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Scaling and asymmetry in an electromagnetically forced dipolar flow structure

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A dipolar flow structure is experimentally studied in a layer of salt solution driven by time-independent electromagnetic forcing. In particular, the response of the flow to the forcing is quantified by measuring the Reynolds number Re as a function of the Chandrasekhar number Ch (the ratio of Lorentz forces to viscous forces) and δ (the ratio of vertical to horizontal length scales of the flow domain). In agreement with theoretical predictions, two scaling regimes are found: $Re \sim Ch/\pi^2$ (viscous regime) and $Re \sim Ch^{1/2} \delta^{-1}$ (advective regime). The transition between the two regimes at $Ch^{1/2} \delta \sim \pi^2$ is reflected in the flow geometry in the form of an asymmetry of the dipolar flow structure.

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I. INTRODUCTION

Electromagnetic (EM) forcing of conducting fluids is nonintrusive, and for this reason it is an unparalleled tool for the study of a large variety of flows. In particular, EM forcing has been used in shallow layers of electrolyte to study quasi-two-dimensional turbulence [1,2], shallow vortices [3], stability of shear flows [4], fully controllable multiscale flows [5], and the principles of stretching and folding in quasi-two-dimensional flows [6]. In addition, research of this type of forcing has been motivated by its applications on EM mixing and stirring in metallurgy [7] and the enhancement of physical mechanisms associated with the different flow behaviors. We quantify the response of the flow by measuring the horizontal velocity field of a simple electromagnetically forced flow—a dipolar flow structure—while exploring the parameter space. Two well-defined regimes were observed: the linear regime discussed previously and a regime where the velocity scales with the magnitude of the forcing to the power 1/2. Furthermore, it is found that the flow depends only on the parameter $Ch \delta^2$ for the whole range of parameters studied.

The article is organized as follows. The experimental setup is described in Sec. II. Section III is devoted to the dimensional analysis of the problem, Section IV presents the experimental results. Then, in Sec. V, the implications for previous and future work are discussed, and finally, the conclusions are outlined in Sec. VI.

II. EXPERIMENTAL SETUP

The experimental setup consists of a water tank with a base of $30 \times 50$ cm$^2$, which is filled with a salt solution with a concentration of 178 g/l to a depth $H$ and covered with a transparent perspex lid to avoid free-surface deformations (Fig. 1). To force the flow, two titanium electrodes (coated with Ir-MMO) are placed along two opposite sides of the tank, and three $28 \times 10 \times 1$ cm$^3$ rectangular permanent magnets are placed 1.1 cm underneath the tank bottom. The electrodes are placed in compartments which are connected to the measurement area of the tank by a system of thin horizontal slits through which the electric current easily passes. The system of slits isolates the chemical reaction products generated at the electrodes from the flow to be studied. As shown in Fig. 2, the magnet at the center has its north pole facing up, while the two side magnets have their north pole facing down. A constant electric current is applied through the fluid using a power supply with a precision of $10^{-2}$ A. Due to the interaction of the electric current and the magnetic field of the magnets, a Lorentz force,

$$ F = J \times B, $$

(1)
is generated (with \( J \) the current density and \( \mathbf{B} \) the magnetic field), by which the fluid is set in motion.

We define a Cartesian coordinate system \( x = (x, y, z) \) with the origin at the center of the tank, \( x \) running parallel to the electrodes, \( y \) across the tank between the electrodes, and \( z \) in the vertical direction. Furthermore, we define the flow velocity \( \mathbf{v} = (u, v, w) \). We consider the electric current to be homogeneous and running only in the \( y \) direction, while the main component of the magnetic field is in the \( z \) direction. Hence, the principal component of the Lorentz force is in the \( x \) direction. To characterize the fluid, we consider two of its properties: the kinematic viscosity \( \nu \), which is considered to be primarily driven by \( \mathbf{v} \times \mathbf{B} \) in Ohm’s law, but in the present work, \( \nu \) is reserved in this study for the characteristic velocity scale.

Particle image velocimetry (PIV) was used to measure the horizontal velocity field of the flow in a plane at mid-depth. The fluid was seeded with 106–150 \( \mu \)m polymethylmethacrylate (PMMA) particles which were illuminated at mid-depth with a double pulsed Nd:YAG laser sheet. Images of the central 30 \( \times \) 30 cm\(^2\) area of the tank (see Fig. 2) were taken, using a Megaplus ES 1.0 camera, at different time intervals (ranging from 10 ms to 1.3 s) depending on the maximum velocity in the flow. These images were then cross-correlated using PIV software from PIVTEC GmbH, Göttingen, Germany, to calculate the horizontal velocity field.

