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A slender-body theory for low-viscosity drops in shear flow between parallel walls

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A slender-body analysis is presented for the deformation and break-up of a highly confined and highly elongated low-viscosity drop in shear flow between two parallel walls that are separated by a distance less than the drop length. The analysis is simplified by the assumption that the drop has a circular cross section. The results show that confinement enhances the alignment of a low-viscosity drop with the imposed flow, thereby reducing its deformation and increasing the critical flow strength required for breakup. In the intermediate limit, where the wall separation is small compared with the drop length but large compared with its width, the dynamics can be related to that of an unconfined drop at a shear rate reduced by a factor of $\sqrt{3}$. Under these corresponding conditions, the drop length and cross-section profile are the same for both cases, whereas the centerline deflection of the confined drop is reduced relative to the unconfined case by $\sqrt{3}$. In the intermediate limit of wall separations, the critical flow strength for a confined drop is $\sqrt{3}$ times larger than that for an unconfined drop. © 2010 American Institute of Physics. [doi:10.1063/1.3379624]

I. INTRODUCTION

Predicting the morphology of a polymer blend is important because the macroscopic properties are governed to a large extent by the microstructure.¹ The two parameters most studied are the capillary number $Ca = \mu E a / \gamma$ and the viscosity ratio $\lambda = \mu_i / \mu$, where μ and μ_i are the viscosities of the matrix and drop-phase fluids, respectively, E is the imposed shear rate, γ is the coefficient of interfacial tension, and $\frac{4}{3}\pi a^3$ is the drop volume. Recently, there has been focus on morphology development in confined geometries as found in microfluidic devices.^{2,3} An interesting observation relates to the breakup behavior of drops in shear flow when confined between two parallel walls. Experiments show that confinement promotes the breakup of drops with high-viscosity ratios (i.e., lower critical capillary number) but hinders the breakup of drops with low-viscosity ratios.⁴ The former observation may be explained by the fact that high-viscosity drops in unbounded shear flow are stabilized by the rotational component of the flow.⁵ The presence of confining walls hinders the rotational component of the flow inside of the drops; thus drops maintain more alignment with the straining component of flow which promotes breakup. An explanation for the suppressed breakup of the low-viscosity drops is, however, lacking.

Boundary-integral methods have been developed to model the dynamics of drops confined between parallel walls under creeping flow conditions;^{6–8} an example is shown in Fig. 1, where the length and orientation of a confined and

unconfined low-viscosity drop in shear flow are depicted. The results indicate that confinement reduces drop deformation and enhances their alignment with the imposed flow. Boundary-integral methods are perhaps the best available numerical method for simulations of deformable drops in Stokes flows, but even they have difficulty describing the breakup of low-viscosity drops due to the characteristically long and slender drop shapes and pointed tips that form under conditions close to breakup. Slender-body theories have been developed to describe the dynamics and breakup of drops in the asymptotic limit $\lambda \rightarrow 0$.^{9–13} In this paper, we develop a slender-body theory for a low-viscosity drop confined between two planar boundaries in the shear flow generated by the tangential motion of these boundaries.

II. PROBLEM FORMULATION

A. Slender-body approximation

We consider a drop with a surfactant-free interface confined between two parallel walls separated by a distance $2W$ for the case $\lambda \ll 1$. Creeping flow conditions are assumed. The walls undergo a relative tangential motion which generates the imposed shear flow,

$$\mathbf{u}^\infty = (Ey, 0, 0), \quad (1)$$

as shown in Fig. 2, where the coordinates and drop-shape parameters are defined. Deformable drops will migrate to the center of the gap between two parallel walls;^{14,15} herein, we assume that the drop is initially centered, as shown in the figure. The drop is assumed to have a slender shape with half-length l and centerline deflection $\eta(x, t) \ll l$, as sketched in Fig. 2. By symmetry, η is an odd function of x with $\eta > 0$ for $x > 0$. Following Hinch and Acrivos,¹² we invoke the *ad hoc* assumption that the drop has a circular cross

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FIG. 1. Boundary-integral simulations (Ref. 8) for evolution of initially spherical drops with radius a in shear flow $\lambda=0.01$ and $Ca=0.8$; (a) half-length l of drops and stationary shapes (trace of drop interface in shear plane) including boundary walls for confined case, (b) alignment, $a/\eta(l)$ (see Fig. 2); isolated drop (solid curves), confined drop $a/W=0.9$ (dashed curves); and steady state predicted by slender-body theory (straight lines, $Et \geq 15$).

section with local radius $R(x,t)$, in order to simplify the analysis. By the assumption that the drop is slender, $R \ll l$, by symmetry, R is an even function of x , and $R=0$ at the ends of the drop $x = \pm l$. Herein, we assume highly confined conditions, where the wall separation is small compared with the drop length,

$$W \ll l. \quad (2)$$

We define fixed Cartesian coordinates (x, y, z) with corresponding unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$; the center of the channel is at $y=0$. Following Hinch and Acrivos,¹² we also use a slightly skew local coordinate system (s, y^*, z^*) centered on the drop axis, where $y^* = y - \eta(x)$, $z^* = z$, and $s \approx x$ is distance along the centerline which is not quite aligned with the x -axis. We define the local cylindrical coordinates (x, r, θ) ,

$$y^* = y - \eta(x) = r \sin \theta, \quad z^* = z = r \cos \theta, \quad (3)$$

where $r=0$ coincides with the centerline, and $\theta=0$ coincides with the z -axis.

B. Scaling analysis

A scaling analysis reveals certain important features of the problem, supports the use of the slender-body approximation, and provides the relevant length and time scales for nondimensionalization. The scaling arguments presented below show that a low-viscosity drop in shear flow becomes highly elongated at high shear rates, and nearly aligned with the imposed velocity. Thus, the drop orientation has only a small projection onto the straining component of the imposed flow. However, this small projection of the drop orientation onto the straining component of the flow, or equivalently, its slight misalignment with respect to the imposed velocity field, $\eta(x)$, is an essential feature of the drop dynamics. A slender low-viscosity drop in shear flow generates negligible normal stresses on its interface if it is completely aligned with the imposed velocity, i.e., $\eta \equiv 0$. High elongation relies on the normal stresses generated by the drop misalignment to stretch the drop.

Pressure p generated in the continuous phase fluid by the presence of the elongated and nearly horizontal drop scales

as $\mu E \beta$, where $\beta \sim \eta/l$ is the small inclination angle of the drop with respect to the flow direction, or equivalently, its projection onto the straining axis of the flow. Under the assumption that the centerline deflection scales with the cross-sectional radius, $\eta \sim R$, we have $p \sim \mu E R/l$. Balancing the pressure and the $O(\gamma/R)$ capillary stresses, we have $a l R^{-2} \sim Ca$. By conservation of drop volume, the length and cross section of a drop are related by

$$l R^2 \sim a^3, \quad (4)$$

thus, we obtain

$$l/a \sim Ca^{1/2}, \quad (5)$$

which indicates that low-viscosity drops (or bubbles) in shear flow are much shorter than in straining flow where $l/a \sim Ca^2$ under strong flow conditions, $Ca \gg 1$.^{10,12}

The flow inside of the slender drop is approximately unidirectional and quasisteady, thus

$$\lambda \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{u}_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \hat{u}_x}{\partial \theta^2} \right] = \hat{p}', \quad (6)$$

where $\hat{\mathbf{u}} = \hat{u}_x(x, r, \theta) \mathbf{e}_x$ and $\hat{p} = \hat{p}(x)$ are the velocity and pressure inside of the drop, and the prime denotes an x -derivative. By the continuity of the velocity at the drop interface, we have $\hat{u}_x \sim E \eta$, and by the assumption that $\eta \sim R$, we have

FIG. 2. Slender drop in shear flow generated by tangential motion of equidistant parallel confining walls. Drop has a circular cross section with local radius $R(x, t)$ and centerline deflection $\eta(x, t)$. Far-field wall-correction velocity, $\hat{u}$.

