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A Hilton-Milner Theorem for Vector Spaces

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Abstract

We show for \(k \geq 2\) that if \(q \geq 3\) and \(n \geq 2k + 1\), or \(q = 2\) and \(n \geq 2k + 2\), then any intersecting family \(F\) of \(k\)-subspaces of an \(n\)-dimensional vector space over \(GF(q)\) with \(\bigcap_{F \in F} F = 0\) has size at most \(\binom{n-1}{k-1} - q^{k(k-1)} \binom{n-k-1}{k-1} + q^k\). This bound is sharp as is shown by Hilton-Milner type families. As an application of this result, we determine the chromatic number of the corresponding \(q\)-Kneser graphs.

1 Introduction

1.1 Sets

Let \(X\) be an \(n\)-element set and, for \(0 \leq k \leq n\), let \(\binom{X}{k}\) denote the family of all subsets of \(X\) of cardinality \(k\). A family \(F \subset \binom{X}{k}\) is called intersecting if for all \(F_1, F_2 \in F\) we have \(F_1 \cap F_2 \neq \emptyset\). Erdős, Ko, and Rado [5] determined the maximum size of an intersecting family, and introduced the so-called shifting technique.
Theorem 1.1 (Erdős-Ko-Rado) Suppose $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Excepting the case $n = 2k$, equality holds only if $\mathcal{F} = \{ F \in \binom{X}{k} : x \in F \}$ for some $x \in X$.

For any family $\mathcal{F} \subset \binom{X}{k}$, the covering number $\tau(\mathcal{F})$ is the minimum size of a set that meets all $F \in \mathcal{F}$. Theorem 1.1 shows that if $\mathcal{F} \subset \binom{X}{k}$ is an intersecting family of maximum size and $n > 2k$, then $\tau(\mathcal{F}) = 1$.

Hilton and Milner [15] determined the maximum size of an intersecting family with $\tau(\mathcal{F}) \geq 2$. Later, Frankl and Füredi [9] gave an elegant proof of Theorem 1.2 using the shifting technique.

Theorem 1.2 (Hilton-Milner) Let $\mathcal{F} \subset \binom{X}{k}$ be an intersecting family with $k \geq 2$, $n \geq 2k + 1$, and $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Equality holds only if

(i) $\mathcal{F} = \{ F \cup \{ G \in \binom{X}{k} : x \in G, F \cap G \neq \emptyset \} \}$ for some $k$-subset $F$ and $x \in X \setminus F$.

(ii) $\mathcal{F} = \{ F \in \binom{X}{a} : |F \cap S| \geq 2 \}$ for some $3$-subset $S$ if $k = 3$.

1.2 Vector spaces

Theorem 1.1 and Theorem 1.2 have natural extensions to vector spaces. We let $V$ always denote an $n$-dimensional vector space over the finite field $GF(q)$. For $k \in \mathbb{Z}^+$, we write $\binom{V}{k}_q$ to denote the family of all $k$-dimensional subspaces of $V$. For $a, k \in \mathbb{Z}^+$, define the Gaussian binomial coefficient by

$$\left[ \begin{array}{c} a \\ k \end{array} \right]_q := \prod_{0 \leq i < k} \frac{q^{a-i} - 1}{q^{k-i} - 1}.$$ 

A simple counting argument shows that the size of $\left[ \begin{array}{c} V \\ k \end{array} \right]_q$ is $\left[ \begin{array}{c} n \\ k \end{array} \right]$. From now on, we will omit the subscript $q$.

If two subspaces of $V$ intersect in the zero subspace, then we say they are disjoint or that they trivially intersect; otherwise we say the subspaces non-trivially intersect. A family $\mathcal{F} \subset \left[ \begin{array}{c} V \\ k \end{array} \right]$ is called intersecting if any two $k$-spaces in $\mathcal{F}$ non-trivially intersect. The maximum size of an intersecting family of $k$-spaces was first determined by Hsieh [16]. For alternate proofs of Theorem 1.3, see [4] and [11]. We remark that there is as yet no analog of the shifting technique for vector spaces.

Theorem 1.3 (Hsieh) Suppose $\mathcal{F} \subset \left[ \begin{array}{c} V \\ k \end{array} \right]$ is intersecting and $n \geq 2k$. Then $|\mathcal{F}| \leq \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]$. Equality holds if and only if $\mathcal{F} = \{ F \in \left[ \begin{array}{c} V \\ k \end{array} \right] : v \subset F \}$ for some one-dimensional subspace $v \subset V$, unless $n = 2k$.

