The Zak transform and some counterexamples in time-frequency analysis
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underflow. From the definition of \( \mathcal{I} \) it follows that

\[
L(s) = \frac{L(s')}{K},
\]

and that the amounts of information will change by \( (1/K)q_i(0) \) in the nodes of \( V_s \), and by zero in the nodes of \( V' \). That is, with schedule \( \mathcal{I} \) the information in the source nodes will be reduced by \( 1/K \) of the amount by which it would be reduced if \( s' \) was applied and the length of the schedule \( \mathcal{I} \) will be \( K \) times less than that of \( s' \). In addition, (2.5) ensures that underflow does not occur during the execution of \( S_{\mathcal{I}} \). Also, since the information in the intermediate nodes (i.e., those of \( V' \)) remains unchanged during the execution of the schedule \( S_{\mathcal{I}} \), that schedule can be repeated. Consider now a schedule that consists of \( K \) repetitions of the schedule \( \mathcal{I} \); obviously this repetition schedule has length equal to \( L(s') \) and transfers all of the information from the source nodes to the destinations; furthermore, it has link activation vector equal to \( f \). What is left is the amount of information \( \delta_i \) that we assumed was residing in each node of \( V' \). Since the only assumption about that information was that it be greater than zero, we can take it to be arbitrarily small. Thus, we can consider that these amounts \( \delta_i \) are transferred to the nodes of \( V' \) from the origin nodes \( V_s \) before we start the execution of \( \mathcal{I} \) and that we transfer them afterwards to the destinations to complete the total evacuation via an arbitrary trivial schedule that has length arbitrarily small (since \( \delta_i \) can be as small as desired) and link activation vector \( f \). Thus, the final schedule \( \mathcal{S} \) is the schedule that consists of the concatenation of the schedule that transfers the initial amounts of information to the nodes of \( V' \) from the source nodes, the \( K \) repetitions of \( \mathcal{I} \) and the schedule that transfers to the destinations the information remaining in the nodes of \( V' \).

We can proceed now with the proof of the theorem.

**Proof of Theorem 1:** Since every \( q_{admissible} \)-admissible schedule satisfies (2.3), we have

\[
S_{q_{admissible}} = \bigcup_{f \in F} S_{q_{admissible}}(f).
\]

Hence, we have

\[
\inf_{s \in S_{q_{admissible}}} L(s) = \inf_{f \in F} \left\{ \inf_{s \in S_{q_{admissible}}(f)} L(s) \right\}.
\]

From (2.6) and in view of Lemmas 1 and 2, we obtain

\[
\inf_{s \in S_{q_{admissible}}} L(s) = \inf_{f \in F} \left\{ \inf_{s \in S_{q_{admissible}}(f)} L(s) \right\}.
\]

The infimum on the right-hand side of (2.7) can be actually achieved by a schedule in \( S_f \) as we show in the following. Consider all possible transmission sets \( T_1, \ldots, T_N \) of the network. For every schedule \( s' \in S_f \), \( s' = \{ (\tau_i, T_i) | i = 1, \ldots, M' \} \), consider the schedule \( s = \{ (\tau_i, T_i) | i = 1, \ldots, M' \} \), where \( \tau_i \) is the sum of those \( \tau'_{i'} \)’s for which the corresponding \( T'_{i'} \)'s are the same set \( T_i \). Clearly, we have

\[
L(s) = L(s') \quad \text{and} \quad f_s = f_{s'}.
\]

Therefore, \( s \in S_f \). Thus, it follows that the solution of the optimiza-

**The Zak Transform and Some Counterexamples in Time-Frequency Analysis**

A. J. E. M. Janssen

**Abstract**—It is shown how the Zak transform can be used to find nontrivial examples of functions \( f, g \in L^2(\mathbb{R}) \) with \( f \cdot g = 0 \neq F \cdot G \), where \( F, G \) are the Fourier transforms of \( f, g \), respectively. This is then used to exhibit a nontrivial pair of functions \( h, k \in L^2(\mathbb{R}) \), \( h \neq k \), such that \( |h| = |k|, |H| = |K| \). A similar construction is used to find an abundance of nontrivial pairs of functions \( h, k \in L^2(\mathbb{R}) \), \( h \neq k \), with \( |A_h| = |A_k| \) or with \( |W_h| = |W_k| \), where \( A_h, A_k \) and \( W_h, W_k \) are

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the ambiguity functions and Wigner distributions of \( h, k \), respectively. One of the examples of a pair of \( h, k \in L^2(\mathbb{R}), h \neq k \), with \( |A_h| = |A_k| \) is F. A. Grünbaum's example given previously. We find, in addition, nontrivial examples of functions \( g \) and signals \( f_1 \neq f_2 \) such that \( f_1 \) and \( f_2 \) have the same spectrogram when using \( g \) as window.

