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Derivatives of Markov kernels and their Jordan decomposition

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Abstract

We study a particular class of transition kernels that stems from differentiating Markov kernels in the weak sense. Sufficient conditions are established for this type of kernels to admit a Jordan-type decomposition. The decomposition is explicitly constructed.

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1 Introduction

Let $P_\theta$ be a family of Markov kernels from a measurable space $(X, \mathcal{X})$ to a locally compact space $Y$ (a precise definition will be given later in the text), with $\theta \in \Theta \subset \mathbb{R}$, and let $\mathcal{C}_c(Y)$ denote the set of continuous real–valued mappings with compact support on $Y$. The Markov kernel $P_\theta$ is called \textit{weakly differentiable} at $\theta$ if for any $x \in X$ a finite signed measure $P_\theta'(x; \cdot)$ on $(Y, \mathcal{Y})$ exists such that for any $g \in \mathcal{C}_c(Y)$:

$$
\frac{d}{d\theta} \int g(y) P_\theta(x; dy) = \int g(y) P_\theta'(x; dy).
$$

This definition of weak differentiability is slightly more general than the original one in [4]: there (1) has to hold for any continuous bounded mapping $g$. Weak differentiability has been successfully applied to the theory of Markov chains. See [1] for an application to a problem in maintenance theory and [2] for an application to option pricing. The concept of weak differentiation is also related to finding optimal statistical tests, see [7]. For Markov chains, the following result is of particular interest: let $\pi_\theta$ denote the (unique) invariant distribution of $P_\theta$ (existence is assumed here), then it can be shown that

$$
\pi_\theta' = \pi_\theta \sum_{n=0}^{\infty} P_\theta^n P_\theta',
$$

where $P_\theta'$ is defined through (1) and $P_\theta^n$ denotes the $n$ fold product of $P_\theta$, see [4, 3] for a proof and more details on weak differentiability. If $P_\theta'$ exists, then the fact that $P_\theta'(x; \cdot)$ fails to be a probability measure poses the problem of sampling from $P_\theta'$. For $x \in X$ fixed, we can represent $P_\theta'(x; \cdot)$ by its Jordan decomposition as a difference between two probability measures as follows. For a finite signed measure $\mu$ denote its Jordan decomposition by $[\mu]^+$ and $[\mu]^-$, i.e., $\mu = [\mu]^+ - [\mu]^-$ and $[\mu]^+, [\mu]^-$ are positive measures. Let

$$
c_{P_\theta}(x) = [P_\theta']^+(x; X) = [P_\theta']^-(x; X)
$$

and

$$
P_\theta^+(x; \cdot) = \frac{[P_\theta']^+(x; \cdot)}{c_{P_\theta}(x)}, \quad P_\theta^-(x; \cdot) = \frac{[P_\theta']^-(x; \cdot)}{c_{P_\theta}(x)},
$$

then it holds, for all $g \in \mathcal{C}_c(Y)$, that

$$
\int g(y) P_\theta'(x; dy) = c_{P_\theta}(x) \left( \int g(y) P_\theta^+(x; dy) - \int g(y) P_\theta^-(x; dy) \right). \tag{4}
$$

For the above line of argument we fixed $x$. For $P_\theta^+$ and $P_\theta^-$ to be Markov kernels, we have to consider $P_\theta^+$ and $P_\theta^-$ as functions in $x$ and have to establish
measurability of $P_\theta^+ (\cdot; A)$ and $P_\theta^- (\cdot; A)$ for any $A \in \mathcal{Y}$. The solution of this problem implies that $c_{P_\theta} (\cdot)$ in (3) is measurable as a mapping from $X$ to $\mathbb{R}$. A representation of $P_\theta'$ through $(c_{P_\theta} (\cdot), P_\theta^+, P_\theta^-)$, with $c_{P_\theta}$ measurable and $P_\theta^\pm$ Markov kernels, is called a weak derivative of $P_\theta$. The existence of a weak derivative is of key importance for the statistical interpretation of (2) and for obtaining efficient unbiased gradient estimators.

