi, t) + \cos \psi(s, t) \mathbf{b}_3(s, \phi, t). \end{aligned}$$

Thus $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ form a right-handed orthonormal basis. (See Figure 3.) We immediately obtain the simpler representations

$$(2.14) \quad \begin{aligned} \mathbf{c}_{2s} \equiv \mathbf{d}_s &= -(\chi_s \sin \psi \sin \theta + \theta_s \cos \psi) \mathbf{c}_1 + (\psi_s + \chi_s \cos \theta) \mathbf{c}_3, \\ \mathbf{c}_{2\phi} \equiv \mathbf{d}_\phi &= -\sin \psi \sin \theta \mathbf{c}_1 + \cos \theta \mathbf{c}_3. \end{aligned}$$

Likewise, we find

$$(2.15) \quad \begin{aligned} \mathbf{c}_{3s} &= -(\psi_s + \chi_s \cos \theta) \mathbf{c}_2 + (\theta_s \sin \psi - \chi_s \cos \psi \sin \theta) \mathbf{c}_1, \\ \mathbf{c}_{3\phi} &= -\cos \psi \sin \theta \mathbf{c}_1 - \cos \theta \mathbf{c}_2. \end{aligned}$$

In summary,

$$(2.16) \quad \begin{aligned} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}, & \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} &= \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}, \\ \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}, & \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}, \\ \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} &= \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\cos \psi \cos \theta & \cos \psi \cos \theta & \sin \psi \\ \sin \psi \sin \theta & -\sin \psi \cos \theta & \cos \psi \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}, \\ \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} &= \begin{bmatrix} \cos \theta & -\cos \psi \sin \theta & \sin \psi \sin \theta \\ \sin \theta & \cos \psi \cos \theta & -\sin \psi \cos \theta \\ 0 & \sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix}. \end{aligned}$$

Again we get t -derivatives by analogy with the s -derivatives. In particular, we find

$$\begin{aligned}
(2.17) \quad \mathbf{d}_t &= -(\chi_t \sin \psi \sin \theta + \theta_t \cos \psi) \mathbf{c}_1 + (\psi_t + \chi_t \cos \theta) \mathbf{c}_3, \\
\mathbf{r}_t &= z_t \mathbf{a}_1 + r_t \mathbf{a}_2 + r \chi_t \mathbf{a}_3 \\
&\equiv (r_t \sin \theta + z_t \cos \theta) \mathbf{b}_1 + (r_t \cos \theta - z_t \sin \theta) \mathbf{b}_2 + r \chi_t \mathbf{b}_3 \\
&\equiv (r_t \sin \theta + z_t \cos \theta) \mathbf{c}_1 \\
&\quad + (r_t \cos \theta - z_t \sin \theta) (\cos \psi \mathbf{c}_2 - \sin \psi \mathbf{c}_3) + r \chi_t (\sin \psi \mathbf{c}_2 + \cos \psi \mathbf{c}_3).
\end{aligned}$$

(The formulas for $\mathbf{d}_{tt}$ and $\mathbf{r}_{tt}$ are much more complicated, but we shall avoid using them in our computations by formulating the equations of motion as systems of first-order equations in t -derivatives.)

We set

$$(2.18) \quad \mathbf{r}_s =: \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \beta_3 \mathbf{b}_3 =: \gamma_1 \mathbf{c}_1 + \gamma_2 \mathbf{c}_2 + \gamma_3 \mathbf{c}_3, \quad \beta_1 = \gamma_1, \quad \beta_3 = r \chi_s,$$

so that

$$(2.19) \quad \gamma_2 = \beta_2 \cos \psi + r \chi_s \sin \psi, \quad \gamma_3 = -\beta_2 \sin \psi + r \chi_s \cos \psi.$$

To ensure that there is neither total compression nor total shear, we require that

$$(2.20) \quad \delta(s, t) := [\mathbf{r}_s(s, \phi, t) \times \mathbf{r}_\phi(s, \phi, t)] \cdot \mathbf{d}(s, \phi, t) \equiv r \gamma_1 \cos \psi > 0$$

except at poles where r is prescribed to vanish.

The strain variables for this theory of shells (most of which do not vanish in the natural reference configuration) consist of any independent set of functions from the set of all dot and triple scalar products of $\mathbf{r}_s, \mathbf{r}_\phi, \mathbf{d}, \mathbf{d}_s, \mathbf{d}_\phi$. It can be shown [2, Sec. 17.8] that such a set consists of

$$\begin{aligned}
(2.21) \quad \mathbf{r}_s \cdot \mathbf{r}_s &= \beta_1^2 + \beta_2^2 + r^2 \chi_s^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \\
\mathbf{r}_s \cdot \mathbf{r}_\phi &= r^2 \chi_s, \\
\mathbf{r}_\phi \cdot \mathbf{r}_\phi &= r^2, \\
\mathbf{r}_s \cdot \mathbf{d} &= \beta_2 \cos \psi + r \chi_s \sin \psi = \gamma_2, \\
\mathbf{r}_\phi \cdot \mathbf{d} &= r \sin \psi, \\
\mathbf{r}_s \cdot \mathbf{d}_s &= -\gamma_1 (\theta_s \cos \psi + \chi_s \sin \psi \sin \theta) + \gamma_3 (\theta_s \sin \psi - \chi_s \cos \psi \sin \theta), \\
\mathbf{r}_s \cdot \mathbf{d}_\phi &= -(\gamma_1 \sin \theta + \beta_2 \cos \theta) \sin \psi + r \chi_s \cos \psi \cos \theta \\
&= -\gamma_1 \sin \psi \sin \theta + \gamma_3 \cos \theta, \\
\mathbf{r}_\phi \cdot \mathbf{d}_s &= r \cos \psi (\psi_s - \chi_s \cos \theta), \\
\mathbf{r}_\phi \cdot \mathbf{d}_\phi &= -r \cos \psi \cos \theta, \\
(\mathbf{r}_s \times \mathbf{r}_\phi) \cdot \mathbf{d} &= \delta = r \gamma_1 \cos \psi.
\end{aligned}$$

If the terms of (2.21) are given, then they determine

$$(2.22) \quad r, \quad \beta_1 = \gamma_1, \quad \beta_2, \quad \gamma_2, \quad \gamma_3, \quad \theta, \quad \psi, \quad \chi_s, \quad \theta_s, \quad \psi_s.$$

The specification of (2.22) as functions of s for fixed t determines a rotationally symmetric configuration $\mathbf{r}, \mathbf{d}$ unique to within a rigid rotation about the axis and a translation along it.

3 Equations of motion

We give a direct derivation of the equations of motion for an axisymmetric shell in polar coordinates and then specialize them to motions invariant about $\mathbf{k}$. A purpose of this exercise is to obtain the weight functions associated with the curvilinear coordinates without invoking the apparatus of tensor analysis. This development complements that of [2, Chap. 17].

