FINITE DEFORMATION THEORY OF HIERARCHICALLY ARRANGED POROUS SOLIDS—II. CONSTITUTIVE BEHAVIOUR

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Abstract—A constitutive formulation for finite deformation of porous solids, including an hierarchical arrangement of the pores is presented. An extended Darcy equation is derived by means of a formal averaging procedure. The procedure transforms the discrete network of pores into a continuum, without sacrificing essential information about orderly intercommunication of the pores. The distinction between different hierarchical levels of pores is achieved by means of a hierarchical parameter. The macroscopic equations are derived assuming that the pores are a network of cylindrical vessels in which Poiseuille-type pressure–flow relations are valid. The relationships between stress, strain, strain rate, fluid volume fraction, fluid volume fraction rate and time are derived from Lagrange equations of irreversible thermodynamics. The theory has applications, particularly in the field of the mechanics of blood perfused soft tissues, where the distinction between arterioles, capillaries and venules is essential for a correct quantification of regional blood perfusion of the tissue. Conductance of the medium depends on the local state of tissue deformation which is assumed to cause stretching and buckling of the vessels. Deformations are assumed quasi-static and isothermal. Both solid and fluid are assumed incompressible. It is shown that the theory is consistent with Biot’s finite deformation theory of porous solids for the limiting case where the pore structure has no hierarchy.

INTRODUCTION

The very start of the theory of flow through porous media are the experiments of Darcy and Ritter [1]. They measured that the creep flow through a soil specimen saturated with water is proportional to the pressure difference across the specimen, proportional to the cross-section of the specimen and inversely proportional to the length of the specimen. Darcy’s law has been generalized to steady-state three-dimensional flow of a Newtonian incompressible fluid through saturated porous media according to:

\[ Q = -K \cdot \nabla p \]  

(1)

in which:

- \( Q \) is the specific flow vector
- the permeability tensor \( K \) is symmetric and inversely proportional to the viscosity of the fluid
- \( \nabla p \) is the (averaged) pressure gradient.

The pressure and flow as used in equation (1) are not measured at the level of the individual pores but rather as averages over a number of pores [2]. Further generalizations to flow through deformable porous media, to transient flow, to flow of compressible fluids through porous media are extensively used in many fields of engineering. These generalizations call for experimental and theoretical verification. The need for theoretical verification has led to the setting up of a mathematical theory which allows the derivation of macroscopic laws—such as Darcy’s law—from a law valid on the microscopic level of the individual pore. In this context, the averaging procedure developed by Slattery [3] and Whitaker [4] plays an important role. Several attempts have been made to derive the macroscopic equation (1) for fluid motion
through a saturated porous medium from different assumptions on the level of the individual pores [5–9]. Biot [10] shows how to describe the interaction between large deformation and fluid motion in a deformable viscoelastic porous medium.

In this paper the mathematical micro–macro transformation theory or formal averaging procedure, developed by Whitaker [4] and Slattery [3] is applied to the specific situation where the pores of the medium are arranged in an hierarchical sequence. An example of such an arrangement is the microcirculatory bed of biological tissues. In a microcirculatory bed blood flows from arteries to arterioles, to capillaries, venules and veins. Darcy's law as such is not able to describe microcirculatory flow. The very definition of pressure and flow as averages over a number of pores, makes it impossible to distinguish between arterial, capillary and venous pressures and flows. This is the reason why a different macroscopic law is developed in which pressure and flow are selectively averaged according to the prevailing hierarchical pore structure: the extended Darcy equation. In the context of the theory of fractured porous media the extended Darcy equation can be viewed upon as a generalization of the constitutive approach of Wilson and Aifantis [11] for the two porosity model in that it includes a spectrum of porosities which intercommunicate across anisotropically oriented interfaces. The first steps towards experimental verification of the present theory in the limiting case of a flow through a rigid porous medium are presented elsewhere [12, 13].

DEFINITIONS

Lagrangian averaging

In the companion paper an Eulerian formal averaging procedure using a representative elementary volume in the current configuration has been defined. An alternative averaging procedure consists in defining a representative elementary volume $R$ in the initial configuration. A vector $X$ points towards the centroid of $R$. The different constituents subdivide the volume $R$ in the subvolumes $R^F$ and $R^S$, similar to those defined in the volume $r$. Initial volume fractions are defined as:

$$N^X = \frac{R^X}{R}. \quad (2)$$

According to the transformation defined in equation (10) of the companion paper, the volume $R$ is mapped onto a volume $\chi(R)$ in the current configuration (Fig. 1):

$$R \rightarrow \chi(R). \quad (3)$$

Note that the transformation $\chi$ does not necessarily map the volume $R$ onto the volume $r$, nor does it map each phase of $R$ onto the corresponding phase of $\chi(R)$. At this point, it is necessary to assume the existence of a second mapping $\chi'$ which maps each phase of $R$ onto the corresponding phase of $\chi(R)$:

$$R^W \rightarrow \chi'(R^W) \quad R^F \rightarrow \chi'(R^F) \quad R^M \rightarrow \chi'(R^M). \quad (4)$$

