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The Residues modulo m
of Products of Random Integers
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The Residues modulo $m$ of Products of Random Integers

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Abstract

For two (possibly stochastically dependent) random variables $X$ and $Y$ taking values in $\{0, \ldots, m-1\}$ we study the distribution of the random residue $U = XY \mod m$. In the case of independent and uniformly distributed $X$ and $Y$ we provide an exact solution in terms of generating functions that are computed via $p$-adic analysis. We show also that in the uniform case it is stochastically smaller than (and very close to) the uniform distribution. For general dependent $X$ and $Y$ we prove an inequality for the distance $\sup_{x \in [0,1]} |F_U(x) - x|$.

1 Introduction

Let $X$ and $Y$ be two (possibly dependent) random variables taking values in $\{0, 1, \ldots, m-1\}$, where $m \geq 2$ is some fixed integer. In this note we study the distribution of the random residue of the product

$$U = XY \mod m.$$ 

We consider first the case when $X$ and $Y$ are independent and uniformly distributed, i.e. $P(X = i, Y = j) = m^{-2}$ for $i, j \in \{0, \ldots, m-1\}$. In Section 2 it is shown that the problem for general $m$ can be reduced to that for $m = p^n$, where $p$ is some prime number and $n \in \mathbb{N}$, and that in this case it is sufficient to determine the cardinalities

$$N_p(l, n) = \# \{(x, y) \in (\mathbb{Z}/p^n\mathbb{Z}) \times (\mathbb{Z}/p^n\mathbb{Z}) \mid xy = p^{n-l}\}.$$
We prove that for every prime number $p$ the generating function $H_p(T, Z) = \sum_{n,l} N_p(l, n)T^n Z^l$ of the double sequence $N_p(l, n)$ is given by

$$H_p(T, Z) = \frac{(1 - pT)^2(1 - p^{-1}Z) - p^2(1 - p^{-1}T)T(1 - Z)}{(1 - Z)(1 - p^{-1}Z)(1 - pT)^2(1 - p^2T)}. \quad (1.1)$$

In the case $p = 2$ we derive a neat explicit formula for the distribution function of $U$. It is given by

$$P(U \leq k) = (k + 1)2^{-n} + 2^{-n+1} \sum_{i=0}^{n-1} (1 - \delta_i) \quad (1.2)$$

for $k = 0, \ldots, 2^{n-1}$, where $\delta_0, \ldots, \delta_{n-1} \in \{0, 1\}$ are the binary digits of $k$, defined by $k = \delta_0 + 2\delta_1 + 4\delta_2 + \cdots + 2^{n-1}\delta_{n-1}$.

It follows from (1.2) that the random 'fractional residue' $2^{-n}U$ is stochastically smaller than a uniform random variable on $[0, 1)$, i.e. $P(U/2^n < u) \geq u$ for all $u \in [0, 1]$ and that the maximal deviation is given by

$$\sup_{0 < u \leq 1} (P(2^{-n}U < u) - u) = (n + 2)2^{-(n+1)}, \quad (1.3)$$

so that the distribution of $2^{-n}U$ tends to the uniform distribution on $[0, 1]$ at an exponential rate (given by (1.3)), as $n \to \infty$. In fact, these stochastic dominance and convergence remain valid for arbitrary $m$.

The rest of the paper is devoted to an extension of this asymptotic equidistribution result to general $m$ and dependent, non-uniform random variables $X$ and $Y$.

We will show that

$$\sup_{0 \leq u \leq 1} \left| P(U/m < u) - u \right| \leq C \left( \frac{\log m}{m} \right)^{1/2} \quad (1.4)$$

if the distribution of $Y$ and the conditional distribution of $X$ given $Y$ do not deviate too much from uniformity and if the latter distribution satisfies a certain Lipschitz condition. Specifically, we assume that

$$P(Y = k) \leq C_0/m \quad p(j|k) = P(X = j \mid Y = k) \leq C_1/m$$

$$\left| \frac{p(j_1|k)}{p(j_2|k)} - 1 \right| \leq C_2 |j_1 - j_2|/m$$
for some constants $C_0, C_1, C_2$. Then (1.4) holds for a certain constant $C$ which depends only on $C_0, C_1$ and $C_2$. From (1.4) we can conclude that $U/m$ is for a large class of joint distributions of $X$ and $Y$ 'almost' uniformly distributed on $[0,1]$ in the sense of weak convergence.