In this paper, we give sufficient conditions for $P_\theta'$ to possess a representation as scaled difference of two Markov kernels. Specifically, we show that uniform boundedness of $P_\theta'$ (i.e., the supremum of $| \int g(y) P_\theta(x; dy) |$ over $g \in \mathcal{C}_c(Y)$ with $|g| \leq 1$ and $x \in X$ is finite) is together with a topological condition on $Y$ sufficient for $c_{P_\theta} (\cdot)$ in (3) to be measurable (and for $P_\theta^+$ and $P_\theta^-$ to be Markov kernels again). In conclusion we will show that uniform boundedness is sufficient for $P_\theta'$ to admit a weak derivative.

The paper is organized as follows. Section 1 introduces the basic concepts and definitions. Section 2 shows that, under suitable conditions, the kernel $P_\theta'$ as defined in (1) can be uniquely extended to the bounded Borel–measurable mappings. In Section 3 an explicit construct of a Jordan–type decomposition of $P_\theta'$ is given.

## 2 Conditional Integrals and Kernels

We say that a topological space is second countable if its topology is generated by a countable basis, i.e., if there exists a countable family of open (or closed) sets which generates the topology. Throughout the paper we let $Y$ always denote a locally compact second countable Hausdorff space. We denote by $\mathcal{Y}$ the $\sigma$–field of Baire measurable subsets of $Y$, i.e., the $\sigma$–field generated by the compact subsets of $Y$.

**Remark 1** On a second countable locally compact space the Borel–field (the $\sigma$–field generated by the open or closed sets) and the Baire–field coincide. (This holds true since any open set in a second countable locally compact space is a countable union of compact sets.) Thus, $\mathcal{Y}$ is the $\sigma$–field generated by the family of open sets in $Y$.

For example, the space $\mathbb{R}^n$ and any submanifold of it constitutes a locally compact second countable space.

**Remark 2** Notice that a metrizable space is second countable if and only if it is separable (see [8] Theorem 16.11). Conversely, a locally compact or even a compact space may be separable but not second countable. An example of
a separable compact space that fails to be second countable is provided by the Stone-Cech compactification of the natural numbers.

Let $X$ be an arbitrary set and let $\mathcal{X}$ be an arbitrary $\sigma$–field on $X$. Let $\mathcal{B}_b(Y)$ be the family of real–valued bounded $\mathcal{Y}$–measurable functions on $Y$, let $\mathcal{C}_c$ the family of continuous functions with compact support on $Y$ and let $\mathcal{B}(X)$ denote the family of real–valued $\mathcal{X}$–measurable functions on $X$.

We call a Baire measurable function, say $g$, simple if and only if an integer $n \in \mathbb{N}$ and, for $i \leq n$, sets $B_i \in \mathcal{Y}$ and constants $\gamma_i \in \mathbb{R}$ exist such that

$$g(y) = \sum_{i=1}^{n} \gamma_i 1_{B_i}(y), \quad y \in Y.$$  

The family of Baire measurable simple functions on $Y$ is denoted by $\mathcal{B}_{\text{simp}}(Y)$.

We note that $\mathcal{C}_c(Y) \subset \mathcal{B}_b(Y)$ and define the supremum norm $\| \cdot \|$ on $\mathcal{B}_b(Y)$ by

$$\|g\| := \sup_{y \in Y} |g(y)|.$$  

We call a set $\mathcal{G} \subset \mathcal{B}_b(Y)$ uniformly bounded or sup–norm bounded if

$$\sup_{g \in \mathcal{G}} \|g\| < \infty.$$  

We say that a sequence $(g_n)_{n \in \mathbb{N}}$ of functions $g_n \in \mathcal{B}_b(Y)$ is uniformly bounded if the set $\{g_n \mid n \in \mathbb{N}\}$ is uniformly bounded.

We say that a linear functional $J : \mathcal{C}_c(Y) \to \mathbb{R}$ is an integral if it is bounded on uniformly bounded subsets of $\mathcal{C}_c(Y)$ (such functionals may also be called sup-norm bounded). We say that a linear functional $\tilde{J} : \mathcal{B}_b(Y) \to \mathbb{R}$ is an extended integral if it is bounded on uniformly bounded subsets $\mathcal{G}$ of $\mathcal{B}_b(Y)$.

We say that a sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n$ from some set $S$ to a Hausdorff space $V$ converges point–wise if $\lim_{n \to \infty} f_n(s)$ exists for any $s \in S$.