Let $f$ be some property pertaining to phase $\chi'(R^X)$ of the volume $\chi(R) = \chi'(R)$ in the current configuration. An alternative definition of the real-volume and bulk-volume average of $f$ is:

$$\theta(f) = \frac{1}{R^X} \int_{R^X} f(\chi'(X)) \, dR \quad \theta(f) = \frac{1}{R} \int_{R^X} f(\chi'(X)) \, dR. \quad (5)$$
The above averages are defined in terms of the initial configuration, whereas the averages defined in the companion paper are in terms of the current configuration. In both cases, however, the current values of the property $f$ are averaged. It is reasonable to assume that real-volume averages are independent of the size and shape of the representative elementary volume within a fairly large range of sizes and shapes. Therefore, we write:

\[ \langle f \rangle^o_x = \langle f \rangle^*_x \] (7)

\[ \langle f \rangle_x = N^{-1}(f)^*_x = N^x(\langle f \rangle^*_x) = \frac{N^x}{N}(f)_x. \] (8)

**Lagrangian velocities**

The local material time derivative is linked to the partial time derivative according to:

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \dot{\chi} \cdot \nabla \] (9)

in which $\dot{\chi}$ represents the velocity of the local material particle. In particular, we can apply equation (9) to the local value of:

\[ \chi = \chi^{-1}(\chi) \] (10)

of a particle (fluid or solid). The vector $\chi$ in equation (10) represents the current position vector of the particle. The transformation $\chi$ is the transformation defined in the companion paper. The vector $\dot{\chi}$ does not represent the position vector of the particle at time $t = 0$ but represents the average initial position vector of the solid particles currently surrounding the particle (Fig. 2). Substitution of equation (10) in equation (9) yields:

\[ \dot{\chi} = \frac{\partial \chi}{\partial t} + \dot{\chi} \cdot F^{-C} \] (11)

in which $\dot{\chi} = \lim_{\Delta t \to 0} \Delta \chi/\Delta t$ represents the absolute velocity of the particle and $\ddot{\chi} = \lim_{\Delta t \to 0} \Delta \dot{\chi}/\Delta t$ is the time derivative of the initial position vector for an observer fixed to the particle. Equation (11) can be applied to a solid particle and to a fluid particle. In particular,
Initial configuration of the solid 

Current configuration

Fig. 2. Change in time of the current position vector $\mathbf{x}$ of a particle is transformed back to the initial configuration. Note that the transformation $\chi$ is different at different times.

Equation (11) is averaged over the solid phase $r^s$ of the elementary volume $r$ centred around the particle:

$$\langle \dot{X} \rangle^s = \langle \frac{\partial X}{\partial t} \rangle^s + \langle \dot{X} \rangle^s \cdot F^{-c}. \tag{12}$$

This time, $\dot{X}$ represents the time derivative of $X$ for an observer fixed to the local solid particles. Due to the very definition of the initial position vector $X$, we know that:

$$\langle \dot{X} \rangle^s = 0. \tag{13}$$

Substituting equation (13) into equation (12) and substracting the resulting equation from equation (11), we obtain:

$$\dot{X} = (\dot{\chi} - \langle \dot{X} \rangle^s) \cdot F^{-c} \tag{14}$$

or

$$\dot{\chi} = (\dot{\chi})^s + \dot{X} \cdot F^c. \tag{15}$$

Equation (15) relates the absolute velocity $\dot{X}$ of a single particle to the velocity of the surrounding solid $\langle \dot{\chi} \rangle^s$.

**Lagrangian flows**

To each infinitesimal compartment $R^l(x_0) \, dx_0$ corresponds a volume fraction $N^l$ per unit $x_0$:

$$N^l = \frac{R^l(x_0) \, dx_0}{R \, dx_0} = \frac{R^l}{R}. \tag{16}$$

It follows that:

$$\int_a^b N^l \, dx_0 = N^F = \frac{R^F}{R}. \tag{17}$$

Any property $f$ of the fluid can be averaged over the infinitesimal fluid compartments $R^l(x_0) \, dx_0$:

$$\langle f \rangle^l = \frac{1}{R^l(x_0) \, dx_0} \int_{R^l(x_0) \, dx_0} f(\chi'(X)) \, dR \tag{18}$$
These averages and those defined by equations (21, 22) of the companion paper interrelate according to:

\[ \langle f \rangle_t = n^t \, dx_0 \langle f \rangle^* \]
\[ \langle f \rangle^*_t = \langle f \rangle^* \]
\[ \langle f \rangle_t = N^t \, dx_0 \langle f \rangle^* = N^t \, dx_0 \langle f \rangle^* = \frac{N^t}{n^t} \langle f \rangle_t. \] (19)

In the companion paper an Eulerian flow vector \( q^* \) has been defined for each compartment \( r^t(x_0) \, dx_0 \) in terms of the instantaneous configuration \( x \). In this paper, we define a Lagrangian flow vector \( Q^* \) in terms of the initial configuration \( X \).

