Uncertainty modelling and structured singular-value computation applied to an electromechanical system

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Abstract: The investigation of closed-loop systems subject to model perturbations is an important issue to assure stability robustness of a control design. A large variety of model perturbations can be described by norm-bounded uncertainty models. A general approach for modelling structured complex and real-valued parametric perturbations is presented. The resulting robustness analysis problem is solved nonconservatively using real and complex-structured singular-value calculations. The uncertainty modelling and robustness analysis are shown for a high-accuracy 5D electromechanical positioning device to be used in optical (Compact Disc) recording.

1 Introduction

To ensure that a model-based control system design will work well with the actual system it is necessary to analyse the closed-loop robustness properties for model perturbations, such as unmodelled parasitic dynamics, linearisation errors and parametric uncertainties. In past years, much research effort has been spent to solve the multivariable robustness analysis problem. An important development is based on the description of model uncertainties as transfer functions which are norm-bounded but otherwise unknown, and using singular values as indicators [1]. Owing to the use of norms the singular-value analysis method is appropriate for all situations with little knowledge about the perturbations. Its major disadvantage is its conservatism, as indicated by Doyle and others [2] in the sense that the uncertainty model set is much larger than necessary and does not account for structure of perturbations. For that reasons, Doyle [3] introduced the structured singular-value analysis. Recently, Fan and others [4, 5] have given an extension to include real-valued uncertainties.

This paper presents a general procedure to model norm-bounded perturbations and some computational aspects of real and complex-structured singular-value analysis. A general concept which is very useful for norm-bounded uncertainty modelling, and especially for robustness analysis with the structured singular value, is the linear fractional transformation (LFT). As an example, consider a system with uncertainty, Fig. 1. The transfer function \( M(s) \) represents the transfer function from the exogenous signals \( u \) (references, disturbances, control inputs, etc.) and the uncertainty outputs \( u_A \) to the controlled variables \( y \) (tracking error, measured signals, etc.) and the uncertainty inputs \( y_A \). The uncertainty is denoted in Fig. 1 as the transfer function \( \Delta(s) \).

\[
\begin{bmatrix}
\Delta \\
M(s) \\
y_A \\
y 
\end{bmatrix}
\begin{bmatrix}
u_A \\
u 
\end{bmatrix}
\begin{bmatrix}
y_A \\
y 
\end{bmatrix}
\]

Fig. 1 System with uncertainty feedback

The upper linear fractional transformation on \( M \) and \( \Delta \) is denoted as \( F_r(M, \Delta) \) and is defined to be equal to the transfer function from \( u \) to \( y \): \( F_r(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1} \Delta M_{12} \).

2 Complex norm-bounded uncertainty modelling

Complex-valued model uncertainties are often used to describe unmodelled dynamics, for instance actuator and sensor dynamics or parasitic system dynamics. Such uncertainties can be described as input/output transfer functions. Well known complex uncertainty descriptions are the multiplicative input and output uncertainty and the additive structure, Fig. 2.

Definition 2.1: A \( p \times p \) complex-valued norm-bounded unstructured perturbation \( \Delta \) is the set of \( p \times p \) transfer functions \( \Delta(s) : C \rightarrow C^p \) which are analytic in the closed right half-plane and have a norm-bound less than or equal to 1.

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equal to some given positive function \( \delta(t) < 1 \) for all \( t \in (-\infty, \infty) \):

\[
\Delta = \{ \Delta(s) \mid |\Delta(j\omega)| \leq \delta(t), \ \omega \in (-\infty, \infty) \}
\]

with \( \delta \) denoting the maximum singular value. The normalised uncertainty set is given by \( B\Delta = \{ \Delta(s) \mid |\Delta(j\omega)| \leq 1, \ \omega \in (-\infty, \infty) \} \).

### 3.1 Parametric uncertainty modelling

**Definition 3.1**: A scalar real-valued norm-bounded perturbation \( \Delta \) is the set of real numbers \( \Delta \) which are bounded in magnitude to some real number \( \delta \in \mathbb{R}^+ \):

\[
\Delta = \{ |\Delta| \in [\delta, +\delta] \}
\]

The normalised version of \( \Delta \) is \( B\Delta = \{ |\Delta| \in [-1, +1] \} \).