Let the covering number $\tau(\mathcal{F})$ of a family $\mathcal{F} \subset \left[ \begin{array}{c} V \\ k \end{array} \right]$ be defined as the minimum dimension of a subspace of $V$ that intersects all elements of $\mathcal{F}$ nontrivially. Theorem 1.3 shows that, as in the set case, if $\mathcal{F}$ is a maximum intersecting family of $k$-spaces, then $\tau(\mathcal{F}) = 1$. Families satisfying $\tau(\mathcal{F}) = 1$ are known as point-pencils.
In this paper, we will extend Theorem 1.2 to vector spaces, and determine the maximum size of an intersecting family $\mathcal{F} \subset \binom{V}{k}$ with $\tau(\mathcal{F}) \geq 2$. For two subspaces $S, T \subseteq V$, we let $S + T \subseteq V$ denote their linear span. We observe that for a fixed 1-subspace $E \subseteq V$ and a $k$-subspace $U$ with $E \nsubseteq U$, the family

$$\mathcal{F}_{E,U} = \{ U \} \cup \{ W \in \binom{V}{k} : E \subseteq W, \dim(W \cap U) \geq 1 \}$$

is not maximal as we can add all subspaces in $\binom{E + U}{k}$ that are not in $\mathcal{F}_{E,U}$. We will say that $\mathcal{F}$ is an HM-type family if

$$\mathcal{F} = \{ W \in \binom{V}{k} : E \subseteq W, \dim(W \cap U) \geq 1 \} \cup \binom{E + U}{k}$$

for some $E \in \binom{V}{1}$ and $U \in \binom{V}{k}$ with $E \nsubseteq U$. If $\mathcal{F}$ is an HM-type family, then its size is

$$|\mathcal{F}| = f(n,k,q) := \binom{n-1}{k-1} - q^{k(k-1)} \binom{n-k-1}{k-1} + q^k. \quad (1.1)$$

The main result of the paper is the following theorem.

**Theorem 1.4** Suppose $k \geq 3$, and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$. For any intersecting family $\mathcal{F} \subseteq \binom{V}{k}$ with $\tau(\mathcal{F}) \geq 2$, we have $|\mathcal{F}| \leq f(n,k,q)$ (with $f(n,k,q)$ as in (1.1)). Equality holds only if

(i) $\mathcal{F}$ is an HM-type family,

(ii) $\mathcal{F} = \mathcal{F}_3 = \{ F \in \binom{V}{k} : \dim(S \cap F) \geq 2 \}$ for some $S \in \binom{V}{3}$ if $k = 3$.

Furthermore, if $k \geq 4$, then there exists an $\epsilon > 0$ (independent of $n,k,q$) such that if $|\mathcal{F}| \geq (1-\epsilon)f(n,k,q)$, then $\mathcal{F}$ is a subfamily of an HM-type family.

If $k = 2$, then a maximal intersecting family $\mathcal{F}$ of $k$-spaces with $\tau(\mathcal{F}) > 1$ is the family of all 2-subspaces of a 3-subspace, and the conclusion of the theorem holds.

After proving Theorem 1.4 in Section 2, we apply this result to determine the chromatic number of $q$-Kneser graphs. The vertex set of the $q$-Kneser graph $qK_{n,k}$ is $\binom{V}{k}$. Two vertices of $qK_{n,k}$ are adjacent if and only if the corresponding $k$-subspaces are disjoint. In [3], the chromatic number of the $q$-Kneser graph $qK_{n,2}$ is determined, and the minimum colorings are characterized. In [18], the chromatic number of the $q$-Kneser graph is determined in general for $q > q_k$. In Section 4, we prove the following theorem.

**Theorem 1.5** If $k \geq 3$, and either $q \geq 3$ and $n \geq 2k+1$, or $q = 2$ and $n \geq 2k+2$, then the chromatic number of the $q$-Kneser graph is $\chi(qK_{n,k}) = \binom{n-k+1}{1}$. Moreover, each color class of a minimum coloring is a point-pencil and the points determining a color are the points of an $(n-k+1)$-dimensional subspace.

In Section 5, we prove the non-uniform version of the Erdős-Ko-Rado theorem.

**Theorem 1.6** Let $\mathcal{F}$ be an intersecting family of subspaces of $V$. 

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(i) If $n$ is even, then $|\mathcal{F}| \leq \left\lfloor \frac{n-1}{2} \right\rfloor + \sum_{i>n/2} \left\lfloor \frac{n}{i} \right\rfloor$.

(ii) If $n$ is odd, then $|\mathcal{F}| \leq \sum_{i>n/2} \left\lceil \frac{n}{i} \right\rceil$.

For even $n$, equality holds only if $\mathcal{F} = \left\lceil \frac{V}{n/2} \right\rceil \cup \{F \in \left\lceil \frac{V}{n/2} \right\rceil : E \leq F\}$ for some $E \in \left\lceil \frac{V}{1} \right\rceil$, or if $\mathcal{F} = \left\lceil \frac{V}{n/2} \right\rceil \cup \left\lfloor \frac{V}{n/2} \right\rfloor$ for some $U \in \left\lfloor \frac{V}{n-1} \right\rfloor$. For odd $n$, equality holds only if $\mathcal{F} = \left\lceil \frac{V}{n/2} \right\rceil$.

Note that Theorem 1.6 follows from the profile polytope of intersecting families which was determined implicitly by Bey [1] and explicitly by Gerbner and Patkós [12], but the proof we present in Section 5 is simple and direct.

2 Proof of Theorem 1.4

This section contains the proof of Theorem 1.4 which we divide into two cases.

2.1 The case $\tau(\mathcal{F}) = 2$

For any $A \leq V$ and $\mathcal{F} \subseteq \left\lceil \frac{V}{k} \right\rceil$, let $\mathcal{F}_A = \{F \in \mathcal{F} : A \leq F\}$. First, let us state some easy technical lemmas.