**Definition 1** A kernel $P(\cdot, \cdot)$ from $X$ to $Y$ is a function $P : X \times Y \to \mathbb{R}$ such that $P(x, \cdot)$ is for any $x \in X$ a finite signed measure on $(Y, \mathcal{Y})$ and $x \mapsto P(x, B)$ is for any $B \in \mathcal{Y}$ a $\mathcal{X}$–measurable function on $X$. We say that the kernel is Markov (or a Markov kernel) if for any $x \in X$ the measure $P(x, \cdot)$ is a probability measure. We denote the space of all kernels from $X$ to $Y$ by $\mathcal{P}(X, Y)$.

**Definition 2** A conditional integral $I(\cdot, \cdot)$ from $X$ to $\mathcal{C}_c(Y)$ is a function $I : X \times \mathcal{C}_c(Y) \to \mathbb{R}$ such that
• $I(x, \cdot)$ is an integral (i.e. a linear functional on $\mathcal{C}_c(Y)$ which is sup-norm bounded) and
• $x \mapsto I(x, f)$ is for any $f \in \mathcal{C}_c(Y)$ a $\mathcal{X}$- measurable function on $X$.

We denote the space of conditional integrals from $X$ to $\mathcal{C}_c(Y)$ by $\mathcal{I}(X, Y)$.

**Definition 3** Let $Z$ denote an arbitrary Hausdorff space. We say that a function $F : \mathcal{B}_b(Y) \rightarrow Z$ is point-wise sequentially continuous on uniformly bounded subsets of $\mathcal{B}_b(Y)$ if for any uniformly bounded point-wise convergent sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{B}_b(Y)$ with limit $g \in \mathcal{B}_b(Y)$ we have that $\lim F(g_n) = F(g)$.

Given a function space $\mathcal{F} \subseteq \mathbb{R}^X$. We say that a set $S \subseteq \mathcal{F}$ is point-wise sequentially closed if $S$ contains all the limits which are in $\mathcal{F}$ of point-wise convergent sequences $(g_n)_{n \in \mathbb{N}}$ whose elements $g_n$ are in $S$. We say that a set $\overline{S}$ is the point-wise sequential closure of a set $S$ if $\overline{S}$ is the smallest point-wise sequentially closed set containing $S$. A set $S$ is point-wise sequentially dense in a set $T$ if $T$ is a subset of the sequential closure $\overline{S}$ of $S$. (For more details on sequential continuity and measurable functions see [5] Section 3.2.)

**Proposition 1** Let $K \subseteq Y$ be compact and let $O \subset Y$ be open with compact closure such that $K \subset O$. Then there exists a continuous function $f : Y \rightarrow [0, 1]$ such that $f(K) = 1$ and $f(Y \setminus O) = 0$.

**Proof.** This follows by an application of the Urysohn Lemma (see [8] 15.6) to $K$ and $Y \setminus O \cup \{\infty\}$ in the one-point compactification (see [8] 19.2 and 19A) $Y \cup \{\infty\}$ of $Y$, since any compact space is normal (see [8] 17.10). \qed

**Lemma 1** It holds that:

(a) The space $\mathcal{B}(X)$ is point–wise sequentially closed in $\mathbb{R}^X$.

(b) The function-space $\mathcal{B}_{\text{simp}}(Y)$ is point–wise sequentially dense in $\mathcal{B}_b(Y)$.

(c) The function-space $\mathcal{C}_c(Y)$ is point–wise sequentially dense in $\mathcal{B}_b(Y)$.

**Proof.** (a) Is the well known fact that a limit of a point–wise convergent sequence of measurable functions is again measurable.

(b) Is a re–formulation of the fact that any measurable function is the point wise limit of a sequence of simple functions. (See for example Corollary 3.2.1 of [5].)

(c) Given an arbitrary compact set $K$ we can by second countability and local compactness of $Y$ choose a sequence $(O_n)_{n \in \mathbb{N}}$ of open sets such that
On \( \bigcap_{n} O_{n} = K \) and the closures \( \overline{O_{n}} \) are compact. By Proposition 1 we find continuous functions \( f_{n} \) such that \( f_{n}(K) = 1 \) and \( f_{n}(Y \setminus O_{n}) = 0 \). Since \( \overline{O_{n}} \) is compact these functions \( f_{n} \) possess compact support. Thus, \( 1_{K} = \lim_{n \in \mathbb{N}} f_{n}(x) \), and \( 1_{K} \) lies in the point-wise sequential closure of \( C_{c}(Y) \). Since any open set \( O \) is the countable union of compact sets, we see that also \( 1_{O} \) and thus especially the function \( 1_{Y} \) belongs to the sequential closure of \( C_{c}(Y) \). (That \( 1_{Y} \) belongs to the sequential closure of \( C_{c}(Y) \) can also be easily seen using a countable partition of unity.) Hence, any linear combination of function \( 1_{A} \) with \( A \in \mathcal{Y} \) belongs to the sequential closure of \( C_{c}(Y) \). So we obtain (c) from (b).