Deterministic sequences of integers whose residues are uniformly distributed are treated in Narkiewicz [10] and Kuipers and Niederreiter [8]. They play an important role in random number generation (Ripley [12]). In the realm of stochastic sequences already Dvoretzky and Wolfowitz [5] studied weak convergence of residues for sums of independent, $\mathbb{Z}_+$-valued random variables; more recent papers on related questions are Brown [3], Barbour and Grübel [1], and Grübel [6]. The distribution of the fractional part of continuous random variables, in particular its closeness or convergence to the uniform distribution on $[0,1)$, has been studied by many authors (e.g. Schatte [13], Stadje [14, 15], Qi and Wilms [11]).

2 The uniform case

We start by deriving the exact probability distribution of $U$ in the case $m = 2^n$, $n \in \mathbb{N}$. For $x \in \mathbb{R}_+$ let $\text{frac}(x)$ be the fractional part of $x$.

**Proposition 1** We have

$$P(U \leq k) = (k + 1)2^{-n} + 2^{-(n+1)} \sum_{i=0}^{n-1} (1 - \delta_i),$$

(2.1)

for every $k \in \{0, 1, \ldots, 2^n - 1\}$, where $\delta_0, \ldots, \delta_{n-1} \in \{0, \ldots, n - 1\}$ are the binary digits of $k$, i.e. $k = \delta_0 + 2\delta_1 + 4\delta_2 + \cdots + 2^{n-1}\delta_{n-1}$.

**Proof.** Obviously,

$$P(U = k) = \sum_{i=0}^{2^n-1} 2^{-2n} \text{card}\{j \in I_n \mid \text{frac}(ij2^{-n}) = k2^{-n}\}.$$  

(2.2)

Let

$$A_m = \begin{cases} \{i \in I_n \mid i2^{-m} \text{ is odd}\}, & \text{if } m < n \\ \{0\}, & \text{if } m = n. \end{cases}$$

It is easily seen that

$$\text{card } A_m = \begin{cases} 2^{n-m-1}, & \text{if } m \in \{0, \ldots, n - 1\} \\ 1, & \text{if } m = n. \end{cases}$$
Consider $i \in A_m$ and $k \in A_l$ for some $m, l \in \{0, \ldots, n-1\}$, say $i = (2p+1)2^m$ and $k = (2q + 1)2^l$. Then for any $j \in I_n$,

$$\frac{n}{2^n} = k2^{-n} \quad (2.3)$$

is equivalent to

$$(2p+1)j - (2q+1)2^{l-m} = N2^{n-m} \text{ for some integer } N. \quad (2.4)$$

For $l < m$ the lefthand side of (2.4) is not integer, so there is no solution $j$ of (2.3). Now let $l \geq m$. Since $2p+1$ and $2^n$ are relatively prime, a simple result on residues implies that the numbers $(2p+1)j - (2q+1)2^{l-m}$ run through a complete set of residues mod $2^n$ if $j$ runs through (the complete set of residues) $0, 1, \ldots, 2^n - 1$. But $N2^{n-m}$ gives different residues mod $2^n$ for $N = 0, \ldots, 2^m - 1$, while for larger values of $N$ one only gets replications of these residues. Thus, the number of solutions $j$ of (2.3) is $2^n$ if $l \geq m$.

The same result also holds for $m \in A_s$, i.e. $m = 0$.

From (2.2) it now follows that if $k \in A_l$ for some $l < n$ we obtain

$$P(U = k2^{-n}) = \sum_{m=0}^{n-1} \sum_{i \in A_m} 2^{-2n} \frac{n}{2^n} + 2^{-n} \delta_{0k}$$

$$= \sum_{m=0}^{n-1} 2^{-2n} \text{card}(A_m)2^n$$

$$= \sum_{m=0}^{n-1} 2^{-n} 2^{n-m-1}$$

$$= (l+1)2^{-(n+1)}, \quad (2.5)$$

while if $k \in A_n$,

$$P(U = 0) = \sum_{m=0}^{n-1} 2^{-2n} \text{card}(A_m)2^n + 2^{-n}$$

$$= (n+2)2^{-(n+1)}. \quad (2.6)$$

In particular, $k \mapsto P(U = k)$ is constant on $A_l$ for every $l$. Therefore, the probability $P(U \in (2^m \alpha, 2^m \alpha + 2^{m-1}])$ is the same for every $\alpha \in \{0, \ldots, 2^{n-m}-1\}$.
1). It follows that

\[ P(U \leq k) = P(U = 0) + P(0 < U < \delta_{n-1} 2^n) + \sum_{l=1}^{n-1} P \left( \sum_{i=l}^{n-1} \delta_i 2^i < U \leq \sum_{i=l-1}^{n-1} \delta_i 2^i \right) \]

\[ = P(U = 0) + \sum_{l=0}^{n-1} P(0 < U \leq \delta_l 2^l). \tag{2.7} \]