**Lemma 2.1** Let $a \geq 0$ and $n \geq k \geq a+1$ and $q \geq 2$. Then

$$\begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-a-1 \\ k-a-1 \end{bmatrix} < \frac{1}{(q-1)q^{n-2k}} \begin{bmatrix} n-a \\ k-a \end{bmatrix}.$$ 

Proof. The inequality to be proved simplifies to

$$q^{k-a-1}(q^k-1)q^{n-2k} < q^{n-a-1}.$$ 

$\square$

**Lemma 2.2** Let $E \in \left\lfloor \frac{V}{1} \right\rfloor$. If $E \nsubseteq L \leq V$, where $L$ is an $l$-subspace, then the number of $k$-subspaces of $V$ containing $E$ and intersecting $L$ is at least $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ (with equality for $l = 2$), and at most $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}$.

Proof. The $k$-spaces containing $E$ and intersecting $L$ in a 1-dimensional space are counted exactly once in the first term. Those subspaces that intersect $L$ in a 2-dimensional space are counted $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = q+1$ times in the first term and $-q$ times in the second term, thus once overall. If a subspace intersects $L$ in a subspace of dimension $i \geq 3$, then it is counted $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ times in the first term and $-q\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ times in the second term, and hence a negative number of times overall.

Our next lemma gives bounds on the size of an HM-type family that are easier to work with than the precise formula mentioned in the introduction.

**Lemma 2.3** Let $n \geq 2k+1$, $k \geq 3$ and $q \geq 2$. If $\mathcal{F} \subseteq \left\lfloor \frac{V}{k} \right\rfloor$ is an HM-type family, then

$$1 \leq (1-q^{n-2k}q^{k-2}) \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix} - q \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$$

$$\leq f(n, k, q) = |\mathcal{F}| \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}.$$
Proof. Since \( q^2/(2^2) = \binom{k}{1} \binom{k}{n-k} \) and \( n \geq 2k + 1 \), the first inequality follows from Lemma 2.1. Let \( \mathcal{F} \) be the HM-type family defined by the 1-space \( E \) and the \( k \)-space \( U \). Then \( \mathcal{F} \) contains all \( k \)-subspaces of \( V \) containing \( E \) and intersecting \( U \), so that the second inequality follows from Lemma 2.2. For the last inequality, Lemma 2.2 almost suffices, but we also have to count the \( k \)-subspaces of \( \binom{E+U}{k} \) that do not contain \( E \). Each \((k-1)\)-subspace \( W \) of \( U \) is contained in \( q+1 \) such subspaces, one of which is \( E+W \). On the other hand, \( E+W \) was counted at least \( q+1 \) times since \( k \geq 3 \). This proves the last inequality. \( \square \)

**Lemma 2.4** If a subspace \( S \) does not intersect each element of \( \mathcal{F} \subset \binom{V}{k} \), then there is a subspace \( T \supset S \) with \( \dim T = \dim S + 1 \) and \( |\mathcal{F}_T| \geq |\mathcal{F}_S|/\binom{k}{1} \).

**Proof.** There is an \( F \in \mathcal{F} \) such that \( S \cap F = 0 \). Average over all \( T = S + E \) where \( E \) is a 1-subspace of \( F \). \( \square \)

**Lemma 2.5** If an \( s \)-dimensional subspace \( S \) does not intersect each element of \( \mathcal{F} \subset \binom{V}{k} \), then \( |\mathcal{F}_S| \leq \binom{k}{s-s-1} \binom{n-s-1}{k-s-1} \).

**Proof.** There is an \((s+1)\)-space \( T > S \) with \( \binom{V}{k} \geq |\mathcal{F}_T| \geq |\mathcal{F}_S|/\binom{k}{1} \). \( \square \)

**Corollary 2.6** Let \( \mathcal{F} \subset \binom{V}{k} \) be an intersecting family with \( \tau(\mathcal{F}) \geq s \). Then for any \( i \)-space \( L \leq V \) with \( i \leq s \) we have \( |\mathcal{F}_L| \leq \binom{k}{s-i} \binom{n-s}{k-s} \). \( \square \)

**Proof.** If \( i = s \), then clearly \( |\mathcal{F}_L| \leq \binom{n-s}{k-s-1} \). If \( i < s \), then there exists an \( F \in \mathcal{F} \) such that \( F \cap L = 0 \); now apply Lemma 2.4 \( s-i \) times.

Before proving the \( q \)-analogue of the Hilton-Milner theorem, we describe the essential part of maximal intersecting families \( \mathcal{F} \subset \binom{V}{k} \) with \( \tau(\mathcal{F}) = 2 \).

**Proposition 2.7** Let \( n \geq 2k \) and let \( \mathcal{F} \subset \binom{V}{k} \) be a maximal intersecting family with \( \tau(\mathcal{F}) = 2 \). Define \( \mathcal{T} \) to be the family of 2-spaces of \( V \) that intersect all subspaces in \( \mathcal{F} \). One of the following three possibilities holds:

(i) \( |\mathcal{T}| = 1 \) and \( \binom{n-2}{k-2} < |\mathcal{F}| < \binom{n-2}{k-2} + (q+1) \binom{k}{1} \binom{n-3}{k-3} \);

(ii) \( |\mathcal{T}| > 1 \), \( \tau(\mathcal{T}) = 1 \), and there is an \((l+1)\)-space \( W \) (with \( 2 \leq l \leq k \)) and a 1-space \( E \leq W \) so that \( \mathcal{T} = \{ M : E \leq M \leq W, \ \dim M = 2 \} \). In this case, \( \binom{k}{1} \binom{n-2}{k-2} - q \binom{n-3}{k-3} \leq |\mathcal{F}| \leq \binom{k}{1} \binom{n-2}{k-2} + \binom{k}{1} \binom{n-3}{k-3} + q \binom{n-3}{k-3} \).