At point \( x \) of the fluid phase, and at time \( t \), the local velocity of fluid with respect to an observer fixed in space is \( \dot{x}^t(x, t) \). The local velocity of fluid with respect to an observer fixed to the solid surrounding \( x \) is:

\[ \dot{v} = \dot{x}^t - \langle \dot{x} \rangle^* \] (20)

which according to equation (15) equals:

\[ \dot{v} = F \cdot \dot{X}^t \] (21)

in analogy to the Eulerian flows \( q_0 \) and \( q \), their Lagrangian equivalents \( Q_0 \) and \( Q \) can be expressed as:

\[ Q_0 = q_0 = n^t \langle \dot{x}_0 \rangle^* \] (22)
\[ Q = n^t \langle \dot{X} \rangle^*. \] (23)

The relationship between the spatial flows \( q \) and \( Q \) is obtained by averaging equation (21) over \( r^t(x_0) \, dx_0 \):

\[ \langle \dot{v} \rangle^* = F \cdot \langle \dot{X} \rangle^* \] (24)

or, using equation (26) of the companion paper and equation (23):

\[ \dot{q} = F \cdot \dot{Q}. \] (25)

The hierarchic flow \( Q_0 \) and the spatial flows \( Q \) form a 'four-dimensional' flow vector:

\[ Q^4 = \left( \begin{array}{c} Q_0 \\ Q \end{array} \right) = \left( \begin{array}{c} q_0 \\ q \end{array} \right). \] (26)

THE SLATTERY-WHITAKER AVERAGING THEOREM

In the companion paper, the Slattery–Whitaker theorem has been formulated as

\[ \langle \nabla f \rangle = \nabla \langle f \rangle + \frac{1}{r} \int_{\partial r^t \cap \{\partial r\}} f \, d\mathbf{a} \] (27)

which in its Lagrangian form reads:

\[ \langle \nabla \rangle^* = \nabla^* \langle f \rangle + \frac{1}{R} \int_{\partial R^t \cap \{\partial R\}} f \, d\mathbf{A}. \] (28)

An essential condition for the validity of equations (27, 28) is that the averaging volume is kept constant in size and shape, and is not rotated when translated from one point of the domain to another. For equation (27), the averaging volume is \( r \), which is constant in size and shape within the current configuration, which is also the configuration in which the gradient \( \nabla \) is
defined. For equation (28) the averaging volume is \( R \), which is constant in size and shape within the initial configuration, which is also the configuration in which the gradient \( \nabla \) is defined.

THE EXTENDED DARCY EQUATION

For the derivation of the extended Darcy equation we assume that the pore geometry consists of cylindrical vessels, which may be curved and interlacing. Defining:

- \( d \): diameter of the deformed vessel
- \( D \): diameter of the undeformed vessel
- \( s \): curvilinear coordinate representing the distance along the vessel axis in its deformed state
- \( S \): curvilinear coordinate representing the distance along the vessel axis in its undeformed state
- \( n^f dx_0 \): fraction of a deformed mixture unit volume, occupied by the cylindrical fluid volumes \( d^2 \pi /4 \frac{\partial s}{\partial x_0} dx_0 \).
- \( N^f dx_0 \): fraction of an undeformed mixture unit volume occupied by the cylindrical fluid volumes \( D^2 \pi /4 \frac{\partial S}{\partial x_0} dx_0 \).

We can establish the following relation between volume changes, length changes and diameter changes:

\[
\frac{d^2 s}{D^2 \partial S} = \frac{\chi'(R^f dx_0)}{R^f dx_0} = \frac{\chi'(R^f dx_0)}{R^f dx_0/R} = \frac{n^f}{N^f} \tag{29}
\]

hereby assuming circular cross-sections to remain circular, and fluid volume fraction changes to spread equally over all the volumes \( d^2 \pi /4 \partial s/\partial x_0 \), irrespective of their orientation.