Lemma 2

Any conditional integral \( I \in \mathcal{I}(X,Y) \) extends uniquely to a conditional integral \( \tilde{I} : X \times B_{b}(Y) \mapsto \mathbb{R} \) such that for any \( x \in X \) the function \( \tilde{I}(x,\cdot) \) is point-wise sequentially continuous on uniformly bounded subsets of \( B_{b}(Y) \). Moreover, there exists a one-one correspondence between kernels and conditional integrals \( G : \mathcal{P}(X,Y) \rightarrow \mathcal{I}(X,Y) \) given by

\[
[G(P)](x,f) = \int f(y) \, P(x,dy) \quad \text{for all} \quad f \in C_{c}(Y),
\]

or, if we prefer to consider the extensions \( \tilde{I} \) of the conditional integrals \( I \), by

\[
[\tilde{G}(P)](x,g) = \int g(y) \, P(x,dy),
\]

for all \( g \in B_{b}(Y) \).

We call the above extension \( \tilde{I} \) of a conditional integral \( I \) the extended conditional integral. By Lemma 1 there is a one–one correspondence between conditional integrals \( I \) and their extensions \( \tilde{I} \).

Proof of Lemma 2: The proof consists of 3 steps. First we show that for a given conditional integral \( I \in \mathcal{I}(X,Y) \) there exists for any \( x \in X \) a unique measure \( P(x,\cdot) \) on \( (Y,\mathcal{Y}) \). Then we show that the integrals \( I(x,\cdot) \) on \( C_{c}(Y) \) extend for arbitrary \( x \in X \) uniquely to extended integrals \( \tilde{I}(x,\cdot) \) on \( B_{b}(Y) \).

Step 1: Let \( I \) be a given conditional integral. According to the Riesz representation theorem, there exists for any \( x \in X \) a unique measure \( P(x,\cdot) \) on \( (Y,\mathcal{Y}) \), such that

\[
I(x,f) = \int f(y) \, P(x,dy) \quad \text{for all} \quad f \in C_{c}(Y).
\]

(6)
Thus, there exists for any $x \in X$ a unique extended integral $\tilde{I}(x, \cdot)$ such that

$$\tilde{I}(x, g) = \int g(y) \, P(x, dy) \quad \text{for all } g \in \mathcal{B}_b(Y). \quad (7)$$

Note that, by the dominated convergence theorem, $\tilde{I}(x, \cdot)$ is sequentially point-wise continuous on uniformly bounded sets. $\tilde{I}(x, \cdot)$ is also the unique extension of $I(x, \cdot)$ from $\mathcal{C}_c(Y)$ to $\mathcal{B}_b(Y)$ which is sequentially point-wise continuous on uniformly bounded sets, since $\{f \in \mathcal{C}_c(Y) \mid -1 \leq f \leq 1\}$ is point-wise sequentially dense in $\{g \in \mathcal{B}_b(Y) \mid -1 \leq g \leq 1\}$ (The fact that $\{f \in \mathcal{C}_c(Y) \mid -1 \leq f \leq 1\}$ is point-wise sequentially dense in $\{g \in \mathcal{B}_b(Y) \mid -1 \leq g \leq 1\}$ is proved completely analogous as we proved (c) in Lemma 1.)

**Step 2:** In the second step we show that the functions $x \mapsto \tilde{I}(x, g)$ are $\mathcal{X}$-measurable, for $g \in \mathcal{B}_b(Y)$ arbitrary, i.e., we show that $\tilde{I}$ is a conditional integral. Further we show that the unique corresponding function $P : X \times \mathcal{Y}$, defined in the first step, is a kernel.

