

## A dual theory for decision under risk and ambiguity

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# A dual theory for decision under risk and ambiguity\*

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## Abstract

This paper axiomatizes, in a two-stage setup, a new theory for decision under risk and ambiguity. The axiomatized preference relation  $\succeq$  on the space  $\tilde{V}$  of random variables induces an ambiguity index  $c$  on the space  $\Delta$  of probabilities on the states of the world and a probability weighting function  $\psi$ , generating the measure  $\nu_\psi$  by distorting an objective probability measure, such that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}$ ,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q \in \Delta} \left\{ \mathbb{E}_Q \left[ \int \tilde{v} d\nu_\psi \right] + c(Q) \right\} \geq \min_{Q \in \Delta} \left\{ \mathbb{E}_Q \left[ \int \tilde{u} d\nu_\psi \right] + c(Q) \right\}.$$

Our theory is *dual* to the theory of variational preferences introduced by [Maccheroni, Marinacci and Rustichini \(2006\)](#), in the same way as the theory of [Yaari \(1987\)](#) is dual to expected utility of [Von Neumann and Morgenstern \(1944\)](#). As a special case, we obtain a preference axiomatization of a decision theory that is dual to the popular maxmin expected utility theory of [Gilboa and Schmeidler \(1989\)](#). We characterize risk and ambiguity aversion in our theory.

**Keywords:** Risk; ambiguity; dual theory; risk and ambiguity aversion; model uncertainty; robustness; numerical representation; multiple priors; variational and multiplier preferences.

**AMS 2000 Subject Classification:** 91B06, 91B16.

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# 1 Introduction

The distinction between risk (probabilities given) and ambiguity (probabilities unknown), after Keynes (1921) and Knight (1921), has become a central aspect in decision-making under uncertainty. Already since Ellsberg (1961) the importance of this distinction had been apparent: while in the classical subjective expected utility (SEU) model of Savage (1954) the distinction between risk and ambiguity was nullified through the assignment of subjective probabilities (Ramsey, 1931, de Finetti, 1931), the Ellsberg (1961) paradox showed experimentally that decisions under ambiguity could not be reconciled with any such assignment of subjective probabilities. It took, however, until the 1980s before decision models were developed that could account for ambiguity without the assignment of subjective probabilities.

Among the most popular models for decision under ambiguity today are maxmin expected utility (MEU, Gilboa and Schmeidler, 1989), also called multiple priors, and Choquet expected utility (CEU, Schmeidler, 1986, 1989). The former model is a decision-theoretic foundation of the classical decision rule of Wald (1950) in (robust) statistics; see also Huber (1981). More recently, Maccheroni, Marinacci and Rustichini (2006) axiomatized the broad and appealing class of variational preferences (VP), which includes MEU and the multiplier preferences of Hansen and Sargent (2000, 2001) as special cases. Multiplier preferences have been widely used in macroeconomics, to achieve “robustness” in settings featuring model uncertainty.

In the Anscombe and Aumann (1963) setting, all the aforementioned decision models reduce to the classical Von Neumann and Morgenstern (1944) expected utility (EU) model under risk, a property that is undesirable from a descriptive perspective: it means, for example, that the Allais (1953) paradox and the common ratio and common consequence effects are still present under risk; see e.g., Machina (1987). Furthermore, Machina (2009) shows that decision problems in the style of Ellsberg (1961) lead to similar paradoxes for CEU as for SEU, arising from event-separability properties that CEU retains in part from SEU.

In this paper, we introduce and axiomatize, in a two-stage setup similar to, but essentially different from, the Anscombe and Aumann (1963) setting, a new theory for decision under risk and ambiguity. As we will explicate, our theory is *dual* to VP of Maccheroni, Marinacci and Rustichini (2006), in the same way as the theory of Yaari (1987) is dual to EU for decision under risk. Thus, this paper may be viewed as an extension of the dual theory (DT) of Yaari (1987) for decision under risk to settings involving risk *and* ambiguity, just like the theory of Maccheroni, Marinacci and Rustichini (2006) is a significant extension to risk and ambiguity of the EU model for risk. As a special case, we obtain a preference axiomatization of a decision theory that is dual to the popular MEU model of Gilboa and Schmeidler (1989). See Table 1.

Table 1: Primal and Dual Decision Theories

	<i>Primal</i>	<i>Dual</i>
Risk	EU (vNM, 1944)	DT (Yaari, 1987)
Risk and ambiguity	MEU (GS, 1989)	<b>This paper</b>
Risk and ambiguity	VP (MMR, 2006)	<b>This paper</b>

The development of the DT of choice under risk of Yaari (1987) was methodologically motivated by the fact that, under EU, the decision-maker’s (DM’s) attitude towards wealth, as represented by the utility function, completely dictates the attitude towards risk. However, attitude towards wealth and attitude towards risk should arguably be treated separately, since

they are “horses of different colors” (Yaari, 1987). This is achieved within the DT model, in which attitude towards wealth and attitude towards risk are disentangled. From an empirical perspective, the DT model naturally rationalizes various behavioral patterns that are inconsistent with EU, such as, perhaps most noticeably, the Allais (1953) paradox and the common ratio effect. DT serves as a building block in more general theories for decision under risk such as rank-dependent utility of Quiggin (1982) and prospect theory of Tversky and Kahneman (1992).

Similarly, our results are both of theoretical (methodological) interest and potentially also empirically relevant. At the theoretical level, our theory separates attitude towards wealth from attitude towards risk *and* attitude towards ambiguity. We also characterize notions of ambiguity and risk aversion in our decision model. From an empirical perspective, an important and distinctive feature of our theory is that it accounts for ambiguity and yet does not collapse to EU under risk, as would be the case if the Anscombe and Aumann (1963) setting would apply, hence is not subject to the objective phenomena of the Allais paradox and the common ratio effect. Inevitably, our theory has of course its own paradoxes, some of which may be rationalized by VP.

The numerical representation of the decision theory we axiomatize induces that the DM considers, for each random variable to be evaluated in the face of risk and ambiguity, a set of potential probabilistic models rather than a single probabilistic model. In recent years, we have seen increasing interest in optimization, macroeconomics, finance, and other fields to account for the possibility that an adopted probabilistic model is an approximation to the true probabilistic model and may be misspecified. Models that explicitly recognize potential misspecification provide a “robust” approach. With the MEU model, one assigns the same plausibility to each probabilistic model in a set of probabilistic models under consideration. The multiplicity of the set of probabilistic models then reflects the degree of ambiguity. The VP model significantly generalizes the MEU model by allowing to attach a plausibility (or ambiguity) index to each probabilistic model. This plausibility index also appears in the numerical representation of our decision model.

More specifically, our numerical representation  $U$  of the preference relation  $\succeq$  on the space of random variables  $\tilde{V}$  takes the form

$$U(\tilde{v}) = \min_{Q \in \Delta} \left\{ \mathbb{E}_Q \left[ \int \tilde{v} d\nu_\psi \right] + c(Q) \right\}, \quad \tilde{v} \in \tilde{V}, \quad (1.1)$$

with  $\Delta$  a set of probabilities on the states of the world,  $c : \Delta \rightarrow [0, \infty]$  the ambiguity index,  $\psi : [0, 1] \rightarrow [0, 1]$  a probability weighting function, and  $\nu_\psi$  an objective measure distorted according to  $\psi$ . Special cases of interest occur when we specify the ambiguity index as the well-known relative entropy (or Kullback-Leibler divergence), or, more generally, as a  $\phi$ -divergence measure (Csiszár, 1975, Ben-Tal, 1985, and Laeven and Stadje, 2013), or simply as an indicator function that takes the value zero on a subset of  $\Delta$  and  $\infty$  otherwise. It is directly apparent from (1.1) that in the absence of uncertainty about the state of the world (i.e., in the case of risk) our decision model reduces to the DT of Yaari (1987). The familiar probability weighting function determines the attitude towards risk. In our general model, we will characterize ambiguity and risk aversion in terms of properties of the ambiguity index and the probability weighting function.

Contrary to the linearity in probabilities that appears in EU, the DT of Yaari (1987), to which our decision model (1.1) reduces under risk, features linearity in wealth. Yaari (1987)

suggests the behavior of a profit maximizing firm as a prime example in which linearity in wealth seems particularly suitable. Other theories that stipulate linearity in wealth, and which can be viewed to occur as special cases of our general theory, are provided by convex measures of risk (Föllmer and Schied, 2004, Chapter 4), encompassing many classical insurance premium principles, and by robust expectations (see e.g., Riedel, 2009, and the references therein). Despite the popularity of these theories, neither linearity in wealth nor linearity in probabilities as in EU is considered fully *empirically* viable for individual decision-making. Instead a more general decision theory for risk and ambiguity in which preferences under risk are represented by a more general measure on the wealth-probability plane, such as rank-dependent utility (Quiggin, 1982) or prospect theory (Tversky and Kahneman, 1992), would be more realistic from a descriptive perspective. Because our theory is dual to VP just like DT is dual to EU, and because DT is a building block for rank-dependent utility and prospect theory, our results may be viewed as a necessary building block for the future development of such general decision theories for risk *and* ambiguity.

In essence, our axiomatization is based on a modification of two axioms stipulated by Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006). First, we postulate a type of ambiguity aversion (Axiom A6 below) with respect to convex combinations of random variables rather than with respect to probabilistic mixtures of lotteries. Consider two ambiguous random variables between which the DM is indifferent. Then our axiom requires that the DM prefers a fraction of the first ambiguous random variable and a remaining fraction of the second (such that the fractions sum to unity) over either one in full. This constitutes a preference for *diversification* induced by convex combinations of ambiguous random variables. In the primal theories of Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006), ambiguity aversion instead takes the form of a preference for *randomization*. Translated to our setting of preferences over random variables, this requires that the DM prefers receiving two random variables between which he is indifferent with probabilities  $p$  and  $1 - p$ ,  $0 < p < 1$ , over getting one of them with certainty.

Second, we replace the (weak) certainty independence axiom of Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006) by a comonotonic type of independence axiom (Axioms A7 and A7<sup>0</sup> below) that pertains to addition of random variables instead of probabilistic mixtures. As is well-known, the independence axiom and its various alternatives are key to obtaining, and empirically verifying, preference representations. Our approach is based on “dual independence” as in Yaari (1987). However, in our general setting that allows for a set of probabilistic models, the implications of comonotonicity, and its interplay with ambiguity, need to be reconsidered: while preferences over random variables may well be invariant to the addition of comonotonic random variables when probability distributions are given (i.e., under risk) as stipulated by Yaari (1987), this implication seems no longer appropriate under ambiguity, essentially because such addition may impact the “level” of ambiguity (see Ex. 3.1).

Therefore, we postulate the following two versions of the dual independence axiom to extend the DT to a setting featuring risk and ambiguity: (i) preferences over random variables are invariant to the addition of a comonotonic random variable with an objectively given probability distribution (Axiom A7); or (ii) preferences over random variables are invariant to convex combinations of random variables and a comonotonic random variable with an objectively given probability distribution (Axiom A7<sup>0</sup>). The former yields a decision theory that is dual to VP, the latter yields a decision theory that is dual to MEU. The mathematical details in the proofs of our characterization results are delicate.

This paper is organized as follows. In Section 2, we introduce our setting and notation. In Section 3, we review some preliminaries, introduce our new axioms, and state our main representation results. Section 4 explicates that our theory is dual to the theory of [Maccheroni, Marinacci and Rustichini \(2006\)](#). Section 5 characterizes ambiguity and risk aversion in our theory. Section 6 axiomatizes, as a special case, the dual theory of [Gilboa and Schmeidler \(1989\)](#). Proofs of our main results are relegated to the Appendix.

## 2 Setup

We adopt a two-stage setup as in the [Anscombe and Aumann \(1963\)](#) approach, with a state space that is a product space, allowing a two-stage decomposition. Different from the [Anscombe and Aumann \(1963\)](#) model, however, we don't assume nor induce EU for risk. Our theory defines a preference relation over random variables just like [Yaari \(1987\)](#) for risk by which our notation is inspired. We now formalize our two-stage setup.

Consider a possibly infinite set  $W$  of states of the world with  $\sigma$ -algebra  $\Sigma'$  of subsets of  $W$  that are events and, similarly, consider a possibly infinite set  $S$  of outcomes with  $\sigma$ -algebra  $\Sigma$  of subsets of  $S$  that are events. Their product space is  $W \times S$ . We suppose that:

- (a) Every (fixed)  $w \in W$  induces a probability measure  $P^w$  on  $(S, \Sigma)$ .
- (b) For every  $A \in \Sigma$ , the mapping  $w \mapsto P^w[A]$  is  $\Sigma'$ -measurable.
- (c) For every  $w \in W$ ,  $P^w$  is non-atomic, i.e., there exists a random variable  $U^w$  on  $S$  that is uniformly distributed on the unit interval under  $P^w$ .<sup>1</sup>

The well-known interpretation of such a two-stage approach is that the objective probability with which outcome  $s$  occurs depends on the state of the world  $w$ , which is subject to resolution of the first-stage uncertainty. Each  $w \in W$  induces a different state of the world with different associated probabilities for certain events  $A \in \Sigma$ .