The sets \( \Delta \) and \( \Delta \) share the property of being bounded in maximum singular value. However, there are three important differences. First, the elements of \( \Delta \) are scalars, while \( \Delta \) may have matrices as its (block) elements. Secondly, the set \( \Delta \) contains only real numbers. Thirdly, the maximum singular value of the elements in \( \Delta \) can vary with frequency while the maximum singular value of a real perturbation, which is equal to the maximum absolute value, is fixed.

One special structure, which is important in the application in Section 5, is the real repeated uncertainty for one parameter.

**Definition 3.2**: A \( p \times p \) real-valued repeated perturbation \( \Delta_r \) is defined as

\[
\Delta_r = \{ |\Delta| = \delta I, \ \delta \in [-\delta, +\delta] \}
\]

where \( \delta \in \mathbb{R}^+ \) is some real number. The normalised version is \( B\Delta_r = \{ |\Delta| = \delta I, \ \delta \in [-1, +1] \} \). In both equations \( I \) is the \( p \times p \) identity matrix.

The starting point for parametric uncertainty modelling is a state-space description of an uncertain system. A procedure is described which can be used to derive an LFT form of a model with parametric uncertainties in the entries of its state-space matrices. This procedure involves three steps: (i) scaling the parameter variations such that they belong to \( B\Delta_r \), (ii) uncertainty extraction resulting in a separation between the nominal (constant) part of a system and a varying (part) and (iii) obtaining an LFT description.

#### 3.2 General case

Consider a vector \( p = (p_1, \ldots, p_l) \in \mathbb{R}^l \) containing \( l \) scalar parameters, for example spring stiffness, resistance etc.

Let the model of the perturbed system be given as a state-space realisation in which the entries of the matrices depend on the parameter vector \( p \):

\[
\begin{align*}
\dot{x} &= A(p)x + B(p)u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \\
y &= C(p)x + D(p)u, \quad y \in \mathbb{R}^l
\end{align*}
\]

Restrict attention to the case of 'smooth' perturbations in the form of parametric uncertainties. More specifically, assume that each entry of the matrices in eqn. 6 is described as a rational multidimensional (ND) polynomial function of the parameters \( p \). For example, the \( i \)th entry of the \( A \)-matrix can have the form \( A_i(p) = [p_1 + a_0 + p_2 + a_1 + p_3 + a_2] \) in which \( a_0 \) and \( a_1 \) are constants.

For this general class of systems the following procedure provides a way to derive an LFT uncertainty description.

**Step 1**: Scaling: Let the parameter vector \( p \) be given with lower and upper bound vectors \( p_{\text{min}} \) and \( p_{\text{max}} \) respectively: \( p_{\text{min}} \leq p_i \leq p_{\text{max}} \) for \( i = 1, \ldots, l \). Define \( p_{\text{rel}} = (p_{\text{max}} - p_{\text{min}})^{1/2}, \ s = \{p_{\text{rel}}, \ldots, p_{\text{rel}}\}, \ \delta = \{\delta_1, \ldots, \delta_l\}, \ \delta_1 \in [-1, +1] \) then \( p_i = p_{\text{rel}} + s_i \delta_i \). In this way the varying parameter vector \( p \) is decomposed into a nominal part \( p_{\text{nom}} \), the constant scaling factors \( s_i \), and the normalised real-valued perturbations \( \delta_i \) collected in the vector \( \delta \).
Step 2: Uncertainty extraction: Let the state-space model eqn. 6 be given and assume the parameter vector $p$ has been scaled. Define the $(n + l) \times (n + m)$ matrix

$$
S(p) = \begin{pmatrix}
A(p) & B(p) \\
C(p) & D(p)
\end{pmatrix}
$$

(7)

The nominal part of the state-space model is given by $S(p_{\text{nom}})$. The uncertain part of the state-space model is defined as an $(n + l) \times (n + m)$ matrix $S(\delta)$ with entries

$$
(S_{i,j}(\delta)) = S_i(p_{\text{nom}}) - S_j(p_{\text{nom}})
$$

(8)

Hence, $(S_{i,j}(\delta)) = 0$ if no uncertain parameter enters the $(i,j)$th entry of $S$, for $i = 1, \ldots, n + l$ and $j = 1, \ldots, n + m$. Using this definition the perturbed state-space model eqn. 6 can be written as

$$
\dot{\tilde{x}} = S(p_{\text{nom}})\tilde{x} + S(\delta)\tilde{u}
$$

from which it is clear that the uncertain part is now separated from the nominal part.

