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Capacity of weakly \((d, k)\)-constrained sequences

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Abstract — In the presentation we find an analytic expression for the maximum of the normalized entropy
\[- \sum_{i\in T} p_i \ln p_i / \sum_{i\in T} p_i,\]
where the set \(T\) is the disjoint union of sets \(S_n\) of positive integers that are assigned probabilities \(P_n, \sum_n P_n = 1\). This result is applied to the computation of the capacity of weakly \((d, k)\)-constrained sequences that are allowed to violate the \((d, k)\)-constraint with small probability.

I. PROBLEM DESCRIPTION AND RESULTS

Let \(T\) be a set of positive integers, and assume that \(T\) is the disjoint union of a (finite or infinite) number of non-empty sets \(S_n, n \in M\). Also assume that there are given numbers \(P_n \geq 0, n \in M\), with \(\sum_n P_n = 1\). We show the following result.

Theorem: The maximum of
\[ H := - \sum_{i\in T} p_i \ln p_i / \sum_{i\in T} p_i \]  
(in : natural logarithm) under the constraints that \(p_i \geq 0, \sum_{i\in S_n} p_i = P_n, n \in M\), equals \(z_0\), where \(z_0 > 0\) is the unique solution \(z\) of the equation
\[ - \sum_{n \in M} P_n \ln Q_n(z) = - \sum_{n \in M} P_n \ln P_n \]  
with for \(z > 0\)
\[ Q_n(z) := \sum_{i \in S_n} e^{-iz}, n \in M. \]  
Moreover, the optimal \(p_i\) are given by
\[ p_i = \frac{P_n}{Q_n(z_0)} e^{-iz_0}, i \in S_n, n \in M, \]
and for these \(p_i\) we have that
\[ \sum_{i \in T} i p_i = \frac{d}{dz} \left[ - \sum_{n \in M} P_n \ln Q_n(z) \right] (z_0). \]

As an application of this result we consider weakly constrained \((d,k)\) sequences [1]. A binary \((d,k)\)-constrained sequence has by definition at least \(d\) and at most \(k\) 'zeros' between consecutive 'ones'. Weakly constrained codes produce sequences that violate the specified constraints with a small probability. It is argued that if the channel is not free of errors, it is pointless to feed the channel with perfectly constrained sequences. A \((d,k)\)-constrained sequence can be thought to be composed of 'phrases' \(1^i0^j\), \(d \leq i \leq k\), where \(0^j\) means a series of \(j\) 'zeros'. In order to compute the channel capacity, i.e. the maximum \(z_0/\ln 2\) of the entropy \(H/\ln 2\), we define
\[ T = \{1,\ldots,d\} \cup \{d+1,\ldots,k+1\} \]
\[ \cup \{k+2, k+3, \ldots\} =: S_1 \cup S_2 \cup S_3, \]
where \(d = 0, 1, \ldots, k = d+1, d+2, \ldots\) are given, and we compute the capacity for the case that the probabilities \(P_1, P_3\) assigned to the sets \(S_1, S_3\) are both small. Clearly, the quantities \(P_1\) and \(P_3\) denote the probabilities that phrases are transmitted that are either too short or too long, respectively. We find that the familiar capacities of \((d,k)\)-constrained sequences [2] are approached from above as \(P_1, P_3 \to 0\) with an error \(A(P_1 \ln P_1 + P_3 \ln P_3)\), where we can evaluate the \(A\) explicitly. We obtain a similar result for the case that \(T\) is as in (6) with \(S_1, S_3\) merged into a single set \(S_1 \cup S_3\). Further results are published in [3].

Conclusions

We have presented an analytic expression for the maximum of the normalized entropy \(- \sum_{i\in T} p_i \ln p_i / \sum_{i\in T} p_i\) under the condition that \(T\) is the disjoint union of sets \(S_n\) of positive integers that are assigned probabilities \(P_n, \sum_n P_n = 1\). We computed the capacity of weakly \((d,k)\)-constrained sequences that are allowed to violate the \((d,k)\)-constraint with given probability.

References