Index Terms—Zak transform, ambiguity function, spectrogram.

I. INTRODUCTION

In [1], F. A. Grünbaum presents a nontrivial example of two functions \( h, k \in L^2(\mathbb{R}) \) such that \( |A_h| = |A_k| \). By nontrivial we mean that \( h \) and \( k \) cannot be obtained from one another by a time-frequency translate or by multiplication by a \( c \in \mathbb{C}, |c| = 1 \). Here, \( A \) refers to the ambiguity function: when \( f, g \in L^2(\mathbb{R}) \), the ambiguity function \( A_{f,g} \) of \( f \) and \( g \) is defined by

\[
A_{f,g}(\theta, \tau) = \int \int e^{-2\pi i s\tau} f(s) g(\tau) ds, \quad \tau, \theta \in \mathbb{R}.
\]

(1.1)

When \( f = g \), we write \( A_f \) instead of \( A_{f,f} \). The purpose of this note is to show that Grünbaum's example is a particular case of a whole class of such examples that can be constructed by using the Zak transform. A second purpose is to present similar examples, by similar constructions, of Fourier pairs, spectrograms and Wigner distributions.

The Zak transform of an \( f \in L^2(\mathbb{R}) \) is defined by

\[
(Zf)(\tau, \Omega) = \int \int e^{-2\pi i s\tau} f(s) ds, \quad \tau, \Omega \in \mathbb{R}.
\]

(1.2)

We recall here the properties of the Zak transform needed for the present purposes. We have the following, cf. [2].

1) \( Z \) is a Hilbert space isometry of \( L^2(\mathbb{R}) \) onto \( L^2([0, \frac{1}{2}]) \).

More precisely, when \( Z(\tau, \Omega) \) is a function satisfying

\[
Z(\tau, \Omega + 1) = Z(\tau, \Omega), \quad \tau, \Omega \in \mathbb{R}. \tag{1.3}
\]

and \( Z \in L^2([0, \frac{1}{2}]) \), there is exactly one \( f \in L^2(\mathbb{R}) \) such that \( Z = Zf \). Conversely, \( Z \in L^2([0, \frac{1}{2}]) \) and \( Zf \) satisfies the (quasi) periodicity relations (1.3) when \( f \in L^2(\mathbb{R}) \). And

\[
(Zf, Zg) = (f, g), \quad f, g \in L^2(\mathbb{R}), \tag{1.4}
\]

where the left-hand side inner product is that in \( L^2([0, \frac{1}{2}]) \) and the right-hand side inner product is that in \( L^2(\mathbb{R}) \).

2) For \( f \in L^2(\mathbb{R}) \) we have the formulas

\[
f(\tau) = \int \int (Zf)(\tau, \Omega) d\Omega, \quad F(\omega) = \int \int e^{-2\pi i \omega\tau} (Zf)(\tau, \omega) d\tau, \quad \tau, \omega \in \mathbb{R}. \tag{1.5}
\]

where \( F \) denotes the Fourier transform of \( f \).

\[
F(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} f(t) dt, \quad \omega \in \mathbb{R}. \tag{1.6}
\]

3) For \( f, g \in L^2(\mathbb{R}) \) we have the formula

\[
(Zf)(\tau, \Omega)(Zg)^*(\tau, \Omega) = \sum_{n, m} \left( f, R_{-m} T_{-n} g \right) e^{-2\pi i n\Omega + 2\pi i m\tau}, \tag{1.7}
\]

where for \( a, b \in \mathbb{R} \) the operators \( T_a, R_b \) are time, frequency shifts defined by

\[
(T_a f)(t) = f(t + a), \quad (R_b f)(t) = e^{-2\pi i b} f(t), \quad t \in \mathbb{R}. \tag{1.8}
\]