Step 3. Obtaining a linear fractional transformation: The third step is to rewrite eqn. 9 into a linear fractional form. We construct this by defining a new input vector $\tilde{x}_d$ and a new output vector $y_d$. The output $y_d$ is fed back to the input $u_d$ through a diagonal perturbation $\Delta(\delta) = \text{diag}(\delta_1 I_1, \ldots, \delta_l I_l)$. Furthermore, constant matrices $B_d$, $C_d$ and $D_d$ are defined which contain information on how the uncertainties affect the nominal model:

$$
\begin{pmatrix}
\tilde{x} \\
y_d
\end{pmatrix} = S(p_{\text{nom}})\begin{pmatrix}
\tilde{x} \\
y_d
\end{pmatrix} + B_d u_d
$$

$$
y_d = C_d\begin{pmatrix}
\tilde{x} \\
y_d
\end{pmatrix} + D_d u_d
$$

(10)

$$
u_d = \Delta(\delta)y_d
$$

where $\Delta(\delta) = \text{diag}(\delta_1 I_1, \ldots, \delta_l I_l)$, in which $I_l$ denotes an identity matrix with dimensions related to the repeatedness of perturbation $\delta_l$ (see also Definition 3.2).

Rewriting eqn. 9 as an LFT involves finding the constant matrices $B_d$, $C_d$ and $D_d$ such that eqn. 9 is equivalent to eqn. 10. Eliminating $u_d$ and $y_d$ in eqn. 10 yields

$$
\begin{pmatrix}
\tilde{x} \\
y_d
\end{pmatrix} = S(p_{\text{nom}})\begin{pmatrix}
\tilde{x} \\
y_d
\end{pmatrix} + B_d (I - \Delta(\delta)D_d)^{-1}\Delta(\delta)C_d \begin{pmatrix}
\tilde{x} \\
y_d
\end{pmatrix}
$$

(11)

which must be equivalent to eqn. 9. This implies that the following realisation problem has to be solved.

General problem definition: Find constant matrices $B_d$, $C_d$ and $D_d$ and $\Delta(\delta) = \text{diag}(\delta_1 I_1, \ldots, \delta_l I_l)$ with dimensions as small as possible such that

$$
B_d(I - \Delta(\delta)D_d)^{-1}\Delta(\delta)C_d = S(\delta)
$$

(12)

where $S(\delta)$ is the matrix from eqn. 8.

Eqn. 12 can be interpreted as follows. Consider only the nontrivial case that $\delta_i \neq 0$, $i = 1, \ldots, l$. In that case, $\Delta(\delta) = \text{diag}(\delta_1 I_1, \ldots, \delta_l I_l)$ is invertible and eqn. 12 can be rewritten as $B_d\Delta(\delta)^{-1}(I - \Delta(\delta)^{-1}C_d = S(\delta)$. Defining $\rho_\delta = 1/\delta_i$ yields

$$
B_d\begin{pmatrix}
\rho_\delta I_1 \\
0
\end{pmatrix}^{-1}C_d = S(\delta)
$$

(13)

which can be considered as a multidimensional (minimal) realisation problem. Note that in general there may be freedom in choosing $(D_d, C_d, B_d)$ as a minimal realisation. This means that an LFT for a given analysis problem need not be unique.

General problem solution: Eqn. 12 is solvable for the general case. In this paper we will make this statement tractable, without giving a rigorous proof.

Recall that the uncertain model has rational ND polynomial parameter-dependent entries of the state-space matrices. Hence, every entry in $S(\delta)$ can be written as a scalar function of the parameter vector $\delta$: $[S_{i,j}(\delta)] = (k(\delta))/(1 + k(\delta))$, with $k(0) = 0$ and $k(0) = 0$ and with the specific structure of the denominator because $S(0) = 0$. The denominator can be represented as an LFT being a negative feedback $k(\delta)$ over a gain 1. The numerator $k(\delta)$ and the function $k(\delta)$ consist both of several terms with products and powers of the parameters $\delta_i$. Each of these terms can be represented by an LFT and hence also the sum of them. This gives an LFT for $k(\delta)$ and one for $1/(1 + k(\delta))$. The product of two LFTs is another LFT, so that we have found an LFT for $S(\delta)$.