$$\hat{u}_x \sim ER. \quad (7)$$

In flows sufficiently strong to induce breakup, variations of the internal pressure $\delta\hat{p}$ become comparable to capillary stresses γ/R . Inserting these estimates into Eq. (6), and scaling x by l and r by R , we obtain $\lambda\text{Ca}(l/a) \sim 1$. Combining this result with Eq. (5), we obtain an estimate for the critical capillary number

$$\text{Ca}_{\text{cr}} \sim \lambda^{-2/3}, \quad (8)$$

and accordingly introduce the flow strength parameter

$$G = \text{Ca}\lambda^{2/3}. \quad (9)$$

Henceforth, $G=O(1)$ is assumed. Thus, from Eqs. (4) and (5), we obtain

$$R/a \sim \lambda^{1/6}, \quad l/a \sim \lambda^{-1/3}. \quad (10)$$

The result indicates that $l/R=O(\lambda^{-1/2})$ which supports the use of the slender-body approximation for $\lambda \ll 1$.¹⁶ According to the scaling estimates (10), we shall nondimensionalize lengths in the flow direction by $a\lambda^{-1/3}$ and lengths in the cross-flow directions by $a\lambda^{1/6}$. The time for convection over the length of the drop, $l/\hat{u}_x$, sets the time scale for the drop dynamics, thus we shall nondimensionalize time by $E^{-1}\lambda^{-1/2}$, according to Eqs. (7) and (10). We thus introduce the dimensionless variables,

$$\bar{x} = \frac{x}{a\lambda^{-1/3}}, \quad \bar{r} = \frac{r}{a\lambda^{1/6}}, \quad \bar{t} = \frac{t}{E^{-1}\lambda^{-1/2}}, \quad (11)$$

and the dimensionless drop-shape and confinement parameters

$$\bar{R} = \frac{R}{a\lambda^{1/6}}, \quad \bar{\eta} = \frac{\eta}{a\lambda^{1/6}}, \quad \bar{l} = \frac{l}{a\lambda^{-1/3}}, \quad \bar{W} = \frac{W}{a\lambda^{1/6}}. \quad (12)$$

Given the scales for the drop-shape and confinement parameters, the assumption of highly confined conditions (2) is equivalent to

$$\bar{W} \ll \lambda^{-1/2}. \quad (13)$$

From the variables (11), it follows that the components of the dimensionless velocity and stress are defined

$$\bar{u}_x = \frac{u_x}{Ea\lambda^{1/6}}, \quad \bar{u}_r = \frac{u_r}{Ea\lambda^{2/3}}, \quad (14)$$

$$\bar{u}_\theta = \frac{u_\theta}{Ea\lambda^{2/3}}, \quad \bar{p} = \frac{p}{\mu E\lambda^{1/2}},$$

$$\bar{\sigma}_{rr} = \frac{\sigma_{rr}}{\mu E\lambda^{1/2}}, \quad \bar{\sigma}_{\theta\theta} = \frac{\sigma_{\theta\theta}}{\mu E\lambda^{1/2}}, \quad \bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\mu E\lambda^{1/2}}, \quad (15)$$

$$\bar{\sigma}_{r\theta} = \frac{\sigma_{r\theta}}{\mu E\lambda^{1/2}}, \quad \bar{\sigma}_{xr} = \frac{\sigma_{xr}}{\mu E}, \quad \bar{\sigma}_{x\theta} = \frac{\sigma_{x\theta}}{\mu E}. \quad (16)$$

By the symmetry of the stress tensor, we have

$$\bar{\sigma}_{rx} = \bar{\sigma}_{xr}, \quad \bar{\sigma}_{\theta x} = \bar{\sigma}_{x\theta}, \quad \bar{\sigma}_{\theta r} = \bar{\sigma}_{r\theta}. \quad (17)$$

By continuity of the velocity and the normal stress balance, the velocity and pressure in the drop-phase fluid have the same characteristic scales as the corresponding quantities in the external phase fluid, thus we define

$$\hat{\bar{u}}_x = \frac{\hat{u}_x}{Ea\lambda^{1/6}}, \quad \hat{\bar{p}} = \frac{\hat{p}}{\mu E\lambda^{1/2}}. \quad (18)$$

However, the components of the viscous stress tensor in the drop-phase fluid are $O(\lambda)$ weaker than the corresponding quantities in the external phase fluid. Thus, the stress tensor in the drop-phase fluid is dominated by the isotropic contribution of pressure and to $O(\lambda)$ can be approximated as

$$\hat{\bar{\sigma}} = -\hat{\bar{p}}\mathbf{I}. \quad (19)$$

C. External flow

Following Hinch and Acrivos,¹² the disturbance velocity associated with the slender low-viscosity drop is approximated by distribution of low-order Stokes flow singularities along the centerline of the drop. Thus, the total velocity field at a point $\mathbf{x}$ outside of the drop is expressed in unscaled variables as

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^\infty(\mathbf{x}) + \int_{-l}^{+l} \left[\frac{3}{2}f(s)\mathbf{T}_{xy}(\hat{\mathbf{x}}; W) + \frac{1}{2}g(s)\mathbf{S}(\hat{\mathbf{x}}; W) + \frac{1}{2}h(s)\mathbf{D}_y(\hat{\mathbf{x}}; W) \right] ds, \quad (20)$$

where $\mathbf{u}^\infty$ is the undisturbed flow (1), $\mathbf{x}_0=(x_0, y_0, z_0)$ is a singular point, and $\hat{\mathbf{x}}=\mathbf{x}-\mathbf{x}_0$. Given that the singularities are located on the drop centerline, we have $\mathbf{x}_0=(s, \eta(s), 0)$, where $-l \leq s \leq l$ is the distance along the centerline. The singular velocities $\mathbf{T}_{xy}$, $\mathbf{S}$, and $\mathbf{D}_y$ are, respectively, the xy -column of the stresslet, a point source, and the y -column of the source-dipole. These solutions decay away from the drop and satisfy no-slip boundary conditions at $y = \pm W$. The stresslet, source, and source-dipole distributions on the drop centerline are $4\pi\mu f$, $2\pi g$, and $2\pi h$. Note that unscaled variables are used in Eq. (20) and in the remainder of this subsection; the different scales for the velocity components given by Eq. (14) make the use of dimensionless variables inconvenient.

The singular velocities are conveniently expressed as

$$\mathbf{v}(\hat{\mathbf{x}}; W) = \mathbf{v}^{(0)}(\hat{\mathbf{x}}) + \mathbf{v}^{2W}(\hat{\mathbf{x}}; W), \quad (21)$$

where $\mathbf{v}$ is any one of the singular velocities $\mathbf{T}_{xy}$, $\mathbf{S}$, or $\mathbf{D}_y$. Here, $\mathbf{v}^{(0)}(\hat{\mathbf{x}})$ is the free-space singular velocity field for an unbounded fluid and $\mathbf{v}^{2W}(\hat{\mathbf{x}}; W)$ is the nonsingular two-wall velocity field correction such that $\mathbf{v}^{(0)}(\hat{\mathbf{x}}) + \mathbf{v}^{2W}(\hat{\mathbf{x}}; W)$ satisfies no-slip boundary conditions at $y = \pm W$. Herein, we employ the superposition approximation of the two-wall velocity field correction

$$\mathbf{v}^{2W}(\hat{\mathbf{x}}; W) \approx \mathbf{v}^W(\hat{\mathbf{x}}; +W) + \mathbf{v}^W(\hat{\mathbf{x}}; -W), \quad (22)$$

where $\mathbf{v}^W(\hat{\mathbf{x}}; \pm W)$ is the one-wall velocity field correction such that $\mathbf{v}^{(0)}(\hat{\mathbf{x}}) + \mathbf{v}^W(\hat{\mathbf{x}}; \pm W)$ satisfies the no-slip boundary

condition at $y = \pm W$. The superposition approximation is rigorous for wall separations large compared with the width of the drop, $W \gg R$, which is compatible with the assumption that the drop is long compared with the wall separation (2) given that the drop is slender. Often, however, approximation (22) is accurate even when this assumption is not satisfied.^{6,8,17,18} In general, the superposition approximation is reliable provided that the wall correction velocity $\mathbf{v}^W(\hat{\mathbf{x}}; W)$ is dominated by the primary velocity $\mathbf{v}^{(0)}(\hat{\mathbf{x}})$.