To characterize the response of the flow, we focus on the velocity in the \( x \) direction—i.e., in the direction of the principal component of the forcing—at \( y = 0 \). From now on, we refer to this velocity distribution as \( \bar{u}(x) \) which, as we shall see, corresponds to the velocity distribution along the symmetry axis of the dipolar structure at \( y = 0 \). Furthermore, we consider the mean value of this distribution

\[
\langle \bar{u}(x) \rangle = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} \bar{u}(x) \, dx
\]

as the characteristic velocity scale.

### III. Dimensional Analysis

Dimensional analysis shows that four independent dimensionless parameters can be defined for this flow problem. The geometry of the tank is represented by two aspect ratios:

\[
\delta = \frac{H}{L_x} \quad \text{and} \quad \delta_L = \frac{L_x}{L_y},
\]

whereas the Chandrasekhar number

\[
\text{Ch} = \frac{IBH}{\rho \nu^2}
\]

defines the ratio of Lorentz to viscous force. Note that the definition of Ch introduced here differs from the one originally used by Chandrasekhar [17]. In the original definition, the current density \( \mathbf{J} \) is considered to be primarily driven by \( \mathbf{v} \times \mathbf{B} \) in Ohm’s law, but in the present work, \( \mathbf{J} \) is given by \( \mathbf{J} = I/(L_x, H) \) (with \( I \) the unit vector in the \( x \) direction) since the induced effects can be neglected and the imposed Lorentz force dominates in electrolytes [15]. This is not the case for other conductive fluids such as liquid metals (see, e.g., Ref. [18]).

A similar definition for the ratio of Lorentz to viscous forces (4) is commonly referred to as the Reynolds number based on the external force (see, e.g., Ref. [19]). However, the term Reynolds number is reserved in this study for the parameter characterizing the response of the flow, which is here defined as

\[
\text{Re} = \frac{(\bar{u}) L_x}{\nu},
\]

with \( (\bar{u}) \) as the mean velocity and which represents the ratio of inertia forces to viscous forces.

In the experiments, the aspect ratio \( \delta \) was set by the depth of the fluid, yielding the values \( \delta = 0.040, 0.067, \) and \( 0.107 \). For each value of \( \delta \), the Chandrasekhar number Ch was varied by changing the magnitude of the electric current. The magnitude of the magnetic field at mid-depth above the center of each magnet \( \mathbf{B} \) then takes the values \( B = 0.018, 0.017, \) and \( 0.015 T \) for the different values of \( \delta \), respectively. In the experiments described here, the horizontal aspect ratio \( \delta_L = 1 \) is kept constant.

We consider the flow to be governed by the Navier-Stokes equation including the Lorentz force

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \rho \nu \nabla^2 \mathbf{v} + \mathbf{J} \times \mathbf{B},
\]
where \( \rho \) is the pressure. From here on, the first term will be neglected since the flow is stationary in the range of parameters studied.

To analyze the possible balances of forces in the flow, we consider the order of magnitude of each term in (6), where the pressure term is considered to be of the same order as the largest term in the equation. From this analysis, two different regimes are obtained.

### A. Viscous regime

Due to the small depth, and hence, the predominance of friction at the bottom and the lid, we assume initially a Poiseuille-like vertical profile for the horizontal velocity field, i.e.,

\[
\nu(x, y, z) = u^*(x, y) \sin \left( \frac{\pi z}{H} \right),
\]

(7)

where \( \sin(\pi z/H) \) is the first term of the Fourier expansion of a Poiseuille profile. In this way, the order of magnitude of the viscous force is given by

\[
[\rho \nu \nabla^2 u] \approx \left[ \rho \nu \frac{\partial^2 u}{\partial z^2} \right] \sim \frac{\pi^2 \rho \nu \langle \langle u \rangle \rangle}{H^2},
\]

(8)

considering that \( \partial / \partial z \gg \partial / \partial x \).

The magnitude of the Lorentz force is given by

\[
[J \times B] = \frac{IB}{L_x H},
\]

(9)

as mentioned before.