After the combination of all entries of $S(\delta)$ into one large LFT structure, a minimal realisation step is necessary for each individual element of $\delta$. For more details see Reference 7.

3.3 Special cases

3.3.1 One varying parameter: Important examples of uncertainty models for the one parameter case are those where entries of the model depend as rational functions on one varying parameter, for instance the operating condition for linearised systems. Consider the system of eqn. 9 with $\delta$ a scalar (i.e. $t = 1$).

Lemma 3.3: Define $\rho = 1/\delta$. Then $S(\rho^{-1})$ is strictly proper in $\rho$.

Proof: According to eqn. 8, for $\delta = 0$, $[S_{i,j}(0)] = 0$ if no uncertain parameter enters entry $(i, j)$ in eqn. 6 and $\lim_{\rho \to 0} [S_{i,j}(\rho)] = 0$, implying that $\lim_{\rho \to 0} S(\rho) = S(0) = 0$.

Theorem 3.4: Assuming that $S(\rho^{-1})$ is rational and strictly proper, the uncertainty modelling problem is to find constant matrices $B_d$, $C_d$ and $D_d$ and $\Delta(\delta) = \delta I_\delta$, $\delta$ scalar, with dimensions as small as possible such that eqn. 12 holds for $\delta$ being scalar. This is equivalent to the realisation problem: $B_d(I - D_d)^{-1}C_d = S(\rho^{-1})$.

Proof: Follows immediately from eqns. 12 and 13 for $\rho = 1$.

Lemma 3.3 shows that the uncertainty modelling for the one parameter case can always be carried out such that $S(\rho^{-1})$ is strictly proper. Therefore a solution always exists, since the problem is equivalent to a standard state-space realisation problem [8].

Corollary 3.5: If $(B_d, C_d, B_d)$ is a minimal realisation, the solution to Theorem 3.4 yields $\Delta(\delta) = \delta I_\delta$ with the smallest possible dimensions for which an LFT can be found.

Remark 3.6: The connection between state-space realisation and parametric uncertainty modelling can also be reversed: a state-space model as an uncertainty. In Reference 9 this has been worked out by defining in discrete time the $z$-variable as a repeated block perturbation (state-space $\mu$).
Example 3.7: Suppose a first-order system has a state-space A-matrix which can be written as \( A = A_{\text{nom}} + \delta^2 \), then
\[
\dot{x} = A_{\text{nom}} x + \delta^2 x
\] (14)
and constructing an LFT is fairly simple in this case:
\[
\dot{x} = A_{\text{nom}} x + B_{\delta} u_{\delta}
\]
\( y_a = C_{\delta} x + D_{\delta} u_{\delta} \)
\( u_{\delta} = \Delta(\delta) y_{\delta} \)
This set equals eqn. 14 if
\[
\Delta(\delta) = \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix}, \quad D_{\delta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\delta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
and
\( B_{\delta} = \begin{bmatrix} 0 & 1 \end{bmatrix} \)
Owing to the structure of \( D_{\delta} \) a polynomial in \( \delta \) is created.

In this example \( \phi = \delta^2 \) could have been modelled and \( \phi \) treated as a simple linear perturbation. However, when this concept is applied more generally for example if \( \delta \) appears somewhere else in the state equation as another polynomial, a procedure as in the example is necessary.

3.3.2 Linear parametric uncertainties: If the parameters \( \delta = (\delta_1, \ldots, \delta_k) \) enter the state-space matrices in a linear way \([10, 11]\), \( D_{\delta} \) can be taken as identically zero, as is clear from eqn. 12. For this case it is obvious that \( S_\delta(\delta) = \delta \sum_{i=1}^k \delta_i S_{\delta_i} \).

Theorem 3.8: Let \( S_\delta(\delta) = \sum_{i=1}^k \delta_i S_{\delta_i} \). The problem to find constant matrices \( B_{\delta}, C_{\delta}, \) and \( \Delta(\delta) = \text{diag}(\delta_i I_{l_i}, \ldots, \delta_i I_{l_i}) \) with dimensions as small as possible such that eqn. 12 holds with \( D_{\delta} = 0 \), is always solvable. The solution is given by the solution to
\[
B_{\delta} \Delta(\delta) C_{\delta} = \sum_{i=1}^k \delta_i S_{\delta_i}
\] (15)
with \( \Delta(\delta) = \text{diag}(\delta_1 I_{l_1}, \ldots, \delta_k I_{l_k}) \).