To compute the righthand side of (2.7), note that the number of integers \( i \in A_m \) satisfying \( 0 < i \leq 2^l \) is equal to \( 2^l - m - 1 \) for \( m = 0, \ldots, l-1 \) and equal to 1 for \( m = l \). Hence, by (2.5),

\[ P(0 < U \leq 2^l) = \sum_{m=0}^{l} P(U \in A_m \cap \{0, \ldots, 2^l\}) \]

\[ = \sum_{m=0}^{l-1} (l + 1)2^{-(n+1)}2^{l-m-1} + (l + 1)2^{-(n+1)} \]

\[ = 2^{-(n+1)}(2^{l+1} - 1). \tag{2.8} \]

Inserting (2.8) and (2.6) in (2.7) now yields (2.1).

**Proposition 2**

1) For arbitrary \( m \) \( U \) is stochastically smaller than a uniform random variable on \([0, 1] \);

2) For arbitrary \( m \)

\[ \sup_{0 < u \leq 1} (P(U < u) - u) = O(m^{-1+\epsilon}), \tag{2.9} \]

for any \( \epsilon > 0 \);

and

3) For \( m = 2^n \),

\[ \sup_{0 < u \leq 1} (P(U < u) - u) = (n + 2)2^{-(n+1)}. \tag{2.10} \]

**Proof.** We start with 1). It is clear that

\[ \#\{0 \leq j < m : ij \mod m \leq k\} = \gcd(i, m) \left( \left\lfloor \frac{k}{\gcd(i, m)} \right\rfloor + 1 \right). \tag{2.11} \]
This implies

\[ P(U \leq k) = \frac{1}{m^2} \sum_{i=0}^{m-1} \gcd(i, m) \left( \left\lfloor \frac{k}{\gcd(i, m)} \right\rfloor + 1 \right) > k/m \] (2.12)

for all \( 0 \leq k < m \), and hence proves 1).

Further, estimating (2.12) in an obvious way from above, we obtain

\[
P(U \leq k) \leq \frac{1}{m^2} \sum_{i=0}^{m-1} \gcd(i, m) \left( \frac{k}{\gcd(i, m)} + 1 \right)
\leq k/m + \frac{1}{m^2} \sum_{i=0}^{m-1} \gcd(i, m)
\leq k/m + \frac{1}{m^2} \sum_{i=0}^{m-1} \left\lfloor \frac{i}{m} \right\rfloor \sum_{i=0}^{m-1} \frac{1}{i}
\leq k/m + \frac{d(m)}{m},
\] (2.13)

where \( d(m) \) denotes the number of divisors of \( m \). It is known that \( d(m) = O(m^\epsilon) \) for all \( \epsilon > 0 \), which implies 2).

To prove 3) define for \( 0 < u \leq 1 \) the integer \( k(u) \) by \( k(u)2^{-n} < u \leq (k(u) + 1)2^{-n} \) and let \( \delta_0, \ldots, \delta_{n-1} \) be its binary digits. By (2.1) we can write

\[
P(U < u) - u = (k(u)2^{-n} + 2^{-n} - u) + 2^{-(n+1)} \sum_{i=0}^{n-1} (1 - \delta_i),
\] (2.14)

which is nonnegative by the definition of \( k(u) \). Further it is clear from (2.14) that \( \sup_{0 < u \leq 1}(P(U < u) - u) \) is approached as \( u \downarrow 0 \), yielding (2.10).

Now we derive the exact formulae for \( P(U = k) \) in the case of general \( m \in \mathbb{N} \).

Let \( X \) and \( Y \) be independent and uniform on the set \( \{0, \ldots, m-1\} \), which we identify with \( \mathbb{Z}/m\mathbb{Z} \). Then \( P(U = a) \) is equal to \( m^{-2} \) times the number of solutions \( (x, y) \in (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) \) of the equation

\[ xy \equiv a \mod m. \]

Let \( m = \prod p_i^{n_i} \) be the prime factorization of \( m \) (\( p_i \) primes, \( n_i \in \mathbb{N} \)). For \( a \in \mathbb{Z}/m\mathbb{Z} \) we define \( a(i) \in \mathbb{Z}/p_i^{n_i}\mathbb{Z} \) as the (unique) solution of

\[ a(i) \equiv a \mod p_i^{n_i}. \]