For relatively weak forcing and correspondingly small Reynolds numbers, inertia can be neglected, and the dominant balance is between the Lorentz and the viscous forces, i.e.,

\[
\frac{IB}{L_x H} \sim \frac{\pi^2 \rho \nu \langle \langle u \rangle \rangle}{H^2},
\]

(10)

which is equivalent to

\[
\text{Re} \sim \frac{\text{Ch}}{\pi^2}.
\]

(11)

### B. Advective regime

The order of magnitude of the advective term is given by

\[
[(v \cdot \nabla)u] \sim \frac{\langle \langle u \rangle \rangle^2}{L_x},
\]

(12)

where the velocity is considered to be of order \( \langle \langle u \rangle \rangle \) and \( L_x \) is taken as the characteristic length scale since advection takes place mainly in the horizontal plane.

For relatively strong forcing and correspondingly large Reynolds numbers, we may assume that the Lorentz force is of the same order as the inertia forces, so

\[
\frac{IB}{L_x H} \sim \frac{\rho \langle \langle u \rangle \rangle^2}{L_x},
\]

(13)

which is equivalent to

\[
\text{Re} \sim \frac{\text{Ch}^{1/2}}{\delta}.
\]

(14)

In addition, we should recall that the flow is stationary, and hence, the input of energy due to the forcing has to be balanced by viscous dissipation. This means that the viscous forces in (6) must be also of the same order as the Lorentz forces, which cannot be achieved if the velocity has a Poiseuille-like vertical profile. That is, the presumption of (7) that the vertical gradient of the velocity is proportional to \( \langle \langle u \rangle \rangle \) is not valid in the advective regime. Hence, we assume that the velocity varies on a scale \( h \) such that

\[
\nu \nabla^2 u \equiv \left[ \frac{\nu}{H^2} \frac{\partial^2 u}{\partial z^2} \right] \sim \frac{\nu \langle \langle u \rangle \rangle}{h^2},
\]

(15)

where \( \nu' = z/h \) and \( h < H/\pi \). Finally, the balance of inertia and viscous forces yields the typical value

\[
h \sim \frac{H}{\text{Re}^{1/2} \delta}
\]

(16)

for the vertical length scale \( h \).

### C. Transition between the viscous and the advective regimes

The transition between the viscous regime and the advective regime is characterized by a change in the scaling of the Reynolds number as a function of the forcing. In this transition,

\[
\text{Re} \sim \frac{\text{Ch}^{1/2}}{\delta} \sim \frac{\text{Ch}}{\pi^2},
\]

(17)

which implies that the transition occurs when

\[
\text{Re} \delta^2 \sim \text{Ch}^{1/2} \delta \sim \pi^2.
\]

(18)

Note that, at this point, the critical value for the length scale \( h \),

\[
h \sim \frac{H}{\pi},
\]

(19)

can be obtained by comparing the magnitudes of the viscous forces in the advective and viscous regimes.

It is then convenient to define the normalized length scale

\[
h^* = \frac{\pi h}{2},
\]

(20)

which can be regarded as the thickness of the boundary layers that form next to the bottom and the lid in the advective regime. This would imply that the transition occurs when the thickness of the boundary layer \( h^* \) is of the same order of half the total depth \( H \), i.e., when

\[
\frac{h^*}{H} \sim \frac{\pi}{2 \text{Re}^{1/2} \delta} \sim \frac{1}{2},
\]

(21)

and that in the advective regime, the thickness of the boundary layer \( h^* \) is smaller than half the total fluid depth.

### IV. EXPERIMENTAL RESULTS

Figure 3(a) shows characteristic flow lines tangential to the instantaneous horizontal velocity components in the measurement plane for \( \text{Ch} = 1.3 \times 10^7 \) and \( \delta = 0.067 \). As can be seen, the forcing generates a dipolar structure with a symmetry axis \( y = 0 \). Apparently, for this value of Ch, the dipole is also nearly symmetric with respect to the line \( x = 0 \).

Figure 3(b) shows the flow lines for \( \text{Ch} = 8.8 \times 10^5 \), with the other parameters unchanged. A clear difference is observed between the flow lines in Figs. 3(a) and 3(b). In particular, the
flow lines for strong forcing spiral out of the vortex cores, in contrast with the flow lines for weak forcing, which are closed. This suggests that there is a strong horizontal divergence for large Ch values. This divergence is due to pumping of fluid from the Bödewadt boundary layers at the bottom and at the lid to the inside of the vortex cores [20]. In addition, there is a strong asymmetry with respect to the line $x = 0$ for Ch = $8.8 \times 10^3$. This asymmetry can be seen, for example, in the positions of the centers of the two cells which are no longer close to $x = 0$ but displaced in the positive $x$ direction.