For \( l = 2 \), the upper bound can be strengthened to \( |\mathcal{F}| \leq (q+1) \binom{n-2}{k-2} - q \binom{n-3}{k-3} + \binom{k}{1} \binom{n-3}{k-3} + q^2 \binom{k}{1} \binom{n-3}{k-3} \);

(iii) \( \mathcal{T} = \binom{A}{2} \) for some 3-subspace \( A \) and \( \mathcal{F} = \{ U \in \binom{V}{k} : \dim(U \cap A) \geq 2 \} \). In this case, \( |\mathcal{F}| = (q^2 + q + 1)(\binom{n-2}{k-2} - \binom{n-3}{k-3}) + \binom{n-3}{k-3} \).
Proof. Let $\mathcal{F} \subset \binom{V}{k}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. By maximality, $\mathcal{F}$ contains all $k$-spaces containing a $T \in \mathcal{T}$. Since $n \geq 2k$ and $k \geq 2$, two disjoint elements of $\mathcal{T}$ would be contained in disjoint elements of $\mathcal{F}$, which is impossible. Hence, $\mathcal{T}$ is intersecting.

Observe that if $A, B \in \mathcal{T}$ and $A \cap B < C < A + B$, then $C \in \mathcal{T}$. As an intersecting family of 2-spaces is either a family of 2-spaces containing some fixed 1-space $E$ or a family of 2-subspaces of a 3-space, we get the following:

(i): $\mathcal{T}$ is either a family of all 2-subspaces containing some fixed 1-space $E$ that lie in some fixed $(l + 1)$-space with $k \geq l \geq 1$, or $\mathcal{T}$ is the family of all 2-subspaces of a 3-space.

(ii): Assume that $\tau(\mathcal{T}) = 1$ and $|\mathcal{T}| > 1$. By (i), $\mathcal{T}$ is the set of 2-spaces in an $(l + 1)$-space $W$ (with $l \geq 2$) containing some fixed 1-space $E$. Every $F \in \mathcal{F} \setminus \mathcal{F}_E$ intersects $W$ in a hyperplane. Let $L$ be a hyperplane in $W$ not on $E$. Then $\mathcal{F}$ contains all $k$-spaces on $E$ that intersect $L$. Hence the lower bound and the first term in the upper bound come from Lemma 2.2. The second term comes from using Lemma 2.5 to count the $k$-spaces of $\mathcal{F}$ that contain $E$ and intersect a given $F \in \mathcal{F}$ (not containing $E$) in a point of $F \setminus W$. If $l \geq 3$, then there are $q^l$ hyperplanes in $W$ not containing $E$ and there are $\binom{n - l}{k - 1}$ $k$-spaces through such a hyperplane; this gives the last term. For $l = 2$, we use the tight lower bound in Lemma 2.2 to count the number of $k$-spaces on $E$ that intersect $L$. There are $q^2$ hyperplanes in $W$, and they cannot be in $\mathcal{T}$, so Lemma 2.5 gives the bound.

(iii): This is immediate. \hfill \Box

Corollary 2.8 Let $\mathcal{F} \subset \binom{V}{k}$ be a maximal intersecting family with $\tau(\mathcal{F}) = 2$. Suppose $q \geq 3$ and $n \geq 2k + 1$, or $q = 2$ and $n \geq 2k + 2$. If $\mathcal{F}$ is at least as large as an HM-type family and $k \geq 3$, then $\mathcal{F}$ is an HM-type family. If $k = 3$, then $\mathcal{F}$ is an HM-type family or an $\mathcal{F}_3$-type family.

There exists an $\epsilon > 0$ (independent of $n, k, q$) such that if $k \geq 4$ and $|\mathcal{F}|$ is at least $(1 - \epsilon)$ times the size of an HM-type family, then $\mathcal{F}$ is an HM-type family.

Proof. Apply Proposition 2.7. Note that the HM-type families are precisely those from case (ii) with $l = k$.

Let $n = 2k + r$ where $r \geq 1$. We have $|\mathcal{F}|/\binom{n - 2}{k - 2} < 1 + \frac{q + 1}{(q - 1)q} \binom{k}{l}$ in case (i) of Proposition 2.7 by Lemma 2.1. We have $|\mathcal{F}|/\binom{n - 2}{k - 2} < \left(\frac{1}{q} + \frac{1}{(q - 1)q}\right) \binom{k}{l} + \frac{q^2}{(q - 1)q}$ in case (ii) when $l < k$. In both cases, for $q \geq 3$ and $k \geq 3$, or $q = 2$, $k \geq 4$, and $r \geq 2$, this is less than $(1 - \epsilon)$ times the lower bound on the size of an HM-type family given in Lemma 2.3. Using the stronger estimate in Lemma 2.3, we find the same conclusion for $q = 2, k = 3$, and $r \geq 2$. 