A quasi-steady state approach allows us to apply Poiseuille’s law for the average fluid velocity \( \bar{\nu} \), relative to the solid, in the deformed vessel piece \( ds \) (Fig. 3):

\[
\|\bar{\nu}\| = - \frac{d^2 \partial p^f}{32 \mu \partial s} \tag{30}
\]

where \( \mu \) is the fluid viscosity. In the case that the fluid is blood, it is necessary to fit an apparent viscosity into equation (30). The apparent viscosity depends on the local vessel diameter \( d \) [14] and is the viscosity which fitted into Poiseuille’s equation yields the real pressure gradient/flow relation for a vessel with diameter \( d \). Assuming that \( \mu \) does not change when the vessels deform, we may write:

\[
\mu(d) = \mu(D). \tag{31}
\]

Fig. 3. The initial arc length \( (S) \) along a vessel changes into the current arc length \( (s) \) because of finite deformation of the solid skeleton.
Using equations (29) and (31), equation (30) becomes:

\[ \dot{x}_0 = -\frac{D^2 n f^4}{32 \mu N^4} \frac{\partial S}{\partial s} \frac{\partial S}{\partial s} \frac{\partial x_0}{\partial s} \]  

in which \( \dot{x}_0 \) is the average change per time unit in hierarchical parameter in the deformed vessel piece \( ds \)

\[ \dot{x}_0 = -\frac{D^2 n f^4}{32 \mu N^4} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial x_0}{\partial s} \frac{\partial x}{\partial s} \cdot \nabla p. \]  

Multiplying both sides of equation (33) with \( \partial X/\partial x_0 \) we obtain:

\[ \dot{X} = -\frac{D^2 n f^4}{32 \mu N^4} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial X}{\partial s} \frac{\partial x}{\partial s} \cdot \nabla p. \]  

Equations (33, 34) can be joined together in the following notation:

\[ \dot{X}^4 = -\frac{D^2 n f^4}{32 \mu N^4} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial X^4}{\partial s} \frac{\partial x}{\partial s} \cdot \nabla p. \]  

in which:

\[ X^4 = \left( \frac{\dot{x}_0}{X} \right). \]  

Averaging equation (35) over \( R^t(x_0) \) \( dx_0 \), yields:

\[ \langle X^4 \rangle^* = -\frac{n f^4}{32 N^4} \left( \frac{D^2}{\mu} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial X^4}{\partial s} \frac{\partial x}{\partial s} \cdot \nabla p \right)^t. \]  

or, using equations (22, 23) and (26):

\[ Q^4 dx_0 = -\frac{(n f)^4}{32 (N^4)^2} \left( \frac{D^2}{\mu} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial X^4}{\partial s} \frac{\partial x}{\partial s} \cdot \nabla p \right)^t. \]  

In the Appendix, equation (38) is transformed to the extended Darcy equation:

\[ Q^4 = -K^4 \cdot \nabla (p^4)^* \]  

with:

\[ \nabla^4 = \left( \begin{array}{c} \frac{\partial}{\partial x_0} \\ \frac{\partial}{\partial X} \end{array} \right) \]  

and the conductance tensor \( K^4 \) defined as:

\[ K^4 = \frac{(n f)^4}{32 N^4} \left( \frac{D^2}{\mu} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial X^4}{\partial s} \frac{\partial x}{\partial s} \cdot \nabla p \right)^t. \]  

or:

\[ K^4 = \frac{(n f)^4}{32 N^4} \left( \frac{D^2}{\mu} \left( \frac{\partial S}{\partial s} \right)^3 \frac{\partial x_0}{\partial s} \frac{\partial X}{\partial s} \cdot \nabla p \right)^t. \]  

Note that the conductance tensor \( K^4 \) is symmetric and uniquely defined by the geometry of the
pores and by the viscosity of the fluid. The axial vessel stretch ratio $\partial s/\partial S$ is a function of the stretch ratio of the surrounding solid along the axis of the vessel:

$$\frac{\partial s}{\partial S} = h\left(\| F \cdot \frac{\partial X}{\partial S} \|^2\right)$$  \hspace{1cm} (42)

where $\| F \cdot \frac{\partial X}{\partial S} \|^2$ is the stretch ratio of the surrounding solid, and $h$ is a scalar function. In the special case of the undeformed state, the conductance tensor equals:

$$K^i = \frac{N^f}{32} \left( D - \frac{\partial X}{\partial S} \right)$$  \hspace{1cm} (43)

The cross terms occurring in the conductance tensor indicate that spatial pressure gradients can generate flow between hierarchical compartments and that pressure differences between hierarchical compartments can generate spatial flow. Non-zero cross-terms implies anisotropy of the interfaces between neighbouring compartments; e.g. in a tree of a forest, flow of sap from the trunk to the branches (hierarchical flow) is associated with the upward direction. The extended Darcy equation (39) describes the relation between average pressure and flows in a fluid compartment. Equations (41) and (42) illustrate how in the deformed solid this pressure–flow relation is independent on the fluid volume of the compartment ($N^f$), on the deformation tensor ($F$), on the undeformed reference geometry of the fluid compartment ($\partial X/\partial S$, $\partial x_0/\partial S$, $D$, $N^f$) and on the fluid viscosity ($\mu$).