We take the linear and angular momentum at (s, ϕ, t) per unit reference area to be

$$(3.1) \quad \begin{aligned} & (\rho A)(s) \mathbf{r}_t(s, \phi, t) + (\rho I)(s) \mathbf{d}_t(s, \phi, t), \\ & (\rho A)(s) \mathbf{r}(s, \phi, t) \times \mathbf{r}_t(s, \phi, t) + (\rho J)(s) \mathbf{d}(s, \phi, t) \times \mathbf{d}_t(s, \phi, t) \\ & + (\rho I)(s) [\mathbf{r}(s, \phi, t) \times \mathbf{d}_t(s, \phi, t) + \mathbf{d}(s, \phi, t) \times \mathbf{r}_t(s, \phi, t)] \end{aligned}$$

where $\rho A, \rho I, \rho J$ are given with

$$(3.2) \quad (\rho A)(s) > 0, \quad (\rho J)(s) > 0, \quad (\rho A)(s)(\rho J)(s) - (\rho I)(s)^2 > 0 \quad \forall s \in (s_1, s_2).$$

These functions are independent of ϕ in consonance with the axisymmetry of the reference configuration. ρA , ρI , and ρJ may be interpreted as the mass, first moment of mass, and second moment of mass per unit reference area of a fiber in the reference configuration normal to the base surface at material point (s, ϕ) . (See (7.8) below.) The conditions (3.2) are necessary and sufficient for the kinetic energy to be a positive-definite quadratic form.

A 3-dimensional interpretation of these inertias, given in Section 7 and the use of a variant of the argument of [2, Ex. 8.4.8] shows that it is always possible to take $\rho I = 0$. For problems in which the spinning shell is in contact with a frictional resistance, e.g., offered by an ambient fluid [5, 6, 7], it is often most convenient to take the base surface to be in contact with the fluid, so that the force on this surface does not produce a moment about the surface. In this case the simplification that $\rho I = 0$ is not available.

Let $\mathbf{n}^s(s_0, \phi, t)$ and $\mathbf{m}^s(s_0, \phi, t)$ denote the internal contact force and couple per unit (reference) length of the material circle $\phi \mapsto \mathbf{r}^\circ(s_0, \phi)$ exerted at (s_0, ϕ) at time t by the material with $s \geq s_0$ on that with $s < s_0$. Likewise, let $\mathbf{n}^\phi(s, \phi_0, t)$ and $\mathbf{m}^\phi(s, \phi_0, t)$ denote the internal contact force and couple per unit (reference) length of the material curve $s \mapsto \mathbf{r}^\circ(s, \phi_0)$ exerted at (s, ϕ_0) at time t by the material with $\phi \in (\phi_0, \phi_0 + \varepsilon)$ on that with $\phi \in (\phi_0 - \varepsilon, \phi_0)$ where ε is any small positive number. Let $\mathbf{f}(s, \phi, t)$ and $\mathbf{l}(s, \phi, t)$ be the applied force and couple per unit reference area of $\mathbf{r}^\circ$. (These might be composite functions depending on the motion.)

Fundamental theory [2, Chap 17] (part of which is reproduced in (7.7)) says that

$$(3.3) \quad \mathbf{m}^s \cdot \mathbf{d} = 0 = \mathbf{m}^\phi \cdot \mathbf{d}, \quad \mathbf{l} \cdot \mathbf{d} = 0.$$

The balances of linear momentum and angular momentum for the material of $\{(\xi, \varphi) : s_1 \leq \xi \leq s, 0 \leq \varphi \leq \phi\}$ are

$$(3.4) \quad \begin{aligned} & \int_0^\phi [\mathbf{n}^s(s, \varphi, t) - \mathbf{n}^s(s_1, \varphi, t)] r^\circ(s_1) d\varphi \\ & + \int_{s_1}^s [\mathbf{n}^\phi(\xi, \phi, t) - \mathbf{n}^\phi(\xi, 0, t)] d\xi \\ & + \int_{s_1}^s \int_0^\phi \mathbf{f}(\xi, \varphi, t) r^\circ(\xi) d\varphi d\xi \\ & = \frac{d}{dt} \int_{s_1}^s \int_0^\phi [(\rho A)(\xi) \mathbf{r}_t(\xi, \varphi, t) + (\rho I)(\xi) \mathbf{d}_t(\xi, \varphi, t)] r^\circ(\xi) d\varphi d\xi, \end{aligned}$$

$$\begin{aligned}
& \int_0^\phi [\mathbf{m}^s(s, \varphi, t) - \mathbf{m}^s(s_1, \varphi, t)] r^\circ(s_1) d\varphi \\
& + \int_0^\phi [\mathbf{r}(s, \varphi, t) \times \mathbf{n}^s(s, \varphi, t) - \mathbf{r}(s_1, \varphi, t) \times \mathbf{n}^s(s_1, \varphi, t)] r^\circ(s_1) d\varphi \\
& + \int_{s_1}^s [\mathbf{m}^\phi(\xi, \phi, t) - \mathbf{m}^\phi(\xi, 0, t)] d\xi \\
(3.5) \quad & + \int_{s_1}^s [\mathbf{r}(\xi, \phi, t) \times \mathbf{n}^\phi(\xi, \varphi, t) - s \mathbf{r}(\xi, 0, t) \times \mathbf{n}^\phi(\xi, 0, t)] d\xi \\
& + \int_{s_1}^s \int_0^\phi [\mathbf{l}(\xi, \varphi, t) + \mathbf{r}(\xi, \varphi, t) \times \mathbf{f}(\xi, \varphi, t)] r^\circ(\xi) d\varphi d\xi \\
& = \frac{d}{dt} \int_{s_1}^s \int_0^\phi \{(\rho A)(\xi) \mathbf{r}(\xi, \varphi, t) \times \mathbf{r}_t(\xi, \varphi, t) \\
& \quad + (\rho I)(\xi) [\mathbf{r}(\xi, \varphi, t) \times \mathbf{d}_t(\xi, \varphi, t) + \mathbf{d}(\xi, \varphi, t) \times \mathbf{r}_t(\xi, \varphi, t)] \\
& \quad + (\rho J)(\xi) \mathbf{d}(\xi, \varphi, t) \times \mathbf{d}_t(\xi, \varphi, t)\} r^\circ(\xi) d\varphi d\xi.
\end{aligned}$$

We differentiate (3.4) with respect to s and ϕ to obtain

$$\begin{aligned}
(3.6) \quad & \partial_s [r^\circ(s) \mathbf{n}^s(s, \phi, t)] + \partial_\phi [\mathbf{n}^\phi(s, \phi, t)] + r^\circ(s) \mathbf{f}(s, \phi, t) \\
& = r^\circ(s) [(\rho A)(s) \mathbf{r}_{tt}(s, \phi, t) + (\rho I)(s) \mathbf{d}_{tt}(s, \phi, t)],
\end{aligned}$$

and differentiate (3.5) with respect to s and ϕ and use (3.6) to obtain

$$\begin{aligned}
(3.7) \quad & \partial_s [r^\circ(s) \mathbf{m}^s(s, \phi, t)] + \partial_\phi [\mathbf{m}^\phi(s, \phi, t)] + r^\circ(s) \mathbf{l}(s, \phi, t) \\
& + r^\circ(s) \mathbf{r}_s(s, \phi, t) \times \mathbf{n}^s(s, \phi, t) + \mathbf{r}_\phi(s, \phi, t) \times \mathbf{n}^\phi(s, \phi, t) \\
& = r^\circ(s) \mathbf{d}(s, \phi, t) \times [(\rho I)(s) \mathbf{r}_{tt}(s, \phi, t) + (\rho J)(s) \mathbf{d}_{tt}(s, \phi, t)].
\end{aligned}$$