4) We have for \( f \), \( g \in L^2(\mathbb{R}) \), \( a, b \in \mathbb{R} \),

\[
(ZT_a f)(\tau, \Omega) = (Zf)(\tau + a, \Omega), \quad (ZR_b f)(\tau, \Omega) = e^{-2\pi i b} (Zf)(\tau, \Omega + b). \tag{1.9}
\]

Formula (1.7) provides an important link between the Zak transform and the ambiguity function since

\[
A_{f,g}(\theta, \tau) = e^{i\theta \tau} (f, R_{-\theta} g), \quad \theta, \tau \in \mathbb{R}. \tag{1.10}
\]

II. THE EXAMPLES

Example 1: \( f, g \in L^2(\mathbb{R}) \) such that \( f, g = 0 = F \cdot G \).

Let \( U \) and \( V \) be two subsets of \( [\frac{1}{2}, \frac{1}{2}] \) such that for any \( \tau, \Omega \in [\frac{1}{2}, \frac{1}{2}] \)

\[
\mu(U) \mu(V) = \mu(U) \mu(V) = 0. \tag{2.1}
\]

Here, \( U = \{ \Omega | (\tau, \Omega) \in U \}, \quad V = \{ \tau | (\tau, \Omega) \in V \}, \quad \text{etc.}, \tag{2.2} \)

and \( \mu \) is Lebesgue measure on \( [\frac{1}{2}, \frac{1}{2}] \). Let \( \psi \in L^2([\frac{1}{2}, \frac{1}{2}]) \) have their supports in \( U, V \), respectively, and extend \( \psi \) quasi-periodically according to (1.3) to all of \( \mathbb{R} \). Then \( \psi = Zf, \psi = Zg \) for some \( f, g \in L^2(\mathbb{R}) \), and, as readily follows from 2), we have \( f, g = 0 = F \cdot G \).

Note: In terms of ambiguity functions we have here an example of \( f, g \) such that \( A_{f,g}(\theta, \tau) = A_{f,g}(\theta, 0) \) for all \( \theta, \tau \in \mathbb{R} \). That \( A_{f,g} \) cannot vanish identically follows from

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A_{f,g}(\theta, \tau)|^2 d\theta d\tau = \|f\|^2 \|g\|^2. \tag{2.3}
\]

Example 2: \( h, k \in L^2(\mathbb{R}), h \neq k \), such that \( |h| = |k|, |H| = |K| \).

It is easy to find such \( h, k \) as follows. Let \( h \in L^2(\mathbb{R}) \) be such that \( |h(t)| = |h^*(t - t)| \), and set \( k(t) = h^*(t - t) \). Then \( K(\omega) = H^*(\omega) \), so that \( |K| = |H| \). A less trivial example is obtained by setting \( h = f + g, k = f - g \), with \( f, g \) as in Example 1, so that \( |h| = |f| + |g| = |k|, |H| = |f| + |G| = |K| \).

Example 3: For the next set of examples we need a lemma on the supporting sets of ambiguity functions.

Lemma 1: Denote for \( f \in L^2(\mathbb{R}) \) by \( S_f \) the supporting set of \( Zf \) (by (1.3) this set is periodic in both variables). Furthermore, denote
for \( f \in L^2(\mathbb{R}) \) and \( \tau_0, \Omega_0 \in [-\frac{1}{2}, \frac{1}{2}] \) by \( S_f(\tau_0, \Omega_0) \) the set
\[
S_f(\tau_0, \Omega_0) = \{(\tau + \tau_0, \Omega + \Omega_0) | (\tau, \Omega) \in S_f\}.
\]  
(2.4)

Finally, denote for \( f, g \in L^2(\mathbb{R}) \) by \( \Sigma_{f,g} \) the set
\[
\Sigma_{f,g} = \{(\tau_0, \Omega_0) \in [-\frac{1}{2}, \frac{1}{2}] | \mu_2(S_f \cap S_g(\tau_0, \Omega_0)) \neq 0\}.
\]  
(2.5)

Here, \( \mu_2 \) is Lebesgue measure in \( \mathbb{R}^2 \). Then \( A_{f,g} \) is supported by the set \( V_{f,g} \) given by
\[
V_{f,g} = \{(n + \tau_0, m + \Omega_0) | n, m \in \mathbb{Z}, (\tau_0, \Omega_0) \in \Sigma_{f,g}\}.
\]  
(2.6)