Let  $\tilde{V}$  be the space of all bounded random variables  $\tilde{v}$  defined on the space  $(W \times S, \Sigma' \otimes \Sigma)$ , i.e.,  $\tilde{v}$  is a mapping from  $W \times S$  to  $\mathbb{R}$ . Similar to [Yaari \(1987\)](#), realizations of the random variables in  $\tilde{V}$  will be viewed as payments denominated by some monetary units. The random variable  $\tilde{v}^w : S \rightarrow \mathbb{R}$  is viewed as the outcome  $s$  contingent payment that the DM receives if he lives in state of the world  $w$ . This makes  $\tilde{v}^w$  also interpretable as a roulette lottery, in neo-Bayesian nomenclature. Henceforth,  $\tilde{v}^w$  and its associated roulette lottery are often identified.<sup>2</sup> We denote by  $\tilde{V}_0$  the subspace of all random variables in  $\tilde{V}$  that take only finitely many values.

For fixed  $w \in W$ , we define the conditional cumulative distribution function (CDF)  $F_{\tilde{v}}(w, t)$  of the  $\Sigma$ -measurable random variable  $\tilde{v}^w$ , given by  $s \mapsto \tilde{v}^w(s)$ , by  $F_{\tilde{v}}(w, t) = P^w[\tilde{v}^w \leq t]$ . (We sometimes omit the dependence on  $t$ , i.e., we sometimes write  $F_{\tilde{v}}(w)$ .) From the assumptions above it follows that for every  $t \in \mathbb{R}$ ,  $F_{\tilde{v}}(\cdot, t)$  is  $\Sigma'$ -measurable. Formally:

**Definition 2.1** *We call a function  $F : W \times \mathbb{R} \rightarrow [0, 1]$  a conditional CDF if, for all  $w \in W$ ,  $F(w)$  is a CDF and, for every  $t \in \mathbb{R}$ ,  $F(\cdot, t)$  is  $\Sigma'$ -measurable.*

<sup>1</sup>The dual theory needs a richness assumption as in (c) to guarantee that all the probability distributions considered can be generated.

<sup>2</sup>We have assumed here that in every state of the world  $w$  the possible outcomes are the same. This can simply be achieved by adding additional outcomes with associated probability zero to each state of the world. Only the probabilities with which certain events  $A \in \Sigma$  occur differ.



Every  $\tilde{v} \in \tilde{V}$  induces a conditional CDF and hence can be identified with a horse (race) lottery, in neo-Bayesian nomenclature. Furthermore, for fixed  $w$ , every horse lottery  $f$  given by  $w \mapsto \mu^w$ , for (roulette) lotteries  $\mu^w$  defined on the probability space  $(S, \Sigma, P^w)$ , induces a CDF  $F(w)$ . Let  $q(w)$  be the left-continuous inverse of  $F(w)$ , i.e.,

$$q(w, \lambda) = \inf\{t \in \mathbb{R} | F(w, t) \geq \lambda\}, \quad \lambda \in (0, 1).$$

Then we can define  $\tilde{v}^w(s) = q(w, U^w(s))$  and it is easy to see that, for every  $w \in W$ ,  $\tilde{v}^w$  has the same probability distribution as  $f(w)$ . Hence, there is a one-to-one correspondence between equivalence classes of random variables  $\tilde{v} \in \tilde{V}$  with the same conditional distributions, and horse lotteries  $f$ .

For some  $\tilde{v} \in \tilde{V}$  (e.g., those that represent payoffs from games such as flipping coins) the DM may actually know the objective probability distribution. As in [Anscombe and Aumann \(1963\)](#), [Gilboa and Schmeidler \(1989\)](#) and [Maccheroni, Marinacci and Rustichini \(2006\)](#) the associated so-called objective lotteries will play a special role for our theory. In this case, the probability distribution of  $\tilde{v}$  does not depend on  $w$ , i.e., for all  $w_1, w_2 \in W$ ,  $F_{\tilde{v}}(w_1) = F_{\tilde{v}}(w_2)$ . We denote the corresponding space of all these random variables in  $\tilde{V}$  and  $\tilde{V}_0$  that carry no ambiguity by  $V$  and  $V_0$ , respectively. For random variables in  $v \in V$  we usually omit the  $w$ , i.e., we just write  $F_v(t)$  instead of  $F_v(w, t)$ . In the space  $V$ ,  $v_n$  converges in distribution to  $v$  if  $F_{v_n}$  converges to  $F_v$  for all continuity points of  $F_v$ .

Furthermore, let  $V'$  be the space defined by

$$V' = \left\{ \tilde{v} \in \tilde{V} \mid \tilde{v} \text{ is independent of } s \in S, \right. \\ \left. \text{i.e., for } s_1, s_2 \in S : \tilde{v}^w(s_1) = \tilde{v}^w(s_2) \right\}.$$

Clearly, the space  $V'$  of all random variables in  $\tilde{V}$  that carry no risk may be identified with the space of bounded measurable functions on  $(W, \Sigma')$ .  $V'_0$  is defined as the corresponding subspace of bounded measurable functions that take only finitely many values.

Let  $\Delta(W, \Sigma')$  be the space of all finitely additive measures on  $(W, \Sigma')$  with mass one. Denote by  $\Delta_\sigma(W, \Sigma')$  the space of all probability measures on  $(W, \Sigma')$ .

### 3 Representation

#### 3.1 Preliminaries

We define a preference relation  $\succeq$  on  $\tilde{V}_0$ . As usual,  $\succ$  stands for strict preference and  $\sim$  for indifference. The preference relation  $\succeq$  on  $\tilde{V}_0$  induces a preference order, also denoted by  $\succeq$ , over random variables  $s \mapsto \tilde{v}^w(s)$  through those random variables in  $\tilde{V}_0$  that are associated with objective lotteries (i.e., are in  $V_0$ ) by defining  $\tilde{v}^w \succeq \tilde{u}^w$  iff  $v \succeq u$  with  $v, u \in V_0$  and  $F_v(t) = F_{\tilde{v}}(w, t)$ ,  $F_u(t) = F_{\tilde{u}}(w, t)$  for all  $t \in \mathbb{R}$ . (This, in turn, induces, similarly, a preference relation over monetary payments.) We suppose that  $\succeq$  satisfies the following properties:

*AXIOM A1–Weak and Non-Degenerate Order:*  $\succeq$  is complete, transitive, and non-degenerate. That is:

- (a)  $\tilde{v} \succeq \tilde{u}$  or  $\tilde{u} \succeq \tilde{v}$  for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ .
- (b) If  $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$ ,  $\tilde{v} \succeq \tilde{u}$  and  $\tilde{u} \succeq \tilde{r}$ , then  $\tilde{v} \succeq \tilde{r}$ .



(c) There exist  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  such that  $\tilde{v} \succ \tilde{u}$ .

While [Yaari \(1987\)](#) assumes that the preference relation is complete on the space of all  $\tilde{v} \in \tilde{V}$  including those taking infinitely many values, we only assume in A1(a) that  $\succeq$  is complete on the space of all  $\tilde{v} \in \tilde{V}_0$  that take finitely many values. We will see later that  $\succeq$  and our representation results may be uniquely extended to the entire space  $\tilde{V}$ . For the dual theory of [Yaari \(1987\)](#) without the completeness axiom, see [Maccheroni \(2004\)](#).

*AXIOM A2–Neutrality:* Let  $\tilde{v}$  and  $\tilde{u}$  be in  $\tilde{V}_0$  and have the same conditional CDF's,  $F_{\tilde{v}}$  and  $F_{\tilde{u}}$ . Then  $\tilde{v} \sim \tilde{u}$ .

Axiom A2 states that  $\succeq$  depends only on the (finite-valued) conditional distributions. In particular,  $\succeq$  induces a preference relation on the space of all (finite-valued) conditional CDF's by defining  $F(\succeq)G$  if and only if there exist two random variables  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  such that  $\tilde{v} \succeq \tilde{u}$  and  $F_{\tilde{v}}(w) = F(w)$ ,  $G_{\tilde{u}}(w) = G(w)$  for all  $w \in W$ . To simplify notation, we will henceforth use  $\succeq$  for preferences over random variables and for preferences over conditional CDF's.

*AXIOM A3–Continuity:* For every  $v, u \in V_0$  such that  $v \succ u$ , and uniformly bounded sequences  $v_n$  and  $u_n$  converging in distribution to  $v$  and  $u$ , there exists an  $n$  from which onwards  $v_n \succ u$  and  $v \succ u_n$ . Furthermore, for every  $\tilde{v} \in \tilde{V}_0$ , the sets

$$\left\{ m \in \mathbb{R} \mid m \succ \tilde{v} \right\} \quad \text{and} \quad \left\{ m \in \mathbb{R} \mid \tilde{v} \succ m \right\}$$

are open.

When restricted to  $V_0$ , our continuity condition A3 is equivalent to the one employed in [Yaari \(1987\)](#). It is a little stronger than that of [Maccheroni, Marinacci and Rustichini \(2006\)](#) or [Gilboa and Schmeidler \(1989\)](#).

*AXIOM A4–Certainty First-Order Stochastic Dominance:* For all  $v, u \in V_0$ : If  $F_v(t) \leq F_u(t)$  for every  $t \in \mathbb{R}$ , then  $v \succeq u$ .

*AXIOM A5–Monotonicity:* For all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ : If  $\tilde{v}^w \succeq \tilde{u}^w$  for every  $w \in W$ , then  $\tilde{v} \succeq \tilde{u}$ .

We postulate Axioms A1-A5 for a preference relation  $\succeq$  defined on the space of finite-valued random variables  $\tilde{V}_0$ . However, by A2, in view of the one-to-one correspondence explicated in the previous section, it is straightforward to verify that this preference relation induces a preference relation, also denoted by  $\succeq$ , on the space of horse lotteries, satisfying the same axioms. Consequently, all axioms considered so far, which will be maintained in our setting, are common; see [Yaari \(1987\)](#), [Schmeidler \(1989\)](#), [Gilboa and Schmeidler \(1989\)](#) and [Maccheroni, Marinacci and Rustichini \(2006\)](#). To (strictly speaking: a subset of) the collection of axioms above, [Gilboa and Schmeidler \(1989\)](#) added the following two axioms:

*AXIOM A6MEU–Uncertainty Aversion:* If  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  and  $\alpha \in (0, 1)$ , then  $F_{\tilde{v}} \sim F_{\tilde{u}}$  implies  $\alpha F_{\tilde{v}} + (1 - \alpha)F_{\tilde{u}} \succeq F_{\tilde{v}}$ .

*AXIOM A7MEU–Certainty Independence:* If  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  and  $v \in V_0$ , then  $F_{\tilde{v}} \succeq F_{\tilde{u}} \Leftrightarrow \alpha F_{\tilde{v}} + (1 - \alpha)F_v \succeq \alpha F_{\tilde{u}} + (1 - \alpha)F_v$  for all  $\alpha \in (0, 1)$ .

With these axioms at hand, one obtains the maxmin expected utility representation, as follows:

**Theorem 3.0.(i) ([Gilboa and Schmeidler, 1989](#))**

A preference relation  $\succeq$  satisfies A1-A5 and A6MEU-A7MEU if, and only if, there exist a

non-decreasing and continuous function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and a non-empty, closed and convex set  $C \subset \Delta(W, \Sigma')$  such that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q \in C} \mathbb{E}_Q \left[ \int \phi(t) F_{\tilde{v}}(\cdot, dt) \right] \geq \min_{Q \in C} \mathbb{E}_Q \left[ \int \phi(t) F_{\tilde{u}}(\cdot, dt) \right]. \quad (3.1)$$

Furthermore,  $\succeq$  has a unique extension to  $\tilde{V}$  which satisfies the same assumptions (over  $\tilde{V}$ ).

More recently, [Maccheroni, Marinacci and Rustichini \(2006\)](#) have obtained a more general representation result, which includes the maxmin expected utility representation of [Gilboa and Schmeidler \(1989\)](#) as a special case, but covers also the multiplier preferences employed in robust macroeconomics; see, for instance, [Hansen and Sargent \(2000, 2001\)](#). If, in the certainty independence axiom A7MEU,  $\alpha$  is close to zero, then  $\alpha F_{\tilde{v}} + (1 - \alpha)F_v$  carries ‘‘almost no ambiguity’’. Hence, if a DM prefers  $\tilde{v}$  over  $\tilde{u}$  (as the axiom presumes), but merely because  $\tilde{v}$  carries less ambiguity than  $\tilde{u}$ , then he may actually prefer  $\alpha F_{\tilde{u}} + (1 - \alpha)F_v$  over  $\alpha F_{\tilde{v}} + (1 - \alpha)F_v$  when  $\alpha$  is small and ambiguity has almost ceased to be an issue. Therefore, [Maccheroni, Marinacci and Rustichini \(2006\)](#) suggest to replace the certainty independence axiom by the following weaker axiom:

*AXIOM A7'MEU-Weak Certainty Independence:* If  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,  $v, u \in V_0$  and  $\alpha \in (0, 1)$ , then  $\alpha F_{\tilde{v}} + (1 - \alpha)F_v \succeq \alpha F_{\tilde{u}} + (1 - \alpha)F_v \Rightarrow \alpha F_{\tilde{v}} + (1 - \alpha)F_u \succeq \alpha F_{\tilde{u}} + (1 - \alpha)F_u$ .