The one-wall correction field corresponding to a given velocity field $\mathbf{v}^{(0)}(\hat{\mathbf{x}})$ is given by^{17,19}

$$\mathbf{v}^W(\hat{\mathbf{x}}; \pm W) = P(\mathbf{R}_0 + H\mathbf{R}_1 + H^2\mathbf{R}_2) \cdot \mathbf{v}^{(0)}(\hat{\mathbf{x}}). \quad (23)$$

Here, the operators are defined

$$\begin{aligned} \mathbf{R}_0 &= -\mathbf{I}_y - 2\hat{y} \nabla \mathbf{e}_y + \hat{y}^2 \nabla^2 \mathbf{I}, \\ \mathbf{R}_1 &= -2 \nabla \mathbf{e}_y + 2\hat{y} \nabla^2 \mathbf{I}, \quad \mathbf{R}_2 = \nabla^2 \mathbf{I}, \end{aligned} \quad (24)$$

where $\mathbf{I}$ is the identity tensor, $\mathbf{I}_y = \mathbf{I} - 2\mathbf{e}_y \mathbf{e}_y$, $H = y_0 \mp W$, and $\hat{y} = y - y_0$. The mirror operator P is defined

$$P(\mathbf{w})(x, y, z) = \mathbf{I}_y \cdot \mathbf{w}(x, \pm 2W - y, z). \quad (25)$$

The requisite singularity solutions $\mathbf{v}^{(0)}(\hat{\mathbf{x}})$ are

$$\begin{aligned} \mathbf{T}_{xy}^{(0)}(\hat{\mathbf{x}}) &= \frac{(x - x_0)(y - y_0)\hat{\mathbf{x}}}{|\hat{\mathbf{x}}|^5}, \quad \mathbf{S}^{(0)}(\hat{\mathbf{x}}) = \frac{\hat{\mathbf{x}}}{|\hat{\mathbf{x}}|^3}, \\ \mathbf{D}_y^{(0)}(\hat{\mathbf{x}}) &= \frac{\mathbf{e}_y}{|\hat{\mathbf{x}}|^3} - 3 \frac{(y - y_0)\hat{\mathbf{x}}}{|\hat{\mathbf{x}}|^5}. \end{aligned} \quad (26)$$

Based on the superposition approximation (22), we can express external velocity (20) as

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^\infty(\mathbf{x}) + \mathbf{u}^{(0)}(\mathbf{x}) + \mathbf{u}^W(\mathbf{x}; +W) + \mathbf{u}^W(\mathbf{x}; -W), \quad (27)$$

where we separately define the contributions to the disturbance velocity from the distributions of free-space singularities and the corresponding one-wall correction velocity

$$\begin{aligned} \mathbf{u}^{(0)}(\mathbf{x}) &= \int_{-l}^{+l} \left[\frac{3}{2} f(s) \mathbf{T}_{xy}^{(0)}(\hat{\mathbf{x}}) + \frac{1}{2} g(s) \mathbf{S}^{(0)}(\hat{\mathbf{x}}) \right. \\ &\quad \left. + \frac{1}{2} h(s) \mathbf{D}_y^{(0)}(\hat{\mathbf{x}}) \right] ds, \\ \mathbf{u}^W(\mathbf{x}; \pm W) &= \int_{-l}^{+l} \left[\frac{3}{2} f(s) \mathbf{T}_{xy}^W(\hat{\mathbf{x}}; \pm W) \right. \\ &\quad \left. + \frac{1}{2} g(s) \mathbf{S}^W(\hat{\mathbf{x}}; \pm W) \right. \\ &\quad \left. + \frac{1}{2} h(s) \mathbf{D}_y^W(\hat{\mathbf{x}}; \pm W) \right] ds. \end{aligned} \quad (28)$$

The kernel functions $\mathbf{T}_{xy}^{(0)}$, $\mathbf{S}^{(0)}$, $\mathbf{D}_y^{(0)}$, $\mathbf{T}_{xy}^W$, $\mathbf{S}^W$, and $\mathbf{D}_y^W$ are defined by Eqs. (23)–(26).

D. Internal flow

By the disparity of the length scales in the flow and cross-flow directions, a lubrication approximation is appropriate for the flow inside of the slender drop; accordingly, the quasisteady unidirectional approximation (6) applies.¹² The internal velocity field is obtained by integrating Eq. (6) and enforcing continuity of the tangential component of the velocity at the drop interface. It is convenient to decompose the internal velocity field as

$$\hat{\mathbf{u}}_x = \hat{\mathbf{u}}_p + \hat{\mathbf{u}}_t, \quad (30)$$

where

$$\hat{\mathbf{u}}_p = -\frac{1}{4}(\bar{R}^2 - \bar{r}^2)\hat{\mathbf{p}}', \quad (31)$$

is the pressure-driven component of the flow that obeys no-slip boundary conditions at the drop interface. The prime denotes a derivative with respect to $\bar{x}$. The tangentially driven component of the velocity, $\hat{\mathbf{u}}_t(\bar{r}, \theta)$, is the homogeneous solution of Eq. (6) (Laplace's equation) that satisfies the boundary condition

$$\hat{\mathbf{u}}_t(\bar{R}, \theta) = \bar{\mathbf{u}}_x(\bar{R}, \theta), \quad (32)$$

where $\bar{\mathbf{u}}_x(\bar{R}, \theta)$ is the x -component of the external velocity field evaluated on drop interface. The cross-sectional area average of the pressure-driven velocity component is

$$\langle \hat{\mathbf{u}}_p \rangle = -\frac{\bar{R}^2}{8} \hat{\mathbf{p}}'. \quad (33)$$

Using Eq. (A4), derived in Appendix A, we obtain the cross-sectional area average of the tangentially driven velocity as the perimeter average of the x -component of the external velocity,

$$\langle \hat{\mathbf{u}}_t \rangle = \langle \bar{\mathbf{u}}_x \rangle_R = \frac{1}{2\pi} \int_0^{2\pi} \bar{\mathbf{u}}_x(\bar{R}, \theta) d\theta. \quad (34)$$

The axial volume flux inside the drop, $\bar{q} = \pi \bar{R}^2 \langle \hat{\mathbf{u}}_x \rangle$, is governed by the conservation relation

$$2\pi \bar{R} \dot{\bar{R}} + \bar{q}' = 0, \quad (35)$$

where the dot denotes a derivative with respect to $\bar{t}$. Integrating this relation with symmetry condition $\bar{q}(0) = 0$ yields

$$\langle \hat{\mathbf{u}}_x \rangle = -\frac{2}{\bar{R}^2} \int_0^{\bar{x}} \bar{R} \dot{\bar{R}} d\bar{x}. \quad (36)$$

Combining this result with Eqs. (30), (33), and (34), and integrating yields the internal pressure,

$$\hat{\mathbf{p}} = \hat{\mathbf{p}}_0(\bar{t}) - 8 \int_0^{\bar{x}} \bar{R}^{-2} [\langle \hat{\mathbf{u}}_x \rangle - \langle \bar{\mathbf{u}}_x \rangle_R] d\bar{x}, \quad (37)$$

where the uniform pressure $\hat{\mathbf{p}}_0(t)$ is determined from the constraint imposed by the conservation of drop volume,

$$\frac{3}{4\pi a^3} \int_{-l}^{+l} \pi R^2 dx = \frac{3}{4} \int_{-l}^{+l} \bar{R}^2 d\bar{x} = 1. \quad (38)$$

E. Boundary conditions on interface

Under the assumption that surface tension gradients are absent, the stress balance at the interface is given by

$$[\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}] \cdot \mathbf{n} = 2\kappa\gamma\mathbf{n}. \quad (39)$$

Here, $\boldsymbol{\sigma}$ and $\hat{\boldsymbol{\sigma}}$ are the stress tensors in the continuous- and drop-phase fluids; $\mathbf{n}$ and κ are the outward normal vector and mean curvature on the drop interface which are given by

$$\mathbf{n} = \frac{[-(R' + \eta' \sin \theta), 1, 0]}{\sqrt{1 + (R' + \eta' \sin \theta)^2}}, \quad 2\kappa = \frac{1}{R\sqrt{1 + R'^2}} - R''. \quad (40)$$

Using the dimensionless variables (11) and (12), we obtain the approximations

$$\mathbf{n} = [-\lambda^{1/2}(\bar{R}' + \bar{\eta}' \sin \theta), 1, 0] + O(\lambda), \quad (41)$$

$$2\bar{\kappa} = \bar{R}^{-1} + O(\lambda),$$

where $\bar{\kappa} = a\lambda^{1/6}\kappa$ is the dimensionless curvature.