THE CONSTITUTIVE BEHAVIOUR

Following Biot [10], we assume that the free energy of the medium per unit initial volume is a function of $E$, $n^f$ and a large number of internal generalized coordinates $e_i$, which describe irreversible micro-deformations of the medium:

$$W = W(E, n^f, e_i).$$  \hspace{1cm} (44)

The dependency of the free energy upon the volume fraction $n^f$ per unit $x_0$ implies that the contribution of a pore volume change to the free energy may be different at different levels of hierarchy. In the case of blood perfused soft tissue, e.g. it is well known that arterial blood vessel walls behave differently than venous blood vessel walls. The internal generalized coordinates $e_i$ represent the numerous deformations on the microlevel which are averaged out by the averaging procedure and therefore do not show up in the averaged Green strain tensor $E$ or the volume fractions $n^f$. They may, e.g. represent the opening angles of microcracks along the solid–fluid interface where fluid seeps in and out. They may or may not be dependent upon the level of hierarchy. The rate of energy dissipation per unit initial volume is described by a dissipation function $D$:

$$D = D(E, n^f, e_i, \tilde{E}, \tilde{n}^f, \tilde{e}_i).$$  \hspace{1cm} (45)

Assuming that the state of the system is in the neighbourhood of equilibrium, $D$ is a quadratic function of $\tilde{E}$, $\tilde{n}^f$ and $\tilde{e}_i$ and is positive definite [15]. The Lagrangian equations of the system are:

$$\frac{\partial W}{\partial E} + \frac{\partial D}{\partial \tilde{E}} = 0$$  \hspace{1cm} (46)

$$\frac{\partial W}{\partial n^f} + \frac{\partial D}{\partial \tilde{n}^f} = \langle p^n \rangle^* - \langle p^n \rangle^*$$  \hspace{1cm} (47)

$$\frac{\partial W}{\partial e_i} + \frac{\partial D}{\partial \tilde{e}_i} = 0$$  \hspace{1cm} (48)
in which $\mathcal{S}$ is the second effective Piola–Kirchhoff stress. In the special case for which $W$ and $D$ take the form:

$$W = W_1(E, n^1) + A_k(E, n^1)e_k + \frac{1}{2} a_{ij} e_i e_j$$  \hspace{1cm} (49)$$

$$D = D_1(E, n^1, \dot{E}, \dot{n}^1) + B_k(E, n^1, \dot{E}, \dot{n}^1)e_k + \frac{1}{2} b_{ij} e_i e_j$$  \hspace{1cm} (50)$$

where $a_{ij}$ and $b_{ij}$ are constants, equations (45–48) can be solved [16]:

$$\mathcal{S} = \frac{\partial W_1}{\partial E} + \frac{\partial A_k}{\partial E} e_k + \frac{\partial D_1}{\partial \dot{E}} \dot{e}_k$$  \hspace{1cm} (51)$$

$$\langle p \rangle^e_t - \langle p \rangle^e_s = \frac{\partial W_1}{\partial \mathcal{J}^1} + \frac{\partial A_k}{\partial \mathcal{J}^1} e_k + \frac{\partial D_1}{\partial \mathcal{J}^1} \dot{e}_k + \frac{\partial B_k}{\partial \mathcal{J}^1} \dot{e}_k$$  \hspace{1cm} (52)$$

$$e_k = \sum C_{ij} \int_0^t \left( -A_k - B_k \right) e^\lambda(t' - t) dt'.$$  \hspace{1cm} (53)$$

The relaxation constants $\lambda$, are non-negative, because $a_{ij}$ and $b_{ij}$ are symmetric matrices, respectively semi-positive-definite and positive-definite. Equation (51) is the stress–strain–strain rate–time relationship of the mixture, equation (52) the pressure–volume relationship of the hierarchical fluid compartments. The terms $\partial D_1/\partial \dot{E}$ and $\partial D_1/\partial \mathcal{J}^1$ represent a viscous resistance to deformation of the porous medium while the terms containing $e_k$ represent a relaxation.

**SUMMARY OF THE EQUATIONS**

The equations derived thus far are:

—Global equilibrium:

$$\nabla \cdot \mathbf{g}^\sigma - \nabla p^S = 0$$  \hspace{1cm} (54)$$

—Fluid continuity:

$$\frac{\partial n^f}{\partial t} + \nabla \cdot (\mathbf{q}^f + n^f \mathbf{u}^f) = 0$$  \hspace{1cm} (55)$$

—Total continuity:

$$\int_a^b \nabla \cdot q \, dx_0 + \nabla \cdot \mathbf{u} = 0$$  \hspace{1cm} (56)$$