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In case (iii), \(|\mathcal{F}_3| = \binom{n-2}{k-2} - \frac{q^2 - q}{q-1} \binom{n-3}{k-3}\). For \(k \geq 4\), this is much smaller than the size of the HM-type families. For \(k = 3\), the two families have the same size. \(\square\)

**Proposition 2.9** Suppose that \(k \geq 3\) and \(n \geq 2k\). Let \(\mathcal{F} \subseteq \binom{V}{k}\) be an intersecting family with \(\tau(\mathcal{F}) \geq 2\). Let \(3 \leq l \leq k\). If there is an \(l\)-space that intersects each \(F \in \mathcal{F}\) and

\[
|\mathcal{F}| > \binom{l}{1} \binom{k}{1}^{l-1} \binom{n-l}{k-1}, \tag{2.2}
\]

then there is an \((l-1)\)-space that intersects each \(F \in \mathcal{F}\).

**Proof.** By averaging, there is a 1-space \(P\) with \(|\mathcal{F}_P| \geq |\mathcal{F}|/\binom{l}{1}\). If \(\tau(\mathcal{F}) = l\), then by Corollary 2.6, \(|\mathcal{F}| \leq \binom{l}{1} \binom{k}{1}^{l-1} \binom{n-l}{k-1}\), contradicting the hypothesis. \(\square\)

**Corollary 2.10** Suppose \(k \geq 3\) and either \(q \geq 3\) and \(n \geq 2k+1\), or \(q = 2\) and \(n \geq 2k+2\). Let \(\mathcal{F} \subseteq \binom{V}{k}\) be an intersecting family with \(\tau(\mathcal{F}) \geq 2\). If \(|\mathcal{F}| > \binom{3}{1} \binom{k}{1}^2 \binom{n-3}{k-3}\), then \(\tau(\mathcal{F}) = 2\); that is, \(\mathcal{F}\) is contained in one of the systems in Proposition 2.7, which satisfy the bound on \(|\mathcal{F}|\).

**Proof.** By Lemma 2.1 and the conditions on \(n\) and \(q\), the right hand side of (2.2) decreases as \(l\) increases, where \(3 \leq l \leq k\). Hence, by Proposition 2.9, we can find a 2-space that intersects each \(F \in \mathcal{F}\). \(\square\)

**Remark 2.11** For \(n \geq 3k\), all systems described in Proposition 2.7 occur.

### 2.2 The case \(\tau(\mathcal{F}) > 2\)

Suppose that \(\mathcal{F} \subseteq \binom{V}{k}\) is an intersecting family and \(\tau(\mathcal{F}) = l > 2\). We shall derive a contradiction from \(|\mathcal{F}| \geq f(n, k, q)\), and even from \(|\mathcal{F}| \geq (1-\epsilon)f(n, k, q)\) for some \(\epsilon > 0\) (independent of \(n, k, q\)).

#### 2.2.1 The case \(l = k\)

First consider the case \(l = k\). Then \(|\mathcal{F}| \leq \binom{k}{1}^k\) by Corollary 2.6. On the other hand,

\[
|\mathcal{F}| \geq \left(1 - \frac{1}{q-1}\right) \binom{k}{1} \binom{n-2}{k-2} > \left(1 - \frac{1}{q-1}\right) \binom{k}{1}^{k-1} \left((q-1)q^{n-2k}\right)^{k-2}
\]

by Lemma 2.3 and Lemma 2.1. If either \(q \geq 3\), \(n \geq 2k+1\) or \(q = 2\), \(n \geq 2k+2\), then either \(k = 3\), \((n, k, q) = (9, 4, 3)\), or \((n, k, q) = (10, 4, 2)\). If \((n, k, q) = (9, 4, 3)\) then \(f(n, k, q) = 3837721\), and \(40^4 = 2560000\), which gives a contradiction. If \((n, k, q) = (10, 4, 2)\), then \(f(n, k, q) = 153171\), and \(15^4 = 50625\), which again gives a contradiction. Hence \(k = 3\).

Now \(|\mathcal{F}| \geq (1 - \frac{1}{q-1}) \binom{k}{1} \binom{n-2}{k-2}\) gives a contradiction for \(n \geq 8\), so \(n = 7\). Therefore, if we assume that \(n \geq 2k+1\) and either \(q \geq 3\), \((n, k) \neq (7, 3)\) or \(q = 2\), \(n \geq 2k+2\) then we are not in the case \(l = k\).

It remains to settle the case \(n = 7\), \(k = l = 3\), and \(q \geq 3\). By Lemma 2.4, we can choose a 1-space \(E\) such that \(|\mathcal{F}_E| \geq |\mathcal{F}|/\binom{3}{1}\) and a 2-space \(S\) on \(E\) such that \(|\mathcal{F}_S| \geq |\mathcal{F}_E|/\binom{3}{1}|\).
Then $|\mathcal{F}_S| > q+1$ since $|\mathcal{F}| > \binom{m}{1}^2$. Pick $F' \in \mathcal{F}$ disjoint from $S$ and define $H := S + F'$. All $F \in \mathcal{F}_S$ are contained in the 5-space $H$. Since $|\mathcal{F}| > \binom{m}{3}$, there is an $F_0 \in \mathcal{F}$ not contained in $H$. If $F_0 \cap S = 0$, then $F \in \mathcal{F}_S$ is contained in $S + (H \cap F_0)$; this implies $|\mathcal{F}_S| \leq q + 1$, which is impossible. Thus, all elements of $\mathcal{F}$ disjoint from $S$ are in $H$.