Proof: From the general problem definition (eqn. 12) eqn. 15 results for the linear parameter case. That a solution to this problem always exists can be seen as follows. Suppose that \( S_{\delta_i} \) has rank \( r_i \), then there exist matrices \( P_i \) and \( Q_i \) where \( P_i \) is \((n+1) \times (r_i)\) and \( Q_i \) is \((r_i) \times (n+m)\) such that \( \delta_i S_{\delta_i} = \delta_i P_i Q_i = P_i[\delta_i I_{l_i}, \delta_i I_{l_i}]Q_i \). Choosing \( \Delta = \text{diag}(\delta_1 I_{l_1}, \ldots, \delta_k I_{l_k}) \), \( B_{\delta} = (P_1, \ldots, P_k) \), and \( C_{\delta} = (Q_1, \ldots, Q_k) \) yields eqn. 15.

From Theorem 3.8 it follows that generically the uncertainty \( \Delta(\delta) = \text{diag}(\delta_1 I_{l_1}, \ldots, \delta_k I_{l_k}) \) for which a solution exists has at least dimension \( \sum_{i=1}^k r_i \), where \( r_i \) is the rank of \( S_{\delta_i} \). However, in some cases perturbations can be taken together which is formulated in the following corollary.

Corollary 3.9: The dimension of an uncertainty \( \Delta(\delta) = \text{diag}(\delta_1 I_{l_1}, \ldots, \delta_k I_{l_k}) \) can be made smaller than \( \sum_{i=1}^k r_i \) if rank \( \sum_{i=1}^k \delta_i S_{\delta_i} \leq \sum_{i=1}^k r_i \), with \( \delta_i \) any nonzero real number. In such a case, some \( \delta_i \) are perturbing the system in a similar way and can be taken together. This is called a reducible uncertainty model. An example has been worked out in Reference 6; see also Reference 7.

3.3.3 Other special problems: The two special cases described previously are formalised with Theorem 3.4 and Theorem 3.8. Two examples are presented to show solutions for the case where products and quotients of parameters appear. In both, the following relations for the varying parameters are assumed \( a = a_{\text{nom}} + \delta_1 \delta_2 \), \( b = b_{\text{nom}} + \delta_3 \delta_4 \).

Example 3.10: Consider the state equation
\[
\dot{x} = a b x = a_{\text{nom}} b_{\text{nom}} x + (\gamma_1 \delta_1 + \gamma_2 \delta_2 + \gamma_3 \delta_3) x
\]
where \( \gamma_1 = s_4 b_{\text{nom}}, \gamma_2 = s_5 \delta_4, \gamma_3 = s_6 \delta_{\text{nom}} \) (uncertainty extraction). The problem is to find matrices \((B_{\delta}, C_{\delta}, D_{\delta})\) such that \( \dot{x} = A_{\text{nom}} x + S_{\delta} \delta_1, \delta_4 x \) is equivalent to the linear fractional form:
\[
\dot{x} = A_{\text{nom}} x + B_{\delta} u_{\delta}
\]
\( y_a = C_{\delta} x + D_{\delta} u_{\delta} \)
\( u_{\delta} = \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix} y_{\delta} \)
The equivalence is satisfied for
\[
B_{\delta} = (\gamma_1, \gamma_2) C_{\delta} = \begin{bmatrix} 1 \\ \gamma_1 \gamma_2 \end{bmatrix} D_{\delta} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]
Example 3.11: Consider
\[
\dot{x} = a b x = a_{\text{nom}} b_{\text{nom}} x + (\gamma_1 \delta_1 + \gamma_2 \delta_2) x
\]
where \( \gamma_1 = s_4 b_{\text{nom}}, \gamma_2 = -s_5 \delta_4 \). Again we are looking for matrices \((B_{\delta}, C_{\delta}, D_{\delta})\) such that \( \dot{x} = A_{\text{nom}} x + S_{\delta} \delta_1, \delta_4 x \) is equivalent to eqn. 16. This is satisfied for
\[
B_{\delta} = (1, 1) C_{\delta} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} D_{\delta} = \begin{bmatrix} 0 & 0 \\ -\gamma_2 & -\gamma_2 \end{bmatrix}
\]

3.4 General \( \mu \)-interconnection structure
For practical problems in general both complex and parametric uncertainties have to be taken into account. This can be done by deriving LFTs for each of the perturbations, and collecting these models into one \( \mu \)-interconnection structure. The uncertainty matrix \( \Delta \) then consists of complex and real-valued entries, as defined in the following general block structure.