—Extended Darcy equation:

$$\mathbf{Q}^f = -\mathbf{K}^f \cdot \nabla^f p$$  \hspace{1cm} (57)$$

—Constitutive relations

$$\mathcal{S} = \frac{\partial W_1}{\partial E} + \frac{\partial A_k}{\partial E} e_k + \frac{\partial D_1}{\partial \dot{E}} \dot{e}_k$$  \hspace{1cm} (58)$$

$$p - p^S = \frac{\partial W_1}{\partial \mathcal{J}^1} + \frac{\partial A_k}{\partial \mathcal{J}^1} e_k + \frac{\partial D_1}{\partial \mathcal{J}^1} \dot{e}_k + \frac{\partial B_k}{\partial \mathcal{J}^1} \dot{e}_k$$  \hspace{1cm} (59)$$

$$e_k = \sum C_{ij} \int_0^t \left( -A_k - B_k \right) e^\lambda(t' - t) dt'.$$  \hspace{1cm} (60)$$
Equations (54)-(60) correspond to equations (47), (57) and (63) of the companion paper and to equations (39), (51–53) respectively, except that the following abbreviated notation is used:

\[ p^s = \langle p^s \rangle \]  
\[ p = \langle p' \rangle \]  
\[ q^t = n^t (\langle \dot{x}^t \rangle^s - (\dot{x}^t)^s) \]  
\[ \dot{x}^t = \left( \begin{array}{c} 0 \\ \dot{u} \end{array} \right) = \left( \begin{array}{c} 0 \\ (\langle \dot{x} \rangle^s) \end{array} \right) . \]

**DERIVATION OF BIOT'S THEORY**

Biot's equations [10] for a two-phase viscoelastic deforming medium are obtained from equations (54–60) by neglecting all transmural pressure differences across the vessel walls:

\[ p - p^s = 0 \]  
and assuming that the ratio \( n^f/N^f \) is independent from the hierarchic parameter \( x_0 \), whence:

\[ \frac{n^f}{N^f} = \frac{\int_a^b n^f \, dx_0}{\int_a^b N^f \, dx_0} = \frac{n^F}{N^F} \quad \forall x_0 \]  

where \( n^F/N^F \) is the ratio of the porosity of the deformed medium over the porosity of the undeformed medium. From equation (88) and with \( p^s \) being independent of \( x_0 \), hierarchic pressure differences vanish:

\[ \frac{\partial p}{\partial x_0} = 0. \]  

The equations resulting from assumptions (64, 65) are:

—Global equilibrium:

\[ \nabla \cdot \sigma = \nabla p = 0 \]  

—Total continuity:

\[ \nabla \cdot \int_a^b q \, dx_0 + \nabla \cdot \dot{u} = 0 \]  

—The integrated form of the extended Darcy equation

\[ \int_a^b Q^t \, dx_0 = - \int_a^b K^t \, dx_0 \cdot \nabla p \]  

—The constitutive relations:

\[ S = \frac{\partial W_1}{\partial E} + \frac{\partial A_k}{\partial E} e_k + \frac{\partial D_1}{\partial E} \dot{\varepsilon}_k + \frac{\partial B_k}{\partial E} \ddot{\varepsilon}_k \]  
\[ e_k = \sum C_{ij} \int_0^t (- A_k - B_k) e^{A(t'-t)} \, dt' \]
in which the values of \( Jn^t \) are considered as some of the internal thermodynamical variables \( e_k \).

Equation (70) can also be written as:

\[
\int_{a}^{\beta} \left( \frac{Q_0}{Q} \right) dx_0 = -\int_{a}^{\beta} K^t dx_0 \cdot \left( \frac{\partial p}{\partial x_0} \right)_{\nabla p} \tag{73}
\]

or, due to equation (67):

\[
\int_{a}^{\beta} \left( \frac{Q_0}{Q} \right) dx_0 = -\int_{a}^{\beta} K^t dx_0 \cdot \left( 0 \right)_{\nabla p} \tag{74}
\]

As the hierarchic fluid flow \( Q_0 \) is of no value in this context, we reduce equation (74) to:

\[
\int_{a}^{\beta} Q dx_0 = -\int_{a}^{\beta} K dx_0 \cdot \nabla p \tag{75}
\]

with, due to equation (41):

\[
K = \frac{(nF)^2 F^2}{32N^F} \left( \frac{D^2}{\mu} \frac{\partial S}{\partial S} \frac{\partial X}{\partial S} \right)_{t} \tag{76}
\]

Equation (76) relates the reduced conductance tensor \( K \) to the geometry of the pores and the viscosity \( \mu \). Equation (75) can be written as:

\[
\dot{Q} = -\dot{K} \cdot \nabla p \tag{77}
\]

with:

\[
\dot{Q} = \int_{a}^{\beta} Q dx_0 \tag{78}
\]

\[
\dot{K} = \frac{(nF)^2 F^2}{32N^F} \int_{a}^{\beta} \left( \frac{D^2}{\mu} \frac{\partial S}{\partial S} \frac{\partial X}{\partial S} \right)_{t} dx_0. \tag{79}
\]