To quantify some of the differences in the flow at different values of the forcing, we focus on $\tilde{u}(x)$, the velocity distribution along the symmetry axis of the dipolar structure at $y = 0$. Figure 4 shows the velocity distributions $\tilde{u}(x)$ for different values of Ch and $\delta = 0.04$. The magnitude of $\tilde{u}$ increases with increasing Ch value, and the asymmetry in this velocity distribution with respect to $x = 0$ becomes more pronounced for large Ch values.

Measured values of the Reynolds number $Re$, based on (5), are plotted in Fig. 5 as a function of the Chandrasekhar number $Ch$ for different values of the aspect ratio $\delta$. The axes have been rescaled with $\delta^2$, and as can be seen, the curves for the different values of $\delta$ collapse. Furthermore, the experimental results are compared with the theoretical predictions $Re \sim Ch/\delta^2$ and $Re \sim Ch^{1/2}/\delta$. The graph clearly shows the existence of the two characteristic scaling regimes: (i) $Re \sim Ch/\delta^2$ for $Ch^{1/2}/\delta < \pi^2$, and (ii) $Re \sim Ch^{1/2}/\delta$ for $Ch^{1/2}/\delta = Re\delta^2 > \pi^2$.

As the value of Ch increases, $q$ sharply increases until $Ch \approx 10^3$. For $Ch > 10^3$, the asymmetry in the flow lines tangential to the horizontal velocity components at mid-depth in the central $30 \times 30$ cm$^2$ region of the tank for (a) $Ch = 1.3 \times 10^3$, (b) $Ch = 8.8 \times 10^3$, and $\delta = 0.067$. The dashed line represents the line $x = 0$.
flow remains almost constant with \( q \approx 1.7 \). This saturation of \( q \) is probably due to the presence of the lateral boundary at \( x = L_x/2 \) since the maximum of \( \tilde{u}(x) \) must remain at a finite distance away from this boundary because \( \tilde{u}(L_x/2) = 0 \).

The transition between \( q \approx 1 \) and \( q \approx 1.7 \) corresponds to the change in scaling between the viscous and the advective regimes shown in Fig. 5. Even though the change in the asymmetry is smoother than the change in the scaling, it can be concluded that the increase in asymmetry is due to advection, which gradually becomes more important as the forcing magnitude is increased.

\section*{V. IMPLICATIONS FOR PREVIOUS AND FUTURE WORK}

Shallow flows are generally modeled with the quasi-two-dimensional Navier-Stokes equation \cite{4}:

\[
\frac{\partial \mathbf{v}_H}{\partial t} + (\mathbf{v}_H \cdot \nabla) \mathbf{v}_H = -\frac{1}{\rho} \nabla p + \lambda \mathbf{v}_H + \frac{f}{\rho},
\]

and the continuity equation

\[
\nabla \cdot \mathbf{v}_H = 0,
\]

where \( \mathbf{v}_H \) is the horizontal velocity, \( \nabla \) is the horizontal gradient operator, \( f \) is an external force, and \( \lambda \) is a constant known as the external friction parameter or the Rayleigh friction parameter. Over the years, many different expressions have been suggested for the friction parameter: \( \lambda = 2\nu/H^2 \) \cite{22}, \( \lambda = \pi^2 \nu/(4H^2) \) \cite{21,23}, or \( \lambda = 2\kappa \nu/H^2 \), where \( \kappa \) is a fitting parameter that depends on the velocity field \cite{4,9,13}.

In these cases, \( H \) is the total depth of the fluid between a solid bottom and a free surface instead of between a solid bottom and a solid lid, as considered in the current article. In general, a good agreement has been found between theory and experiments. These observations have led to believe that the use of a linear damping term to parametrize the effect of bottom friction in shallow flows is well supported.

However, in the current article, we have found the well-defined limit \( \text{Ch}^2 = \pi^4 \) for the use of (23) and (24) to model electromagnetically forced shallow flows. Above this limit, the damping rate due to bottom friction depends on the thickness of the boundary layer which, in turn, depends on the magnitude and distribution of the horizontal flow velocity.

Previous experiments have been usually carried out in fluid layers with a depth \( H \approx 0.2-0.3 \) cm, while the fluid depth is one order of magnitude larger in the experiments presented in the current article. Despite this difference, the corresponding nondimensional parameters have similar values, implying that the flows are dynamically similar.