**Proof:** We combine (1.7), (1.9), and (1.10) to obtain
\[
e^{-2\pi i0_0(r)(\tau, \Omega)}(zg)^*(\tau - \tau_0, \Omega - \Omega_0)
\]
\[
= \sum_{n,m} e^{-2\pi i n m + 0_0(n + \tau_0)} A_{f,g}(m + \Omega_0, n + \tau_0) e^{2\pi i n m - 0_0(n, \Omega_0)}
\]  
(2.7)

for \( \tau, \tau_0, \Omega, \Omega_0 \in \mathbb{R} \). Now we have for \( \tau_0, \Omega_0 \in [-\frac{1}{2}, \frac{1}{2}] \) that
\[
A_{f,g}(m + \Omega_0, n + \tau_0) = 0, \quad \forall n, m \in \mathbb{Z},
\]  
(2.8)

if and only if
\[
\mu_2(S_f \cap S_g(\tau_0, \Omega_0)) = 0.
\]  
(2.9)

Since any point \( (\theta, \tau) \in \mathbb{R}^2 \) can be written as \( (m + \Omega_0, n + \tau_0) \) for some \( n, m \in \mathbb{Z}, \tau_0, \Omega_0 \in [-\frac{1}{2}, \frac{1}{2}] \), the lemma follows.

To some insight how the lemma can be used to construct counterexamples, we present Figs. 1-4. Observe that \( \Sigma_{f,f} = -\Sigma_{f,f} \) so that \( \Sigma_{f,f} \) is symmetric about the origin.

With the aid of Lemma 1, one can construct functions \( f \) whose ambiguity function \( A_f \) has, in the terminology of Price and Hofstetter [3], volume-clearance around the origin arbitrarily close to 4. That is, for any \( \delta > 0 \), \( \varepsilon > 0 \), there is an \( f \in L^2(\mathbb{R}) \) and a convex set \( C \) with \( \mu_2(C) \geq 4 - \delta \) such that \( A_f(\theta, \tau) = 0 \) for \( \theta, \tau \in C, \theta^2 + \tau^2 \leq \varepsilon^2 \). One can take for \( f \) any function whose Zak transform is concentrated in a small disk around the origin. The volume-clearance result in [3] says that \( \mu_2(C) \) cannot exceed 4. As a limiting case, when \( \delta > 0, \varepsilon \to 0 \), one can take \( f = \sum_{k} \delta_k \) so that \( A_f(\theta, \tau) = \sum_{k,m} \delta_k(\theta) \delta_m(\tau) \) (here, \( \delta_k \) is the delta function at \( k \)).

*Example 4* \( h, k \in L^2(\mathbb{R}), n \neq k, \) such that \( |A_h| = |A_k| \).

It is easy to see that we have \( |A_f| = |A_g| \) when \( f, g \in L^2(\mathbb{R}) \) and \( k = cR_\theta T_\tau h \) for some \( a, b \in \mathbb{R}, c \in \mathbb{Z}, |c| = 1 \). Less trivial examples can be constructed as follows. Take \( f = f_{f,g}, g = g \) as in Fig. 3. Now we have
\[
\sum_{f} \sum_{g} \sum_{f} \sum_{g} = \sum_{f} \sum_{g} \sum_{f} \sum_{g} = 0.
\]  
(2.10)

Hence, \( A_{f,f} = A_{g,g} = 0 \), etc. When we set \( h = f + g, k = f - g \) and observe that
\[
A_{f+g,f+g} = A_{f,f} + A_{g,g} = \mu_2(S_f \cap S_g(\tau_0, \Omega_0)) \neq 0.
\]  
(2.11)

we readily see that \( |A_h| = |A_k| \). An example of this situation of the Grünbaum type is given in Fig. 4. Grünbaum considers functions \( f \) and \( g \) with support in \( \{|t| \leq 6 \} \) and \( \{|t| \leq 5 \} \), respectively. By appropriate translation and scaling, it can be achieved that \( f \) and \( g \) have their supports in intervals \( (e, 2e) \) and \( (\frac{3}{2} - 2\theta, \frac{3}{2} - 2\tau) \). It follows from the definition of the Zak transform that \( S_f \) and \( S_g \) are as in Fig. 4. Again, we have a situation in which (2.10) holds.

*Example 5*: \( h, k \in L^2(\mathbb{R}), h \neq k, \) such that \( W_h = W_k \).