Denote by  $m_{\tilde{v}}$  the certainty equivalent of  $\tilde{v}$ , that is,  $m_{\tilde{v}} \sim \tilde{v}$ ,  $m_{\tilde{v}} \in \mathbb{R}$ . Replacing A7MEU by A7'MEU (*ceteris paribus*) yields the following theorem:

**Theorem 3.0.(ii) ([Maccheroni, Marinacci and Rustichini, 2006](#))**

A preference relation  $\succeq$  satisfies A1-A5 and A6MEU-A7'MEU if, and only if, there exist a non-decreasing and continuous function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  and a grounded<sup>3</sup>, convex and lower-semicontinuous function  $c : \Delta(W, \Sigma') \rightarrow [0, \infty]$  such that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,

$$\begin{aligned} \tilde{v} \succeq \tilde{u} &\Leftrightarrow \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \phi(t) F_{\tilde{v}}(\cdot, dt) \right] + c(Q) \right\} \\ &\geq \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \phi(t) F_{\tilde{u}}(\cdot, dt) \right] + c(Q) \right\}. \end{aligned}$$

Furthermore, there exists a unique minimal  $c_0$  given by

$$c_0(Q) = \sup_{v' \in V'_0} \{m_{v'} - \mathbb{E}_Q[v']\}.$$

### 3.2 New Axioms

We replace the uncertainty aversion axiom of [Gilboa and Schmeidler \(1989\)](#) and [Maccheroni, Marinacci and Rustichini \(2006\)](#) (Axiom A6MEU) by the following assumption:

*AXIOM A6-Ambiguity-No-Risk Aversion:* If  $v', u' \in V'_0$  and  $\alpha \in (0, 1)$ , then  $v' \sim u'$  implies  $\alpha v' + (1 - \alpha)u' \succeq v'$ .

Observe that, different from Axiom A6MEU (and Axioms A7MEU and A7'MEU), Axiom A6 considers convex combinations of random variables rather than mixtures of conditional

<sup>3</sup>We say that  $c$  is grounded if  $\min_{Q \in \Delta(W, \Sigma')} c(Q) = 0$ .

CDF's. We refer to this axiom as the ambiguity-no-risk aversion axiom: the DM takes convex combinations of random variables that carry no risk (are in  $V_0'$ ) and that he is indifferent to. An ambiguity-no-risk averse DM, then, prefers the “diversified” combination of random variables  $(\alpha v' + (1 - \alpha)u')$  over the original non-diversified random variable ( $v'$  or  $u'$ ). Further discussion of Axiom A6 is deferred to Section 5.

Subsequently, we say that two random variables  $\tilde{v}, \tilde{r} \in \tilde{V}$  are *comonotonic* if, for every  $w \in W$  and every  $s, s' \in S$ ,

$$(\tilde{v}^w(s') - \tilde{v}^w(s))(\tilde{r}^w(s') - \tilde{r}^w(s)) \geq 0.$$

Comonotonic random variables don't provide hedging potential because their realizations move in tandem without generating offsetting possibilities (Schmeidler, 1986, 1989, Yaari, 1987).

Now consider  $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$  and suppose that the DM prefers  $\tilde{v}$  over  $\tilde{u}$ . Is it natural to require that the DM then also prefers  $\tilde{v} + \tilde{r}$  over  $\tilde{u} + \tilde{r}$ , or  $\alpha\tilde{v} + (1 - \alpha)\tilde{r}$  over  $\alpha\tilde{u} + (1 - \alpha)\tilde{r}$  with  $\alpha \in (0, 1)$ , in general (without comonotonicity imposed)? If  $\tilde{u}$  and  $\tilde{r}$  are not comonotonic, then the DM may try to employ  $\tilde{r}$  to hedge against adverse realizations of  $\tilde{u}$ . As a result,  $\tilde{u} + \tilde{r}$  can conceivably be better diversified than  $\tilde{v} + \tilde{r}$  (depending on the joint stochastic nature of  $\tilde{u}, \tilde{r}$  on the one hand and that of  $\tilde{v}, \tilde{r}$  on the other), and the DM may instead prefer  $\tilde{u} + \tilde{r}$  over  $\tilde{v} + \tilde{r}$ .

Yaari (1987), in the context of decision under risk, asserts that a preference of  $v$  over  $u$  induces a preference of  $\alpha v + (1 - \alpha)r$  over  $\alpha u + (1 - \alpha)r$ ,  $\alpha \in (0, 1)$ , in case  $v, r$  and  $u, r$  are *pairwise comonotonic* (pc). (As we will see in Section 6, this assertion also implies (*ceteris paribus*) a preference of  $v + r$  over  $u + r$ .) In particular, Yaari (1987) replaces the independence axiom of EU by the following assumption, restricted to decision under risk:

*AXIOM A7D–Dual Independence:* Let  $v, u, r \in V_0$  and assume that  $v, r$  and  $u, r$  are pc. Then, for every  $\alpha \in (0, 1)$ ,  $v \succeq u \Rightarrow \alpha v + (1 - \alpha)r \succeq \alpha u + (1 - \alpha)r$ .

This yields the following representation theorem:<sup>4</sup>

**Theorem 3.0.(iii) (Yaari, 1987)**

A preference relation  $\succeq$  satisfies A1-A5 on the space  $V_0$  and A7D if, and only if, there exists a non-decreasing and continuous function  $\psi : [0, 1] \rightarrow [0, 1]$  with  $\psi(0) = 0$  and  $\psi(1) = 1$  such that, for all  $v, u \in V_0$ ,

$$v \succeq u \Leftrightarrow \int_{-\infty}^0 (\psi(1 - F_v(t)) - 1)dt + \int_0^{\infty} \psi(1 - F_v(t))dt. \quad (3.2)$$

Yaari (1987) referred to this result as “the dual theory of choice under risk”, because for non-negative random variables bounded by one in  $V_0$ , Axiom A7D can be obtained by replacing probabilistic mixtures of distribution functions in the independence axiom of EU for risk by convex combinations of associated *inverse* distribution functions. The function  $\psi$  may be viewed as the reciprocal of the von Neumann-Morgenstern utility function  $\phi$ . It is applied to distribution functions instead of to corresponding monetary payments. From the numerical representation (3.2), it becomes readily apparent that A7D (jointly with A1-A5) implies that, for all  $v, u \in V_0$ ,

$$v \succeq u \Leftrightarrow \lambda v + m \succeq \lambda u + m, \quad \text{for every } \lambda \geq 0, m \in \mathbb{R}.$$

---

<sup>4</sup>Yaari (1987) proved the theorem under the additional condition that the random variables involved are non-negative and bounded by one. However, we will see later that the theorem also holds on the space of all bounded random variables.

Thus, while the independence axiom of EU entails linearity in probabilities, Axiom A7D entails linearity in monetary payments.

It seems natural and important to consider extending the dual theory of choice under risk to a setting featuring ambiguity, similar to what Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006) have achieved for the EU model under risk. The importance of this extended theory is even more evident in view of the role played by DT in more general decision theories such as those of Quiggin (1982) and Tversky and Kahneman (1992).

But what would be the appropriate version of dual independence in our setting with risk *and* ambiguity to replace the certainty independence axiom (A7MEU) or the weak certainty independence axiom (A7'MEU)? To answer this question, consider  $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$  and suppose that  $\tilde{v} \succeq \tilde{u}$ , as before. Suppose furthermore that  $\tilde{v}, \tilde{r}$  and  $\tilde{u}, \tilde{r}$  are pc. Should a DM then also prefer  $\tilde{v} + \tilde{r}$  over  $\tilde{u} + \tilde{r}$ ? (Note that this is not implied by Axiom A7D, which requires the random variables to live in the space  $V_0$ .) Even though adding  $\tilde{r}$  does, in view of the pc assumption, not induce any discriminatory hedging potential, it may still impact the ambiguity “level”, in a discriminatory manner, leading to a preference reversal.

Consider the following example:

**Example 3.1** Consider two urns,  $A$  and  $B$ , and 50 balls, 25 of which are red and 25 of which are black. Every urn contains 25 balls. The exact number of balls per color in each urn is unknown. Furthermore, consider two urns,  $C$  and  $D$ , and 50 balls, 30 of which are red and 20 of which are black. As for  $A$  and  $B$ , every urn contains 25 balls, but the exact number of balls per color in each urn is unknown.

Denote by  $p_i$  the (unknown) probability of drawing a red ball from urn  $i$ ,  $i \in \{A, B, C\}$ . Draw a random number  $U$  from the unit interval. Consider:

- (i) the random variable  $\tilde{v}$  that represents a payoff of \$100 if  $U \leq p_C$  and 0 otherwise;
- (ii) the random variable  $\tilde{u}$  that represents a payoff of \$100 if  $U \leq p_A$  and 0 otherwise;
- (iii) the random variable  $\tilde{r}$  that represents a payoff of \$100 if  $U \leq p_B$  and 0 otherwise.

Note that  $\tilde{v}, \tilde{r}$  and  $\tilde{u}, \tilde{r}$  are pc. Typically,  $\tilde{v} \succeq \tilde{u}$  because  $30 > 25$ . At the same time, the DM may prefer  $\tilde{u} + \tilde{r}$  over  $\tilde{v} + \tilde{r}$ , because the former combination is, loosely speaking, less ambiguous than the latter combination. In particular, the unknown probability of drawing red from  $A$  is connected (complementary) to the unknown probability of drawing red from  $B$ : with certainty,  $p_A + p_B = 1$ . By contrast, the probability of drawing red from  $B$  (or  $A$ ) is not connected to the probability of drawing red from  $C$ . Mathematically,  $\tilde{u} + \tilde{r}$  yields at least \$100 with probability  $\max\{p_A, 1 - p_A\} \geq 50\%$ , and it yields exactly \$200 with probability  $\min\{p_A, 1 - p_A\} = 1 - \max\{p_A, 1 - p_A\}$ . On the contrary,  $\tilde{v} + \tilde{r}$  has potential realizations of \$0, \$100, and \$200 with unknown probabilities, where no non-trivial upper or lower bounds can be given.  $\nabla$

We will assert that, if  $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$ ,  $\tilde{v} \succeq \tilde{u}$ , and  $\tilde{v}, \tilde{r}$  and  $\tilde{u}, \tilde{r}$  are pc, then the implication  $\tilde{v} + \tilde{r} \succeq \tilde{u} + \tilde{r}$  only holds if  $\tilde{r}$  carries no ambiguity (i.e., is in  $V_0$ ), hence cannot impact the ambiguity level, in a discriminatory manner. This motivates to replace the weak certainty independence axiom by the following assumption:

*AXIOM A7-Weak Certainty Dual Independence:* Let  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  and  $r \in V_0$ . Suppose that  $\tilde{v}, r$  and  $\tilde{u}, r$  are pc. Then  $\tilde{v} \succeq \tilde{u} \Rightarrow \tilde{v} + r \succeq \tilde{u} + r$ .

### 3.3 Main Results

Let  $\psi : [0, 1] \rightarrow [0, 1]$  be a non-decreasing and continuous function satisfying  $\psi(0) = 0$  and  $\psi(1) = 1$ . We refer to  $\psi$  as a distortion or probability weighting function. For  $v \in V$ , we define the distortion measure  $\nu_\psi$  through

$$\int v d\nu_\psi = \int_{-\infty}^0 (\psi(1 - F_v(t)) - 1) dt + \int_0^\infty \psi(1 - F_v(t)) dt.$$

One readily verifies that, for  $a, b \in \mathbb{R}$  with  $a > 0$ ,  $\int (av + b) d\nu_\psi = a \int v d\nu_\psi + b$ . We now state our main result, which provides a representation theorem characterizing a preference relation satisfying Axioms A1-A7:

**Theorem 3.2** *( $\alpha$ ) A preference relation  $\succeq$  satisfies A1-A7 if, and only if, there exist a non-decreasing and continuous function  $\psi : [0, 1] \rightarrow [0, 1]$  with  $\psi(0) = 0$  and  $\psi(1) = 1$  and a grounded, convex and lower-semicontinuous function  $c : \Delta(W, \Sigma') \rightarrow [0, \infty]$  such that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,*

$$\begin{aligned} \tilde{v} \succeq \tilde{u} &\Leftrightarrow \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_\psi \right] + c(Q) \right\} \\ &\geq \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{u}^* d\nu_\psi \right] + c(Q) \right\}. \end{aligned} \quad (3.3)$$

Furthermore, there exists a unique minimal  $c_{\min}$  satisfying (3.3) given by

$$c_{\min}(Q) = \sup_{v' \in V'_0} \left\{ m_{v'} - \mathbb{E}_Q[v'] \right\}.$$

( $\beta$ ) There exists a unique extension of  $\succeq$  to  $\tilde{V}$  satisfying A1-A7 on  $\tilde{V}$  and (3.3).

(Here,  $\tilde{v}^*$  denotes the random variable given by  $s \mapsto \tilde{v}^*(s)$ .) The proof will be deferred to the Appendix.

The numerical representation (3.3) and Lemma I.4 below show that the weak certainty dual independence axiom (A7) is equivalent to Yaari's Axiom A7D when  $\succeq$  is restricted to  $V_0$ . Thus, our theory is a true extension of Yaari (1987) to a setting in which the DM potentially faces ambiguity; our theory reduces to Yaari's theory on the space  $V_0$ . Theorem 3.2 also immediately yields the following corollary, which proves useful in the next section:

**Corollary 3.3** *Suppose that  $\tilde{v}, \tilde{u}$  are pc and  $\tilde{v} \sim \tilde{u}$ . Then, for every  $\alpha \in (0, 1)$ ,  $\alpha\tilde{v} + (1 - \alpha)\tilde{u} \succeq \tilde{u}$ .*

### 3.4 Interpretation

Define  $U$  as the numerical representation in (3.3), i.e.,

$$U(\tilde{v}) = \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_\psi \right] + c(Q) \right\}.$$

A special feature of the dual theory is that, because  $U(U(\tilde{v})) = U(\tilde{v})$ , the value obtained by applying the numerical representation to  $\tilde{v} \in \tilde{V}_0$  is equal to the certainty equivalent of  $\tilde{v}$ . Therefore,  $U$  may also be adopted as primitive rather than as binary preference.