Then as \( \mathbb{Z}/m\mathbb{Z} = \prod (\mathbb{Z}/p_i^{n_i}\mathbb{Z}) \) (the Chinese remainder theorem), we have the following decomposition.
**Lemma 1** The number of pairs \((x, y) \in (\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})\) satisfying
\[
xy \equiv a \mod m \tag{2.15}
\]
is equal to the product of the numbers of solutions \((x, y) \in (\mathbb{Z}/p_i^{n_i}\mathbb{Z}) \times (\mathbb{Z}/p_i^{n_i}\mathbb{Z})\) of
\[
xy \equiv a(i) \mod p_i^{n_i} \tag{2.16}
\]

By the Lemma, we only have to determine the number of solutions of (2.15) for \(m\) of the form \(m = p^n\).

Fix a prime number \(p\) and a natural number \(n\). Observe first that the number of solutions \((x, y) \in (\mathbb{Z}/p^n\mathbb{Z}) \times (\mathbb{Z}/p^n\mathbb{Z})\) of \(xy \equiv a \mod p^n\) depends on \(a\) only through the \(p\)-adic norm of \(a\), that is, through the exponent of the maximal power of \(p\) that divides \(a\). Indeed, if there exists an invertible \(b\) in \(\mathbb{Z}/p^n\mathbb{Z}\) satisfying
\[
ab \equiv p^{n-1} \mod p^n
\]
then
\[
\#\{(x, y) \in (\mathbb{Z}/p^n\mathbb{Z}) \times (\mathbb{Z}/p^n\mathbb{Z}) \mid xy \equiv a \mod p^n\} = \#\{(x, y) \mid xyb \equiv p^{n-1} \mod p^n\} = \#\{(x, z) \in (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z}) \mid xz \equiv p^{n-1} \mod p^n\} = N_p(l, n).
\]

To compute \(N_p(l, n)\), we use the following well-known formula from the theory of \(p\)-adic integration (Christol [4, Sect. 7.2.2, p. 466]). Let \(f(x_1, \ldots, x_r)\) be a polynomial with coefficients in \(\mathbb{Z}_p\), the ring of \(p\)-adic integers, and let \(|\cdot|_p\) denote the \(p\)-adic norm. Then for any real \(s > 0\),
\[
\int_{(\mathbb{Z}_p)^r} |f(x_1, \ldots, x_r)|_p^s \mu(dx_1) \cdots \mu(dx_r) = p^s - (p^s - 1)Q(p^{-r-s}), \tag{2.17}
\]
where \(\mu\) is the Haar measure on \(\mathbb{Z}_p\) and \(Q(T)\) is a Poincaré series:
\[
Q(T) = \sum_{k=0}^{\infty} T^k \#\{(x_1, \ldots, x_r) \in (\mathbb{Z}/p^k\mathbb{Z})^r \mid f(x_1, \ldots, x_r) \equiv 0 \mod p^k\}.
\]

**Theorem 1** The generating functions
\[
G_{pl}(T) = \sum_{n=0}^{\infty} N_p(l, n)T^n, \quad H_p(T, Z) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} N_p(l, n)T^nZ^l
\]
are given by

\[
G_{p,i}(T) = \frac{p!(1-pT)^2 - p^2(1-p^{-1})^2T}{p!(1-pT)^2(1-p^2T)}
\] (2.18)

\[
H_p(T, Z) = \frac{(1-pT)^2(1-p^{-1}Z) - p^2(1-p^{-1}T)(1-Z)T}{(1-Z)(1-p^{-1}Z)(1-p)T(1-p^2T)}
\] (2.19)

Proof. We use formula (2.17) for \( r = 2 \) and \( f(x,y) = f_l(x,y) = p^lxy \). For the lefthand side of (2.17) we obtain

\[
\int_{(x_p)^2} |f_l(x,y)|^s_p \mu(dx)\mu(dy) = \int_{(y_p)^2} p^{-l}|x|^s_p |y|^s_p \mu(dx)\mu(dy)
\]

\[
= p^{-l} \left( \int_{z_p} |x|^s_p \mu(dx) \right)^2.
\]

By (2.17),

\[
\int_{z_p} |x|^s_p \mu(dx) = p^s - (p^s - 1) \frac{1}{1-p^{-1-s}} = \frac{1-p^{-1}}{1-p^{-1-s}}.
\]

(Note that here \( Q(T) = 1/(1-T) \), since \( \#\{x \in \mathbb{Z}/p^n \mid x \equiv 0 \mod p^n \} = 1 \) for all \( n \)). Furthermore,

\[
xy \equiv p^{-l} \mod p^n \quad \text{iff} \quad p^lxy \equiv 0 \mod p^n.
\]