The leading-order stress balance at the drop interface is obtained by inserting the pressure-dominated approximation of the drop-phase stress tensor (19) and the geometric approximations (41) into Eq. (39). Expressed in component form and in dimensionless variables (11)–(18), we obtain,

$$\bar{\sigma}_{xr} = O(\lambda), \quad \bar{\sigma}_{rr} = (G\bar{R})^{-1} - \bar{p} + O(\lambda), \quad (42)$$

$$\bar{\sigma}_{r\theta} - (\bar{R}' + \bar{\eta}' \sin \theta)\bar{\sigma}_{x\theta} = O(\lambda),$$

where G is the flow strength parameter (9).

Mass conservation at the drop interface requires the kinematic condition,

$$\dot{R} + \dot{\eta} \sin \theta = \mathbf{u} \cdot \mathbf{n}. \quad (43)$$

In dimensionless variables with $\mathbf{n}$ given by Eq. (41), this becomes

$$\dot{\bar{R}} + \dot{\bar{\eta}} \sin \theta = -(\bar{R}' + \bar{\eta}' \sin \theta)\bar{u}_x + \bar{u}_r, \quad (44)$$

where dot denotes a derivative with respect to $\bar{t}$ and prime denotes a derivative with respect to $\bar{x}$.

For a given drop shape, Eqs. (27)–(29), (37), and (42) provide a closed set of equations for the velocity; the evolution of the drop shape is governed by the kinematic condition (44) subject to the volume constraint (38).

III. SOLUTION OF THE HYDRODYNAMIC PROBLEM

In this section, we evaluate the velocity and associated stress and present the results in terms of dimensionless variables (11)–(18) and the dimensionless singularity distribution functions,

$$\bar{f} = \frac{f}{Ea^2\lambda^{1/3}}, \quad \bar{g} = \frac{g}{Ea^2\lambda^{5/6}}, \quad \bar{h} = \frac{h}{Ea^3\lambda}. \quad (45)$$

The singularity distribution functions are then determined from the stress balance equations (42) and are presented at the end of this section.

A. External flow

It is convenient to express the velocity in terms of local coordinates (3) relative to the drop centerline, noting that in these coordinates, the boundary walls are located at

$$y^* = \bar{W}_{\pm}^* = \pm \bar{W} - \bar{\eta}(\bar{x}). \quad (46)$$

Accordingly, we express the external velocity field (27) as

$$\bar{\mathbf{u}}(\bar{\mathbf{x}}^*) = \bar{\mathbf{u}}^{\infty}(\bar{\mathbf{x}}^*) + \bar{\mathbf{u}}^{(0)}(\bar{\mathbf{x}}^*) + \bar{\mathbf{u}}^W(\bar{\mathbf{x}}^*; \bar{W}_{-}^*) + \bar{\mathbf{u}}^W(\bar{\mathbf{x}}^*; \bar{W}_{+}^*), \quad (47)$$

and similarly express the pressure and stress,

$$\bar{p}(\bar{\mathbf{x}}^*) = \bar{p}^{\infty}(\bar{\mathbf{x}}^*) + \bar{p}^{(0)}(\bar{\mathbf{x}}^*) + \bar{p}^W(\bar{\mathbf{x}}^*; \bar{W}_{-}^*) + \bar{p}^W(\bar{\mathbf{x}}^*; \bar{W}_{+}^*), \quad (48)$$

$$\bar{\boldsymbol{\sigma}}(\bar{\mathbf{x}}^*) = \bar{\boldsymbol{\sigma}}^{\infty}(\bar{\mathbf{x}}^*) + \bar{\boldsymbol{\sigma}}^{(0)}(\bar{\mathbf{x}}^*) + \bar{\boldsymbol{\sigma}}^W(\bar{\mathbf{x}}^*; \bar{W}_{-}^*) + \bar{\boldsymbol{\sigma}}^W(\bar{\mathbf{x}}^*; \bar{W}_{+}^*). \quad (49)$$

In presenting the results below, we exploit the symmetry relation (17). In local cylindrical coordinates and dimensionless variables, the imposed shear flow (1) and its associated stress are given by

$$\bar{u}_x^{\infty} = \bar{\eta}(x) + \bar{r} \sin \theta, \quad \bar{u}_r^{\infty} = \bar{u}_{\theta}^{\infty} = 0, \quad (50)$$

$$\bar{\sigma}_{xr}^{\infty} = \sin \theta, \quad \bar{\sigma}_{x\theta}^{\infty} = \cos \theta, \quad (51)$$

$$\bar{p}^{\infty} = \bar{\sigma}_{xx}^{\infty} = \bar{\sigma}_{rr}^{\infty} = \bar{\sigma}_{\theta\theta}^{\infty} = \bar{\sigma}_{r\theta}^{\infty} = 0.$$

The contributions to the disturbance flow from the distributions of free-space singularities, obtained from Eq. (28), are given by

$$\bar{u}_x^{(0)} = \frac{\bar{f}}{\bar{r}} \sin \theta + O(\lambda \log \lambda), \quad (52)$$

$$\bar{u}_r^{(0)} = \frac{\bar{f}\bar{\eta}'}{2\bar{r}} + \frac{\bar{g}}{\bar{r}} + \left(\frac{\bar{h}}{\bar{r}^2} - \bar{f}' \right) \sin \theta + \frac{\bar{f}\bar{\eta}'}{2\bar{r}} \cos 2\theta + O(\lambda \log \lambda), \quad (53)$$

$$\bar{u}_{\theta}^{(0)} = -\frac{\bar{h}}{\bar{r}^2} \cos \theta + \frac{\bar{f}\bar{\eta}'}{2\bar{r}} \sin 2\theta + O(\lambda \log \lambda), \quad (54)$$

where the neglected $O(\lambda \log \lambda)$ terms result from higher-order derivatives of the dimensionless singularity distributions, $\bar{f}'', \bar{g}', \bar{h}'$; finite-length effects enter at $O(\lambda)$.

Henceforth, second- and higher-order harmonic variations in θ will be omitted from the disturbance velocity field because the corresponding tractions with this angular dependence are incompatible with the assumed circular cross section of the drop.¹² The pressure and stress associated with the remaining terms in the velocity field (52)–(54) are

$$\bar{p}^{(0)} = -\frac{2\bar{f}'}{\bar{r}} \sin \theta, \quad \bar{\sigma}_{xr}^{(0)} = -\frac{\bar{f}}{\bar{r}^2} \sin \theta, \quad (55)$$

$$\bar{\sigma}_{x\theta}^{(0)} = \frac{\bar{f}}{\bar{r}^2} \cos \theta, \quad \bar{\sigma}_{r\theta}^{(0)} = \left\{ \frac{4\bar{h}}{\bar{r}^3} - \frac{\bar{f}'}{\bar{r}} \right\} \cos \theta,$$

$$\bar{\sigma}_{xx}^{(0)} = \frac{4\bar{f}'}{\bar{r}} \sin \theta,$$

$$\bar{\sigma}_{rr}^{(0)} = -\frac{2\bar{g}}{\bar{r}^2} - \frac{\bar{f}\bar{\eta}'}{\bar{r}^2} + \left\{ \frac{2\bar{f}'}{\bar{r}} - \frac{4\bar{h}}{\bar{r}^3} \right\} \sin \theta, \quad (56)$$

$$\bar{\sigma}_{\theta\theta}^{(0)} = \frac{2\bar{g}}{\bar{r}^2} + \frac{\bar{f}\bar{\eta}'}{\bar{r}^2} + \frac{4\bar{h}}{\bar{r}^3} \sin \theta,$$

which are compatible with the *ad hoc* assumption that the cross section of the drop is circular. The pressure generated by the free-space singularities is entirely associated with the stresslet distribution because the free-space source and source-dipole are potential flows.