The numerical representation in (3.3) may be given the following interpretation. The function  $c$  is non-negative and grounded, i.e., for every  $Q \in \Delta(W, \Sigma')$ ,  $c(Q) \geq 0$ , and there exists at least one measure  $P' \in \Delta(W, \Sigma')$  such that  $c(P') = 0$ . This measure  $P'$  may be considered to be the DM's approximation, selected from the set of all measures on  $(W, \Sigma')$ . If the DM believes that  $P'$  is a good (reliable) approximation, then he can simply take the ( $P'$ -)expectation over all certainty equivalents of an objective lottery, i.e., calculate  $\mathbb{E}'[\int \tilde{v} \cdot d\nu_\psi]$ , which would correspond to  $c(Q) = \infty$  if  $Q \neq P'$ ,  $Q \in \Delta(W, \Sigma')$ , in (3.3). In many situations, however, the DM may not fully trust his approximation  $P'$  and takes other measures on  $(W, \Sigma')$  into account.

One way to proceed would be to assume a worst case approach and consider  $\min_{Q \in \Delta(W, \Sigma')} \mathbb{E}_Q[\int \tilde{v} \cdot d\nu_\psi] = \min_w \int \tilde{v}^w d\nu_\psi$ , which corresponds to  $c(Q) = 0$  for all  $Q \in \Delta(W, \Sigma')$ . In this case, the DM would consider all measures on  $(W, \Sigma')$  equally plausible. Alternatively, the DM may consider his approximation  $P'$  more plausible than other measures, but still wants to take other measures into account (non-trivially). In this case, he would take the minimum over all measures in  $\Delta(W, \Sigma')$ , and “penalize” every measure  $Q$  not equal to  $P'$  by a penalty  $c(Q)$ . This penalty depends on the degree of plausibility that the DM associates to the measure  $Q$ . The function  $c$  is therefore often referred to as an ambiguity index. Such procedures that explicitly account for the fact that the measure  $P'$  is only an approximation and may deviate from the true measure are often times referred to as robust approaches. They are robust against “malevolent nature”.

In statistics, engineering and optimal control, risk management, and robust macroeconomics, the plausibility of the measure  $Q$  is often expressed by the relative entropy of  $Q$  with respect to the approximation  $P'$ ; see Csiszár (1975), Ben-Tal (1985), Hansen and Sargent (2000, 2001), Maccheroni, Marinacci and Rustichini (2006), Strzalecki (2011a) and Laeven and Stadje (2013, 2014). In our setting, this would lead to  $c(Q) = \theta R(Q|P')$  with  $R(Q|P') = \mathbb{E}_Q \left[ \log \left( \frac{dQ}{dP'} \right) \right]$  and  $\theta$  a non-negative constant. The relative entropy, also referred to as Kullback-Leibler divergence, measures the deviation of  $Q$  from  $P'$  and is zero if and only if  $Q = P'$ . Thus, measures that are close to the approximation  $P'$  are penalized weakly, while measures that deviate strongly from the approximation  $P'$  are penalized strongly. Specifically, for  $v' \in V'_0$ ,

$$\min_{Q \in \Delta(W, \Sigma')} \{ \mathbb{E}_Q[v'] + \theta R(Q|P') \} = -\theta \log \{ \mathbb{E}' [\exp(-v'/\theta)] \}.$$

In general, for  $\tilde{v} \in \tilde{V}_0$ ,

$$\min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v} \cdot d\nu_\psi \right] + c(Q) \right\} = -\theta \log \left( \mathbb{E}' \left[ \exp \left( \frac{\int -\tilde{v} \cdot d\nu_\psi}{\theta} \right) \right] \right).$$

Other ways of penalizing “deviating beliefs” are  $c(Q) = \theta G(Q|P')$  with  $G(Q|P')$  the relative Gini index given by  $\mathbb{E} \left[ \left( \frac{dQ}{dP'} - 1 \right)^2 \right]$ . Again,  $G(Q|P') = 0$  if and only if  $Q = P'$ ;  $G(Q|P')$  measures how much the ratio of  $Q$  and  $P'$  deviates from one; see also Maccheroni, Marinacci, Rustichini and Taboga (2004). Maccheroni, Marinacci and Rustichini (2006) also propose to weight every state of the world  $w$  by a weighting function  $h : W \rightarrow \mathbb{R}_+$  satisfying  $\int_W h(w) P'(dw) = 1$ . The corresponding penalty functions are then given by  $c(Q) = \int_W h(w) \log \left( \frac{dQ}{dP'}(w) \right) Q(dw)$  and  $c(Q) = \int_W h(w) \left( \frac{dQ}{dP'} - 1 \right)^2 Q(dw)$ .<sup>5</sup>

<sup>5</sup>Note that these two penalty functions are not *probabilistically sophisticated* on  $(W, \Sigma')$  unless  $h = 1$ ; see also



## 4 Duality

In this section, we explicate why Theorem 3.2 is a genuine dual version of the representation result obtained by Maccheroni, Marinacci and Rustichini (2006). As in Yaari (1987), we shall refer to  $1 - F(t)$ , with  $F(t)$  a CDF, as a decumulative distribution function (DDF). We suppose throughout this section that  $F$  is supported on the unit interval. The (generalized) inverse of a DDF is a reflected (in  $t=1/2$ ) quantile function,  $F^{-1}(1 - t)$ . In brief, the reason of genuine duality is that the numerical representation obtained in Theorem 3.2 corresponds exactly to the numerical representation obtained by Maccheroni, Marinacci and Rustichini (2006), but with the respective DDF's implicitly appearing in our Axioms A1-A7 and Theorem 3.2 replaced by their inverses. It provides the true analog for Maccheroni, Marinacci and Rustichini (2006) of the dual theory of choice under risk in Yaari (1987) for EU under risk.

To verify, first note that  $\succeq$  also induces a preference relation on the subspace of  $\tilde{V}_0$  that is restricted to all non-negative random variables bounded by one. We will henceforth denote this subspace by  $\tilde{V}_0^{[0,1]}$ . Next, in view of the neutrality axiom (Axiom A2),  $\succeq$  also induces a preference relation on the space of conditional reflected quantile functions (inverse DDF's): given  $\tilde{v} \in \tilde{V}_0^{[0,1]}$  with conditional quantile function  $q_{\tilde{v}}$ , we can define its conditional reflected quantile function  $\tilde{G}_{\tilde{v}}$  by

$$\tilde{G}_{\tilde{v}}(\cdot, t) = q_{\tilde{v}}(\cdot, 1 - t), \quad t \in [0, 1]. \quad (4.1)$$

Now define

$$\tilde{G}_{\tilde{v}}(\succeq)\tilde{G}_{\tilde{u}} \quad \text{if and only if} \quad \tilde{v} \succeq \tilde{u}.$$

With this definition, we have defined a preference relation,  $(\succeq)$ , on the convex space

$$\begin{aligned} \tilde{\Gamma} = \{ & \tilde{G} : W \times [0, 1] \rightarrow [0, 1] \text{ For every fixed } t \in [0, 1], \\ & \tilde{G}(\cdot, t) \text{ is } \Sigma' \text{-measurable. For every fixed } w \in W, \\ & \tilde{G}(w, \cdot) \text{ is a decreasing and right-continuous step function with} \\ & \tilde{G}(w, 1) = 0\}. \end{aligned} \quad (4.2)$$

Indeed, every conditional reflected quantile function is in  $\tilde{\Gamma}$  and for every element  $G \in \tilde{\Gamma}$ , there exists a random variable  $\tilde{v} \in \tilde{V}_0^{[0,1]}$  such that  $G = \tilde{G}_{\tilde{v}}$ .<sup>6</sup> For simplicity, we will henceforth denote the preference relations on the spaces  $\tilde{\Gamma}$  and  $\tilde{V}_0^{[0,1]}$  both by  $\succeq$ , too. We define  $\Gamma$  as the subspace of all elements in  $\tilde{\Gamma}$  that carry no ambiguity, i.e.,

$$\Gamma = \{G \in \tilde{\Gamma} \mid \text{for all } w_1, w_2 \in W : G(w_1, \cdot) = G(w_2, \cdot)\}.$$

By Axioms A1-A7,  $\succeq$  on the space  $\tilde{V}_0^{[0,1]}$  induces a preference relation on  $\tilde{\Gamma}$  that satisfies:

- (i) *Weak and Non-Degenerate Order:*  $\succeq$  is complete, transitive and non-degenerate.

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Section 5.

<sup>6</sup>The latter statement can be verified as follows. Recall that if  $U^*$  is uniformly distributed on the unit interval and  $q_X$  is the quantile function of the random variable  $X$ , then  $q_X(U^*)$  has the same distribution as  $X$ . Thus, if we define the random variable  $\tilde{v} \in \tilde{V}_0^{[0,1]}$  through  $\tilde{v}_{\tilde{G}} = \tilde{G}(w, 1 - U^w)$ , then the conditional reflected quantile function of  $\tilde{v}$  is equal to  $\tilde{G}$ . Furthermore, by neutrality,  $\tilde{G}_1(\succeq)\tilde{G}_2$  if and only if  $\tilde{v}_{\tilde{G}_1} \succeq \tilde{v}_{\tilde{G}_2}$ .



(ii) *Continuity*: For every  $\tilde{G} \in \tilde{\Gamma}$ , the sets

$$\left\{ m \in \mathbb{R} \mid m \succ \tilde{G} \right\} \quad \text{and} \quad \left\{ m \in \mathbb{R} \mid \tilde{G} \succ m \right\}$$

are open. Furthermore, for all  $1 \geq c > 0$ , the restriction of  $\succeq$  to  $\Gamma$  is continuous with respect to the topology of uniformly bounded convergence in distribution.

(iii) *Certainty First-Order Stochastic Dominance*: For all  $G_1, G_2 \in \Gamma$ : If  $G_2(t) \geq G_1(t)$  for every  $t \in [0, 1]$ , then  $G_2 \succeq G_1$ .

(iv) *Monotonicity*: For all  $\tilde{G}_1, \tilde{G}_2 \in \tilde{\Gamma}$ : If  $\tilde{G}_2(w, \cdot) \succeq \tilde{G}_1(w, \cdot)$  for every  $w \in W$ , then  $\tilde{G}_2 \succeq \tilde{G}_1$ .

(v) *Uncertainty Aversion*: If  $\tilde{G}_1, \tilde{G}_2 \in \tilde{\Gamma}$  and  $\alpha \in (0, 1)$ , then  $\tilde{G}_1 \sim \tilde{G}_2$  implies  $\alpha\tilde{G}_1 + (1 - \alpha)\tilde{G}_2 \succeq \tilde{G}_1$ .

(vi) *Weak Certainty Independence*: If  $\tilde{G}_1, \tilde{G}_2 \in \tilde{\Gamma}$ ,  $G_3, G_4 \in \Gamma$  and  $\alpha \in (0, 1)$ , then  $\alpha\tilde{G}_1 + (1 - \alpha)G_3 \succeq \alpha\tilde{G}_2 + (1 - \alpha)G_3 \Rightarrow \alpha\tilde{G}_1 + (1 - \alpha)G_4 \succeq \alpha\tilde{G}_2 + (1 - \alpha)G_4$ .

**Remark 4.1** *To see that these properties are indeed satisfied by  $\succeq$  on  $\tilde{\Gamma}$ , note that (i), (ii), (iii) and (iv) for  $\succeq$  on  $\tilde{\Gamma}$  follow directly from the corresponding axioms for  $\succeq$  on  $\tilde{V}_0$ . To see (v) and (vi), first observe that, for  $\tilde{G}_1, \tilde{G}_2 \in \tilde{\Gamma}$ ,*

$$\tilde{v}_{\alpha\tilde{G}_1 + (1-\alpha)\tilde{G}_2} = \alpha\tilde{v}_{\tilde{G}_1} + (1 - \alpha)\tilde{v}_{\tilde{G}_2}. \quad (4.3)$$

Now let us prove that  $\succeq$  on  $\tilde{\Gamma}$  satisfies (v). Let  $\tilde{G}_1, \tilde{G}_2 \in \tilde{\Gamma}$  be such that  $\tilde{G}_1 \sim \tilde{G}_2$ . By definition,  $\tilde{v}_{\tilde{G}_1} \sim \tilde{v}_{\tilde{G}_2}$ . Since  $\tilde{v}_{\tilde{G}_1}$  and  $\tilde{v}_{\tilde{G}_2}$  are comonotonic, (4.3) and Corollary 3.3 imply that

$$\tilde{v}_{\alpha\tilde{G}_1 + (1-\alpha)\tilde{G}_2} = \alpha\tilde{v}_{\tilde{G}_1} + (1 - \alpha)\tilde{v}_{\tilde{G}_2} \succeq \tilde{v}_{\tilde{G}_1}.$$

Thus,  $\alpha\tilde{G}_1 + (1 - \alpha)\tilde{G}_2 \succeq \tilde{G}_1$ . This proves (v).