Thus, the coefficients on the righthand side of (2.17) are just the \( N_p(l,n) \). It follows that

\[
p^s - (p^s - 1) \sum_n N_p(l,n)(p^{-2-s})^n = p^{-l} \left( \frac{1-p^{-1}}{1-p^{-1-s}} \right)^2.
\]

Setting \( T = p^{-2-s} \), so that \( p^{-s} = p^2T \) we get

\[
\frac{1}{p^2T} - \left( \frac{1}{p^2T} - 1 \right) G_{p,i}(T) = p^{-l} \left( \frac{1-p^{-1}}{1-pT} \right)^2
\] (2.20)

and (2.18) follows from (2.20) by a short calculation. Similarly, multiplying (2.20) by \( Z^l \) and summing over \( l \) yields (2.19).
For example, if $p = 2$ the numbers $N_p(0, n)$ of solutions $(x, y)$ of $(x, y) \equiv 0 \mod 2^n$ is $(n + 2)2^{n-1}$, as

$$G_{2,0}(T) = \sum_{n=0}^{\infty} N_p(0, n) T^n = \frac{(1 - 2T)^2 - T}{(1 - 2T)^2(1 - 4T)} = \frac{1 - T}{(1 - 2T)^2} = \sum_{n=0}^{\infty} (n + 2)2^{n-1}T^n.
$$

3 The inequality for dependent random variables

We will now prove (1.4). For this we need some basic theory of continued fractions (see e.g. Hardy and Wright [7], Billingsley [2]) and a probability estimate due to Lévy [9]).

Any $x \in [0, 1]$ has a continued fraction expansion $x = [a_1(x), a_2(x), \ldots]$ providing a sequence of fractions usually denoted by

$$p_n(x)/q_n(x) = [a_1(x), \ldots, a_n(x)].$$

For two positive numbers $\rho_0 < \rho_1$ let

$$B(\rho_0, \rho_1) = \{x \in [0, 1] \mid \rho_0 < q_k(x) < \rho_1 \text{ for some } k \in \mathbb{N}\}.$$

**Lemma 2** $\lambda(B(\rho_0, \rho_1)) \geq 1 - \frac{2\rho_0}{\rho_1 - \rho_0}(1 + 2 \log_2 \rho_0) - \rho_1^{-1}$.

**Proof.** Let $Q$ be the set of all finite sequences $\vec{q} = (q_1, \ldots, q_k)$, $k \in \mathbb{N}$, of denominators of possible continued fraction expansions satisfying $q_k \leq \rho_0$. We set $x(\vec{q}) = p_k/q_k$, where $p_k$ is the $k$th numerator corresponding to $q_1, \ldots, q_k$, and

$$I(\vec{q}) = \{x \in [0, 1] \mid (q_1(x), \ldots, q_k(x)) = \vec{q}\},$$

$$J(\vec{q}) = I(\vec{q}) \cap \{x \in [0, 1] \mid q_{k+1}(x) \geq \rho_1 \text{ or } x = x(\vec{q})\},$$

$$J(0) = \{x \in [0, 1] \mid q_1(x) \geq \rho_1\}.$$  

The sets $J(\vec{q}), \vec{q} \in Q$, and $J(0)$ are pairwise disjoint intervals and

$$B(\rho_0, \rho_1) = [0, 1] \setminus \left( J(0) \cup \bigcup_{\vec{q} \in Q} J(\vec{q}) \right).$$
Thus,
\[
\lambda([0,1]\setminus B(\rho_0, \rho_1)) = \lambda(J(0)) + \sum_{q \in Q} \lambda(J(q))
\]
\[
= \lambda(J(0)) + \sum_{k=1}^{k_0} \sum_{\substack{q \in Q \, 
\delta(|q| = k)}} \lambda(J(q)), \tag{3.1}
\]
where $|q|$ denotes the length of the sequence $q$ and $k_0$ is the maximum length of sequences in $Q$. Since
\[
\rho_0 > q_k \geq 2^{(k-1)/2}
\]
for every $(q_1, \ldots, q_k) \in Q$, it follows that
\[
k_0 < 1 + 2 \log_2 \rho_0. \tag{3.2}
\]

Now let $U$ be a random variable that is uniformly distributed on $[0,1]$. Then if $q \in Q, |q| = k$, it follows that
\[
\lambda(J(q)) = P(q_{k+1} \geq \rho_1, U \in I(q))
\]
\[
= P(U \in I(q)) P(q_{k+1} \geq \rho_1 | U \in I(q))
\]
\[
\leq P(U \in I(q)) P(a_{k+1} > \rho_1 - \rho_0 | U \in I(q)) \tag{3.3}
\]
\[
\leq P(U \in I(q)) 2 \left( \frac{\rho_1 - \rho_0}{\rho_0} \right)^{-1}.
\]
For the first inequality in (3.3) we have used the recursion $q_{k+1} = q_k a_{k+1} + q_{k-1}$ which for $q \in Q, |q| = k$, implies that $a_{k+1} > (\rho_1 - \rho_0)/\rho_0$. The second inequality follows from a result of Lévy [9, p. 296].