Now $F_0$ must meet $F'$ and $S$, so $F_0$ meets $H$ in a 2-space $S_0$. Since $|\mathcal{F}_S| > q + 1$, we can find two elements $F_1, F_2$ of $\mathcal{F}_S$ with the property that $S_0$ is not contained in the 4-space $F_1 + F_2$. Since any $F \in \mathcal{F}$ disjoint from $S$ is contained in $H$ and meets $F_0$, it must meet $S_0$ and also $F_1$ and $F_2$. Hence the number of such $F$’s is at most $q^5$. Altogether $|\mathcal{F}| \leq q^5 + \binom{m}{3}^2$; the first term comes from counting $F \in \mathcal{F}$ disjoint from $S$ and the second term comes from counting $F \in \mathcal{F}$ on a given one-dimensional subspace $E < S$. This contradicts $|\mathcal{F}| \geq (1 - \frac{1}{q^2}) \binom{m}{1} \binom{m}{2}$.

### 2.2.2 The case $l < k$

Assume, for the moment, that there are two $l$-subspaces in $V$ that non-trivially intersect all $F \in \mathcal{F}$, and that these two $l$-spaces meet in an $m$-space, where $0 \leq m \leq l - 1$. By Corollary 2.6, for each 1-subspace $P$ we have $|\mathcal{F}_P| \leq \binom{m}{1}^{l-1} \binom{n-l}{k-l}$, and for each 2-subspace $L$ we have $|\mathcal{F}_L| \leq \binom{m}{1}^{l-2} \binom{n-l}{k-l}$. Consequently,

$$|\mathcal{F}| \leq \binom{m}{1} \binom{m}{1}^{l-1} \binom{n-l}{k-l} + (\binom{m}{1} - \binom{m}{1})^2 \binom{m}{1}^{l-2} \binom{n-l}{k-l}.$$  

(2.3)

The upper bound (2.3) is a quadratic in $x = \binom{m}{1}$ and is largest at one of the extreme values $x = 0$ and $x = \binom{l-1}{1}$. The maximum is taken at $x = 0$ only when $\binom{m}{1} - \frac{1}{2} \binom{m}{1} > \frac{1}{2} \binom{l-1}{1}$; that is, when $k = l$. Since we assume that $l < k$, the upper bound in (2.3) is largest for $m = l - 1$. We find

$$|\mathcal{F}| \leq \binom{l-1}{1} \binom{m}{1}^{l-1} \binom{n-l}{k-l} + (\binom{m}{1} - \binom{l-1}{1})^2 \binom{m}{1}^{l-2} \binom{n-l}{k-l}.$$  

On the other hand,

$$|\mathcal{F}| \geq (1 - \frac{1}{q^2}) \binom{m}{1}^{l-2} \binom{n-2}{k-2} > (1 - \frac{1}{q^2}) \binom{m}{1}^{l-1} \binom{n-2}{k-2} ((q - 1)q^{n-2k})^{l-2}.$$

Comparing these, and using $k > l$, $n \geq 2k + 1$, and $n \geq 2k + 2$ if $q = 2$, we find either $(n, k, l, q) = (9, 4, 3, 3)$ or $q = 2$, $n = 2k + 2$, $l = 3$, and $k \leq 5$. If $(n, k, l, q) = (9, 4, 3, 3)$ then $f(n, k, q) = 3837721$, while the upper bound is 3508960, which is a contradiction. If $(n, k, l, q) = (12, 5, 3, 2)$ then $f(n, k, q) = 183628563$, while the upper bound is 146766865, which is a contradiction. If $(n, k, l, q) = (10, 4, 3, 2)$ then $f(n, k, q) = 153171$, while the upper bound is 116205, which is a contradiction. Hence, under our assumption that there are two distinct $l$-spaces that meet all $F \in \mathcal{F}$, the case $2 < l < k$ cannot occur.

We now assume that there is a unique $l$-space $T$ that meets all $F \in \mathcal{F}$. We can pick a 1-space $E < T$ such that $|\mathcal{F}_E| \geq |\mathcal{F}|/\binom{m}{1}$. Now there is some $F' \in \mathcal{F}$ not on $E$, so $E$ is in $\binom{m}{1}$ lines such that each $F \in \mathcal{F}_E$ contains at least one of these lines. Suppose $L$ is one of these lines and $L$ does not lie in $T$; we can enlarge $L$ to an $l$-space that still does not
meet all elements of \( \mathcal{F} \), so \( |\mathcal{F}_L| \leq \binom{n}{l}^{l-1} \binom{k}{l-l-1} \) by Lemma 2.4 and Lemma 2.5. If \( L \) does lie on \( T \), we have \( |\mathcal{F}_L| \leq \binom{k}{l}^{l-2} \binom{n-l}{k-l-1} \) by Corollary 2.6. Hence,
\[
|\mathcal{F}| \leq \binom{l}{l} \left( \binom{l-1}{l} \binom{k}{l-l}^{l-2} \binom{n-l}{k-l-1} + \left( \binom{l}{l} - \binom{l-1}{l} \right) (\binom{k}{l}^{l-1} \binom{n-l-1}{k-l-1}) \right).
\]