For rotational symmetry, the components of the resultants $\mathbf{n}^s(s, \phi, t)$, $\mathbf{n}^\phi(s, \phi, t)$, $\mathbf{m}^s(s, \phi, t)$, $\mathbf{m}^\phi(s, \phi, t)$, and the loads $\mathbf{f}(s, \phi, t)$, $\mathbf{l}(s, \phi, t)$ with respect to the orthonormal basis $\{\mathbf{a}_j(s, \phi, t)\}$, say, must be independent of ϕ . In this case,

$$(3.8) \quad \partial_\phi \mathbf{n}^\phi(s, \phi, t) = \mathbf{k} \times \mathbf{n}^\phi(s, \phi, t), \quad \partial_\phi \mathbf{m}^\phi(s, \phi, t) = \mathbf{k} \times \mathbf{m}^\phi(s, \phi, t).$$

A basic difficulty in coordinate versions of the dynamics of rods and shells is that the acceleration of a position vector like $\mathbf{r}$ is most easily expressed in a basis fixed in space or, more generally, in a basis with a prescribed motion in space, and that the acceleration of the director $\mathbf{d}$ is most easily expressed in a basis containing $\mathbf{d}$. On the other hand, the stress resultants are most easily expressed in a material basis associated with the deformation, such as $\mathbf{c}_k$. We now get componential versions of (3.6) with respect to the basis $\{\mathbf{a}_k\}$, whose motion is partly prescribed and partly dependent on the deformation through χ , and get componential versions of (3.7) with respect to the basis $\{\mathbf{c}_k\}$. We

substitute (2.17) into (3.6), (3.7) and use (3.8) to get

$$(3.9) \quad \partial_s[r^\circ \mathbf{n}^s \cdot \mathbf{a}_1] + r^\circ \mathbf{f} \cdot \mathbf{a}_1 = r^\circ \rho A z_{tt} + r^\circ \rho I \mathbf{a}_1 \cdot \mathbf{d}_{tt},$$

$$(3.10) \quad \begin{aligned} & \partial_s[r^\circ \mathbf{n}^s \cdot \mathbf{a}_2] - r^\circ \chi_s \mathbf{n}^s \cdot \mathbf{a}_3 - \mathbf{n}^\phi \cdot \mathbf{a}_3 + r^\circ \mathbf{f} \cdot \mathbf{a}_2 \\ & = r^\circ \rho A (r_{tt} - r \chi_t^2) + r^\circ \rho I \mathbf{a}_2 \cdot \mathbf{d}_{tt}, \end{aligned}$$

$$(3.11) \quad \begin{aligned} & \partial_s[r^\circ \mathbf{n}^s \cdot \mathbf{a}_3] + r^\circ \chi_s \mathbf{n}^s \cdot \mathbf{a}_2 + \mathbf{n}^\phi \cdot \mathbf{a}_2 + r^\circ \mathbf{f} \cdot \mathbf{a}_3 \\ & = r^\circ \rho A (2r_t \chi_t + r \chi_{tt}) + r^\circ \rho I \mathbf{a}_3 \cdot \mathbf{d}_{tt}, \end{aligned}$$

$$(3.12) \quad \begin{aligned} & \partial_s[r^\circ \mathbf{m}^s \cdot \mathbf{c}_1] + r^\circ (\theta_s \sin \psi - \chi_s \cos \psi \sin \theta) \mathbf{m}^s \cdot \mathbf{c}_3 - \sin \theta \cos \psi \mathbf{m}^\phi \cdot \mathbf{c}_3 \\ & + r^\circ (\gamma_2 \mathbf{c}_3 - \gamma_3 \mathbf{c}_2) \cdot \mathbf{n}^s - r (\cos \psi \mathbf{c}_2 - \sin \psi \mathbf{c}_3) \cdot \mathbf{n}^\phi + r^\circ \mathbf{l} \cdot \mathbf{c}_1 \\ & = r^\circ \mathbf{c}_3 \cdot [\rho I \mathbf{r}_{tt} + \rho J \mathbf{d}_{tt}], \end{aligned}$$

$$(3.13) \quad \begin{aligned} & \partial_s[r^\circ \mathbf{m}^s \cdot \mathbf{c}_3] - r^\circ (\theta_s \sin \psi - \chi_s \cos \psi \sin \theta) \mathbf{m}^s \cdot \mathbf{c}_1 \\ & + \cos \psi \sin \theta \mathbf{c}_1 \cdot \mathbf{m}^\phi + r^\circ [\gamma_1 \mathbf{c}_2 - \gamma_2 \mathbf{c}_1] \cdot \mathbf{n}^s - r \sin \psi \mathbf{c}_1 \cdot \mathbf{n}^\phi + r^\circ \mathbf{l} \cdot \mathbf{c}_3 \\ & = -r^\circ \mathbf{c}_1 \cdot [\rho I \mathbf{r}_{tt} + \rho J \mathbf{d}_{tt}], \end{aligned}$$

Recall that a companion to the requirement that (3.3) hold is that the $\mathbf{d}$ component of the torque balance reduces to a constitutive identity [2, Chap. 17].

Constitutive equations. For a viscoelastic shell of strain-rate type, we assume that the components of $\mathbf{n}^s, \mathbf{n}^\phi, \mathbf{m}^s, \mathbf{m}^\phi$ with respect to a suitable basis carried along with the motion depend on the strains (2.21) or (2.22) and their time derivatives. We take the equilibrium response to be hyperelastic. Because there is but one independent spatial variable, the requirement that the equations of motion for a hyperelastic shell be hyperbolic endows the constitutive functions for components of the stress resultants with nice monotonicity properties with respect to the components of $\mathbf{r}_s$ and $\mathbf{d}_s$ that are derivatives. (If the shell theory is generated by constraining the 3-dimensional theory, then these monotonicity conditions are consequences of the Strong Ellipticity Condition [2, Prop. 17.4.21]. We assume that the stress resultants are strictly monotone functions of the corresponding components of the strain rate, to ensure that the material response is truly dissipative. These conditions suggest that the mathematical analysis of rotationally symmetric motions of a nonlinear viscoelastic shell, ensuring its existence and uniqueness for all time, can be carried out by using the same methods as [4].

Rather than developing this theory in detail, we shall exhibit in Section 6 a set of constitutive functions to be used in a numerical example. These functions enjoy the monotonicity conditions just described.

Governing equations. The governing equations consist of the geometrical relations of Section 2, especially the strain-configuration equations (2.12) and (2.18), the equations of motion (3.9)–(3.13), and the constitutive equations in which the strains of (2.21) intervene.