To estimate $\lambda(J(0))$, note that $q_1(x) \geq \rho_0$ implies that $x \leq p_1(x)/q_1(x) = 1/\rho_1$. Thus, by (3.1), (3.2) and (3.3).
\[
\lambda([0,1] \setminus B(\rho_0, \rho_1)) \leq \rho_1^{-1} + k_0 \frac{2\rho_0}{\rho_1 - \rho_0} \sum_{q \in Q} P(U \in I(q))
\]
\[
\leq \rho_1^{-1} + (1 + 2 \log_2 \rho_0) \frac{2\rho_0}{\rho_1 - \rho_0}.
\]
The Lemma is proved.
Lemma 3 Let $X$ be uniformly distributed on $\{0, 1, \ldots, m-1\}$. Then
\[
P(X/m \notin B(p_0, p_1)) \leq 2p_0(1 + 2 \log_2 p_0) \left(\frac{1}{p_0} + \frac{p_0}{m}\right) + p_1^{-1} + m^{-1}.
\] (3.4)

Proof. For every half-open or open interval $I$ in $[0, 1]$ we have
\[
|P(X/m \in I) - \lambda(I)| \leq m^{-1}.
\] (3.5)

As $J(0)$ and $J(\tilde{q})$ are half-open intervals, (3.1) and (3.4) yield
\[
P(X/m \notin B(p_0, p_1)) \leq \lambda(J(0)) + \sum_{\tilde{q} \in Q} \lambda(J(\tilde{q})) + m^{-1}(1 + \text{card } Q).
\] (3.6)

It remains to find an upper bound for \text{card } Q. Let $\tilde{Q}$ be the set of sequences in $Q$ having maximal length, i.e., the set of those $(q_1(x), \ldots, q_k(x)) \in Q$ for which $q_{k+1}(x) \geq p_0$. Since
\[
\lambda(I(q_1, \ldots, q_k)) = \frac{1}{q_k(q_k + q_{k-1})} > \frac{1}{2q_k^2} \geq \frac{1}{2p_0^2}
\]
for $(q_1, \ldots, q_k) \in \tilde{Q}$, we clearly have \text{card } $\tilde{Q} < 2p_0^2$. Inequality (3.4) now follows from (3.6), Lemma 2 and
\[
\text{card } Q \leq k_0 \text{card } \tilde{Q} < (1 + \log_2 p_0)(2p_0^2).
\]

Lemma 4 Let
\[
p(j, k) = P(X = j, Y = k), \quad j, k \in \{0, \ldots, m-1\}
\]
be the joint distribution of $X$ and $Y$. Assume that there are constants $C_1$ and $C_2$ such that
\[
p(j|k) = P(X = j|Y = k) \leq C_1/m
\] (3.7)
\[
\left|\frac{p(j_1|k)}{p(j_2|k)} - 1\right| \leq C_2|j_1 - j_2|/m
\] (3.8)

for all $j, k, j_1, j_2 \in \{0, \ldots, m-1\}$. Then
\[
|P(U/m < u|Y = k) - u| \leq \frac{3C_2}{m} + \inf_{n \geq 1} f \left( q_n \left( \frac{k}{m} \right) \right)
\]
for all $k \in \{0, \ldots, m-1\}$, where

$$f(q) = \frac{3}{q} + \frac{(C_1 + C_2)q}{m}, \quad q \in \mathbb{N}.$$  

Proof. Let $p/q$ be an arbitrary fraction from the continued fraction expansion of $k/m$. Let

$$J_i = \{(i-1)q, (i-1)q+1, \ldots, iq-1\}$$

$$J_i(u) = \{j \in J_i \mid \text{frac}(jk/m) < u\},$$

where $\text{frac}(x)$ denotes the fractional part of $x \geq 0$. Then

$$P(U/m < u \mid Y = k) = \sum_{i=1}^{[m/q]} \sum_{j \in J_i(u)} P(X = j \mid Y = k)$$