The leading-order contribution to the x -component of the disturbance flow from the wall-correction velocity, obtained from Eq. (29), is

$$\bar{u}_x^W = \bar{f} \frac{\bar{r} \sin \theta - 2\bar{W}_\pm^*}{4\bar{W}_\pm^{*2} - 4\bar{r}\bar{W}_\pm^* \sin \theta + \bar{r}^2} + O(\bar{W}_\pm^* \lambda), \quad (57)$$

where $O(\bar{W}_\pm^* \lambda)$ terms are negligible by assumption (13). The result satisfies the no-slip boundary condition, i.e., $\bar{u}_x^{(0)} + \bar{u}_x^W = 0$ at $\bar{y}^* = \bar{r} \sin \theta = \bar{W}_\pm^*$; however, it is incompatible with the assumed circular cross section of the drop because it contains higher-order harmonic variation in θ . Higher-order harmonics are purged from $\bar{u}_x^W$ by expanding formula (57) in powers of $\bar{r}/\bar{W}_\pm^*$ and truncating the series for terms beyond the first harmonic to obtain

$$\bar{u}_x^W = -\frac{\bar{f}}{2\bar{W}_\pm^*} \left[1 + \frac{1}{2}(\bar{r}/\bar{W}_\pm^*) \sin \theta + O((\bar{r}/\bar{W}_\pm^*)^2 \cos 2\theta) \right]. \quad (58)$$

This approximation does not satisfy the no-slip boundary condition at $\bar{y}^* = \bar{W}_\pm^*$ but it is a rigorous approximation of the wall-correction velocity (57) in the near field ($\bar{r} \approx 1$), provided that $\bar{W} \gg 1$. In any case, approximation (58) is consistent with and required by the *ad hoc* assumption of a circular drop cross section that was used to eliminate the higher-order harmonics from the velocity field associated with the distribution of free-space singularities (52)–(54).

We similarly obtain the leading-order contributions to the r - and θ -components of the disturbance flow from the wall-correction velocity, expanding the results for $\bar{r}/\bar{W}_\pm^* \ll 1$, and keeping terms only up to the first harmonic variation in θ ,

$$\bar{u}_r^W = \frac{\bar{f}'\bar{r}_W}{4} + \frac{\bar{f}\bar{\eta}'\bar{r}_W}{8\bar{W}_\pm^*} + \left\{ \frac{\bar{f}'}{2} + \frac{\bar{f}\bar{\eta}'}{2\bar{W}_\pm^*} \left(\frac{\bar{r}_W^2}{8} - 1 \right) + \frac{\bar{g}}{\bar{W}_\pm^*} \left(\frac{\bar{r}_W^2}{8} - 1 \right) + \frac{3}{4} \frac{\bar{h}}{\bar{W}_\pm^{*2}} \left(\frac{\bar{r}_W^2}{4} - 1 \right) \right\} \sin \theta, \quad (59)$$

$$\bar{u}_\theta^W = \left\{ \frac{\bar{f}'}{2} \left(1 - \frac{\bar{r}_W^2}{2} \right) - \frac{\bar{f}\bar{\eta}'}{2\bar{W}_\pm^*} \left(1 + \frac{\bar{r}_W^2}{8} \right) + \frac{\bar{g}}{\bar{W}_\pm^*} \left(\frac{3\bar{r}_W^2}{8} - 1 \right) + \frac{3}{4} \frac{\bar{h}}{\bar{W}_\pm^{*2}} \left(\frac{3\bar{r}_W^2}{4} - 1 \right) \right\} \cos \theta, \quad (60)$$

where we define $\bar{r}_W = \bar{r}/\bar{W}_\pm^*$. In obtaining these results, $O(\bar{W}_\pm^{*2}\lambda)$ terms have been neglected which is permissible by assumption (13). The pressure and stress contributions associated with the wall-correction velocity (58)–(60) are

$$\bar{p}^W = \left(\bar{g} + \frac{\bar{f}\bar{\eta}'}{2} \right) \frac{1}{\bar{W}_\pm^{*2}} + \frac{\bar{h}}{\bar{W}_\pm^*} + \left\{ -\frac{\bar{f}'}{2\bar{W}_\pm^{*2}} + (\bar{g} + \bar{f}\bar{\eta}') \frac{1}{\bar{W}_\pm^{*3}} + \frac{3\bar{h}}{2\bar{W}_\pm^{*4}} \right\} \bar{r} \sin \theta, \quad (61)$$

$$\bar{\sigma}_{xr}^W = -\frac{\bar{f}}{4\bar{W}_\pm^{*2}} \sin \theta, \quad \bar{\sigma}_{x\theta}^W = -\frac{\bar{f}}{4\bar{W}_\pm^{*2}} \cos \theta, \quad (62)$$

$$\bar{\sigma}_{r\theta}^W = \left\{ -\frac{\bar{f}'}{4\bar{W}_\pm^{*2}} + \frac{\bar{g}}{2\bar{W}_\pm^{*3}} + \frac{3\bar{h}}{4\bar{W}_\pm^{*4}} \right\} \bar{r} \cos \theta,$$

$$\bar{\sigma}_{xx}^W = -\frac{\bar{f}'}{\bar{W}_\pm^*} - (\bar{g} + \bar{f}\bar{\eta}') \frac{1}{\bar{W}_\pm^{*2}} - \frac{\bar{h}}{\bar{W}_\pm^{*3}} - \left\{ \left(\bar{g} + \frac{3\bar{f}\bar{\eta}'}{2} \right) \frac{1}{\bar{W}_\pm^{*3}} + \frac{3\bar{h}}{2\bar{W}_\pm^{*4}} \right\} \bar{r} \sin \theta, \quad (63)$$

$$\bar{\sigma}_{rr}^W = \frac{\bar{f}'}{2\bar{W}_\pm^*} - \left(\bar{g} + \frac{\bar{f}\bar{\eta}'}{4} \right) \frac{1}{\bar{W}_\pm^{*2}} - \frac{\bar{h}}{\bar{W}_\pm^{*3}} + \left\{ \frac{\bar{f}'}{2\bar{W}_\pm^{*2}} - \left(\frac{\bar{g}}{2} + \frac{3\bar{f}\bar{\eta}'}{4} \right) \frac{1}{\bar{W}_\pm^{*3}} - \frac{3\bar{h}}{4\bar{W}_\pm^{*4}} \right\} \bar{r} \sin \theta, \quad (64)$$

$$\bar{\sigma}_{\theta\theta}^W = \frac{\bar{f}'}{2\bar{W}_\pm^*} - \left(\bar{g} + \frac{\bar{f}\bar{\eta}'}{4} \right) \frac{1}{\bar{W}_\pm^{*2}} - \frac{\bar{h}}{\bar{W}_\pm^{*3}} + \left\{ \frac{\bar{f}'}{\bar{W}_\pm^{*2}} - \left(\frac{3\bar{g}}{2} + \frac{3\bar{f}\bar{\eta}'}{4} \right) \frac{1}{\bar{W}_\pm^{*3}} - \frac{9\bar{h}}{4\bar{W}_\pm^{*4}} \right\} \bar{r} \sin \theta. \quad (65)$$

The wall-correction velocities for the source and source-dipole are not potential flows thus, all of the singularity distributions contribute to the pressure associated with the wall-correction velocity (61).