4 Impossibility of certain axisymmetric motions

A rotationally symmetric motion is *axisymmetric* if

$$(4.1) \quad \mathbf{r}_s \cdot \mathbf{b}_3 = 0 \quad \Leftrightarrow \quad \chi_s = 0, \quad \mathbf{d} \cdot \mathbf{b}_3 = 0 \quad \Leftrightarrow \quad \psi = 0,$$

in which case

$$(4.2) \quad \mathbf{r}_s = r_s \mathbf{a}_2 + z_s \mathbf{k}, \quad \gamma_3 = 0, \quad \mathbf{d} = \mathbf{b}_2 = \mathbf{c}_2, \quad \mathbf{a}_3 = \mathbf{b}_3 = \mathbf{c}_3,$$

and if the resultants satisfy

$$(4.3) \quad \mathbf{n}^s \cdot \mathbf{b}_3 = 0, \quad \mathbf{n}^\phi \times \mathbf{b}_3 = \mathbf{o}, \quad \mathbf{m}^s \times \mathbf{b}_3 = \mathbf{o}, \quad \mathbf{m}^\phi \times \mathbf{b}_1 = \mathbf{o}.$$

If an axisymmetric shell is to execute free axisymmetric motions, then condition (4.3) is effectively a set of constitutive restrictions. Of course, a shell not satisfying such constitutive restrictions could suffer axisymmetric motions under the action of suitable control loads.

Let us study whether our equations of motion for rotationally symmetric motions can sustain axisymmetric motions. We specialize (3.11) and (3.12) to an axisymmetric motion, for which (4.1)–(4.3) hold. Then these equations reduce to

$$(4.4) \quad r^\circ \mathbf{f} \cdot \mathbf{b}_3 = r^\circ \mathbf{b}_3 \cdot [\rho A \mathbf{r}_{tt} + \rho I \mathbf{d}_{tt}],$$

$$(4.5) \quad r^\circ \mathbf{l} \cdot \mathbf{c}_1 = r^\circ \mathbf{b}_3 \cdot [\rho I \mathbf{r}_{tt} + \rho J \mathbf{d}_{tt}],$$

Now the formulas of Section 2 show that

$$(4.6) \quad \begin{aligned} \mathbf{b}_3 \cdot \mathbf{r}_{tt} &\equiv \mathbf{c}_3 \cdot \mathbf{r}_{tt} = 2r_t \chi_t + r \chi_{tt} \equiv r^{-1} (r^2 \chi_t)_t, \\ \mathbf{b}_3 \cdot \mathbf{d}_{tt} &= \cos \theta \chi_{tt} - 2\chi_t \theta_t \sin \theta \equiv \frac{(\chi_t \cos^2 \theta)_t}{\cos \theta} \end{aligned}$$

when (4.1)–(4.3) hold, when $r \neq 0$, and when $\cos \theta \neq 0$, so that the right-hand sides of (4.4) and (4.5) are zero if $\chi_t = 0$. (r can only vanish at a pole for a shell closed at the pole. We tacitly limit the subsequent discussion to those s, t for which $\cos \theta(s, t) \neq 0$; an argument based on continuity can handle the case that $\cos \theta(s, t) = 0$.) Conversely, (3.2) implies that if the right-hand sides of (4.4) and (4.5) vanish, i.e., if the expressions in (4.6) vanish, then

$$(4.7) \quad r(s, t)^2 \chi_t(t) = r(s, 0)^2 \chi_t(0), \quad \cos^2 \theta(s, t) \chi_t(t) = \cos^2 \theta(s, 0) \chi_t(0),$$

so that $\chi_t(t)$ has the same sign as $\chi_t(0)$. If $\chi_t(0) \neq 0$, then the motion must have the form

$$(4.8) \quad r(s, t) = \frac{r(s, 0)}{\cos \theta(s, 0)} \cos \theta(s, t),$$

$$(4.9) \quad r(s, t) = r(s, 0)q(t), \quad \cos \theta(s, t) = \cos \theta(s, 0)q(t), \quad q(t) := \sqrt{\frac{\chi_t(0)}{\chi_t(t)}},$$

the positivity of q implying that $\cos \theta(s, t)$ has the same sign as $\cos \theta(s, 0)$.

Thus if the left-hand sides of (4.4) and (4.5) vanish, then axisymmetric motions with $\chi_t(0) \neq 0$ are severely restricted with r and θ determined by χ_t and their initial conditions. Indeed, the substitution of (4.9) into the forms of (3.9)–(3.10), (3.13) subject to (4.1)–(4.3), (4.9) yield an overdetermined system of three equations for χ_t and z in which the acceleration terms have very special forms. It is highly unlikely for a given $\mathbf{f}$, say $\mathbf{f} = \mathbf{o}$, for given constitutive functions, and for given initial conditions that these equations admit a solution with χ_t independent of s . Consequently, the typical way the right-hand sides of (4.4) and (4.5) can vanish is for $\chi_t = 0$, which means that there is no rotational motion about the axis $\mathbf{k}$.

In this case, (a variant of the existence theory of [4] implies that) a non-linearly viscoelastic shell of strain-rate type under natural constitutive restrictions can execute an axisymmetric motion, consisting of longitudinal and radial motions, provided that the components of body force and body couple on the left-hand sides of (4.4) and (4.5) vanish. On the other hand, if there is rotational motion, then these reduced equations (4.4) and (4.5) say that such motions can only occur if the components $\mathbf{f} \cdot \mathbf{b}_3$, $\mathbf{l} \cdot \mathbf{a}_2$, of the external loads are artificial feedbacks chosen to balance these equations. Without such feedbacks, our material cannot sustain rotational axisymmetric motions.

Such motions, however, can be sustained naturally for *azimuthally unshearable* shells, for which the equations of (4.1) are treated as material constraints, so that they cannot be violated no matter how the shell is loaded. In this case, as we shall now show, the combination of resultants entering the two degenerate equations of motion (3.11), (3.12) (which are the sources of (4.4) and (4.5)) are combinations of Lagrange multipliers maintaining the constraints: They cannot be presumed to vanish and they are determined from the inertia terms in these equations.

To study such constraints we use a Principle of Virtual Power to determine the resultants in duality with the constrained kinematic variables. These resultants are the Lagrange multipliers. The full discussion of the Principle of Virtual Power appropriate for our work is given in [2, Chap. 17]. Here we formally derive that part of the Principle of Virtual Power needed. The test function (virtual displacement or velocity) corresponding to $\mathbf{r}$ of (2.6) is (its first variation)

$$(4.10) \quad \mathbf{r}^\Delta = r^\Delta \mathbf{a}_2 + r\chi^\Delta \mathbf{a}_3 + z^\Delta \mathbf{k}.$$

We take the dot product of (3.6), subject to (3.8) with $r\chi^\Delta \mathbf{a}_3$ (which is the part of $\mathbf{r}^\Delta$ corresponding to the first constraint of (4.1)) and then integrate the resulting product with respect to s over $[s_1, s_2]$. (For our present purposes, it is not necessary to integrate this product by parts.) That part of the integral containing the resultants is

$$(4.11) \quad \int_{s_1}^{s_2} \{\partial_s[r^\circ \mathbf{n}^s \cdot \mathbf{a}_3] + r^\circ \chi_s \mathbf{n}^s \cdot \mathbf{a}_2 + \mathbf{n}^\phi \cdot \mathbf{a}_2\} r\chi^\Delta ds.$$

When (4.1)₁ holds, $\chi_s = 0$, so that χ depends only on t , and the second summand in the braces vanishes. In this case, χ^Δ , which we can take to be independent of t , is a pure constant. By a choice of coordinates, we can in fact choose $\chi^\Delta = 0$. Thus its coefficient in (4.11) is a (combination of) Lagrange multiplier(s) maintaining its constraint. It is precisely this combination that appears in the left-hand side of (3.11). Since it is not required to vanish, it balances this equation and is determined by it.