On the other hand, we have \( |\mathcal{F}| > \left( 1 - \frac{1}{q^2 - 1} \right) \left( (q - 1)q^{n-2k} \right) \binom{k}{l}^{l-1} \binom{n-l}{k-l-1} \). Under our standard assumptions \( n \geq 2k + 1 \) and \( n \geq 2k + 2 \) if \( q = 2 \), this implies \( q = 2, n = 2k + 2, l = 3 \), which gives a contradiction. We showed: If \( q \geq 3 \) and \( n \geq 2k + 1 \) or if \( q = 2 \) and \( n \geq 2k + 2 \), then an intersecting family \( \mathcal{F} \subset \binom{V}{k} \) with \( |\mathcal{F}| \geq f(n, k, q) \) must satisfy \( \tau(\mathcal{F}) \leq 2 \). Together with Corollary 2.8, this proves Theorem 1.4.

## 3 Critical families

A subspace will be called a hitting subspace (and we shall say that the subspace intersects \( \mathcal{F} \)), if it intersects each element of \( \mathcal{F} \).

The previous results just used the parameter \( \tau \), so only the hitting subspaces of smallest dimension were taken into account. A more precise description is possible if we make the intersecting system of subspaces critical.

**Definition 3.1** An intersecting family \( \mathcal{F} \) of subspaces of \( V \) is critical if for any two distinct \( F, F' \in \mathcal{F} \) we have \( F \not\subseteq F' \), and moreover for any hitting subspace \( G \) there is a \( F \in \mathcal{F} \) with \( F \subseteq G \).

**Lemma 3.2** For every non-extendable intersecting family \( \mathcal{F} \) of \( k \)-spaces there exists some critical family \( \mathcal{G} \) such that
\[
\mathcal{F} = \{ F \in \binom{V}{k} : \exists G \in \mathcal{G}, G \subseteq F \}.
\]

**Proof.** Extend \( \mathcal{F} \) to a maximal intersecting family \( \mathcal{H} \) of subspaces of \( V \), and take for \( \mathcal{G} \) the minimal elements of \( \mathcal{H} \). \( \square \)

The following construction and result are an adaptation of the corresponding results from Erdős and Lovász [6]:

**Construction 3.3** Let \( A_1, \ldots, A_k \) be subspaces of \( V \) such that \( \dim A_i = i \) and \( \dim(A_1 + \cdots + A_k) = \binom{k+1}{2} \). Define
\[
\mathcal{F}_i = \{ F \in \binom{V}{k} : A_i \subseteq F, \ \dim A_j \cap F = 1 \text{ for } j > i \}.
\]

Then \( \mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k \) is a critical, non-extendable, intersecting family of \( k \)-spaces, and \( |\mathcal{F}_i| = \binom{i+1}{1} \binom{i+2}{1} \cdots \binom{k}{1} \) for \( 1 \leq i \leq k \).
For subsets Erdős and Lovász proved that a critical, non-extendable, intersecting family of $k$-sets cannot have more than $k^k$ members. They conjectured that the above construction is best possible but this was disproved by Frankl, Ota and Tokushige [10]. Here we prove the following analogous result.

**Theorem 3.4** Let $\mathcal{F}$ be a critical, intersecting family of subspaces of $V$ of dimension at most $k$. Then $|\mathcal{F}| \leq \left[ \binom{k}{1} \right]^k$.

**Proof.** Suppose that $|\mathcal{F}| > \left[ \binom{k}{1} \right]^k$. By induction on $i$, $0 \leq i \leq k$, we find an $i$-dimensional subspace $A_i$ of $V$ such that $|\mathcal{F}_{A_i}| > \left[ \binom{k}{1} \right]^k$. Indeed, since by induction $|\mathcal{F}_{A_i}| > 1$ and $\mathcal{F}$ is critical, the subspace $A_i$ is not hitting, and there is an $F \in \mathcal{F}$ disjoint from $A_i$. Now all elements of $\mathcal{F}_{A_i}$ meet $F$, and we find $A_{i+1} > A_i$ with $|\mathcal{F}_{A_{i+1}}| > |\mathcal{F}_{A_i}|/\left[ \binom{k}{1} \right]$. For $i = k$ this is a contradiction. $\square$

**Remark 3.5** For $l \leq k$ this argument shows that there are not more than $\left[ \binom{k}{1} \right]^2 \left[ \binom{k}{1} \right]^{l-1}$ $l$-spaces in $\mathcal{F}$.

If $l = 3$ and $\tau > 2$ then for the size of $\mathcal{F}$ the previous remark essentially gives $\left[ \binom{3}{1} \right]^2 \left[ \binom{n-3}{k-3} \right]$, which is the bound in Corollary 2.10.

Modifying the Erdős-Lovász construction (see Frankl [7]), one can get intersecting families with many $l$-spaces in the corresponding critical family.

**Construction 3.6** Let $A_1, \ldots, A_l$ be subspaces with $\dim A_1 = 1$, $\dim A_i = k + i - l$ for $i \geq 2$. Define $\mathcal{F}_i = \{ F \in \left[ \binom{k}{1} \right] : A_i \leq F, \dim(F \cap A_j) \geq 1 \text{ for } j > i \}$. Then $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_l$ is intersecting and the corresponding critical family has at least $\left[ \binom{k}{1} \right] \cdot \ldots \cdot \left[ \binom{k}{l} \right]$ $l$-spaces.