$$+ \sum_{k \in J_1[m/q]+1}^{k < m} P(X = j \mid Y = k)$$

$$= I + II.$$  

Clearly, (3.7) yields

$$II \leq C_1 q/m.$$  

Regarding the sum $I$, we can write

$$I = \sum_{i=1}^{[m/q]} \sum_{j \in J_i(u)} p(j \mid k)$$

$$\leq \sum_{i=1}^{[m/q]} A_i \text{card } J_i(u) \sum_{j \in J_i} p(j \mid k),$$

where $A_i = \max_{j \in J_i} p(j \mid k)$ and $a_i = \min_{j \in J_i} p(j \mid k)$. From (3.8) we can conclude that

$$A_i/a_i \leq 1 + (C_2 q/m).$$

Obviously, $\text{card } J_i = q$. We need an upper bound for $\text{card } J_i(u)$. Note that

$$\left| \frac{k}{m} - \frac{p}{q} \right| < q^{-2}.$$
For arbitrary $j \in J_i(u)$ write $j = (i - 1)q + h$, where $h \in J_1$; we obtain

$$\frac{jk}{m} = \frac{(i - 1)q \frac{k}{m} + \frac{hk}{m}}{m} = \frac{(i - 1)q \frac{k}{m} + \frac{hk}{m}}{m}$$

and

$$\frac{hk}{m} = \frac{h \left( \frac{k}{m} - \frac{p}{q} \right) + \frac{hp}{q}}{q} = \frac{\alpha + \frac{hp}{q}}{p}$$

where $|\alpha| < q^{-1}$. Recall that $p$ and $q$ are relatively prime. Thus, as $h$ runs through $J_1$, $\frac{hk}{m}$ runs through the set of all values $\frac{l}{q} + \alpha$, $l \in J_1$. Let $\beta_i = (i - 1)qk/m$.

Let $\tilde{J}_i(u)$ be the number of values $\frac{\beta_i + (l/q)}{m}$ in $[0, u)$ for which $l \in J_1$. Clearly, we have $\tilde{J}_i(u) \in \{[qu], [qu] + 1\}$. Since $|\alpha| < q^{-1}$, it now follows easily that

$$|\tilde{J}_i(u) - \text{card } J_i(u)| \leq 2,$$

so that

$$|qu - \text{card } J_i(u)| \leq 3. \quad (3.13)$$

By (3.12) and (3.13),

$$\frac{A_i \text{ card } J_i(u)}{a_i \text{ card } J_i} \leq \left( 1 + \frac{C_1q}{m} \right) \frac{qu + 3}{q} \leq u + \frac{C_1q}{m} + \frac{3}{q} + \frac{3C_2}{m}. \quad (3.14)$$

Inserting (3.14) and (3.10) in (3.9) we find that

$$P(U/m < u) \leq u + \frac{C_2q}{m} + \frac{3}{q} + \frac{3C_2}{m} + \frac{C_1q}{m}$$

$$= u + \frac{3C_2}{m} + f(q).$$

Minimizing with respect to all possible denominators $q = q_n(k/m)$ we arrive at

$$P(U/m < u) - u \leq \frac{3C_2}{m} + \inf_{n \geq 1} f \left( q_n \left( \frac{k}{m} \right) \right).$$

The analogous lower bound $P(U/m < u) \geq u - (3C_2/m) - f(q)$ is derived along the same lines.
Theorem 2 Assume that the joint distribution of $X$ and $Y$ satisfies conditions (3.7) and (3.8) and that

$$P(Y = k) \leq C_0/m, \quad k = 0, \ldots, m - 1.$$ \hspace{1cm} (3.15)

for some constant $C_0$. Then there is a constant $C$ depending only on $C_0, C_1, C_2$ such that

$$\sup_{0 \leq u \leq 1} |P(U/m < u) - u| \leq C \left( \log \frac{m}{m} \right)^{1/2}.$$ \hspace{1cm} (3.16)

Proof. By the formula of total probability and Lemma 4, we obtain

$$P(U/m < u) = \sum_{k=0}^{m-1} P(Y = k) P(U/m < u | Y = k) \leq u + 3C_2m^{-1} + \sum_{k=0}^{m-1} P(Y = k) \min_{n \geq 1} f \left( q_n \left( \frac{k}{m} \right) \right)$$

$$= u + 3C_2m^{-1} + E \left( \min_{n \geq 1} f \left( q_n \left( \frac{Y}{m} \right) \right) \right).$$ \hspace{1cm} (3.17)