The analogous treatment of the moment equations is a little trickier. We could get the test function corresponding to $\mathbf{d}$ by replacing the s -derivatives in the expression for $\mathbf{d}_s$ in (2.12) with their variations. But this formula suppresses the structure to accommodate (3.3), which implies that the bending couples are cross products with $\mathbf{d}$. Instead, we merely observe that since the $\mathbf{b}_k$ are orthonormal, any derivatives, in particular, any variation has the form

$$(4.12) \quad \mathbf{b}_k^\Delta = \boldsymbol{\omega} \times \mathbf{b}_k.$$

Taking (4.1)₃ as a constraint implies that $\mathbf{b}_2 = \mathbf{d}$, so $\mathbf{d}^\Delta$ has the form (4.12). Let us also take $\chi^\Delta = 0$, in consonance with (4.1)₂, so that (2.5) and (2.10) imply that $\mathbf{b}_3^\Delta \equiv \boldsymbol{\omega} \times \mathbf{b}_3 = \mathbf{o}$. Since $\mathbf{b}_2 \cdot \mathbf{b}_3 = 0$, it then follows that $0 = \mathbf{b}_2^\Delta \cdot \mathbf{b}_3 + \mathbf{b}_2 \cdot \mathbf{b}_3^\Delta \equiv (\boldsymbol{\omega} \times \mathbf{b}_2) \cdot \mathbf{b}_3 + 0 \equiv \boldsymbol{\omega} \cdot \mathbf{b}_1$. (Likewise $\boldsymbol{\omega} \cdot \mathbf{b}_2 = 0$.) We obtain the appropriate version of the Principle of Virtual Power by taking the dot product of (3.7) with $\boldsymbol{\omega}$ and then integrating the resulting product with respect to s over $[s_1, s_2]$, again refraining from integrating by parts. That part of the integral containing the resultants is

$$(4.13) \quad \int_{s_1}^{s_2} \{\partial_s[r^\circ \mathbf{m}^s] + \mathbf{k} \times \mathbf{m}^\phi + r^\circ \mathbf{r}_s \times \mathbf{n}^s + [\mathbf{k} \times \mathbf{r}] \times \mathbf{n}^\phi\} \cdot \boldsymbol{\omega} ds.$$

In particular, we specialize this equation to $\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \mathbf{b}_1)\mathbf{b}_1 \equiv (\boldsymbol{\omega} \cdot \mathbf{c}_1)\mathbf{c}_1$. Since $\boldsymbol{\omega} \cdot \mathbf{b}_1 = 0$ when (4.1) is taken as a constraint, its coefficient in the integrand of (4.13), namely, the dot product of the expression in braces with $\mathbf{c}_1$, is a (combinations of) Lagrange multiplier(s) helping to maintaining the constraint (4.1). This multiplier is precisely the expression for the sum of the resultants appearing in (3.12).

In summary, an unconstrained shell, in particular one capable of undergoing rotationally symmetric motions, typically cannot undergo rotational axisymmetric motions, unless very special feedback loads are applied. An azimuthally unshearable axisymmetric shell can undergo axisymmetric rotational motions. The equations (3.11), (3.12) then serve to restrict the Lagrange multipliers. The remaining three equations then determine the motion.

5 Spatially autonomous problems

We can gain some insight into the complexity of rotationally symmetric motions of homogeneous cylindrical shells, like that treated in Section 6, by seeking solutions governed by ordinary differential equations in time, for which

$$(5.1) \quad r, z_s, \gamma_1, \gamma_2, \gamma_3, \chi_s, \psi \quad \text{are independent of } s, \quad \theta = 0.$$

We take $r^\circ = 1$. Then the equations of motion (3.9)–(3.13) reduce to

$$\begin{aligned} 0 &= \rho A z_{tt} + \rho I \mathbf{a}_1 \cdot \mathbf{d}_{tt}, \\ -\chi_s \mathbf{n}^s \cdot \mathbf{a}_3 - \mathbf{n}^\phi \cdot \mathbf{a}_3 &= \rho A (r_{tt} - r \chi_t^2) + \rho I \mathbf{a}_2 \cdot \mathbf{d}_{tt}, \\ (5.2) \quad \chi_s \mathbf{n}^s \cdot \mathbf{a}_2 + \mathbf{n}^\phi \cdot \mathbf{a}_2 &= \rho A (2r_t \chi_t + r \chi_{tt}) + \rho I \mathbf{a}_3 \cdot \mathbf{d}_{tt}, \\ (\gamma_2 \mathbf{c}_3 - \gamma_3 \mathbf{c}_2) \cdot \mathbf{n}^s - r(\cos \psi \mathbf{c}_2 - \sin \psi \mathbf{c}_3) \cdot \mathbf{n}^\phi &= \mathbf{c}_3 \cdot (\rho I \mathbf{r}_{tt} + \rho J \mathbf{d}_{tt}), \\ (\gamma_1 \mathbf{c}_2 - \gamma_2 \mathbf{c}_1) \cdot \mathbf{n}^s - r \sin \psi \mathbf{c}_1 \cdot \mathbf{n}^\phi &= -\mathbf{c}_1 \cdot (\rho I \mathbf{r}_{tt} + \rho J \mathbf{d}_{tt}), \end{aligned}$$

Since the $\mathbf{a}$'s depend on χ and since χ_t and χ_{tt} appear explicitly in (5.2), these equations are not independent of s unless $\chi_s = 0$. This means that there are no such simple solutions accounting for twisting. They are prevented by the presence of Coriolis acceleration. Consequently, the motions governed by ordinary differential equations are equivalent to the more restricted class treated in [1]. Moreover, this observation underlies the complexity of motions found in Section 6.

6 Example

We specialize the theory developed above to uniform nonlinearly elastic cylindrical shells, using specific constitutive functions described in the next section, and perform a numerical experiment exhibiting coupled breathing, twisting, and shearing motions of the shell. We consider a shell whose natural state is a circular cylinder of radius 1 and length L , which we take to be the reference configuration:

$$(6.1) \quad \mathbf{r}^\circ(s, \phi) = \mathbf{j}_2(\phi) + s\mathbf{k}, \quad \mathbf{d}^\circ(\phi) = \mathbf{j}_2(\phi),$$

where $(s, \phi) \in [0, L] \times [0, 2\pi)$. Then

$$(6.2) \quad r^\circ = 1, \quad z^\circ = s, \quad \chi^\circ = 0, \quad \theta^\circ = 0, \quad \psi^\circ = 0, \quad \gamma_1^\circ = 1, \quad \eta^\circ = 0,$$

$$(6.3) \quad \mathbf{a}_2^\circ = \mathbf{j}_2(\phi), \quad \mathbf{a}_1^\circ = \mathbf{k}, \quad \mathbf{b}_3^\circ = \mathbf{j}_3(\phi), \quad \mathbf{b}_2^\circ = \mathbf{j}_2(\phi) = \mathbf{d}^\circ.$$

We take

$$(6.4) \quad \mathbf{f} = \mathbf{o}, \quad \mathbf{l} = \mathbf{o},$$

and choose ζ_1 and ζ_2 to be constants satisfying $\rho I = 0$ (in accord with the comments in the third paragraph of Section 3) and $\zeta_2 - \zeta_1 = 2h$ so that the shell has constant thickness $2h$.