Finally, we prove (vi). If  $\alpha\tilde{G}_1 + (1 - \alpha)G_3 \succeq \alpha\tilde{G}_2 + (1 - \alpha)G_3$ , then  $\alpha\tilde{v}_{\tilde{G}_1} + (1 - \alpha)\tilde{v}_{G_3} \succeq \alpha\tilde{v}_{\tilde{G}_2} + (1 - \alpha)\tilde{v}_{G_3}$ . First, we prove that this implies that  $\alpha\tilde{v}_{\tilde{G}_1} \succeq \alpha\tilde{v}_{\tilde{G}_2}$ . Suppose that  $\alpha\tilde{v}_{\tilde{G}_2} \succ \alpha\tilde{v}_{\tilde{G}_1}$ . Then, by A7,

$$\alpha\tilde{v}_{\tilde{G}_2} + (1 - \alpha)\tilde{v}_{G_3} \succ \alpha\tilde{v}_{\tilde{G}_1} + (1 - \alpha)\tilde{v}_{G_3},$$

which is a contradiction. Thus, indeed  $\alpha\tilde{v}_{\tilde{G}_1} \succeq \alpha\tilde{v}_{\tilde{G}_2}$ . Next, by A7, this entails that

$$\tilde{v}_{\alpha\tilde{G}_1 + (1-\alpha)G_4} = \alpha\tilde{v}_{\tilde{G}_1} + (1 - \alpha)\tilde{v}_{G_4} \succeq \alpha\tilde{v}_{\tilde{G}_2} + (1 - \alpha)\tilde{v}_{G_4} = \tilde{v}_{\alpha\tilde{G}_2 + (1-\alpha)G_4}.$$

Hence,  $\alpha\tilde{G}_1 + (1 - \alpha)G_4 \succeq \alpha\tilde{G}_2 + (1 - \alpha)G_4$ .

As  $\tilde{\Gamma}$  is composed of decreasing and right-continuous step functions that map from  $[0, 1]$  to  $[0, 1]$  for fixed  $w \in W$  and are zero at one,  $\tilde{\Gamma}$  may also be regarded as the space of conditional DDF's associated with random variables that take only finitely many values on the unit interval. This is the final key to establishing duality. Hence, we can apply the main result of [Maccheroni, Marinacci and Rustichini \(2006\)](#) to obtain a numerical representation of the preference relation

satisfying (i) to (vi) above. This entails that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0^{[0,1]}$ , the numerical representation  $U$  coincides with a representation  $\bar{U}$  on  $\tilde{\Gamma}$  given by

$$\begin{aligned}\bar{U}(\tilde{G}_{\tilde{v}}) &= \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ - \int_0^1 \psi(t) \tilde{G}_{\tilde{v}}(\cdot, dt) \right] + c(Q) \right\} \\ &= \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int_0^1 \psi(\tilde{G}_{\tilde{v}}^{-1}(\cdot, t)) dt \right] + c(Q) \right\} \\ &= \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int_0^1 \psi(1 - F_{\tilde{v}}(\cdot, t)) dt \right] + c(Q) \right\},\end{aligned}$$

for some non-decreasing and continuous function  $\psi$  on the unit interval, unique up to positive affine transformations and normalized such that  $\psi(0) = 0$  and  $\psi(1) = 1$ . (Note that, because  $\tilde{G}_{\tilde{v}}(\cdot, t) = q_{\tilde{v}}(\cdot, 1 - t)$ , we have  $\tilde{G}_{\tilde{v}}^{-1}(\cdot, t) = (1 - F_{\tilde{v}}(\cdot, t))$ .)

## 5 Ambiguity and Risk Aversion

### 5.1 Further Discussion of Axiom A6

We refer to  $P'$  as a *reference measure* on  $(W, \Sigma')$  if the DM is indifferent between random variables that have the same probability distribution under  $P'$ . We say that a DM who adopts a reference measure on  $(W, \Sigma')$  is *probabilistically sophisticated*; see [Machina and Schmeidler \(1992\)](#) and [Epstein \(1999\)](#). Our axioms do not necessarily imply the existence of a reference measure on  $(W, \Sigma')$ . But in case there is a reference measure  $P'$  on  $(W, \Sigma')$ , we define, for a given  $v' \in V'_0$ ,  $F'_{v'}$  by  $F'_{v'}(t) = P'[v' \leq t]$ . With slight abuse of notation we say that  $v' \succeq_2 u'$  if, for every  $t \in \mathbb{R}$ ,  $\int_{-\infty}^t F'_{v'}(\tau) d\tau \leq \int_{-\infty}^t F'_{u'}(\tau) d\tau$ . We call  $\succeq_2$  *second order stochastic dominance* (SSD) on  $V'$  with respect to  $P'$ ; see [Rothschild and Stiglitz \(1970\)](#). The following proposition shows that, if a reference measure  $P'$  is available, postulating that the DM respects Axiom A6 is equivalent to requiring that the DM respects SSD on  $V'_0$ .<sup>7</sup>

**Proposition 5.1** *Suppose that a preference relation  $\succeq$  satisfies A1-A5 and that there exists a reference measure  $P'$  on  $(W, \Sigma')$ . Then the following statements are equivalent:*

- (a)  $\succeq$  respects A6;
- (b)  $\succeq$  respects SSD on  $V'_0$  with respect to  $P'$ .

*Proof.* One may verify that Axioms A1-A3 are already sufficient to guarantee the existence of a numerical representation of  $\succeq$ , denoted by  $U : \tilde{V}_0 \rightarrow \mathbb{R}$ . By Axioms A4-A5,  $U$  is monotonic in the sense that, if  $v' \geq u'$   $P'$ -a.s., then  $v' \succeq u'$  and hence  $U(v') \geq U(u')$ . The proposition now follows from Proposition 2.1 in [Dana \(2005\)](#).  $\square$

Hence, we refer to Axiom A6 as “Ambiguity-No-Risk Aversion”; see also the following subsection.

<sup>7</sup>The preference relation  $\succeq$  respects SSD on  $V'_0$  if, for all  $v', u' \in V'_0$  with  $v' \succeq_2 u'$ ,  $v' \succeq u'$ .

## 5.2 Ambiguity Aversion

Subsequently, we say that  $\succeq$  is *strongly more ambiguity averse* than  $\succeq^*$  if, for all  $\tilde{v} \in \tilde{V}_0$  and  $v \in V_0$ ,

$$\tilde{v} \succeq v \Rightarrow \tilde{v} \succeq^* v.$$

We also introduce the notion of *weakly more ambiguity averse* with which one can separate being more risk averse from being more ambiguity averse in our setting, as we will see in Propositions 5.2-5.6 that follow. We say that  $\succeq$  is weakly more ambiguity averse than  $\succeq^*$  if, for all  $v' \in V'_0$  and  $m \in \mathbb{R}$ ,

$$v' \succeq m \Rightarrow v' \succeq^* m.$$

Similar definitions of *comparative ambiguity aversion* can be found, for instance, in Epstein (1999), Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006); see also the early Yaari (1969), Schmeidler (1989) and Gilboa and Schmeidler (1989).<sup>8</sup> Our notion of strongly more ambiguity averse agrees with the comparative ambiguity aversion concept in Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006).

The following result completely characterizes our two notions of comparative ambiguity aversion in the setting of our main representation result:

**Proposition 5.2** *Consider two preference relations,  $\succeq$  and  $\succeq^*$ , induced by assuming Axioms A1-A7. Then,*

(i)  $\succeq$  is weakly more ambiguity averse than  $\succeq^*$  if, and only if,  $c^* \geq c$ .

(ii)  $\succeq$  is strongly more ambiguity averse than  $\succeq^*$  if, and only if,  $\psi^* = \psi$  and  $c^* \geq c$ .

*Proof.* (i): If  $c^* \geq c$ , then  $v' \succeq m$ , with  $v' \in V'_0$  and  $m \in \mathbb{R}$ , implies that

$$m \leq \min_{Q \in \Delta(W, \Sigma')} \{ \mathbb{E}_Q[v'] + c_{\min}(Q) \} \leq \min_{Q \in \Delta(W, \Sigma')} \{ \mathbb{E}_Q[v'] + c^*(Q) \}.$$

This proves the “if” part. To prove the “only if” part, suppose that  $\succeq$  is weakly more ambiguity averse than  $\succeq^*$ . Then, for every  $v' \in V'_0$ , the certainty equivalent  $m_{v'}$  under  $\succeq$  is smaller than the corresponding certainty equivalent  $m_{v'}^*$  under  $\succeq^*$ . But then, for every  $Q \in \Delta(W, \Sigma')$ ,

$$c_{\min}^*(Q) = \sup_{v' \in V'_0} \{ m_{v'}^* - \mathbb{E}_Q[v'] \} \geq \sup_{v' \in V'_0} \{ m_{v'} - \mathbb{E}_Q[v'] \} = c_{\min}(Q).$$

(ii) If  $\succeq^*$  is strongly more ambiguity averse than  $\succeq$ , then  $\succeq^*$  is also weakly more ambiguity averse than  $\succeq$ . As we have seen in (i), this implies that  $c^* \geq c$ . Furthermore, because the preferences relations necessarily agree on  $V_0$ , we also have  $\psi^* = \psi$ . This proves the “only if”

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<sup>8</sup> The difference between definitions of uncertainty aversion consists primarily in the “factorization” of ambiguity attitude and risk attitude. Schmeidler (1989) and Gilboa and Schmeidler (1989) adopt the Anscombe-Aumann framework with objective unambiguous lotteries. Epstein (1999), by contrast, instead of adopting a two-stage setup and assuming that there exists a space of objective lotteries, models ambiguity by assuming that there exists a set of events  $\mathcal{A}$  that every DM considers to be unambiguous. Then he defines comparative ambiguity aversion through the random variables that are measurable with respect to  $\mathcal{A}$ . The model-free factorization approach of Ghirardato and Marinacci (2002) in principle encompasses both approaches to modeling ambiguity.

part. To prove the “if” part, suppose that  $c^* \geq c$  and  $\psi^* = \psi$ . Then  $\tilde{v} \succeq v$ , with  $\tilde{v} \in \tilde{V}_0$  and  $v \in V_0$ , entails that

$$\begin{aligned} m_v &\leq \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_\psi \right] + c_{\min}(Q) \right\} \\ &\leq \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_{\psi^*} \right] + c_{\min}^*(Q) \right\}. \end{aligned}$$

□

We note that (i) and (ii) of Proposition 5.2 hold similarly in the primal framework of Maccheroni, Marinacci and Rustichini (2006), with the probability weighting function  $\psi$  in (ii) replaced by the utility function  $\phi$ .

In Epstein (1999), Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006) a DM is considered to be *ambiguity averse* if and only if he is more ambiguity averse than an *ambiguity neutral* DM. While in Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006) ambiguity neutrality is equivalent to having SEU preferences, Epstein (1999) identifies ambiguity neutrality with probabilistic sophistication. Ghirardato and Marinacci (2002), however, argue that in full generality (unless the probability space is rich enough) probabilistically sophisticated behavior may still include behavior that can be considered to be ambiguity averse.<sup>9</sup> Consequently, in our setting, instead of identifying ambiguity neutrality ( $\succeq^{\text{AN}}$ ) with probabilistic sophistication, it seems more natural to define  $\succeq^{\text{AN}}$  via a numerical representation that induces computing a plain expectation on the space  $W$  with respect to some measure  $P'$ . In other words, we consider a DM to be *ambiguity neutral* if there exist a measure  $P'$  and a distortion function  $\psi$  such that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \mathbb{E}_{P'} \left[ \int \tilde{v}^* d\nu_\psi \right] \geq \mathbb{E}_{P'} \left[ \int \tilde{u}^* d\nu_\psi \right].$$

Next, we say that a DM with a preference relation  $\succeq$  is (*strongly* or *weakly*) *ambiguity averse* if there exists an ambiguity neutral DM with a preference relation  $\succeq^{\text{AN}}$  such that  $\succeq$  is (strongly or weakly) more ambiguity averse than  $\succeq^{\text{AN}}$ .

**Proposition 5.3** *If  $\succeq$  satisfies A1-A7, then  $\succeq$  is strongly (hence weakly) ambiguity averse.*

*Proof.* Assume Axioms A1-A7. By Theorem 3.2, there exist functions  $\psi$  and  $c$  such that (3.3) hold. Set  $P' = \arg \min_Q c_{\min}(Q)$ . Because  $c_{\min}$  is grounded,  $c_{\min}(P') = 0$ . Denote by  $\succeq^{\text{AN}}$  the ambiguity neutral agent with measure  $P'$  and distortion function  $\psi$ . Suppose that  $\tilde{v} \succeq v$ ,

<sup>9</sup> For instance, if  $W$  has only finitely many elements, identifying ambiguity neutrality with probabilistic sophistication would imply that also a DM with a numerical representation of the form  $U(\tilde{v}) = \min_{Q \in \Delta(W, \Sigma')} \{ \mathbb{E}_Q[\int \tilde{v} d\nu_\psi] \} = \inf_w \int \tilde{v}^w d\nu_\psi$  would be ambiguity neutral, at least, if  $P'$  does not exclude any  $w \in W$ . (That is,  $P'[w] > 0$  for all  $w \in W$ .) This seems counterintuitive, since the “worst ambiguity case” possible is assumed. A worst case DM is also probabilistically sophisticated if  $W$  is a subset of  $\mathbb{R}^d$  and  $P' \sim \text{Leb}$ .  $W \subset \mathbb{R}^d$  is typically satisfied in a Bayesian framework. Strzalecki (2011b), however, proves that in the specific framework of Maccheroni, Marinacci and Rustichini (2006), ambiguity neutrality in the sense of Epstein (1999), with non-trivial no-ambiguity sets, implies that the DM has preferences given by SEU. Marinacci (2002) had proven the same result under MEU.