1. Large wall separations

According to Eqs. (58)–(60), the wall correction produces an $O(1)$ velocity in the y -direction at large wall separations, $\bar{W} \gg 1$,

$$\bar{\mathbf{u}}^W = \bar{u}_y^{W\infty} \mathbf{e}_y + O(\bar{r}/\bar{W}). \quad (66)$$

This far-field flow is produced by the derivative of stresslet distribution on the drop centerline,

$$\bar{u}_y^{W\infty} = \bar{f}'(\bar{x}) \int_{-\infty}^{+\infty} (s - \bar{x}) T_{xyy}^W(\bar{x} - s; \bar{W}) ds = \frac{\bar{f}'}{2}, \quad (67)$$

where

$$T_{xyy}^W(q; W) = \frac{48qW^4 - 3q^3W^2}{(q^2 + 4W^2)^{7/2}}. \quad (68)$$

The nondecaying contribution of the wall-correction velocity is a consequence of the highly confined conditions (13) assumed in our analysis. The far-field form of the wall-correction velocity (66)–(68) applies to wall separations in the intermediate limit,

$$1 \ll \bar{W} \ll \lambda^{-1/2}. \quad (69)$$

At separations large compared with the drop length, $\bar{W} \gg \lambda^{-1/2}$, the wall-correction velocity vanishes as $\bar{u}^W \sim \bar{W}^{-2}$, according to Eqs. (23)–(26). Equations (61)–(65) indicate that stresses associated with the wall-correction velocity decay as $\sigma^W \sim \bar{W}^{-1}$ for $\bar{W} \gg 1$.

B. Internal flow

Collecting the contributions to the x -component of the external velocity from Eqs. (50), (52), and (58) and combining them according to the decomposition (47) yields

$$\bar{u}_x = \bar{\eta} - \frac{\bar{f}w_1}{2} + \left[\bar{r} + \frac{\bar{f}}{\bar{r}} \left(1 - \frac{\bar{r}^2 w_2}{4} \right) \right] \sin \theta, \quad (70)$$

where we define

$$w_k = \left(\frac{1}{\bar{W} - \bar{\eta}} \right)^k + \left(\frac{1}{-\bar{W} - \bar{\eta}} \right)^k. \quad (71)$$

Evaluating the perimeter average (34) of the external velocity (70) and inserting the result into Eqs. (36) and (37) yields the internal pressure field,

$$\hat{p} = \hat{p}_0(\bar{r}) + 8 \int_0^{\bar{x}} \left[\bar{R}^{-2} \left(\bar{\eta} - \frac{\bar{f}w_1}{2} \right) + 2\bar{R}^{-4} \int_0^{\bar{x}} \bar{R} \dot{\bar{R}} d\bar{x} \right] d\bar{x}, \quad (72)$$

where the value of $\hat{p}_0(\bar{r})$ is determined by the volume constraint (38).

The unidirectional internal velocity field is not needed for obtaining the evolution equations because pressure dominates viscous stresses inside the drop. For completeness, we present the solution for the internal velocity field in Appendix B.

C. Singularity distributions

The singularity distributions $\bar{f}$, $\bar{g}$, and $\bar{h}$ are determined from the stress balance at the drop interface. Combining the stress contributions given in Eqs. (51), (55), (56), and (62)–(65) according to the decomposition (49) and inserting them into the stress balance equations (42) and solving the linear system yields

$$\bar{f} = \bar{R}^2 D_{+1}^{-1}, \quad \bar{f}' = [2\bar{R}' - 4\bar{\eta}' R_{w3}] \bar{R} D_{+1}^{-2}, \quad (73)$$

$$\bar{g} = \left[\frac{\bar{R}^2}{2} (\hat{p} - (G\bar{R})^{-1} - \bar{\eta}') - 2\bar{f}\bar{R}' \frac{R_{w3}}{D_2} + \bar{f}'\bar{R} \left(\frac{R_{w1}}{2} - \frac{D_{+1}R_{w3}}{D_2} \right) \right] D_3^{-1}, \quad (74)$$

$$\bar{h} = [\bar{f}(\bar{R}' - \bar{\eta}' R_{w3}) - \bar{g} R_{w3}] \bar{R} D_2^{-1}, \quad (75)$$

where the internal pressure is given by Eq. (72). Here, we have defined

$$R_{wk} = 2^{-k} \bar{R}^k w_k, \quad (76)$$

with w_k defined by Eq. (71), and

$$D_{\pm 1} = 1 \pm R_{w2}, \quad D_2 = 1 + 3R_{w4}, \quad (77)$$

$$D_3 = 1 + 2R_{w2} - 4D_2^{-1} R_{w3}^2.$$

1. Large wall separations

Expanding Eqs. (72)–(77) for $\bar{R}/\bar{W} \ll 1$, we obtain

$$\hat{p} = \hat{p}^{(0)} + \hat{p}^{(2)} (\bar{R}/\bar{W})^2 + O(\bar{R}/\bar{W})^4, \quad (78)$$

$$\bar{f} = \bar{f}^{(0)} \left[1 - \frac{1}{2} (\bar{R}/\bar{W})^2 \right] + O(\bar{R}/\bar{W})^4, \quad (79)$$

$$\begin{aligned} \bar{f}' &= \bar{f}^{(0)'} [1 - (\bar{R}/\bar{W})^2] + O(\bar{R}/\bar{W})^4, \\ \bar{g} &= \bar{g}^{(0)} [1 - (\bar{R}/\bar{W})^2] + \left[\frac{\bar{R}^2 \hat{p}^{(2)}}{2} + \bar{\eta} \bar{R} \bar{R}' \right] (\bar{R}/\bar{W})^2 \\ &\quad + O(\bar{R}/\bar{W})^4, \end{aligned} \quad (80)$$

$$\bar{h} = \bar{h}^{(0)} \left[1 - \frac{1}{2} (\bar{R}/\bar{W})^2 \right] + O(\bar{R}/\bar{W})^4. \quad (81)$$

Here, $\bar{f}^{(0)}$, $\bar{g}^{(0)}$, $\bar{h}^{(0)}$ are the leading-order singularity distributions for $\bar{R}/\bar{W} \ll 1$,

$$\bar{f}^{(0)} = \bar{R}^2, \quad \bar{f}^{(0)'} = 2\bar{R}\bar{R}', \quad (82)$$

$$\bar{g}^{(0)} = \frac{\bar{R}^2}{2}(\hat{p}^{(0)} - (G\bar{R})^{-1} - \bar{\eta}'), \quad \bar{h}^{(0)} = \bar{R}^3\bar{R}',$$

and $\hat{p}^{(0)}$, $\hat{p}^{(2)}$ are the first two terms in the expansion for the internal pressure,

$$\hat{p}^{(0)} = \hat{p}_0(\bar{r}) + 8 \int_0^{\bar{x}} \left[\bar{\eta}\bar{R}^{-2} + 2\bar{R}^{-4} \int_0^{\bar{x}} \bar{R}\bar{R}d\bar{x} \right] d\bar{x}, \quad (83)$$

and

$$\hat{p}^{(2)} = -\frac{8}{\bar{R}^2} \int_0^{\bar{x}} \bar{\eta}d\bar{x}. \quad (84)$$

As discussed below Eq. (69), the stresses associated with the wall-correction velocity vanish for $\bar{W} \gg 1$, thus the leading-order singularity distributions (82) and internal pressure (83) correspond to those for an isolated drop (i.e., $\bar{R}/\bar{W}=0$) and agree with the analysis of Hinch and Acrivos.¹²

By the symmetry depicted in Fig. 2 and discussed in Sec. II A, we have $\hat{p}^{(2)} < 0$ and $\bar{\eta}\bar{R}' < 0$. Accordingly, Eqs. (78)–(81) indicate that for a given drop shape, the presence of the boundaries reduces the strength of all singularity distributions and the internal pressure induced by the flow.