To perform a numerical experiment it is convenient to write the equations of motion (3.9)–(3.13) as a nonlinear hyperbolic system of first-order equations for the velocities $\mathbf{v}$ and the strains $\mathbf{q}$:

$$(6.5) \quad \begin{aligned} \mathbf{v} &\equiv (v_1, v_2, v_3, v_4, v_5) := (z_t, r_t, r\chi_t, \theta_t, \psi_t) \\ \mathbf{q} &\equiv (q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8) := (z_s, r_s, r\chi_s, \theta_s, \psi_s, r, \theta, \psi). \end{aligned}$$

Supplementing the equations of motion (3.9)–(3.13) with the compatibility equations relating $\partial_s \mathbf{v}$ to $\partial_t \mathbf{q}$ yields the system

$$(6.6) \quad \begin{aligned} \partial_t \mathbf{p}(\mathbf{v}, \mathbf{q}) &= \partial_s \mathbf{m}(\mathbf{q}) + \mathbf{g}(\mathbf{v}, \mathbf{q}) + \mathbf{h}(\mathbf{q}), \\ \partial_t \mathbf{q} &= \partial_s \mathbf{C} \mathbf{v} + \mathbf{w}(\mathbf{v}, \mathbf{q}), \end{aligned}$$

with

$$(6.7) \quad \begin{aligned} \mathbf{p} &:= (\rho A v_1, \rho A v_2, \rho A v_3, p_4, p_5), \\ \mathbf{m} &:= (\mathbf{n}^s \cdot \mathbf{a}_1, \mathbf{n}^s \cdot \mathbf{a}_2, \mathbf{n}^s \cdot \mathbf{a}_3, \mathbf{m}^s \cdot \mathbf{c}_1, \mathbf{m}^s \cdot \mathbf{c}_3), \\ \mathbf{g} &:= (0, \rho A r^{-1} v_3^2, -\rho A r^{-1} v_2 v_3, g_4, g_5), \\ \mathbf{h} &:= (0, -\chi_s \mathbf{n}^s \cdot \mathbf{a}_3 - \mathbf{n}^\phi \cdot \mathbf{a}_3, \chi_s \mathbf{n}^s \cdot \mathbf{a}_2 + \mathbf{n}^\phi \cdot \mathbf{a}_2, h_4, h_5), \\ \mathbf{w} &:= (0, 0, (v_2 q_3 - v_3 q_2) r^{-1}, 0, 0, v_2, v_4, v_5), \\ p_4 &:= \rho J (v_5 + r^{-1} v_3 \cos \theta), \\ p_5 &:= \rho J (v_4 \cos \psi + r^{-1} v_3 \sin \psi \sin \theta), \\ g_4 &:= \rho J (v_4 \cos \psi + r^{-1} v_3 \sin \psi \sin \theta) (r^{-1} v_3 \cos \psi \sin \theta - v_4 \sin \psi), \\ g_5 &:= -\rho J (v_5 + r^{-1} v_3 \cos \theta) (r^{-1} v_3 \cos \psi \sin \theta - v_4 \sin \psi), \\ h_4 &:= (\theta_s \sin \psi - \chi_s \cos \psi \sin \theta) \mathbf{m}^s \cdot \mathbf{c}_3 - \sin \theta \cos \psi \mathbf{m}^\phi \cdot \mathbf{c}_3 \\ &\quad + (\gamma_2 \mathbf{c}_3 - \gamma_3 \mathbf{c}_2) \cdot \mathbf{n}^s - r (\cos \psi \mathbf{c}_2 - \sin \psi \mathbf{c}_3) \cdot \mathbf{n}^\phi, \\ h_5 &:= (-\theta_s \sin \psi + \chi_s \cos \psi \sin \theta) \mathbf{m}^s \cdot \mathbf{c}_1 + \sin \theta \cos \psi \mathbf{m}^\phi \cdot \mathbf{c}_1 \\ &\quad + (\gamma_1 \mathbf{c}_2 - \gamma_2 \mathbf{c}_1) \cdot \mathbf{n}^s - r \sin \psi \mathbf{n}^\phi \cdot \mathbf{c}_1, \\ \mathbf{C} &:= \begin{bmatrix} \mathbf{I}_5 \\ \mathbf{O}_{3 \times 5} \end{bmatrix} \end{aligned}$$

where $\mathbf{I}_5$ is the 5×5 identity matrix and $\mathbf{O}_{3 \times 5}$ is the 3×5 zero matrix.

We adopt the boundary conditions:

$$(6.8) \quad \begin{aligned} \mathbf{r}(\sigma, \phi, t) \cdot \mathbf{k} &= \sigma \quad \text{for } \sigma = 0, L, \\ \mathbf{d}(\sigma, \phi, t) \cdot \mathbf{k} &= 0 \quad \text{for } \sigma = 0, L, \\ \mathbf{n}^s(\sigma, \phi, t) \cdot \mathbf{j}_2(\phi) &= 0 = \mathbf{n}^s(\sigma, \phi, t) \cdot \mathbf{j}_3(\phi) \quad \text{for } \sigma = 0, L, \\ \mathbf{m}^s(\sigma, \phi, t) \cdot \mathbf{j}_2(\phi) &= 0 = \mathbf{m}^s(\sigma, \phi, t) \cdot \mathbf{j}_3(\phi) \quad \text{for } \sigma = 0, L. \end{aligned}$$

These conditions say that the ends of the cylindrical shell are welded to lubricated magnets separated by a fixed distance (equal to the natural length).

We adopt the initial conditions:

$$(6.9) \quad \begin{aligned} \mathbf{r}(s, \phi, 0) &= R [\cos(\phi + \gamma(s - L/2)) \mathbf{i}_2 + \sin(\phi + \gamma(s - L/2)) \mathbf{i}_3] + s \mathbf{k}, \\ \mathbf{r}_t(s, \phi, 0) &= \mathbf{o}, \\ \mathbf{d}(s, \phi, 0) &= \cos(\phi + \gamma(s - L/2)) \mathbf{i}_2 + \sin(\phi + \gamma(s - L/2)) \mathbf{i}_3, \\ \mathbf{d}_t(s, \phi, 0) &= \mathbf{o}. \end{aligned}$$

These conditions say that the initial state of the cylindrical shell is inflated or deflated to a cylinder of a radius R , each vertical fiber is twisted into a helix, and each director is pointing radially outward. We actually take $L = 4$ m, $h = 2\pi \times 10^{-3}$ m, and

$$(6.10) \quad R = 0.995 \text{ m}, \quad \gamma = 0.0985 \text{ radians/m}.$$

These values correspond to an equilibrium state for the elastic material described in the next section.