$\tilde{v} \in \tilde{V}_0$  and  $v \in V_0$ . Then,

$$\begin{aligned} m_v &\leq \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v} \cdot d\nu_\psi \right] + c_{\min}(Q) \right\} \\ &\leq \mathbb{E}_{P'} \left[ \int \tilde{v} \cdot d\nu_\psi \right] + c_{\min}(P') = \mathbb{E}_{P'} \left[ \int \tilde{v} \cdot d\nu_\psi \right]. \end{aligned}$$

□

### 5.3 Risk Aversion

Suppose that  $\succeq$  satisfies the following property:

*AXIOM A8–Risk Aversion:* If  $v, u \in V_0$  and  $\alpha \in (0, 1)$ , then  $v \sim u$  implies  $\alpha v + (1-\alpha)u \succeq v$ .

Note the similarity between risk aversion (Axiom A8) and ambiguity-no-risk aversion (Axiom A6). The mere difference between A8 and A6 is that risk aversion is defined on the space of random variables that carry no ambiguity ( $V_0$ ) while ambiguity-no-risk aversion is defined on the space of random variables that carry no risk ( $V'_0$ ). Using these spaces we will be able to completely separate risk aversion from ambiguity aversion, as the following proposition (jointly with Propositions 5.2 and 5.3) makes precise:

**Proposition 5.4** *Suppose that  $\succeq$  satisfies A1-A7. Then the following statements are equivalent.<sup>10</sup>*

- (i)  $\succeq$  satisfies A8;
- (ii)  $\succeq$  respects SSD on  $V_0$ ;
- (iii) the function  $\psi$  in Theorem 3.2 is convex.

*Proof.* (ii)  $\Leftrightarrow$  (iii) follows from Theorem 2 in Yaari (1987); see also Yaari (1986). (i)  $\Leftrightarrow$  (ii) follows from Proposition 2.1 in Dana (2005). □

It follows from Proposition 5.4 that if  $\succeq$  is risk averse, then, for all  $v \in V_0$ , the certainty equivalent is smaller than the expectation, i.e.,  $e(v) = \int_0^1 q_v(t) dt \succeq m_v$ , because  $e(v) \succeq_2 v \sim m_v$ . This property ( $e(v) \geq m_v$ ) is sometimes called *weak risk aversion* and property (ii) of Proposition 5.4 is sometimes referred to as *strong risk aversion*. (As is well-known, these two notions of risk aversion agree under EU.)

Note that if A1-A8 hold, then the expectation under the measure  $P'$  will always be preferred to  $\tilde{v}$ . This is true because

$$\begin{aligned} \mathbb{E}'[e(\tilde{v}^w)] &= \mathbb{E}' \left[ \int_0^1 (1 - F_{\tilde{v}}(\cdot, t)) dt \right] \geq \mathbb{E}' \left[ \int_0^1 \psi(1 - F_{\tilde{v}}(\cdot, t)) dt \right] \\ &\geq \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int_0^1 \psi(1 - F_{\tilde{v}}(\cdot, t)) dt \right] + c(Q) \right\} \sim \tilde{v}. \end{aligned}$$

Now consider two preference relations,  $\succeq$  and  $\succeq^*$ . We say that  $\succeq$  is *more risk averse* than  $\succeq^*$  if, for all  $v \in V_0$  and  $m \in \mathbb{R}$ ,

$$v \succeq m \Rightarrow v \succeq^* m.$$

<sup>10</sup>We say that  $v \succeq_2 u$ ,  $v, u \in V_0$ , if, for every  $t \in \mathbb{R}$ ,  $\int_{-\infty}^t F_v(\tau) d\tau \leq \int_{-\infty}^t F_u(\tau) d\tau$ . We call  $\succeq_2$  SSD on  $V$ . The preference relation  $\succeq$  respects SSD on  $V_0$  if, for all  $v, u \in V_0$  with  $v \succeq_2 u$ ,  $v \succeq u$ .

The following two results characterize comparative risk aversion in our setting. They extend results of Yaari (1986, 1987), Roëll (1987) and Chew, Karni and Safra (1985) to our setting.

**Proposition 5.5** *Suppose that  $\succeq$  and  $\succeq^*$  both satisfy A1-A7. Then the following statements are equivalent:*

- (i)  $\succeq$  is more risk averse than  $\succeq^*$ ;
- (ii)  $\psi \leq \psi^*$ .

*Proof.* Suppose that  $\psi \leq \psi^*$ . Then,  $v \succeq m$ , with  $v \in V_0$ , implies that

$$\begin{aligned} m &\leq \int_{-\infty}^0 (\psi(1 - F_v(t)) - 1)dt + \int_0^{\infty} \psi(1 - F_v(t))dt \\ &\leq \int_{-\infty}^0 (\psi^*(1 - F_v(t)) - 1)dt + \int_0^{\infty} \psi^*(1 - F_v(t))dt. \end{aligned}$$

In particular,  $v \succeq^* m$ . This proves (ii) $\Rightarrow$ (i). To prove (i) $\Rightarrow$ (ii), suppose that there exists a  $p_0 \in [0, 1]$  such that  $\psi(p_0) > \psi^*(p_0)$ . Let  $v$  be the random variable that pays off 1 with probability  $p_0$  and 0 else. Then,  $v \sim \psi(p_0)$ . But because  $\psi(p_0) > \psi^*(p_0) \sim^* v$ , the constant amount  $\psi(p_0)$  is preferred over  $v$  by the DM with preference relation  $\succeq^*$ . Hence, we get a contradiction.  $\square$

**Proposition 5.6** *Suppose that  $\succeq$  and  $\succeq^*$  both satisfy A1-A8. Then the following statements are equivalent:*

- (a)  $\succeq$  is more risk averse than  $\succeq^*$ ;
- (b) There exists a convex function  $f : [0, 1] \rightarrow [0, 1]$  with  $f(0) = 0$  and  $f(1) = 1$  such that  $f \circ \psi^* = \psi$ .

*Proof.* Suppose (b) is true. Then,  $f(x) \leq x$ . In particular,  $\psi \leq \psi^*$  and by Proposition 5.5 it follows that  $\succeq$  is more risk averse than  $\succeq^*$ . This proves (b) $\Rightarrow$ (a). To prove (a) $\Rightarrow$ (b), suppose that  $\succeq$  is more risk averse than  $\succeq^*$ . Then, Proposition 5.5 implies that  $\psi \leq \psi^*$ . Furthermore, by Proposition 5.4,  $\psi$  and  $\psi^*$  are both convex, hence there exists a function  $f$  with the stated properties.  $\square$

## 6 A Dual Gilboa-Schmeidler Representation

In this section, we replace (*ceteris paribus*) Axiom A7 by the following stronger (i.e., more restrictive) assumption:

*AXIOM A7<sup>0</sup>-Certainty Dual Independence:* Let  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  and  $r \in V_0$ . Suppose that  $\tilde{v}, r$  and  $\tilde{u}, r$  are pc. Then  $\tilde{v} \succeq \tilde{u} \Leftrightarrow \alpha \tilde{v} + (1 - \alpha)r \succeq \alpha \tilde{u} + (1 - \alpha)r$  for all  $\alpha \in (0, 1)$ .

As we will see, Axiom A7<sup>0</sup> relates via duality to Axiom A7MEU in the same way as Axiom A7 relates to Axiom A7'MEU. Furthermore, Axiom A7<sup>0</sup> can readily be seen to correspond to Yaari's Axiom A7D, with the difference that A7<sup>0</sup> allows  $\tilde{v}$  and  $\tilde{u}$  to be in  $\tilde{V}_0$ , while A7D assumes all random variables to be in  $V_0$ . At the same time, Lemma I.4 shows that Axiom A7, when restricted to  $V_0$ , corresponds to Axiom A7D. The following two results explicate the difference between Axioms A7 and A7<sup>0</sup>:

**Lemma 6.1**  $A7^0$  implies that, for  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,  $\tilde{v} \succeq \tilde{u}$  if and only if  $\lambda\tilde{v} \succeq \lambda\tilde{u}$  for every  $\lambda \geq 0$ .

*Proof.* The proof of the “if” part is straightforward. Let us prove the “only if” part. So, suppose that  $\tilde{v} \succeq \tilde{u}$ . If  $\lambda \in [0, 1]$ , then  $\lambda\tilde{v} \succeq \lambda\tilde{u}$  follows directly from Axiom  $A7^0$  with  $\alpha = \lambda$  and  $r = 0$ . If  $\lambda > 1$ , then let us suppose that  $\lambda\tilde{u} \succ \lambda\tilde{v}$  would hold. Defining  $\alpha = \frac{1}{\lambda} \in (0, 1)$  yields, by  $A7^0$ ,

$$\tilde{u} = \alpha\lambda\tilde{u} + (1 - \alpha)0 \succ \alpha\lambda\tilde{v} + (1 - \alpha)0 = \tilde{v},$$

which is a contradiction. Hence, indeed  $\lambda\tilde{v} \succeq \lambda\tilde{u}$  for every  $\lambda \geq 0$ .  $\square$

Thus, Axiom  $A7^0$  implies that the preference relation is scale invariant on  $\tilde{V}_0$ , while Axiom  $A7$  only implies scale invariance on  $V_0$  (formally, via Lemma I.4). The next proposition shows explicitly that Axiom  $A7^0$  is stronger than Axiom  $A7$ :

**Proposition 6.2** *Axiom  $A7^0$  implies Axiom  $A7$ .*

*Proof.* Suppose that  $\tilde{v} \succeq \tilde{u}$  and that  $\tilde{v}, \tilde{u}$  and  $r$  are pc. Let  $\alpha \in (0, 1)$ . Then, by Lemma 6.1, under Axiom  $A7^0$ ,  $\frac{1}{\alpha}\tilde{v} \succeq \frac{1}{\alpha}\tilde{u}$ . Next, let  $\bar{r} = \frac{r}{1-\alpha}$ . Then, we obtain from Axiom  $A7^0$  that

$$\tilde{v} + r = \alpha \left( \frac{1}{\alpha} \tilde{v} \right) + (1 - \alpha)\bar{r} \succeq \alpha \left( \frac{1}{\alpha} \tilde{u} \right) + (1 - \alpha)\bar{r} = \tilde{u} + r.$$

Hence,  $A7$  is indeed satisfied.  $\square$

The following theorem shows that if Axioms  $A1$ - $A6$  and  $A7^0$  hold, then we obtain the dual analogue of the popular [Gilboa and Schmeidler \(1989\)](#) maxmin expected utility representation:

**Theorem 6.3** (a) *A preference relation  $\succeq$  satisfies  $A1$ - $A6$  and  $A7^0$  if, and only if, there exist a non-decreasing and continuous function  $\psi : [0, 1] \rightarrow [0, 1]$  with  $\psi(0) = 0$  and  $\psi(1) = 1$  and a convex set  $\mathcal{Q} \subset \Delta(W, \Sigma')$  such that, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,*

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{v} \cdot d\nu_\psi \right] \geq \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{u} \cdot d\nu_\psi \right]. \quad (6.1)$$

*Furthermore, there exists a unique extension of  $\succeq$  to  $\tilde{V}$  satisfying  $A1$ - $A6$  and  $A7^0$  on  $\tilde{V}$  and (6.1).*

(b) *If moreover the numerical representation in (6.1) is continuous from below, then  $\mathcal{Q} \subset \Delta_\sigma(W, \Sigma')$ , i.e., the minimum may be taken over a convex set of probability measures.*

The proof will be deferred to the Appendix.



## I Appendix: Proofs of the Main Results

*Proof of Theorem 3.2.* Our proof does not rely on formalizing the duality relation to the main characterization result of [Maccheroni, Marinacci and Rustichini \(2006\)](#) as explicated in Section 4, but provides a direct construction and insightful derivation of our main representation theorem. The only property that is not straightforward to verify in the “if” part of Theorem 3.2( $\alpha$ ) is the continuity property (Axiom A3). Let  $U$  be the numerical representation in (3.3). This implies that, for all  $v \in V_0$ ,

$$U(v) = \int_{-\infty}^0 (\psi(1 - F_v(t)) - 1)dt + \int_0^{\infty} \psi(1 - F_v(t))dt.$$

The first part of Axiom A3 would follow if we could show that  $U$  is continuous with respect to weak convergence of uniformly bounded sequences. So, suppose that  $v_n$  is a uniformly bounded sequence in  $V_0$ , and  $v_n \rightarrow v$ , in distribution. Then, by definition,  $F_{v_n}$  converges to  $F_v$  at all continuity points of  $F_v$ . Because  $F_v$  and the  $F_{v_n}$ 's are non-decreasing functions, they are continuous, Lebesgue almost everywhere. But this implies that  $F_{v_n}$  converges to  $F_v$ , Lebesgue almost everywhere. Furthermore, because  $v_n$  is uniformly bounded by a constant, say  $M$ ,  $F_{v_n}(t) \in \{0, 1\}$  for  $t \notin [-M, M]$ . In view of the point-wise convergence of  $F_{v_n}$  to  $F_v$ , Lebesgue almost everywhere, this implies that  $F_v(t) \in \{0, 1\}$  for  $t \notin [-M, M]$ , as well. Finally, because  $\psi$  is a continuous function, it is bounded on  $[0, 1]$ . Hence,

$$\begin{aligned} \lim_n U(v_n) &= \lim_n \int_{-\infty}^0 (\psi(1 - F_{v_n}(t)) - 1)dt + \int_0^{\infty} \psi(1 - F_{v_n}(t))dt \\ &= \lim_n \int_{-M}^0 (\psi(1 - F_{v_n}(t)) - 1)dt + \int_0^M \psi(1 - F_{v_n}(t))dt \\ &= \int_{-M}^0 (\psi(1 - F_v(t)) - 1)dt + \int_0^M \psi(1 - F_v(t))dt \\ &= U(v), \end{aligned}$$

as desired. Proving the second part of Axiom A3 is straightforward and will be omitted, as is the verification of Axioms A1-A2 and A4-A7.