Equation (77) is the three-dimensional form of Darcy’s law. Thus, Darcy’s law has been derived from the extended Darcy equation assuming that transmural pressure differences vanish. Similarly, we write instead of equation (69):

\[
\nabla \cdot \dot{q} + \nabla \cdot \dot{\bar{u}} = 0 \tag{80}
\]

with:

\[
\dot{q} = \int_{a}^{\beta} q dx_0 \tag{81}
\]

The relation between \( \dot{q} \) and \( \dot{Q} \) follows from equation (25):

\[
\dot{q} = F \cdot \dot{Q}. \tag{82}
\]

Substituting equations (82) and (77) in equation (80), yields:

\[
\nabla \cdot \dot{q} - \nabla \cdot (F \cdot \dot{K} \cdot \nabla p) = 0. \tag{83}
\]

The equilibrium equation (68), the continuity equation (83) and the constitutive relations (71, 72) are equations corresponding to Biot’s theory [10] for the case of incompressibility of solid and fluid. This results shows that the present theory reduces to Biot’s theory in the special case where the hierarchical arrangement of the pores is irrelevant.
REFERENCES


(Revision received 17 March 1995; accepted 8 May 1995)

APPENDIX

Equation (61) can be written in the form:

$$-\frac{32(N)^2Q^4}{(n)^2J} = \int \frac{d^3X}{dS} \cdot \frac{\partial^2X}{\partial S \partial S} \cdot \frac{\partial^2X}{\partial S \partial S}$$

Applying the Slattery-Whitaker theorem (50):

$$-\frac{32(N)^2Q^4}{(n)^2J} = \int \frac{d^3X}{dS} \cdot \frac{\partial^2X}{\partial S \partial S} \cdot \frac{\partial^2X}{\partial S \partial S}$$

and knowing neither $p$ and

$$\frac{D^2}{\mu} \left( \frac{\partial S}{\partial S} \right)^3 \frac{\partial X}{\partial S}$$

to be statistically correlated, yields:

$$-\frac{32(N)^2Q^4}{(n)^2J} = \int \frac{d^3X}{dS} \cdot \frac{\partial^2X}{\partial S \partial S} \cdot \frac{\partial^2X}{\partial S \partial S}$$

Applying the Slattery-Whitaker theorem once more yields:

$$-\frac{32(N)^2Q^4}{(n)^2J} = \int \frac{d^3X}{dS} \cdot \frac{\partial^2X}{\partial S \partial S} \cdot \frac{\partial^2X}{\partial S \partial S}$$

Noticing that along the vessel wall-fluid interface:

$$\frac{\partial X}{\partial S} \cdot dA = 0$$

(A5)
and that along the fluid–fluid interface,
\[
dq = \frac{\delta r}{\delta x_0} \nabla x_0 \quad \text{for } HP = x_0
\]
(A6)
\[
dq = \frac{\delta r}{\delta x_0} \nabla x_0 \quad \text{for } HP = x_0 + dx_0
\]
(A7)
and hence:
\[
\frac{\partial X}{\partial S} \cdot dA = -\frac{\partial x_0}{\partial S} \frac{\partial R}{\partial x_0} \quad \text{for } HP = x_0
\]
(A8)
\[
\frac{\partial X}{\partial S} \cdot dA = \frac{\partial x_0}{\partial S} \frac{\partial R}{\partial x_0} \quad \text{for } HP = x_0 + dx_0
\]
(A9)

the surface integrals of equation (A4) transform into volume integrals:
\[
- \frac{32(N)^2Q^* dx_0}{(n)^2J} = \frac{1}{\mu} \int_{R_{(x_0+dx_0)\in D}} \frac{D^2}{\partial s^3} \frac{\partial X^4 \partial X^2 \partial X}{\partial S^2 \partial S^2} \delta R
\]
(A10)

Using the definition of the bulk-volume average (16), equation (A10) becomes:
\[
- \frac{32NQ^*}{(n)^2J} = \left( \frac{D^2}{\mu} \frac{\partial S^3}{\partial S} \frac{\partial X^4 \partial X^2 \partial X}{\partial S^2 \partial S^2} \right) \cdot \mathbf{T}(\rho)^*.
\]
(A11)

As \( p^* \) and \( D^2/\mu(\partial S/\partial s)^3 \partial X^4/\partial S \partial x_0/\partial S \) are statistically uncorrelated one can write:
\[
- \frac{32NQ^*}{(n)^2J} = \left( \frac{D^2}{\mu} \frac{\partial S^3}{\partial S} \frac{\partial X^4 \partial X^2 \partial X}{\partial S^2 \partial S^2} \right) \cdot \mathbf{T}(\rho)^*.
\]
(A12)
or:
\[
Q^* = -K^* \cdot \mathbf{T}(\rho)^*
\]
(39)

with:
\[
K^* = \frac{(n)^2J}{32N} \mu \frac{D^2}{\partial s^3} \frac{\partial X^4 \partial X^2 \partial X}{\partial S^2 \partial S^2}.
\]
(41)