In previous experiments on electromagnetically shallow flows, only the linear relationship between the forcing and the velocity has been reported, even though small deviations for strong forcing were also observed (see, e.g., Ref. \cite{10}). This implies that these experiments where performed mostly within the viscous regime, thus supporting previous experimental results.

Due to the success of (23) and (24) in describing shallow flows, this system of equations has been solved numerically to model an electromagnetically forced array of vortices in a shallow layer of an electrolyte \cite{24}. In these simulations, the magnitude of the forcing was varied while \( \lambda \) was kept constant. For small forcing magnitudes, a good agreement with laboratory experiments was obtained, and a threshold equivalent to \( \text{Ch}^2 = \pi^4 \) at which the vortices changed shape was observed. However, for stronger forcing magnitudes the numerical simulations started to differ significantly from the experimental data. This discrepancy can be easily explained since above the threshold \( \text{Ch}^2 = \pi^4 \) the damping rate depends on the forcing, and \( \lambda \) is no longer a constant.

In addition, for \( \text{Ch}^2 > \pi^4 \), the flow lines tangential to the horizontal velocity describe spirals originating at the vortex cores. This shape suggests a strong horizontal divergence, in disagreement with (24). In fact, it is for this type of flows with curvilinear streamlines and an additional secondary motion that Pononame et al. \cite{14} proposed that \( \text{Re} \propto \text{Ch}^{2/3} \). However, in our experiments, a clear regime with this scaling was not observed.

It has been noticed in previous work that experiments in a single shallow layer have some shortcomings for the study of two-dimensional turbulence. This motivated experiments in a two-layer configuration \cite{25,26}, which were later considered as ideal to study two-dimensional flows. The problem of the response of the flow to EM forcing in these two-layer experiments is certainly more complex than in a single layer, e.g., there is a larger number of nondimensional parameters, a deformable interface, and a deformable free-surface. However, the results presented in the current article indicate that there is also a dynamical limit for considering a two-layer shallow-flow as quasi-two-dimensional.

\section*{VI. CONCLUSIONS}

We studied experimentally the response of a generic electromagnetically forced flow. This response was quantified by measuring the Reynolds number \( \text{Re} \) as a function of the Chandrasekhar number \( \text{Ch} \) (the ratio of Lorentz forces to viscous forces). We found two scaling regimes: \( \text{Re} \sim \text{Ch}^{1/2} \) (viscous regime) and \( \text{Re} \sim \text{Ch}^{1/2} \delta^{-1} \) (advective regime), with a transition at \( \text{Re} \delta^2 \sim \text{Ch}^{1/2} \delta^{-1} \). This scaling is in good agreement with our theoretical predictions.

The transition between the two regimes is related to a qualitative change of the vertical velocity profile: from a Poiseuille-like profile in the viscous regime to a profile...
composed, in the advective regime, of an inviscid interior and two boundary layers, one at the bottom and one at the lid, each with a thickness $h^* = \pi H / (2\text{Re}^{1/2}/\delta)$. This transition marks the upper limit for the magnitudes of the forcing and the velocity where the quasi-two-dimensional Navier-Stokes equations (23) and the two-dimensional continuity equation (24) can be used to model shallow flows. Furthermore, it was found that the flow is characterized by a single parameter $\text{Re}\delta^2$. Such a conclusion had already been reached by Dolzhanskii [27] for the viscous regime where the flow is described by (23) and (24). However, it has been shown in the current article that this dependence extends to the advective regime.

In the particular case of the dipolar structure studied here, nonlinear effects are reflected in the form of an asymmetry due to the self-advection of the two vortices composing the dipole, as it had been previously observed [13,15]. These nonlinear effects, in the form of vortex-vortex interactions, can be already observed for $\text{Ch}^2 \gtrsim 30$ as inertia forces become increasingly important, and they are predominant in the advective regime when $\text{Re} \sim \text{Ch}^{1/2}/\delta$, i.e., for $\text{Ch}^2 > \pi^4$.

The current article presents new insight into the structure and dynamics of electromagnetically forced flows in a shallow layer of electrolyte. The results presented can serve as a guideline for future experimental and numerical work on, for example, shallow flows, quasi-two-dimensional turbulence, or the stability of quasi-two-dimensional spatially periodic flows. Another interesting line for future research is the study of the response of the flow to the electromagnetic forcing in other conductive fluids such as liquid metals, which are of interest in metallurgical processing applications.

[16] The kinematic viscosity was measured for the solution used in the experiments at working temperature $21^\circ$C using a capillary viscometer 501 13 from Schott Instruments.