For our purpose of exhibiting the richness of rotationally symmetric motions, almost any reasonable constitutive equations will do. (We actually use equations for elastic shells, avoiding the difficulties with the possible onset of shocks by simply terminating the computations before shocks occur but after interesting motions appear.) The complexity of the equations of (6.5)–(6.7), however, makes reasonable choices of constitutive equations far from obvious. (The linearity of the Reissner-Mindlin theory for plates [13, 11] prevents this theory equations from being useful.) In the next section we accordingly derive the constitutive equations for the shell by integrating the St. Venant-Kirchhoff constitutive equations through the thickness of the plate. (That this process is exempt from early criticism was shown in [12]; cf. [2, Secs. 12.12, 17.2].) Because the curvature of the shell and the form of the 3-dimensional constitutive equations prevent all the actual integrations from being carried out in closed form, these integrations are performed numerically, and form part of the computations of solutions of our initial-boundary-value problem. We exhibit the constitutive equations in Section 7. Such an integration process is particularly convenient for handling several different 3-dimensional constitutive equations (which have zero probability of yielding closed-form constitutive equations for the shell).

System (6.6) is discretized here by using the Leap Frog method (see [10] or [8]), which is an explicit scheme without damping. There are 40 equally spaced mesh points and the time step is 10^{-6} . The numerical solutions are plotted in Figures 4–8. They show that as the twisting angle χ executes an innocuous oscillation, the radius r and the tangent angle θ cease to be constant functions of s , so that deformed images of the cylindrical shell are not cylindrical. At the same time, the shear angle ψ executes small rapid oscillations, which are not shown here. The resulting complicated motion was foreshadowed by the discussion in Section 5.

Figure 4: Initial state of the cylindrical shell.

Figure 5: State at $t = 0.02$.

Figure 6: State at $t = 0.07$.

7 Constitutive properties

To determine the constitutive properties, we formulate our equations as of those for the 3-dimensional theory subject to the constraint (2.8). We obtain constitutive equations in this process. We use the notation of Section 2 and follow Section 17.2 of [2].

Deformation gradient. Let $\tilde{\mathbf{x}} \equiv (\tilde{s}, \tilde{\phi}, \tilde{\zeta})$ denote the inverse of $\tilde{\mathbf{z}}$ defined in (2.7). Then $\mathbf{z} = \tilde{\mathbf{z}}(\tilde{\mathbf{x}}(\mathbf{z}))$, and so

$$\begin{aligned}
 \mathbf{I} &= \frac{\partial \tilde{\mathbf{z}}}{\partial \mathbf{x}}(\mathbf{x}) \cdot \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{z}}(\tilde{\mathbf{z}}(\mathbf{x})) = \tilde{z}_s \tilde{s}_z + \tilde{z}_\phi \tilde{\phi}_z + \tilde{z}_\zeta \tilde{\zeta}_z \\
 &= [1 - \zeta \theta_s^\circ(s)] \mathbf{b}_1^\circ \tilde{s}_z + [r^\circ + \zeta \cos \theta^\circ] \mathbf{b}_3^\circ \tilde{\phi}_z + \mathbf{d}^\circ \tilde{\zeta}_z.
 \end{aligned}
 \tag{7.1}$$

We take the inner product of this identity on the left with $\mathbf{b}_1^\circ, \mathbf{b}_3^\circ, \mathbf{b}_2^\circ$ to obtain

$$\tilde{s}_z = \frac{\mathbf{b}_1^\circ}{1 - \zeta \theta_s^\circ}, \quad \tilde{\phi}_z = \frac{\mathbf{b}_3^\circ}{r^\circ + \zeta \cos \theta^\circ}, \quad \tilde{\zeta}_z = \mathbf{d}^\circ \equiv \mathbf{b}_2^\circ.
 \tag{7.2}$$

Figure 7: State at $t = 0.106$.

Figure 8: State at $t = 0.143$.

Then (2.8) implies that

(7.3)

$$\begin{aligned}
\mathbf{F} &:= \frac{\partial \mathbf{p}}{\partial \mathbf{z}} = \mathbf{p}_s \tilde{s}_z + \mathbf{p}_\phi \tilde{\phi}_z + \mathbf{p}_\zeta \tilde{\zeta}_z \\
&= (\mathbf{r}_s + \zeta \mathbf{d}_s) \frac{\mathbf{b}_1^\circ}{1 - \zeta \theta_s^\circ} + d \mathbf{b}_2^\circ + \mathbf{k} \times (\mathbf{r} + \zeta \mathbf{d}) \frac{\mathbf{b}_3^\circ}{r^\circ + \zeta \cos \theta^\circ} \\
&= [\beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + r \chi_s \mathbf{b}_3 + \zeta \mathbf{d}_s] \frac{\mathbf{b}_1^\circ}{1 - \zeta \theta_s^\circ} \\
&\quad + (\cos \psi \mathbf{b}_2 + \sin \psi \mathbf{b}_3) \mathbf{b}_2^\circ + [r \mathbf{b}_3 - \zeta (\sin \psi \mathbf{a}_2 - \cos \psi \cos \theta \mathbf{b}_3)] \frac{\mathbf{b}_3^\circ}{r^\circ + \zeta \cos \theta^\circ} \\
&=: F_{kl} \mathbf{b}_k \mathbf{b}_l^\circ
\end{aligned}$$

where

$$(7.4) \quad \begin{aligned} F_{11} &= \frac{\beta_1 - \zeta(\chi_s \sin \psi \sin \theta + \theta_s \cos \psi)}{1 - \zeta \theta_s^\circ}, & F_{12} &= 0, & F_{13} &= -\frac{\zeta \sin \psi \sin \theta}{r^\circ + \zeta \cos \theta^\circ}, \\ F_{21} &= \frac{\beta_2 - \zeta \sin \psi (\psi_s + \chi_s \cos \theta)}{1 - \zeta \theta_s^\circ}, & F_{22} &= \cos \psi, & F_{23} &= -\frac{\zeta \sin \psi \cos \theta}{r^\circ + \zeta \cos \theta^\circ}, \\ F_{31} &= \frac{r \chi_s + \zeta \cos \psi (\psi_s + \chi_s \cos \theta)}{1 - \zeta \theta_s^\circ}, & F_{32} &= \sin \psi, & F_{33} &= \frac{r + \zeta \cos \psi \cos \theta}{r^\circ + \zeta \cos \theta^\circ}. \end{aligned}$$

Stress resultants. Let $\mathbf{T}$ denote the first Piola-Kirchhoff stress tensor. Thus $\mathbf{T} \cdot \boldsymbol{\nu}$ is the force per unit reference area exerted on a material surface with unit outer normal $\boldsymbol{\nu}$ in its reference configuration. Then the definitions of $\mathbf{n}^\alpha$ and $\mathbf{m}^\alpha$ imply that

$$(7.5) \quad \begin{aligned} r^\circ \mathbf{n}^s &:= \int_{\zeta_1}^{\zeta_2} \mathbf{T} \cdot (\tilde{\mathbf{z}}_\zeta \times \tilde{\mathbf{z}}_\phi) d\zeta, & r^\circ \mathbf{m}^s &:= \mathbf{d} \times \int_{\zeta_1}^{\zeta_2} \zeta \mathbf{T} \cdot (\tilde{\mathbf{z}}_\zeta \times \tilde{\mathbf{z}}_\phi) d\zeta, \\ \mathbf{n}^\phi &:= \int_{\zeta_1}^{\zeta_2} \mathbf{T} \cdot (\tilde{\mathbf{z}}_s \times \tilde{\mathbf{z}}_\zeta) d\zeta, & \mathbf{m}^\phi &:= \mathbf{d} \times \int_{\zeta_1}^{\zeta_2} \zeta \mathbf{T} \cdot (\tilde{\mathbf{z}}_s \times \tilde{\mathbf{z}}_\zeta) d\zeta \end{aligned}$$

where

$$(7.6) \quad \tilde{\mathbf{z}}_s = \mathbf{r}_s^\circ + \zeta \mathbf{d}_s^\circ \equiv (1 - \zeta \theta_s^\circ) \mathbf{b}_1^\circ, \quad \tilde{\mathbf{z}}_\phi = \mathbf{k} \times (\mathbf{r}^\circ + \zeta \mathbf{d}^\circ) \equiv (r^\circ + \zeta \cos \theta^\circ) \mathbf{b}_3^\circ, \quad \tilde{\mathbf{z}}_\zeta = \mathbf{d}^\circ.$$