Given two non-negative integers \( m_1 \) and \( m_2 \) define a vector \( \kappa \) with length \( m_1 + m_2 \) and with positive integer entries:
\[
\kappa = (k_1, \ldots, k_{m_1}, k_{m_1+1}, \ldots, k_{m_1+m_2})
\] (17)
Definition 3.12: Given the vector \( \kappa \) (eqn. 17), the associated block-diagonal perturbation \( \Delta_\kappa \) is defined by the set
\[
\Delta_\kappa = \{ \Delta_i: \Delta_i = \text{diag}(\Delta_i I_{k_1}, \ldots, \Delta_i I_{k_{m_1}}) \}
\]
\[
\{\Delta_i I_{k_{m_1+1}}, \ldots, \Delta_i I_{k_{m_1+m_2}}\}
\] (18)
where \( \Delta_i \in \Delta \) (Definition 3.1), \( i = 1, \ldots, m_2 \), \( \Delta_i \in \Delta \) (Definition 2.1) but with the additional constraint that \( \Delta_i \) is a scalar if \( k_i > 1 \), \( i = m_1 + 1, \ldots, m_1 + m_2 \), and where \( I_k \) is the \( k \times k \) identity matrix. The normalised block-diagonal perturbation set is denoted as \( B_{\Delta_\kappa} \) with \( \Delta_\kappa \in B_{\Delta_\kappa} \).
Notice that $\Delta I_1$ is a real repeated block and $\Delta I_{2,\ldots}$ is a complex repeated block. If $k = 1$ the uncertainty is nonrepeated and the complex uncertainties are allowed to be matrices in that case. The vector $\kappa$ thus comprises the structure information (real/complex, repeatedness). The following section shows how robustness analysis can be done for general block-structures given by Definition 3.12.

4 Structured singular value analysis

This section briefly describes the structured singular value analysis for both the complex and the real case; for more details see References 3, 4, 12. In the sequel, we assume that $\Delta(s)$ as well as the nominal system $M(s)$ are stable. First, well-known results for the unstructured case are reviewed.

Consider the system of Fig. 1 in which the uncertainty feedback $\Delta$ is assumed to be a full complex uncertainty ($\Delta \notin B_{1,1}$, Definition 2.1). Denote the partition of $M(s)$ which is coupled to $\Delta$ by $M_{11}(s)$. For this unstructured case the well known small-gain theorem provides necessary and sufficient conditions for internal stability of the perturbed system $F_{1}(M(s), \Delta)$; the system in Fig. 1 is internally stable if and only if $\max_{\omega} \sigma_{\max}(M_{11}(j\omega)) < 1$, $\omega \in (-\infty, \infty)$ [13]. Now consider the case that $B_{1,1}^m$ is replaced by $B_{1,1}$ with $B_{1,1}$ some block-diagonal structure. Then the small gain theorem does not necessarily hold. For this reason Doyle [3] introduced the structured singular value $\mu$.

**Theorem 4.1:** Let $M(s)$ be stable and let $\Delta \in B_{1,1}$, then the system $F_{1}(M(s), \Delta)$ in Fig. 1 is internally stable if and only if
\[
\det (I - \Delta(j\omega)M_{11}(j\omega)) \neq 0
\]
for all $\omega \in (-\infty, \infty)$, which holds if and only if
\[
\sup_{\omega} \left\{ \mu(M_{11}(j\omega)) \right\} < 1, \quad \omega \in (-\infty, \infty)
\]

**Proof:** see Reference 3.

The difference between the structured singular-value theorem (Theorem 4.1) and the small-gain theorem is that the maximum singular value of a matrix can be computed easily and exactly, which is not the case for the structured singular value. Computing $\mu$ requires the optimisation of an expression in several independent variables. It is known that this optimisation problem leads to an upper or lower bound for $\mu$ if and only if $\max_{\omega} \sigma_{\max}(M_{11}(j\omega)) < 1$, $\omega \in (-\infty, \infty)$.