IV. EVOLUTION EQUATIONS

Collecting the contributions to the velocity, according to the decomposition (47) and inserting the result into the kinematic condition (44), we obtain the evolution equations for the drop-shape parameters,

$$\dot{\bar{R}} = -\bar{\eta}\bar{R}' - \frac{\bar{R}\bar{\eta}'}{2} + \frac{\bar{f}}{\bar{R}}(\bar{R}'R_{w1} + \eta'R_{w2}) + \bar{f}'\frac{R_{w1}}{2} + \frac{\bar{g}}{\bar{R}}, \quad (85)$$

$$\begin{aligned} \dot{\bar{\eta}} = & -\bar{\eta}\bar{\eta}' - \bar{R}\bar{R}' + (2\bar{u}_y^{W\infty} - \bar{f}') - \frac{\bar{f}}{\bar{R}}\left(\bar{R}'D_{-1} - \bar{\eta}'\frac{R_{w3}}{2}\right) \\ & - \frac{\bar{g}}{\bar{R}}(2R_{w1} - R_{w3}) + \frac{\bar{h}D_4}{\bar{R}^2}, \end{aligned} \quad (86)$$

where dot denotes a derivative with respect to $\bar{t}$ and prime denotes a derivative with respect to $\bar{x}$. The singularity distributions $\bar{f}$, $\bar{g}$ and $\bar{h}$ are given by Eqs. (73)–(75), R_{wk} is defined by Eq. (76), $D_{\pm 1}$ is defined by Eq. (77), and

$$D_4 = 1 - 3R_{w2} + 3R_{w4}. \quad (87)$$

The nonvanishing component of the wall-correction velocity, $\bar{u}_y^{W\infty}$ is given by Eq. (67). Equations (85) and (86) enforce conservation of drop volume, given that the uniform pressure $\hat{p}_0(\bar{r})$ in Eq. (72) is determined by the volume constraint (38).

Equations (85) and (86) were integrated using the numerical method described by Hinch and Acrivos¹² in Sec. IV of their paper. At a given wall separation, the flow strength parameter G was increased by small steps, integrating the

evolution equations in time until a stationary state was achieved. The stationary state was used as the initial condition for the next larger value of G . The flow strength parameter was increased until the time required to reach stationary state began to diverge, signaling $G \approx G_{cr}$; we did not attempt to accurately determine the critical flow strength parameter by extrapolation.^{12,20} We found it convenient to expand the coefficient functions in the evolution equations in powers of $1/\bar{W}$. Only even powers are generated because the drop is centered between the two walls. Numerically converged results for $\bar{W} \geq 4$ are obtained by retaining terms up to $O(1/\bar{W}^4)$. The results are discussed in Sec. IV B.

A. Intermediate limit of wall separations

In the limit of large wall separations, $\bar{R}/\bar{W} \rightarrow 0$, evolution equations (85) and (86) reduce to

$$\dot{\bar{R}} = -\bar{\eta}\bar{R}' - \bar{R}\bar{\eta}' - (2G)^{-1} + \frac{\bar{R}}{2}\hat{p}^{(0)}, \quad (88)$$

$$\dot{\bar{\eta}} = -\bar{\eta}\bar{\eta}' - \bar{R}\bar{R}', \quad (89)$$

where $\hat{p}^{(0)}$ is given by Eq. (83) with $\hat{p}_0(\bar{r})$ determined by the volume constraint (38). These equations are obtained by substituting the singularity distributions (82) and internal pressure (83) for an isolated drop and using the limiting results $w_k \rightarrow 0$, $D_{\pm} \rightarrow 1$, $D_4 \rightarrow 1$. Given the highly confined conditions (13) assumed in our analysis, Eqs. (88) and (89) correspond to the intermediate limit of wall separations (69).

For an isolated drop, the evolution equations, derived by Hinch and Acrivos,¹² are

$$\dot{\bar{R}} = -\bar{\eta}\bar{R}' - \bar{R}\bar{\eta}' - (2G)^{-1} + \frac{\bar{R}}{2}\hat{p}^{(0)}, \quad (90)$$

$$\dot{\bar{\eta}} = -\bar{\eta}\bar{\eta}' - 3\bar{R}\bar{R}', \quad (91)$$

where $\hat{p}^{(0)}$ given by Eq. (83). These equations cannot be recovered from Eqs. (85) and (86) because of the nonvanishing component of the wall-correction velocity, $\bar{u}_y^{W\infty}$, that appears in Eq. (86). The evolution equations for an isolated drop (90) and (91), corresponding to $\bar{W} \rightarrow \infty$, differ from the evolution equations (88) and (89), corresponding to the intermediate limit of wall separations (69). The dynamics of an isolated drop are only recovered at separations that are large compared with the drop length, $\bar{W} \gg \lambda^{-1/2}$.

There is, however, a simple rescaling transformation by which the evolution equations for intermediate wall separations (69) become identical to the evolution equations for an isolated drop. The appropriate transformation is

$$\bar{\eta} = 3^{-1/2}\tilde{\eta}, \quad \bar{t} = 3^{1/2}\tilde{t}, \quad G = 3^{1/2}\tilde{G}, \quad \hat{p} = 3^{-1/2}\hat{\tilde{p}}, \quad (92)$$

with $\bar{x}$ and $\bar{R}$ unchanged. Transforming Eqs. (83), (88), and (89) accordingly yields

$$\dot{\bar{R}} = -\bar{\eta}\bar{R}' - \bar{R}\bar{\eta}' - (2\tilde{G})^{-1} + \frac{\bar{R}}{2}\hat{\tilde{p}}^{(0)}, \quad (93)$$

FIG. 3. Steady state drop length $\bar{l}$ and alignment $\bar{\eta}(\bar{l})^{-1}$ as functions of flow strength parameter $\tilde{G}$ for wall spacing $\bar{W}=4$ (dotted curves), $\bar{W}=10$ (dashed curves), and intermediate limit, Eq. (69) (solid curves).

$$\ddot{\eta} = -\tilde{\eta}\tilde{\eta}' - 3\bar{R}\bar{R}', \quad (94)$$

$$\hat{p}^{(0)} = \hat{p}_0(\bar{l}) + 8 \int_0^{\bar{x}} \left[\tilde{\eta}\bar{R}^{-2} + 2\bar{R}^{-4} \int_0^{\bar{x}} \bar{R}\dot{\bar{R}}d\bar{x} \right] d\bar{x}, \quad (95)$$

where dot denotes a derivative with respect to $\tilde{t}$ and prime denotes a derivative with respect to $\bar{x}$.

1. Discussion

The transformed evolution Eqs. (93)–(95) are identical to the evolution Eqs. (83), (90), and (91) for an isolated drop in shear flow. According to Eq. (92), the dynamics of a confined drop with intermediate wall separation (69) and flow strength parameter $G = \sqrt{3}\tilde{G}$ corresponds to that of an isolated drop in a weaker flow, with flow strength parameter $G = \tilde{G}$. Under these “corresponding conditions,” the drop cross-section profile $\bar{R}(\bar{x})$ and length are the same for the two cases. However, the centerline deflection profile $\bar{\eta}(\bar{x})$ for the confined drop is less than that of the isolated drop by a factor of $\sqrt{3}$. Moreover, the confined drop evolves more slowly than the isolated drop by the same factor under these corresponding conditions.

Beyond a critical flow strength parameter, G_{cr} , stable stationary states cease to exist. According to the definition of corresponding conditions given above, the critical flow strength parameter for a confined drop with intermediate wall separation (69) is given by

$$G_{\text{cr}} = \sqrt{3}G_{\text{cr}}^{(0)}, \quad (96)$$

where $G_{\text{cr}}^{(0)} \approx 0.0541$ is the critical flow strength parameter for an isolated drop in an unbounded shear flow, as computed by Hinch and Acrivos.¹² The critical drop length (length at the critical flow strength) for confined drops with intermediate wall separation is the same as the critical length for isolated drops.

For given flow strength, a confined drop is shorter and more aligned with the velocity than an unconfined drop at steady-state. Here, alignment is characterized by $\bar{\eta}(\bar{l})^{-1}$. This statement follows from the above results for corresponding conditions and the assumption that the stationary drop length and alignment increase monotonically with flow strength.

For wall separations in the intermediate range (69), the leading-order effect of confinement results from the convection by the nondecaying component of the wall-correction velocity, $\bar{u}_y^{W\infty}$. According to Eqs. (67), (79), and (82), we have

$$\bar{u}_y^{W\infty} = \bar{R}\bar{R}'. \quad (97)$$

Accordingly, $\bar{u}_y^{W\infty}$ is an odd function of $\bar{x}$ with $\bar{u}_y^{W\infty} < 0$ for $\bar{x} > 0$, as shown schematically in Fig. 2. We conclude that the far-field wall-correction velocity rotates a drop, enhancing its alignment, and thereby reducing its projection onto the straining component of the imposed flow and reducing its length at a given flow strength.