For $n$ large enough the Erdős-Ko-Rado theorem for vector spaces follows from the obvious fact that no critical, intersecting family can contain more than one 1-dimensional member. The Hilton-Milner theorem and the stability of the systems follow from (*) which was used to describe the intersecting systems with $\tau = 2$. As remarked above, the fact that the critical family has to contain only spaces of dimension 3 or more limits its size to $O(\left[ \binom{n}{k-3} \right])$, if $k$ is fixed and $n$ is large enough. Stronger and more general stability theorems can be found in Frankl [8] for the subset case.

## 4 Coloring $q$-Kneser graphs

In this section, we prove Theorem 1.5. We will need the following result of Bose and Burton [2] and its extension by Metsch [17].

**Theorem 4.1 (Bose-Burton)** If $\mathcal{E}$ is a family of 1-subspaces of $V$ such that any $k$-subspace of $V$ contains at least one element of $\mathcal{E}$, then $|\mathcal{E}| \geq \left[ \binom{n-k+1}{1} \right]$. Furthermore, equality holds if and only if $\mathcal{E} = \left[ \binom{H}{1} \right]$ for some $(n-k+1)$-subspace $H$ of $V$.
Proposition 4.2 (Metsch) If $E$ is a family of $\left[\begin{array}{c} n-k+1 \\ 1 \end{array}\right] - \varepsilon 1$-subspaces of $V$, then the number of $k$-subspaces of $V$ that are disjoint from all $E \in E$ is at least $\varepsilon q^{(k-1)(n-k)}$.

Proof of Theorem 1.5. Suppose that we have a coloring with at most $\left[\begin{array}{c} n-k+1 \\ 1 \end{array}\right]$ colors. Let $G$ (the good colors) be the set of colors that are point-pencils and let $B$ (the bad colors) be the remaining set of colors. Then $|G| + |B| \leq \left[\begin{array}{c} n-k+1 \\ 1 \end{array}\right]$. Suppose $|B| = \varepsilon > 0$.

By Proposition 4.2, the number of $k$-spaces with a color in $B$ is at least $\varepsilon q^{(k-1)(n-k)}$, so that the average size of a bad color class is at least $q^{(k-1)(n-k)}$. This must be smaller than the size of a HM-type family. Thus, by Lemma 2.3,

$$q^{(k-1)(n-k)} \leq \left[\begin{array}{c} k \\ 1 \end{array}\right] \left\lfloor \frac{n-2}{k-2} \right\rfloor.$$  

For $k \geq 3$ and $q \geq 3$, $n \geq 2k+1$ or $q = 2$, $n \geq 2k+2$, this is a contradiction. (The weaker form of Proposition 4.2, as stated in [17], suffices unless $q = 2$, $n = 2k+2$.) If $|B| = 0$, all color classes are point-pencils, and we are done by Theorem 4.1. 

5 Proof of Theorem 1.6

Let $a + b = n$, $a < b$ and let $F_a = F \cap \left[\begin{array}{c} V \\ a \end{array}\right]$ and $F_b = F \cap \left[\begin{array}{c} V \\ b \end{array}\right]$. We prove

$$|F_a| + |F_b| \leq \left[\begin{array}{c} n \\ \frac{b}{a} \end{array}\right]$$  \hspace{1cm} (5.4)

with equality only if $F_a = \emptyset$ and $F_b = \left[\begin{array}{c} V \\ b \end{array}\right]$.

Adding up (5.4) for $n/2 < b \leq n$ gives the bound on $|F|$ in Theorem 1.6 if $n$ is odd; adding the result of Greene and Kleitman [14] that states $|F_{n/2}| \leq \left[\begin{array}{c} n-1 \\ \frac{n}{2} \end{array}\right]$ proves it for even $n$. For the uniqueness part of Theorem 1.6, we only have to note that if $n$ is even then, by results of Godsil and Newman [13], we must have $F_{n/2} = \{F \in \left[\begin{array}{c} V \\ n/2 \end{array}\right] : E \leq F\}$ for some $E \in \left[\begin{array}{c} V \\ 1 \end{array}\right]$ or $F_{n/2} = \left[\begin{array}{c} V \\ n/2 \end{array}\right]$ for some $U \in \left[\begin{array}{c} V \\ n-1 \end{array}\right]$.

Now we prove (5.4). Consider the bipartite graph with vertex set $\left(\left[\begin{array}{c} V \\ a \end{array}\right], \left[\begin{array}{c} V \\ b \end{array}\right]\right)$ and join $A \in \left[\begin{array}{c} V \\ a \end{array}\right]$ and $B \in \left[\begin{array}{c} V \\ b \end{array}\right]$ if $A \cap B = 0$. Observe that $F_a \cup F_b$ is an independent set in this graph. Now, this graph is regular with degree $q^{ab}$. Therefore any independent set in this graph has size at most $\left[\begin{array}{c} n \\ b \end{array}\right]$ by König’s Theorem. Moreover, independent sets of size $\left[\begin{array}{c} n \\ b \end{array}\right]$ can only be $\left[\begin{array}{c} a \\ b \end{array}\right]$ or $\left[\begin{array}{c} V \\ b \end{array}\right]$, but the former is not an intersecting family. This proves (5.4). □

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