**NOMENCLATURE**

<table>
<thead>
<tr>
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<th>Symbol</th>
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<td>Tensor product</td>
<td>( \cdot )</td>
<td>dot product of the vectors ( a ) and ( b )</td>
</tr>
<tr>
<td>Vector product</td>
<td>( \cdot )</td>
<td>dot product of a second order tensor and a vector</td>
</tr>
<tr>
<td>Second order tensor</td>
<td>( \cdot )</td>
<td>dot product of two second order tensors, such that ( \nabla_T (g \cdot b) \cdot c = g \cdot (b \cdot c) )</td>
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<tr>
<td>Dyadic product of vectors</td>
<td>( \otimes )</td>
<td>length of vector ( g )</td>
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**Tensor notation**

- \( \mathbf{a} \) vector in 3D space
- \( \mathbf{a}^* \) vector in 4D space
- \( \mathbf{g} \) second order tensor in 3D space
- \( \mathbf{g}^* \) second order tensor in 4D space
- \( \mathbf{g} \otimes \mathbf{b} \) dyadic product of the vectors \( a \) and \( b \)
\( g^c \)  
\( g^{-1} \)  
\( \det(g) \)  
\( I \)  
\( \triangleq \) conjugate of \( g \)  
inverse of \( g \)  
determinant of \( g \)  
unit second order tensor  

Set notation  
\( A \cap B \) intersection of set \( A \) and set \( B \)  
\( A \cup B \) union of set \( A \) and set \( B \)  
\( -A \) complementary set of set \( A \)  
\( \forall a \) for all \( a \)  

Specific symbols  
\( a \) current vessel cross section  
\( da \) elementary surface in current configuration  
\( dA \) elementary surface in initial configuration  
\( d^* \) local material time derivative  
\( \frac{d}{dt} \) partial time derivative  
\( dV \) boundary surface of volume \( V \)  
\( E \) Green strain tensor  
\( F \) deformation tensor  
\( f \) Jacobian  
\( K^* \) current conductance tensor  
\( K^t \) initial conductance tensor  
\( K^\delta \) current permeability tensor  
\( n^t \) current fluid volume fraction per unit  
\( N^t \) initial fluid volume fraction per unit hierarchic parameter  
\( n^F \) current total fluid volume fraction (current porosity)  
\( N^F \) initial total fluid volume fraction (initial porosity)  
\( n^X \) current volume fraction of phase \( X \)  
\( N^X \) initial volume fraction of phase \( X \)  
\( p \) \( (= p'g) \) average fluid pressure  
\( p' \) local fluid pressure  
\( p^*_t \) local solid pressure  
\( p^*_s \) \( (= (p^*_s)^s) \) average solid pressure  
\( q \) spatial fluid flow vector (Eulerian)  
\( \overrightarrow{Q} \) spatial fluid flow vector (Lagrangean)  
\( \overrightarrow{Q}' \) fluid flow vector (Eulerian)  
\( \overrightarrow{Q}'_t \) fluid flow vector (Lagrangean)  
\( \overrightarrow{Q}^t \) integrated fluid flow vector (Eulerian)  
\( \overrightarrow{Q}'^t \) integrated fluid flow vector (Lagrangean)  
\( \overrightarrow{r} \) representative volume in current configuration  
\( \overrightarrow{R} \) representative volume in initial configuration  
\( \overrightarrow{r}' \) fluid volume in \( r \) per unit hierarchic parameter  
\( \overrightarrow{R}' \) fluid volume in \( R \) per unit hierarchic parameter  
\( r^X \) volume of phase \( X \) in \( r \)  
\( R^X \) volume of phase \( X \) in \( R \)  
\( s \) current arc length along vessel axis  
\( S \) initial arc length along vessel axis  
\( S^t \) effective 2nd Piola-Kirchhoff stress tensor  
\( (= Jf \cdot Q^t \cdot f^c) \)  
\( t \) time  
\( u \) displacement vector  
\( V \) relative fluid velocity  
\( W \) Helmholtz free energy per unit initial volume  
\( \lambda \) hierarchic parameter  
\( x \) current position vector  
\( X \) average initial position vector of the solid  
\( x_0 \)  
\( (x_0) \) transformation from initial to current configuration  
\( \overrightarrow{X} \) part of local Cauchy stress tensor due to deformation of the solid  
\( \sigma^* \) local Cauchy stress tensor  
\( \sigma \) effective Cauchy stress  
\( \mu \) fluid viscosity