Let $\mathbb{R}^X$ be endowed with the topology of point-wise convergence. Define an operator $T : \mathcal{B}_b(Y) \to \mathbb{R}^X$ by

$$[T(g)](x) = \tilde{I}(x, g).$$

The fact that, for arbitrary $x \in X$, the integral $\tilde{I}(x, \cdot)$ is point-wise sequentially continuous on uniformly bounded sets of $\mathcal{B}_b(Y)$ (where we take $M = \mathcal{B}_b(Y)$ and $V = \mathbb{R}$ in Definition 3) implies that $T$ is also point-wise sequentially continuous (where we take $M = \mathcal{B}_b(Y)$ and $V = \mathbb{R}^X$ in Definition 3).

Further, $f \in \mathcal{C}_c(Y)$ implies by definition of $T$ and the fact that $I \in \mathcal{I}(X, Y)$ that

$$T(f) = [x \mapsto I(x, f)] \in \mathcal{B}(X), \quad (8)$$

i.e., we have that $T(\mathcal{C}_c(Y)) \subseteq \mathcal{B}(X)$.

By (8) together with Lemma 1 (c) and the point-wise sequential continuity of $T$, we obtain that $T(\mathcal{B}_b(Y)) \subseteq \mathcal{B}(X)$. In other words, we obtain that $g \in \mathcal{B}$ implies that $x \mapsto I(x, g)$ is $\mathcal{X}$-measurable. The fact that $x \mapsto I(x, g)$ is $\mathcal{X}$-measurable implies in the case that $g$ is the characteristic function of a set $B$ that $x \mapsto P(x, B)$ is $\mathcal{X}$-measurable. Thus, $P$ is a kernel and (as already noted in the first step) by the Riesz representation theorem unique.

In the first two steps we have shown that to an integral $I \in \mathcal{I}(X, Y)$ there corresponds a unique kernel $P \in \mathcal{P}(X, Y)$ and a unique extended integral $\tilde{I}$. Further we know by equation (6) and (5) that this correspondence is given by $G^{-1}$. In the third step we show that to any $P \in \mathcal{P}(X, Y)$ there corresponds a unique $I = G(P) \in \mathcal{I}(X, Y)$. 
Step 3: We show now that any kernel $P$ corresponds to an unique integral $I$. That any kernel $P$ gives us by formula (7) for any $x$ an extended integral $\tilde{I}(x,.)$ is trivial. To show that $\tilde{I}$ is a conditional extended integral note that for any simple function $g = \sum_{i=1}^{n} \gamma_i 1_{B_i} \in B_{simp}$ we have:

$$\tilde{I}(x, g) = \sum_{i} \gamma_i P(x, B_i).$$

So for $g \in B_{simp}$ the function $x \mapsto \tilde{I}(x, g)$ is a finite sum of $\mathcal{X}$-measurable functions and thus itself $\mathcal{X}$-measurable. It remains to be shown that $x \mapsto \tilde{I}(x, g)$ is for any $g \in B_b(Y)$ a $\mathcal{X}$-measurable function. We do this by arguments analogous to the arguments provided in step 2 as will be explained in the following.

Let $T$ denote the operator defined in step 2. Recall that $T$ is point-wise sequentially continuous. Furthermore, $f \in B_{simp}(Y)$ implies (by definition of $T$ and the fact that for $g \in B_{simp}(Y)$ the function $x \mapsto \tilde{I}(x, g)$ is $\mathcal{X}$-measurable) that:

$$T(f) = [x \mapsto \tilde{I}(x, f)] \in \mathcal{B}(X),$$

i.e., we have that $T(B_{simp}(Y)) \subseteq \mathcal{B}(X)$.

By (9) together with Lemma 1 (b) and point-wise sequential continuity of $T$, we obtain that $T(B_b(Y)) = \mathcal{B}(X)$. In other words, we obtain that $g \in \mathcal{B}$ implies that $x \mapsto \tilde{I}(x, g)$ is $\mathcal{X}$-measurable.\]

Now we define weak differentiability of conditional integrals and kernels.

Definition 4 Let $\Theta$ be an open interval in $\mathbb{R}$ and let $\vartheta \mapsto I_\vartheta$ be a path in (mapping from $\Theta$ to) the space $\mathcal{I}(X,Y)$. We say that $\vartheta \mapsto I_\vartheta$ is weakly differentiable if

$$\frac{dI_\vartheta(x,f)}{d\vartheta} \text{ exists for all } (x,f) \in X \times \mathcal{C}_c(Y).$$

If $\vartheta \mapsto I_\vartheta$ is weakly differentiable then we say that it is bounded weakly differentiable if

$$\sup_{f \in \mathcal{C}_c(Y)} \left| \frac{dI_\vartheta(x,f)}{d\vartheta} \right| < \infty,$$

for any $x \in X$.