\[
\mu(M_{11}) \leq \sqrt{\left( \max_{\omega} \left\{ \min_{\omega} \sigma_{\min}(M_{11}(j\omega)) \right\} \right)}
\]

with $M_{11} = D_{11}D_{11}^{-1}$, $D \in D_{s}$, and with $G$ as defined as
\[
G_{1} = \{ G | G = \text{diag}(G_{1}, \ldots, G_{m}, O_{m+1}, \ldots, O_{m+m}) \}
\]
where $G_{i} \in C^{n_{i,m_{i}}}$, $i = 1, \ldots, m$, and with $O_{j}$ the null-matrix with dimension $k_{i} \times k_{i}$ if $k_{i} > 1$ and $p \times p$, $p = \text{dim}(\Delta)$ if $k_{i} = 1$. Notice that $G_{i} = G_{i} \in \mathbb{R}$ if $k_{i} = 1$. If there are no real blocks ($m_{i} = 0$), then $G_{i} = 0$ and eqn. 23 simplifies to 22. This shows that eqn. 23 is based on the same principle as the earlier upper bound for purely complex structures, namely the minimisation of a maximum singular value. Note also that the inequality in eqn. 23 still holds if $G = 0$ is chosen. Hence, the complex structured singular value bound (eqn. 22) is a sufficient condition (an upper bound) for the real case. However, less conservative results (a smaller upper bound for $\mu$) may be obtained for $G \neq 0$. The computation involves a minimisation over the free parameters in $D$ and $G$. From the definitions of $D_{s}$ and $G_{1}$ it follows that the number of parameters involved is given by
\[
D \text{- scaling: } \sum_{i=1}^{m} k_{i} + 2(k_{i} - 1)^{2} - 1
\]
\[
G \text{- scaling: } \sum_{i=1}^{m} k_{i} + 2(k_{i} - 1)^{2}
\]
with $k_{i}$ the entries of $\kappa$. For example, a complex non-repeated problem with three uncertainties, $m_{1} = 0$, $m_{2} = 3$, $k_{1} = (k_{1}, k_{2}, k_{3}) = (1, 1, 1)$, has only two parameters for $D$-scaling and none for $G$-scaling. For a $3 \times 3$ real-repeated one parameter problem, $m_{1} = m_{2} = m_{3} = 1$, $k_{1} = k_{2} = k_{3} = 3$, has 17 parameters for $D$-scaling and 18 for $G$-scaling.

An algorithm has been written to compute the upper bound (eqn. 23). In fact, all possible combinations of real, real repeated, complex and (scalar) complex repeated can be handled with it. The algorithm is used in the following section.

To give some insight into the effect of the $G$-scaling on the value of the upper bound, this section concludes with a simple example.

**Example 4.2:** Suppose we have one real scalar uncertainty $\Delta = \delta \in [-1, 1]$. Denote the related $\mu$-interconnection structure $M_{11}(s)$ as $m(s)$. Let $m(s)$ be evaluated at some frequency $\omega_{0}: m(\omega_{0}) = r + q$ where $r, q \in \mathbb{R}$. In this case, the upper bound (eqn. 23) can be written as ($D = 1$)
\[
\mu(m(\omega_{0})) \leq \sqrt{\left( \max_{\omega} \left\{ 0, \inf_{\omega} |r^2 + q^2 - 2rq| \right\} \right)}
\]
with $G = g$. First suppose that $q \neq 0$, then for any given $(r, q)$ a $g$ can be found such that $(r^2 + q^2 - 2rq) < 0$ and hence $\mu(m(j\omega)) = 0$. Now assume that $q = 0$, i.e. $m(j\omega)$ crosses the real axis, then eqn. 27 gives $\mu(m(j\omega)) \leq |r|$ for any choice of $g$. This is equivalent to the well known amplitude margin of a scalar system.

The example shows that the $G$-scaling in fact pushes the upper bound down in those cases where the interconnection matrix has only complex values, if calculated for a real uncertainty. This result can be generalised for multivariable systems, using the eigenvalues of $DM_1D^{-1}$, see Reference 15. It also shows that the optimisation problem is not continuous on $G$, see also Reference 16. Generically, complex perturbations always prevent this situation.