When \( f, g \in L^2(\mathbb{R}) \) we define the Wigner distribution of \( f \) and \( g \) by
\[
W_{f,g}(t, \omega) = \int_{-\infty}^{\infty} e^{-2\pi i s(t + \frac{1}{2}s)} f(s + \frac{1}{2}s) g^*(s - \frac{1}{2}s) ds,
\]  
(2.12)

Unlike the ambiguity function, \( W_{f,g} \) is always real. We have
\[
W_{f,g}(t, \omega) = 2 A_{f,g}(2\omega, 2t), \quad t, \omega \in \mathbb{R}.
\]  
(2.13)
correspondence proves that Lloyd's method I converges for a large class of error measures, if the density function is continuous, positive, and defined on a finite interval. The proof is done by modeling the behavior of a continuous optimization algorithm by a finite state machine.

Index Terms—Convergence of sequences, optimal quantization, fixed point iteration, finite state machine.

I. INTRODUCTION

The well-known Lloyd's method I [7] for optimal quantization is a fixed point algorithm to compute a locally optimal quantizer. The method was originally derived for the mean-square error measure, but is applicable for a wide range of error measures, as we will see later. After its invention in 1957, Lloyd's method I was extended by Netravali and Saigal [9] for optimal quantization under entropy constraints. Then the fixed point iteration scheme of Lloyd's method I was generalized from scalar quantization to vector quantization, resulting in the popular Linde-Buzo-Gray (LBG) algorithm [6].

Despite its long history in use no one, to the best of author's knowledge, has proven the convergence of Lloyd's method I before. Interestingly, the convergence of the LBG algorithm, the vector version of the scalar Lloyd's method I, was shown by Abaya and Wise [1], by Selim and Ismail [11] when they proved the convergence of the K-means algorithm, and later by Sabin and Gray [10] in a more general setting. It should be noted though that the LBG algorithm is by nature one of discrete optimization. Being iteratively applied to an initial code book the LBG algorithm generates a sequence of ever-improved code books. All these code books contain a finite number of points (in a vector space). A code book which may be perceived as a vector quantizer is a partition of a finite point set, hence the both sets of input and output for the LBG algorithm are finite. The original Lloyd's method I is, on the contrary, a continuous optimization algorithm, trying to partition an infinite number of points obeying a continuous density function $p(x)$ into $K$ sets. Due to this significant difference, the proofs of convergence cited previously for the LBG algorithm cannot be extended to the original Lloyd's method I.

The convergence of Lloyd's method I was previously studied by a number of researchers [2], [5], [12] in the context of uniqueness of a locally optimal quantizer. It was shown that Lloyd's method I converges to the globally optimal quantizer if the density function is continuous and log-concave, and if the error weighting function is convex and symmetric. In this correspondence, we will prove that Lloyd's method I converges for all continuous, positive densities defined on a finite interval under the class of convex and symmetric error measures. This more general result is obtained by a finite state machine that models the behavior of Lloyd's method I and by using a monotonicity property of the method.

II. FORMULATION AND PREPARATION

In order to facilitate the key proof of the correspondence, we need to formulate the problem of optimal quantization, and list some published results about the problem. It is assumed that the signal amplitude density function $p(x)$ is continuous, positive, and defined on a finite interval which is normalized to $\{0,1\}$, that is, $(x, x \in [0,1]) p(x) > 0$. Let $x \in [0,1]$. The vector $q$ partitions the range $[0,1]$ into $K$ intervals: $[0, q_1], (q_1, q_2), \ldots, (q_{K-1}, 1]$. Consequently, a $K$-level quantizer for $p(x)$ may be characterized by two vectors $q \in \mathbb{R}^{K-1}$, $q_{j+1} < q_j, 1 \leq j \leq K$, and $r \in \mathbb{R}^K$. The vector $q$ partitions the range $[0,1]$ into $K$ intervals: $[0, q_1], (q_1, q_2), \ldots, (q_{K-1}, 1]$. Here the conventions $q_0 = 0$ and $q_K = 1$ come naturally and will be used in the sequel.

On Convergence of Lloyd’s Method I

Xiaolin Wu

Abstract—Although Lloyd's method I for optimal quantization was proposed more than thirty years ago and has been frequently referred to in the literature, its convergence has so far not been shown. This