We say that a path $\theta \mapsto P_\theta$ in the space $\mathcal{P}(X,Y)$ of kernels is bounded differentiable if the corresponding path $\theta \mapsto G(P_\theta)$ in the space $\mathcal{I}(X,Y)$ of conditional integrals is bounded weakly differentiable.
Theorem 1 If the path $\vartheta \mapsto P_\vartheta$ in the space $\mathcal{P}(X,Y)$ is bounded weakly differentiable, then the weak derivative can be represented by a path $\vartheta \mapsto P'_\vartheta$ in the space $\mathcal{P}(X,Y)$. The connection between $\vartheta \mapsto P_\vartheta$ and $\vartheta \mapsto P'_\vartheta$ is given by

$$\int f(y) P'_\vartheta(x, dy) = \frac{d\int f(y) P_\vartheta(x, dy)}{d\vartheta}.$$ 

Proof. Let $I_\vartheta = G(P_\vartheta)$ be the corresponding path in the space of conditional integrals. Define for any $(x, f) \in X \times C_c(Y)$ the function $I'_\vartheta(x, f)$ by

$$I'_\vartheta(x, f) = \frac{dI_\vartheta(x, f)}{d\vartheta}.$$

Let $(h_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of positive reals which goes to 0. Then for $f \in C_c$ we have:

$$x \mapsto I'_\vartheta(x, f) = x \mapsto \frac{dI_\vartheta(x, f)}{d\vartheta} = x \mapsto \lim_{n \to \infty} \frac{I_{\vartheta + h_n}(x, f) - I_\vartheta(x, f)}{h_n}.$$

Thus, $x \mapsto I'_\vartheta(x, f)$ is for $f \in C_c(Y)$ a limit of a sequence of $\mathcal{X}$-measurable functions and therefore itself $\mathcal{X}$-measurable. Furthermore, $I'(x, \cdot)$ is by the condition of boundedness in the definition of bounded weakly differentiable for any $x \in X$ norm-bounded; i.e., $I'(x, \cdot)$ is bounded on uniformly bounded subsets of $C_c(Y)$. Thus, $I'(x, \cdot)$ is for any $x \in X$ an integral and $I'(\cdot, \cdot)$ is thus itself a conditional integral. By the correspondence between conditional integrals and kernels we obtain a kernel $P' = G^{-1}(I')$. The formula connecting $P'$ and $P$ is clear from the correspondence between $P'$, $P$ and $I'$, $I$ and the definition of $I'$.

3 Jordan Decomposition of Weak Derivatives of Markov Kernels

Definition 5 Given a kernel $P \in \mathcal{P}(X,Y)$ we define the absolute value $|P|$ of the kernel as follows:

$$|P|(x,B) = \sup_{\substack{A \subseteq \mathcal{Y} \\text{ s.t. } A \subseteq B \subseteq \mathcal{Y}}} 2 \cdot P(x,A) - P(x,B), \quad x \in X, B \in \mathcal{Y}.$$ 

Lemma 3 The absolute value $|P|$ of a kernel $P \in \mathcal{P}(X,Y)$ is again a kernel.

Proof: That the absolute value $|P|(x, \cdot)$ is a finite measure is a well known fact and it remains to be shown that the function

$$x \mapsto |P|(x,B)$$
is $\mathcal{X}$-measurable.