5 Robustness analysis of a 5D actuator

In optical recording (Compact Disc), a very high information density is applied. To detect this information, high precision mechanisms are needed to position the laser spot on the disc with an accuracy $\pm 0.1 \mu m$. Using servoactuators with a high bandwidth (500–1000 Hz) it is possible to keep on the track despite disturbances from outside the mechanism such as mechanical shocks and disc eccentricity. An actuator which makes it possible to achieve a very high bandwidth is the 5D-actuator [17]. This consists of a magnetic ring with a lens in it, which is magnetically positioned by an active system of nine coils, Fig. 3. Using a mirror underneath the magnetic ring, the positions $(z, \beta, \alpha)$ can be detected, while the $x$ (tracking) and $z$ (focusing) positions are measured relative to the disc above. The position of the lens is controllable in 5 degrees of freedom by means of the electromagnetic forces.

A major problem with this system is that it has severe couplings between the magnetic forces as a function of position $z$. This gives interaction problems between the $\alpha$ and $z$-degrees of freedom to be controlled. From a nonlinear model it follows that with the aid of a decoupling matrix the $(x, \beta)$ and $(y, \alpha)$ degrees of freedom can be decoupled from one another. We restrict attention to the 2D problem in the $(y, \alpha)$-direction only. A state-space model has been derived, linearised with respect to vertical position $z$

\[
\begin{bmatrix}
\dot{y} \\
\dot{\beta} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y \\
\beta \\
z
\end{bmatrix}
+ \begin{bmatrix}
1.77 - 0.24z^2 \\
0 \\
5.34z - 1.77z^2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]

and with the outputs $y$ and $z$. The $z$-dependency of the model appears nonlinearly in the input matrix $B$ and is caused by a nonlinear distribution of the magnetic field lines as a function of $z$. The entries of $B$ are polynomial fits on data obtained from a finite element calculation of the magnetic field distribution. The interaction terms are for $z > 0$ of opposite sign compared to those for $z < 0$. Using a multimodel design method [18] a diagonal controller has been designed for three operating points: $z = -1, 0$ and $+1 \, \text{mm}$, resulting in a compensator for $y$: $0.51 \times 10^7 (8.23 \times 10^{-4} s + 1)/(3.26 \times 10^{-3} s + 1)$ and for $z$: $0.30 \times 10^7 (9.54 \times 10^{-4} s + 1)/(2.85 \times 10^{-5} s + 1)$. We are interested if this system is stable in all operating points. To be more precise, whether it is stable for every position $z$, where $z$ can vary between $-1.8 \, \text{mm}$ and $+1.8 \, \text{mm}$. We restrict attention to the variations in $z$ only, hence we have a one-parameter problem. Using the uncertainty modelling procedure described, a $\mu$-interconnection structure can be derived with a real repeated uncertainty $\Delta = zI$ (scaled to $-1, \ldots, +1$, and with $I$ the $4 \times 4$ identity matrix), with the matrices in eqn. 6 model as follows:

\[
\begin{bmatrix}
B_{\alpha} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

For this case, the following calculations have been done: (i) small-gain theorem, (ii) structured singular-value computation (eqn. 22) assuming that $\Delta$ is a $4 \times 4$ diagonal
complex uncertainty: $\kappa = (1, 1, 1, 1)$, $n_c = 0$, $m_c = 4$, $p = 4$ (four times), i.e. three $D$-scalings, (iii) $A$ assumed to be complex repeated (eqn. 22), $\kappa = 4$, $n_c = 0$, $m_c = 1$, i.e. 21 $D$-scalings, and (iv) real-repeated (eqn. 23): $\kappa = 4$, $m_c = 1$, $m_r = 0$, i.e. 21 $D$-scalings and 22 $G$-scalings. Results of the computations are given in Fig. 4.

From the figure it follows that the system is indeed on the edge of stability for these operating points.

6 Conclusions
Robustness analysis for systems with complex and real-valued uncertainties consists of uncertainty modelling and computing stability bounds. A procedure has been described to model complex and real perturbations, comprising scaling of the individual perturbations, extracting the varying part from the constant part of a system and creating a linear fractional form. For those types of models, recent developments of structured singular value computation for complex and real, possibly repeated, uncertainties are applicable. An electromechanical positioning device, to be used in optical recording, has been analysed for stability over a range of operating conditions.

7 References