B. Numerical results

The results that we obtained by numerical integration of Eqs. (85) and (86) are depicted in Fig. 3 in terms of the transformed variables (92). As explained above, in these variables the limiting results for $\bar{R}/\bar{W} \rightarrow 0$ correspond to intermediate wall separations (69) with flow strength parameter given by $G = \sqrt{3}\tilde{G}$ and to an isolated drop with $G = \tilde{G}$. The results shown in Fig. 3 indicate that the evolution equations for intermediate wall separations (93)–(95) are accurate for $\bar{W} \geq 10$. The terminal points of curves shown in Fig. 3 correspond approximately to $G \approx G_{\text{cr}}$, as discussed in Sec. IV. Accordingly, the results indicate that the critical flow strength parameter exceeds the limiting result (96) for finite values of wall separation.

The boundary-integral simulations depicted in Fig. 1 correspond to conditions characterized by $\tilde{G} = 0.0214$. Results are shown for an unconfined drop and a confined drop with $\bar{W} = 2.4$. The shape profiles depicted in Fig. 1(a) indicate that the drops are not slender, therefore our slender-body description does not apply; a quantitative comparison would require smaller values of λ , with $G = O(1)$ and thus $\text{Ca} \sim \lambda^{-2/3}$, but this is computationally impractical. The results of our boundary-integral simulations are nevertheless qualitatively consistent with the prediction that, for a given flow strength, confined drops are more aligned with the imposed flow and therefore shorter than isolated drops.

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APPENDIX A: TANGENTIALLY DRIVEN COMPONENT OF INTERNAL VELOCITY FIELD

Here, we present the solution for the tangentially driven component of the unidirectional internal velocity field, $\hat{u}_t(\bar{r}, \theta)$, defined in Sec. II D. Accordingly, we seek a nonsingular solution of Laplace's equation on a circular disk $\bar{r} \leq \bar{R}$, $0 \leq \theta < 2\pi$ with boundary condition (32) prescribed on the boundary $\bar{r} = \bar{R}$. Under these conditions, we have

$$\hat{u}_t(\bar{r}, \theta) = a_0 + \sum_{n=1}^{\infty} \bar{r}^n [a_n \cos n\theta + b_n \sin n\theta], \quad (\text{A1})$$

where the constants a_n ($n=0, 1, 2, \dots$) and b_n ($n=1, 2, \dots$) are given by integrals of $\bar{u}_x$ on the perimeter.

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \bar{u}_x(\bar{R}, \theta) d\theta, \quad (\text{A2})$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \bar{u}_x(\bar{R}, \theta) \cos n\theta d\theta, \quad (\text{A3})$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \bar{u}_x(\bar{R}, \theta) \sin n\theta d\theta; \quad n = 1, 2, \dots, \infty.$$

A useful result that follows from Eqs. (A1) and (A2) is the equality of the area-averaged value of $\hat{u}_t(r, \theta)$ over the drop cross section and its perimeter-averaged value,

$$\frac{1}{\pi \bar{R}^2} \int_0^{2\pi} \int_0^{\bar{R}} \hat{u}_t(\bar{r}, \theta) \bar{r} d\bar{r} d\theta = a_0 = \frac{1}{2\pi} \int_0^{2\pi} \bar{u}_x(\bar{R}, \theta) d\theta. \quad (\text{A4})$$

APPENDIX B: SOLUTION FOR INTERNAL VELOCITY FIELD

Here, we present the solution for the unidirectional internal velocity field defined by decomposition (30). Inserting the pressure field (72) into Eq. (31), we obtain the pressure-driven component of the velocity,

$$\hat{u}_p = -2[1 - (\bar{r}/\bar{R})^2] \left(\bar{\eta} - \frac{\bar{f}w_1}{2} + 2\bar{R}^{-2} \int_0^{\bar{x}} \bar{R} \bar{R} d\bar{x} \right). \quad (\text{B1})$$

The tangentially driven component of the internal velocity field is obtained by inserting the $\bar{x}$ -component of the external

velocity (70) into Eqs. (A2) and (A3). Accordingly, Eq. (A1) yields

$$\hat{u}_t = a_0 + b_1 \bar{r} \sin \theta, \quad (\text{B2})$$

where

$$a_0 = \bar{\eta} - \frac{\bar{f}w_1}{2}, \quad b_1 = \left[1 + \frac{\bar{f}}{\bar{R}^2} \left(1 - \frac{\bar{R}^2 w_2}{4} \right) \right]. \quad (\text{B3})$$

Inserting the stresslet distribution function (73) into Eqs. (B1)–(B3) yields a solution for the internal velocity in terms of the drop-shape and confinement parameters.

¹C. Tucker III and P. Moldenaers, "Microstructural evolution in polymer blends," *Annu. Rev. Fluid Mech.* **34**, 177 (2002).

²V. Cristini and Y.-C. Tan, "Theory and numerical simulation of droplet dynamics in complex flows—A review," *Lab Chip* **4**, 257 (2004).

³H. Stone, A. Stroock, and A. Ajdari, "Engineering flows in small devices: Microfluidics toward a lab-on-a-chip," *Annu. Rev. Fluid Mech.* **36**, 381 (2004).

⁴A. Vananroye, P. van Puyvelde, and P. Moldenaers, "Effect of confinement on droplet breakup in sheared emulsions," *Langmuir* **22**, 3972 (2006).

⁵J. Rallison, "Note on the time-dependent deformation of a viscous drop which is almost spherical," *J. Fluid Mech.* **98**, 625 (1980).

⁶A. Griggs, A. Zinchenko, and R. Davis, "Low-Reynolds-number motion of a deformable drop between two parallel plane walls," *Int. J. Multiphase Flow* **33**, 182 (2007).

⁷P. Janssen and P. Anderson, "Boundary-integral method for drop deformation between parallel plates," *Phys. Fluids* **19**, 043602 (2007).

⁸P. Janssen and P. Anderson, "A boundary-integral model for drop deformation between two parallel plates with non-unit viscosity ratio drops," *J. Comput. Phys.* **227**, 8807 (2008).

⁹J. Buckmaster, "The bursting of pointed drops in slow viscous flow," *Trans. ASME, J. Appl. Mech.* **40**, 18 (1973).

¹⁰A. Acrivos and T. Lo, "Deformation and breakup of a single slender drop in an extensional flow," *J. Fluid Mech.* **86**, 641 (1978).

¹¹E. Hinch and A. Acrivos, "Steady long slender droplets in two-dimensional straining motion," *J. Fluid Mech.* **91**, 401 (1979).

¹²E. Hinch and A. Acrivos, "Long slender drops in a simple shear flow," *J. Fluid Mech.* **98**, 305 (1980).

¹³D. Khakhar and J. Ottino, "Deformation and breakup of slender drops in linear flows," *J. Fluid Mech.* **166**, 265 (1986).

¹⁴P.-H. Chan and L. Leal, "The motion of a deformable drop in a second-order fluid," *J. Fluid Mech.* **92**, 131 (1979).

¹⁵M. King and D. T. Leighton, "Measurement of shear-induced dispersion in a dilute emulsion," *Phys. Fluids* **13**, 397 (2001).

¹⁶E. Hinch and A. Acrivos (Ref. 12) obtained these scalings by nondimensionalizing their model equations based on the slender-body approximation. Note that the characteristic pressure that they provide has a typographical error; the characteristic pressure is $\mu E \lambda^{1/2}$, according to Eq. (14).

¹⁷S. Bhattacharya and J. Bławdziewicz, "Image system for Stokes-flow singularity between two parallel planar walls," *J. Math. Phys.* **43**, 5720 (2002).

¹⁸S. Bhattacharya, J. Bławdziewicz, and E. Wajnryb, "Many-particle hydrodynamic interactions in parallel-wall geometry: Cartesian-representation method," *Physica A* **356**, 294 (2005).

¹⁹S. Kim and S. Karrila, *Microhydrodynamics: Principles and Selected Applications* (Dover, New York, 2005).

²⁰J. Bławdziewicz, V. Cristini, and M. Loewenberg, "Critical behavior of drops in linear flows. I. Phenomenological theory for drop dynamics near critical stationary states," *Phys. Fluids* **14**, 2709 (2002).