The proof of the “only if” part of Theorem 3.2( $\alpha$ ) consists of the following four steps:

1. We show first that  $\succeq$  has a numerical representation  $U$  on  $\tilde{V}_0$  satisfying certain properties.
2. Next, we prove that, for all  $v \in V_0$ ,

$$U(v) = \int v d\nu_\psi.$$

3. Then, we show that, for all  $v' \in V'_0$ ,

$$U(v') = \min_{Q \in \Delta(W, \Sigma')} \{ \mathbb{E}_Q[v'] + c(Q) \}.$$

4. Finally, we derive from Steps 2 and 3 that (3.3) holds on  $\tilde{V}_0$ .

Before proceeding to Step 1, we state the following preliminary lemmata, assuming Axioms A1-A7 hold:

**Lemma I.1** *Let  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  and  $m \in \mathbb{R}$ . If  $\tilde{v} \succ \tilde{u}$  and  $\tilde{v}, \tilde{u}$  are pc, then  $\tilde{v} + m \succ \tilde{u} + m$ .*

*Proof.* By A7,  $\tilde{v} + m \succeq \tilde{u} + m$ . Suppose that  $\tilde{v} + m \sim \tilde{u} + m$  would hold. Then, again by A7,  $\tilde{v} = \tilde{v} + m - m \sim \tilde{u} + m - m = \tilde{u}$ , which is a contradiction.  $\square$

**Lemma I.2** *For every  $\tilde{v} \in \tilde{V}_0$  there exists a certainty equivalent  $m_{\tilde{v}} \in \mathbb{R}$  such that  $\tilde{v} \sim m_{\tilde{v}}$ .*

*Proof.* Suppose that the lemma does not hold. Then, the sets  $\{m \in \mathbb{R} | m \succ \tilde{v}\}$  and  $\{m \in \mathbb{R} | \tilde{v} \succ m\}$  are disjoint, jointly contain the whole real line, and are non-empty, by A4. They must, for example, contain the constants  $\|\tilde{v}\|_\infty + 1$  and  $-\|\tilde{v}\|_\infty - 1$ , respectively.<sup>11</sup> But, by A3, both sets are open, leading to a contradiction. Thus, there exists  $m_{\tilde{v}} \in \mathbb{R}$  such that  $\tilde{v} \sim m_{\tilde{v}}$ .  $\square$

**Lemma I.3** *Let  $x, y \in \mathbb{R}$ . If  $x > y$ , then  $x \succ y$ .*

*Proof.* First, we prove that if  $m > 0$ , then  $m \succ 0$ . By non-degeneracy (Axiom A1), there exist  $\tilde{v}, \tilde{u} \in \tilde{V}_0$  such that  $\tilde{v} \succ \tilde{u}$ , hence, there exist certainty equivalents  $m_{\tilde{v}}, m_{\tilde{u}} \in \mathbb{R}$  such that

$$m_{\tilde{v}} \sim \tilde{v} \succ \tilde{u} \sim m_{\tilde{u}}. \quad (\text{I.1})$$

By A4 and reflexivity (for all  $\tilde{v} \in \tilde{V}_0$ ,  $\tilde{v} \sim \tilde{v}$ , as implied by A1), (I.1) immediately yields that  $m_{\tilde{v}} > m_{\tilde{u}}$ . In view of Lemma I.1, this entails that  $\varepsilon = m_{\tilde{v}} - m_{\tilde{u}} \succ 0$ . By A4, for  $\lambda \geq 1$ ,  $\lambda\varepsilon \succeq \varepsilon \succ 0$ . At the same time, for  $1/2 \leq \lambda < 1$ , if  $0 \succeq \lambda\varepsilon$  would hold, then, again by A4,  $0 \succeq \lambda\varepsilon \succeq (1 - \lambda)\varepsilon \succeq 0$ , so  $\lambda\varepsilon \sim (1 - \lambda)\varepsilon \sim 0$ . But in that case A7 implies that  $\varepsilon = \lambda\varepsilon + (1 - \lambda)\varepsilon \sim 0 + (1 - \lambda)\varepsilon \sim 0$ , which is a contradiction. Hence, for all  $\lambda > 0$ ,  $\lambda\varepsilon \succ 0$ , and therefore, for all  $m > 0$ ,  $m \succ 0$ .

Next, let  $x, y \in \mathbb{R}$  with  $x > y$ . Then, because  $x - y > 0$ ,  $x - y \succ 0$ . By Lemma I.1 this finally entails that  $x \succ y$ .  $\square$

### Step 1:

We prove first that  $\succeq$  has a numerical representation  $U : \tilde{V}_0 \rightarrow \mathbb{R}$ , i.e., for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow U(\tilde{v}) \geq U(\tilde{u}),$$

satisfying the following properties:

- (i) *Conditional Law Invariance:*  $U(\tilde{v})$  depends only on  $F_{\tilde{v}}$ .
- (ii) *Continuity:* Suppose that  $v_n$  is a uniformly bounded sequence in  $V_0$  converging in distribution to  $v$ , then  $\lim_n U(v_n) = U(v)$ .
- (iii) *Certainty First-Order Stochastic Dominance:* For all  $v, u \in V_0$ : If  $F_v(t) \leq F_u(t)$  for every  $t \in \mathbb{R}$ , then  $U(v) \geq U(u)$ .
- (iv) *Monotonicity:* For all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ : If  $U(\tilde{u}^w) \leq U(\tilde{v}^w)$  for every  $w \in W$ , then  $U(\tilde{u}) \leq U(\tilde{v})$ .
- (v) *Translation Invariance:* For all  $\tilde{v} \in \tilde{V}_0$  and  $m \in \mathbb{R}$ ,  $U(\tilde{v} + m) = U(\tilde{v}) + m$ .

<sup>11</sup>We define  $\|\tilde{v}\|_\infty = \sup_{w \in W} \inf\{c | P^w[|\tilde{v}^w| \leq c] = 1\}$ .

- (vi) *Lipschitz Continuity*: For all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,  $|U(\tilde{v}) - U(\tilde{u})| \leq \|\tilde{v} - \tilde{u}\|_\infty$ .
- (vii) *Ambiguity-No-Risk Concavity*: If  $v', u' \in V'_0$  and  $\alpha \in (0, 1)$ , then  $U(\alpha v' + (1 - \alpha)u') \geq \alpha U(v') + (1 - \alpha)U(u')$ .
- (viii) *Certainty Comonotonic Additivity*: Let  $\tilde{v} \in \tilde{V}_0$  and  $r \in V_0$ . Suppose that  $\tilde{v}, r$  are pc. Then  $U(\tilde{v} + r) = U(\tilde{v}) + U(r)$ .
- (ix) *Certainty Positive Homogeneity*: For all  $v \in V_0$  and  $\lambda \geq 0$ ,  $U(\lambda v) = \lambda U(v)$ .

Assume Axioms A1-A7 hold. For  $\tilde{v} \in \tilde{V}_0$ , set  $U(\tilde{v}) = m_{\tilde{v}}$ . By Lemma I.2,  $U$  is well-defined and, by Lemma I.3,  $m_{\tilde{v}}$  is unique. Note that with this definition, for all  $m \in \mathbb{R}$ ,  $U(m) = m$ . Furthermore, it follows from the strict monotonicity on  $\mathbb{R}$  proved in Lemma I.3 that  $U(\tilde{v}) > U(\tilde{u})$  if and only if  $\tilde{v} \succ \tilde{u}$ . Thus,  $U$  is a numerical representation of  $\succeq$ .

Next, let us show that  $U$  satisfies properties (i)-(ix). Properties (i)-(iv) follow directly from Axioms A1-A5 and the fact that  $U$  is a numerical representation of  $\succeq$ . Furthermore, Axiom A7 implies that, for all  $m \in \mathbb{R}$ ,

$$\tilde{v} \sim \tilde{u} \Leftrightarrow \tilde{v} + m \sim \tilde{u} + m. \quad (\text{I.2})$$

We claim that this implies that  $U$  is translation invariant (property (v)). This can be seen as follows. If  $\tilde{v} \sim m_{\tilde{v}}$ , then, by (I.2),  $\tilde{v} + m \sim m_{\tilde{v}} + m$ . But this implies that

$$U(\tilde{v} + m) = m_{\tilde{v}} + m = U(\tilde{v}) + m,$$

as desired. Property (vi) holds because, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0$ ,

$$U(\tilde{v}) \leq U(\tilde{u} + \|\tilde{v} - \tilde{u}\|_\infty) = U(\tilde{u}) + \|\tilde{v} - \tilde{u}\|_\infty,$$

where we used properties (iii)-(iv) in the inequality and property (v) in the equality.

Next, to prove property (vii), let  $\alpha \in (0, 1)$  and let  $v', u' \in V'_0$ . Without loss of generality we may assume that  $v' \succeq u'$ . Thus,  $U(v') \geq U(u')$ . Let  $m = U(v') - U(u') \geq 0$ . Then  $U(u' + m) = U(u') + m = U(v')$ . In particular,  $u' + m \sim v'$ . Hence, A6 yields

$$\begin{aligned} U(\alpha v' + (1 - \alpha)u') &= U(\alpha v' + (1 - \alpha)(u' + m) - (1 - \alpha)m) \\ &= U(\alpha v' + (1 - \alpha)(u' + m)) - (1 - \alpha)m \\ &\geq \alpha U(v') + (1 - \alpha)U(u' + m) - (1 - \alpha)m \\ &= \alpha U(v') + (1 - \alpha)U(u'), \end{aligned}$$

where we used property (v) in the second and in the last equalities. Thus,  $U$  is concave on  $V'_0$ .

To prove property (viii), let  $r \in V_0$  and  $\tilde{v} \in \tilde{V}_0$  be pc. Because  $r \sim m_r$ , it follows from A7 that  $\tilde{v} + r \sim \tilde{v} + m_r$ . Thus,

$$U(\tilde{v} + r) = U(\tilde{v} + m_r) = U(\tilde{v}) + m_r = U(\tilde{v}) + U(r).$$

Hence,  $U$  satisfies (viii).

Finally, to prove property (ix), let  $v \in V_0$  and notice that  $v, v$  is pc. Thus,  $U(2v) = U(v + v) = 2U(v)$ , by property (viii). Iterating this argument yields that  $U(\lambda v) = \lambda U(v)$  for all rational non-negative  $\lambda$ . Now the continuity of  $U$  on  $V_0$  implies that  $U(\lambda v) = \lambda U(v)$  for all  $\lambda \geq 0$ .

**Step 2:**

We prove that there exists a non-decreasing and continuous function  $\psi : [0, 1] \rightarrow [0, 1]$  with  $\psi(0) = 0$  and  $\psi(1) = 1$  such that, for all  $v \in V_0$ ,

$$U(v) = \int_{-\infty}^0 (\psi(1 - F_v(t)) - 1)dt + \int_0^{\infty} \psi(1 - F_v(t))dt. \quad (\text{I.3})$$

We will use definitions and notation introduced in Section 4. The proof of Step 2 consists of the following three parts:

- (a) First, we show that it is sufficient to prove (I.3) for  $v \in V_0$  such that  $v \geq 0$ .
- (b) Next, we show that it is also sufficient to prove (I.3) for  $v \in V_0$  such that  $\|v\|_{\infty} \leq 1$ .
- (c) Then, we show that our Axiom A7, when restricted to  $V_0$ , corresponds to Axiom A7D used by Yaari (1987). Thus, we conclude that (I.3) holds.

Part (a): It is sufficient to prove (I.3) for  $v \in V_0$  such that  $v \geq 0$  because, then, for possibly negative  $v$ ,

$$U(v + \|v\|_{\infty}) - \|v\|_{\infty} = \int (v + \|v\|_{\infty})d\nu_{\psi} - \|v\|_{\infty} = \int v d\nu_{\psi} = U(v).$$

Part (b): Suppose that, for a given  $v \in V_0$  such that  $v \geq 0$ ,  $\sup_s v(s) > 1$  and define  $\lambda = \frac{1}{\sup_s v(s)}$ . Then, in view of positive homogeneity ((ix) of Step 1 above),

$$\lambda U\left(\frac{v}{\lambda}\right) = \lambda \int \frac{v}{\lambda} d\nu_{\psi} = \int v d\nu_{\psi} = U(v).$$

Hence, it is sufficient to prove (I.3) for non-negative  $v \in V_0$  that are bounded by one.