Let $\mathcal{A}$ be the set-field generated by a countable basis of the topology of $Y$. Then, $\mathcal{A}$ is countable and generates the $\sigma$-field $\mathcal{Y}$. For any set $B \in \mathcal{Y}$ and any measure $\mu$ on $(Y, \mathcal{Y})$ there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of sets $A_n \in \mathcal{A}$ such that $\lim \mu(A_n \triangle B) = 0$ (see [6] Lemma A.24). Thus, the function

$$x \mapsto |P|(x, B)$$

is the point-wise supremum over the countable family

$$\left\{ x \mapsto 2 \cdot P(x, A) - P(x, B) : A \in \mathcal{A} \text{ and } A \subseteq B \right\}$$

of $\mathcal{X}$-measurable functions and thus itself a $\mathcal{X}$-measurable function on $X$. $\square$

**Definition 6** We say that a kernel is positive if $P(x, B) \geq 0$ for all $(x, B) \in X \times Y$. We say that a pair of kernels $(P^+, P^-)$ forms a decomposition of a kernel $P$ if $P^+$ and $P^-$ are positive kernels and $P(x, B) = P^+(x, B) - P^-(x, B)$. We say that this decomposition is minimal or Jordan if for any other decomposition $(Q^+, Q^-)$ of $P$ we have $P^+(x, B) \leq Q^+(x, B)$ and $P^-(x, B) \leq Q^-(x, B)$.

**Corollary 1** Any kernel $P \in \mathcal{P}(X, Y)$ possesses a Jordan decomposition.

**Proof:** For $(x, B) \in X \times Y$ define

$$P^+(x, B) := \frac{|P|(x, B) + P(x, B)}{2}$$

and

$$P^-(x, B) := \frac{|P|(x, B) - P(x, B)}{2}.$$ 

Then, $P^+(x, B), P^-(x, B) \geq 0$ and $P^+(x, \cdot), P^-(x, \cdot)$ are measures, and $x \mapsto P^+(x, B)$ as well as $x \mapsto P^+(x, B)$ are $\mathcal{X}$-measurable functions on $X$. It is also clear that the decomposition is minimal. $\square$

**Theorem 2** Suppose that the path $\vartheta \mapsto P_{\vartheta}$ in the space $\mathcal{P}(X, Y)$ is bounded weakly differentiable and that for any $\theta$ the kernel $P_{\theta}$ is Markov. Then there exist for any $\vartheta$ Markov kernels $Q_{\vartheta}^+$ and $Q_{\vartheta}^-$ from $X$ to $Y$ and a $\mathcal{X}$-measurable function $c_{\vartheta} : X \to \mathbb{R}$ such that the weak derivative $P'_{\vartheta}$ of $P_{\vartheta}$ decomposes in the form

$$P_{\vartheta}(x, B) = c_{\vartheta}(x) \left( Q_{\vartheta}^+(x, B) - Q_{\vartheta}^-(x, B) \right) \forall (x, B) \in X \times Y.$$
**Proof:** By Theorem 1, the weak derivative \( P'_\vartheta \) is for any \( \vartheta \) a kernel and by the Corollary 1, \( P'_\vartheta \) possesses a Jordan decomposition \((P^+_\vartheta, P^-_\vartheta)\), i.e., \( P'_\vartheta = P^+_\vartheta - P^-_\vartheta \) and \( P^+_\vartheta, P^-_\vartheta \) are positive kernels. Since the \( P_\vartheta \) are Markov kernels we have \( P'_\vartheta(x, Y) = P^-_\vartheta(x, Y) \). Let \( c_\vartheta : X \to \mathbb{R} \) be defined by

\[
c_\vartheta(x) := P^+_\vartheta(x, Y) = P^-_\vartheta(x, Y).
\]

Since \( P^+_\vartheta \) is a kernel, the function \( c(\cdot) \) is \( X \)- measurable. Let

\[
Q^+_\vartheta(x, B) := \frac{1}{c(x)} P^+_\vartheta(x, B) \text{ for all } x \text{ with } c(x) > 0,
\]

\[
Q^-_\vartheta(x, B) := \frac{1}{c(x)} P^-_\vartheta(x, B) \text{ for all } x \text{ with } c(x) > 0
\]

and let for an arbitrary fixed probability measure \( \mu \), arbitrary \( x \) with \( c_\vartheta(x) = 0 \) and arbitrary \( B \in \mathcal{Y} \)

\[
Q^+_\vartheta(x, B) = Q^-_\vartheta(x, B) = \mu(B).
\]

Then \( Q^+_\vartheta \) as well as \( Q^-_\vartheta \) are Markov kernels. □

**Remark 3** This specific decomposition \((c_\vartheta(\cdot), Q^+_\vartheta, Q^-_\vartheta)\) is only possible because the kernels \( P'_\vartheta \) stem from weak differentiation of a Markov kernel valued function \( \vartheta \mapsto P_\vartheta \).

**References**