Note that the right side of (3.17) is equal to $\int_0^1 (1 - G(x)) dx$, where

$$G(x) = P \left( \min_{n \geq 1} f \left( q_n \left( \frac{Y}{m} \right) \right) < x \right).$$

Let $C_3 = C_1 + C_2$. The function $f(t) = 3t^{-1} + C_3m^{-1}t$, $t > 0$, is strictly convex, has the unique minimum $t_0 = (3m/C_3)^{1/2}$ and $x_0 = f(t_0) = 2t_0^{-1}$. Thus the equation $f(t) = x$ has no solution for $x < x_0$ and exactly two solutions $t_1(x) < t_2(x)$ for $x > x_0$. If $x > x_0$, a short calculation yields

$$f(6/x) = f(mx/2C_3) = \frac{x}{2} + \frac{6C_3}{mx} < x,$$

and consequently $t_1(x) < 6/x < mx/2C_3 < t_2(x)$. These observations show that

$$G(x) = P(t_1(x) < q_n(Y/m) < t_2(x) \text{ for some } n \in \mathbb{N})$$

$$\geq P(6/x < q_n(Y/m) < mx/2C_3 \text{ for some } n \in \mathbb{N})$$

$$= P(Y/m \in B(6/x, \ mx/2C_3)).$$ \hspace{1cm} (3.18)

From (3.15) and Lemma 3 it now follows that

$$1 - G(x) \leq H(x) + m^{-1}, \quad x \in (0, 1]$$

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where the function $H$ is defined by

$$H(x) = \frac{2C_3}{mx} + 2C_0 \left( \frac{(6/x)^2 m^{-1}}{m x^2} + \frac{12C_3}{m x^2 - 12C_3} \right) (1 + 2 \log_2 (6/x)), \quad x > x_0.$$  

Thus, for any $y \in (x_0, 1]$ we have the following estimate:

$$E(\min[1, f(qn(Y/m))]) = \int_0^1 (1 - G(x)) \, dx \leq y + \int_y^1 H(x) \, dx.$$  

On $(x_0, \infty)$ the function $H(x)$ is positive and strictly decreasing from infinity at zero. Further,

$$H(x) \geq 2 \left( \frac{36}{mx^2} + \frac{12C_3}{mx^2} \right) (1 + 2 \log_2 (6/x)) \geq 12 \cdot \frac{48}{mx^2}, \quad x \in (x_0, 1]$$  

as $C_0 \geq 1$ and $C_3 \geq 1$. Let $x_1$ be the solution of $H(x) = 1$ in $(x_0, \infty)$. For sufficiently large $m$ we have $x_1 < 1$ and then, by (3.20),

$$x_1 \geq \max[12(C_3/m)^{1/2}, (576/m)^{1/2}].$$  

Hence if $x_1 \leq x \leq 1$, $H(x)$ can be bounded as follows:

$$H(x) \leq \frac{2C_3}{mx} + 2C_0 \left( \frac{36}{mx^2} + \frac{12C_3}{mx^2 (1 - (12C_3/m x_1^2))} \right) (1 + \log_2 (36/x_1^2))$$

$$\leq \frac{2C_3}{mx} + \frac{2C_0}{mx^2} \left( 36 + \frac{144}{11} C_3 \right) (1 + \log_2 (36m/576))$$

$$\leq \frac{2C_3}{mx} + \frac{2C_0}{mx^2} (36 + 14C_3) (\log_2 m - 3).$$  

For any $y \in [x_1, 1]$ we now find that

$$y + \int_y^1 H(x) \, dx \leq y + \frac{2C_3}{my} + \frac{2C_0 (36 + 14C_3)(\log_2 m - 3)}{my}.  \quad (3.21)$$  

Over $y \in (0, \infty)$ the right-hand side of (3.21) is minimized for

$$y_0 = [2C_3 + 2C_0 (36 + 14C_3)(\log_2 m - 3)]^{1/2} m^{-1/2},$$  

the corresponding minimum being equal to $2y_0$. A short calculation shows that $H(y_0) \to (9 + 3C_3)/(9 + 4C_3) < 1$, as $m \to \infty$. Thus, $y_0 > x_1$ for sufficiently large $m$. Hence we may insert the value $y_0$ in (3.21) for all but finitely many $m$. To summarize, it is now proved that

$$P(U/m < u) \leq u + C \sqrt{\frac{\log m}{m}}.$$  

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for some constant $C$ depending only on $C_0, C_1,$ and $C_2$. Similarly it can be shown that $P(U/m < u) \geq u - C((\log m)/m)^{1/2}$.

References