Part (c): We need the following lemma:

**Lemma I.4** *Maintain Axioms A1-A6. On the space  $V_0$ , Axioms A7D and A7 are equivalent, i.e., for  $v, u, r \in V_0$  with  $v, r$  and  $u, r$  pc,*

$$\text{for every } \alpha \in (0, 1), v \succeq u \Rightarrow \alpha v + (1 - \alpha)r \succeq \alpha u + (1 - \alpha)r \quad (\text{I.4})$$

*if, and only if,*

$$v \succeq u \Rightarrow v + r \succeq u + r. \quad (\text{I.5})$$

*Proof.* For every  $\alpha \in (0, 1)$ , upon making the transformation  $r' = \frac{r}{1 - \alpha}$ , we see that (I.4) (with  $r$  replaced by  $r'$ ) is equivalent to

$$v \succeq u \Rightarrow \alpha v + r \succeq \alpha u + r.$$

In particular, if (I.4) holds, then, for every fixed  $r \in V_0$  comonotonic to  $v$  and  $u$ , with  $v, u \in V_0$ , and every  $\alpha \in (0, 1)$ ,

$$v \succeq u \Rightarrow \alpha v + r \succeq \alpha u + r.$$

Next, letting  $\alpha$  approach (converge to) 1 and using the continuity axiom (A3), we arrive at

$$v \succeq u \Rightarrow v + r \succeq u + r.$$

This proves the “only if” part.

To prove the “if” part, suppose that  $v \succeq u$  and that the implication (I.5) holds. Let  $\alpha \in (0, 1)$ . Note that  $\alpha v + (1 - \alpha)r' \succeq \alpha u + (1 - \alpha)r'$  would follow directly from (I.5) if we could show that  $\alpha v \succeq \alpha u$ . But this is an immediate consequence of the fact that  $U$  is a numerical representation of  $\succeq$  on  $V_0$  and satisfies (ix) of Step 1 above.  $\square$

Lemma I.4 implies that, on the space of non-negative random variables in  $V_0$  bounded by one, Axioms A1-A4 and A7D hold. Similar to Yaari (1987), by neutrality (A2), this induces a preference relation on the space of conditional reflected quantile functions,  $\Gamma$ , simply denoted by  $\succeq$ , that satisfies weak and non-degenerate order, continuity, certainty first-order stochastic dominance and the independence axiom;<sup>12</sup> see Section 4 for the exact definitions. Therefore, by the mixture space theorem (Herstein and Milnor, 1953), there exists a non-decreasing and continuous function  $\psi : [0, 1] \rightarrow [0, 1]$  such that the numerical representation is given by

$$U(v) = - \int_0^1 \psi(t) G_v(\cdot, dt) = \int_0^1 \psi(G_v^{-1}(\cdot, t)) dt = \int_0^1 \psi(1 - F_v(\cdot, t)) dt,$$

where  $G_v$  is defined by (4.1). Hence, (I.3) holds. Finally, it is straightforward to verify that  $U(m) = m$  for all  $m \in [0, 1]$  (see Step 1 above) implies that we must have  $\psi(0) = 0$  and  $\psi(1) = 1$ .

#### Steps 3+4:

Recall Step 1. By construction,  $U(0) = 0$ . As  $U$  satisfies (i)-(ix),  $U$  may be identified with a concave and normalized niveloid<sup>13</sup> on the space of bounded,  $\Sigma'$ -measurable functions on  $W$ . Classical duality results in convex analysis for niveloids (see, for instance, Lemma 26 in Maccheroni, Marinacci and Rustichini, 2006 or Theorem 4.15 and Remark 4.16 in Föllmer and Schied, 2004) then yield that, for all  $v' \in V'_0$ ,

$$U(v') = \min_{Q \in \Delta(W, \Sigma')} \{ \mathbb{E}_Q[v'] - c_{\min}(Q) \}, \quad (\text{I.6})$$

with  $c_{\min}$  defined by  $c_{\min}(Q) = \sup_{v' \in V'_0} \{ U(v') - \mathbb{E}_Q[v'] \} \geq U(0) = 0$  and being the unique minimal function satisfying (I.6). As  $U(m) = m$  for all  $m \in \mathbb{R}$ , there exists a  $Q$  such that  $c(Q) < \infty$ . Now we have

$$0 = U(0) = \min_{Q \in \Delta(W, \Sigma')} c(Q).$$

In particular,  $c$  is grounded, convex and lower-semicontinuous.

For  $\tilde{v} \in \tilde{V}_0$ , define  $m_{\tilde{v}^w}$  as the corresponding *certainty equivalent* of  $\tilde{v}$  in the state of the world  $w$ , i.e.,

$$m_{\tilde{v}^w} = U(\tilde{v}^w) = \int \tilde{v}^w d\nu_\psi.$$

Set  $\bar{v}^w = m_{\tilde{v}^w}$ . Clearly,  $\bar{v}$  is independent of  $s$ . Furthermore, by the Theorem of Tornelli,  $\bar{v}$  is  $\Sigma'$ -measurable. In particular,  $\bar{v}$  is in  $V'_0$ . Observe that, for every  $w \in W$ ,  $U(\bar{v}^w) = U(m_{\tilde{v}^w}) = U(U(\tilde{v}^w)) = U(\tilde{v}^w)$ , where we have used in the last equality that, for all  $m \in \mathbb{R}$ ,  $U(m) = m$ .

<sup>12</sup>The independence axiom asserts that if, for DDF's  $G_1, G_2, G_3 \in \Gamma$ ,  $G_1 \succeq G_2$ , then, for every  $\alpha \in (0, 1)$ ,  $\alpha G_1 + (1 - \alpha)G_3 \succeq \alpha G_2 + (1 - \alpha)G_3$ .

<sup>13</sup>The mapping  $U$  from  $V'_0$  to  $\mathbb{R}$  is a concave and normalized niveloid if it is concave, Lipschitz continuous with respect to the  $\|\cdot\|_\infty$ -norm, and satisfies  $U(m) = m$  for all  $m \in \mathbb{R}$ .

Hence, property (iii) of Step 1 implies that  $U(\bar{v}) = U(\tilde{v})$ . This entails that, for all  $\tilde{v} \in \tilde{V}_0$ ,

$$\begin{aligned}
U(\tilde{v}) &= U(\bar{v}) = \min_{Q \in \Delta(\bar{W}, \Sigma')} \{ \mathbb{E}_Q[\bar{v}] + c(Q) \} \\
&= \min_{Q \in \Delta(\bar{W}, \Sigma')} \left\{ \int U(m_{\tilde{v}^w}) Q(dw) + c(Q) \right\} \\
&= \min_{Q \in \Delta(\bar{W}, \Sigma')} \left\{ \int U(\tilde{v}^w) Q(dw) + c(Q) \right\} \\
&= \min_{Q \in \Delta(\bar{W}, \Sigma')} \left\{ \int \left( \int_{-\infty}^0 (\psi(1 - F_{\tilde{v}^w}(t)) - 1) dt \right. \right. \\
&\quad \left. \left. + \int_0^{\infty} \psi(1 - F_{\tilde{v}^w}(t)) dt \right) Q(dw) + c(Q) \right\} \\
&= \min_{Q \in \Delta(\bar{W}, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_\psi \right] + c(Q) \right\},
\end{aligned}$$

where we have used (I.6) in the second and (I.3) in the fifth equalities. This proves the “only if” part of Theorem 3.2(α).

The proof of Theorem 3.2(β) now follows by defining, for all  $\tilde{v}, \tilde{u} \in \tilde{V}$ ,

$$\begin{aligned}
\tilde{v} \succeq \tilde{u} &\Leftrightarrow \min_{Q \in \Delta(\bar{W}, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_\psi \right] + c(Q) \right\} \\
&\geq \min_{Q \in \Delta(\bar{W}, \Sigma')} \left\{ \mathbb{E}_Q \left[ \int \tilde{u}^* d\nu_\psi \right] + c(Q) \right\}. \quad \square
\end{aligned}$$

*Proof of Theorem 6.3.* The “if” part of (a) follows as in the proof of Theorem 3.2. We prove the “only if” part. For brevity and different from the proof of Theorem 3.2, the proof that we provide below is based on formalizing the duality relation to the main characterization result of Gilboa and Schmeidler (1989). The proof consists of the following five steps:

- (i) First, we show that we may assume that  $\tilde{v} \geq 0$ .
- (ii) Next, we show that we may also assume that  $\|\tilde{v}\|_\infty \leq 1$ .
- (iii) Then, we show that our Axioms A1-A6 and A7<sup>0</sup> correspond to the axioms of Gilboa and Schmeidler (1989), but with the roles of DDF’s and reflected quantile functions switched.
- (iv) Step (iii) enables us to employ well-known results to obtain a dual representation.
- (v) Finally, we show that this dual representation corresponds to the representation in (6.1).

**Step (i):**

It is sufficient to restrict attention to  $\tilde{v} \in \tilde{V}_0$  such that  $\tilde{v} \geq 0$ , because if we have proven that the numerical representation in (6.1) holds for non-negative  $\tilde{v}$ , then, in view of the fact that this numerical representation coincides with its certainty equivalent, for possibly negative  $\tilde{v}$ ,

$$\tilde{v} + \|\tilde{v}\|_\infty \sim \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{v}^* + \|\tilde{v}\|_\infty d\nu_\psi \right] = \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{v}^* d\nu_\psi \right] + \|\tilde{v}\|_\infty,$$

which, because of comonotonicity, is equivalent to

$$\tilde{v} \sim \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{v} \cdot d\nu_\psi \right].$$

Consequently, we may assume that  $\tilde{v} \geq 0$ .

**Step (ii):**

It is also sufficient to restrict attention to  $\tilde{v} \in \tilde{V}_0$  such that  $\tilde{v}$  is bounded by one. To see this, suppose that  $\|\tilde{v}\|_\infty > 1$ . Let  $\lambda$  be a constant such that  $0 < \lambda < 1/\|\tilde{v}\|_\infty$ . Then

$$\lambda \tilde{v} \sim \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \lambda \tilde{v} \cdot d\nu_\psi \right] = \lambda \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{v} \cdot d\nu_\psi \right],$$

which, because of Lemma 6.1, is equivalent to

$$\tilde{v} \sim \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int \tilde{v} \cdot d\nu_\psi \right].$$

Consequently, we may assume that  $\tilde{v} \leq 1$ .

**Step (iii):**

As in Section 4,  $\succeq$  also induces a preference relation on the subspace  $\tilde{V}_0^{[0,1]}$ . Furthermore, since  $\succeq$  is conditional law invariant (by Axiom A2),  $\succeq$  also induces a preference relation on the space of reflected quantile functions. Thus, we can define a preference relation, denoted simply by  $\succeq$ , on the space  $\tilde{\Gamma}$  with  $\tilde{\Gamma}$  defined in (4.2).

As in Section 4, by A1-A6 and A7<sup>0</sup>,  $\succeq$  induces a preference relation on the space  $\tilde{\Gamma}$  that satisfies (i)-(v) of Section 4 and

(vi) *Certainty Independence*: If  $\tilde{G}_1, \tilde{G}_2 \in \tilde{\Gamma}$  and  $G_3 \in \Gamma$ , then

$$\tilde{G}_1 \succeq \tilde{G}_2 \Leftrightarrow \alpha \tilde{G}_1 + (1 - \alpha)G_3 \succeq \alpha \tilde{G}_2 + (1 - \alpha)G_3 \text{ for all } \alpha \in (0, 1).$$

To verify (vi), note that if  $\tilde{G}_1 \succeq \tilde{G}_2$ , then A7<sup>0</sup> implies

$$\begin{aligned} \tilde{v}_{\alpha \tilde{G}_1 + (1 - \alpha)G_3} &= \alpha \tilde{G}_1(\cdot, 1 - U^\bullet) + (1 - \alpha)G_3(1 - U^\bullet) \\ &\succeq \alpha \tilde{G}_2(\cdot, 1 - U^\bullet) + (1 - \alpha)G_3(1 - U^\bullet) = \tilde{v}_{\alpha \tilde{G}_2 + (1 - \alpha)G_3}. \end{aligned}$$

**Steps (iv)+(v):**

The final key to establishing a dual representation is that, as in Section 4, because  $\tilde{\Gamma}$  is composed of decreasing and right-continuous step functions that map from  $[0, 1]$  to  $[0, 1]$  for fixed  $w \in W$  and are zero at one,  $\tilde{\Gamma}$  may also be *regarded* as the space of conditional DDF's associated with random variables that take only finitely many values on the unit interval. Thus, for all  $\tilde{v}, \tilde{u} \in \tilde{V}_0^{[0,1]}$ , the numerical representation  $U$  coincides with a representation  $\bar{U}$  on  $\tilde{\Gamma}$ , which, by the representation theorem of Gilboa and Schmeidler (1989) and (3.1) is given by

$$\bar{U}(\tilde{G}_{\tilde{v}}) = \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ - \int_0^1 \psi(t) \tilde{G}_{\tilde{v}}(\cdot, dt) \right],$$



where the function  $\psi$  is non-decreasing and continuous and unique up to positive affine transformations. Because  $\tilde{G}_{\tilde{v}} \succeq \tilde{G}_{\tilde{u}}$  if and only if  $\tilde{v} \succeq \tilde{u}$ , we also obtain a numerical representation  $U$  of  $\succeq$  on  $\tilde{V}_0$ . This yields

$$\begin{aligned} U(\tilde{v}) &= \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ - \int_0^1 \psi(t) \tilde{G}_{\tilde{v}}(\cdot, dt) \right] = \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int_0^1 \psi(\tilde{G}_{\tilde{v}}^{-1}(\cdot, t)) dt \right] \\ &= \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[ \int_0^1 \psi(1 - F_{\tilde{v}}(\cdot, t)) dt \right]. \end{aligned}$$

Finally, the function  $\psi$  can be selected to satisfy  $\psi(0) = 0$  and  $\psi(1) = 1$ . The unique extension of  $\succeq$  to  $\tilde{V}$  follows as in the proof of Theorem 3.2 from continuity. The proof of (b) follows from Föllmer and Schied (2004), Chapter 4.  $\square$

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