Thus

$$(7.7) \quad \begin{aligned} r^\circ \mathbf{n}^s \cdot \mathbf{b}_j &= \int_{\zeta_1}^{\zeta_2} \mathbf{b}_j \cdot \mathbf{T} \cdot \mathbf{b}_1^\circ (r^\circ + \zeta \cos \theta^\circ) d\zeta, \\ \mathbf{n}^\phi \cdot \mathbf{b}_j &= \int_{\zeta_1}^{\zeta_2} \mathbf{b}_j \cdot \mathbf{T} \cdot \mathbf{b}_3^\circ (1 - \zeta \theta_s^\circ) d\zeta, \\ r^\circ \mathbf{m}^s \cdot \mathbf{b}_j &= \int_{\zeta_1}^{\zeta_2} \zeta (\mathbf{b}_j \times \mathbf{d}) \cdot \mathbf{T} \cdot \mathbf{b}_1^\circ (r^\circ + \zeta \cos \theta^\circ) d\zeta, \\ \mathbf{m}^\phi \cdot \mathbf{b}_j &= \int_{\zeta_1}^{\zeta_2} \zeta (\mathbf{b}_j \times \mathbf{d}) \cdot \mathbf{T} \cdot \mathbf{b}_3^\circ (1 - \zeta \theta_s^\circ) d\zeta. \end{aligned}$$

Note that these equations imply (3.3). The same integration process yields the definitions

$$(7.8) \quad \begin{aligned} (\rho A)(s) &:= \frac{1}{r^\circ(s)} \int_{\zeta_1(s)}^{\zeta_2(s)} \tilde{\rho}(s, \zeta) [1 - \zeta \theta_s^\circ(s)] [r^\circ(s) + \zeta \cos \theta^\circ(s)] d\zeta, \\ (\rho I)(s) &:= \frac{1}{r^\circ(s)} \int_{\zeta_1(s)}^{\zeta_2(s)} \tilde{\rho}(s, \zeta) [1 - \zeta \theta_s^\circ(s)] [r^\circ(s) + \zeta \cos \theta^\circ(s)] \zeta d\zeta, \\ (\rho J)(s) &:= \frac{1}{r^\circ(s)} \int_{\zeta_1(s)}^{\zeta_2(s)} \tilde{\rho}(s, \zeta) [1 - \zeta \theta_s^\circ(s)] [r^\circ(s) + \zeta \cos \theta^\circ(s)] \zeta^2 d\zeta. \end{aligned}$$

Here $\tilde{\rho}(s, \zeta)$ is the density per unit reference volume at material point (s, ϕ, ζ) . (The second definition can be used to show that r° can be chosen to make ρI vanish.)

We specialize these formulas to the cylindrical shell treated in Section 6 by taking $r^\circ = 1$ and $\theta^\circ = \frac{\pi}{2}$. We get the specific constitutive equations for the elastic response for our shell by substituting into (7.7) the St. Venant-Kirchhoff constitutive function from 3-dimensional elasticity:

$$(7.9) \quad \mathbf{T} = \lambda(\text{tr} \mathbf{E}) \mathbf{F} + 2\mu \mathbf{F} \cdot \mathbf{E},$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{F}^* \cdot \mathbf{F} - \mathbf{I})$ and λ and μ are the Lamé coefficients.

For the problem treated in Section 6 we take the elastic modulus $E = 10^{-3}$ GPa and take Poisson's ratio $\nu = 0.49$, so that the Lamé coefficients are $\lambda = 1.644 \times 10^{-2}$ GPa and $\mu = 3.356 \times 10^{-4}$ GPa. We take the density $\tilde{\rho} = 920$ kg/m³. These values correspond to a rubberlike material.

8 Comments

Our entire formalism goes through when the shell is rotationally symmetric, rather than being merely axisymmetric. Such shells could have a curvilinear anisotropy characterized by a section of the natural reference configuration like Figure 1c.

It is conceptually straightforward to enhance our theory by allowing the shell to suffer thickness strains. Here the equations are best derived from the 3-dimensional theory (either by imposing constraints [2] in the manner of Section 7 or by using a modification of the approach of Libai & Simmonds [9]). The resulting equations are quite complicated. The same remarks apply to the theory of incompressible shells [3], and indeed to the 3-dimensional theory.

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References

- [1] S. S. Antman, Breathing oscillations of rotating nonlinearly elastic and viscoelastic rings, in *Advances in the Mechanics of Plates and Shells*, edited by D. Durban, D. Givoli, & J. G. Simmonds, Kluwer, 2001, pp. 1–16.
- [2] S. S. Antman, *Nonlinear Problems of Elasticity*, 2nd. edn., Springer, 2005.
- [3] S. S. Antman & F. Schuricht, Incompressibility in rod and shell theories, *Math. Mod. Num. Anal.* **33** (1999) 289–304.
- [4] S. S. Antman & T. I. Seidman, Parabolic-hyperbolic systems governing the spatial motion of nonlinearly viscoelastic rods, *Arch. Rational Mech. Anal.* **175** (2005) 85–150.
- [5] D. Bourne & S. S. Antman, A non-self-adjoint quadratic eigenvalue problem describing a fluid-solid interaction Part I: Formulation, analysis, and computations, *Comm. Pure Appl. Anal.* **8** (2009) 123–142.
- [6] D. Bourne & S. S. Antman, The Taylor-Couette problem for flow in a deformable cylinder, in preparation.
- [7] D. Bourne, H. Elman, & J. E. Osborn, A non-self-adjoint quadratic eigenvalue problem describing a fluid-solid interaction Part II: Analysis of convergence, *Comm. Pure Appl. Anal.* **8** (2009) 143–160.
- [8] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations*, SIAM, 2007.
- [9] A. Libai & J. G. Simmonds, *The Nonlinear Theory of Elastic Shells*, 2nd edn., Cambridge Univ. Pr., 1998.
- [10] K. W. Morton & D. F. Mayers, *Numerical Solution of Partial Differential Equations*, 2nd edn., Cambridge Univ. Pr., 2005.
- [11] R. D. Mindlin, Influence of rotatory inertia and shear on flexural vibrations of isotropic elastic plates, *J. Appl. Mech.* **18** (1951) 31–38.
- [12] P. Podio-Guidugli, An exact derivation of the thin plate equation, *J. Elasticity* **22** (1989) 121–133.
- [13] E. Reissner, The effect of elastic shear deformation on the bending of elastic plates, *J. Appl. Mech.* **12** (1945) A69–A77.

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