Texture and shape of two-dimensional domains of nematic liquid crystals
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The function $L$ is continuous on $0 < x < \infty$, and slowly oscillating. That is, $L(x) > 0$ for all $x > 0$, and for each fixed $y > 0$ we have $L(qy)L(x) = 1$ if $x \to \infty$. It is a well-known consequence that this holds uniformly with respect to $q$ in every interval $0 < q < M$, provided that $0 < \delta < M < \infty$ (see [4], [5]).

The numbers $a$ and $b$ satisfy $a > 0$, $b > 0$.

Throughout the paper, $\lambda$, $L$, $a$, $b$, are fixed. That is, numbers depending only on $\lambda$, $L$, $a$, $b$, are called constants, and none of our statements is intended to hold uniformly with respect to $\lambda$, $L$, $a$, $b$.

**B. For every fixed $u > 1$ we have**

\[ \lim_{y \to \infty} \sum_{y \leq x \leq uy} \lambda(p) = b \log a, \]

where $p$ runs through the primes.

**C. For $y \to \infty$ we have**

\[ \sum_{n \leq y} \lambda(n) n^s \sim a^{-1} b y^{(s-1)(s+1)} L(log y). \]

**D. For every fixed $i > 2$ and every fixed $u > 1$ we have**

\[ \lim_{y \to \infty} \sum_{y \leq x \leq uy} \lambda(p)^i = 0. \]

**E. If $0 < b < 1$ the following condition holds: for every $i (i = 1, 2, 3, \ldots)$ there is a constant $C_i$ such that**

\[ \sum_{y < x \leq uy} \lambda(p)^i \leq C_i (log y)^{(2i-1)}. \]

**Notations**

For $A$, see (1.4), for $p(n)$ see (1.1), for $\mu(n)$ we use sec. 2. $\Phi(y, u)$ is an abbreviation:

\[ \Phi(y, u) = y^{(s-1)(s+1)} L(log y). \]

A phrase like $C = C(\delta)$ means: $C$ may depend on $\delta$, on $\lambda$, $L$, $a$, $b$, but not on any other parameters or functions. If $p$ is used as a summation index it is assumed to run through prime numbers only.

**The main theorem**

Let $A$, $B$, $C$, $D$, $E$ hold. Let $\delta$ and $M$ be constants, $0 < \delta < M$. Then we have, if $y \to \infty$,

\[ A_i(y^u, y) \sim a^{-1} b y^{(s-1)(s+1)} L(log y), \]

uniformly for $0 < u < M$ (for $\eta$ see sec. 2).
1.4 Remarks

In the case $a = 0$, $b > 0$ (part I) we had a similar, though simpler, result, viz.,
\begin{equation}
A(g^n, y) \sim \theta_n(y) \log y^n L(\log y).
\end{equation}

Note that $\eta(u) - b^{-1}\theta_n(u)$ (see sec. 2).

The constant $u^{-b}$ in (1.6) and (1.10) is irrelevant, of course, since $u^{-b} L$ is also a slowly oscillating function. We only introduced this factor in order to keep (1.6) in harmony with (1.3), as (1.5) can be obtained from (1.6) by a process of summation by parts.

In assumption E we require (1.8) only if $0 < b < 1$. If $b > 1$ we do not need this extra condition. It is not difficult to see from our proofs that (1.5) is not needed either if $b = 1$, $L = 1$, but we did not stress this fact in the form of a theorem.

2. The function $\eta$

For $u = 0$ the function $\eta$ is uniquely defined by the following set of conditions:

(i) $\eta(u)$ is continuous for $u > 0$,

(ii) $\eta(u) = b(u)^{-1}$ for $0 < u < 1$,

(iii) $\eta(u) = (b - 1)\eta(u) - b\eta(u - 1)$ for $u > 1$.

This differential-difference equation can be written in the following integral form. If $a > 1$, we have for $u > 1$
\begin{equation}
\eta(u) = \left(\frac{u}{a}\right)^{b-1} \eta(u) - b \int \eta(u^{0} - 1)^{b(u^{0} - 1)} du.
\end{equation}

The equivalence of (iiii) and (2.1) is easily verified if we write (2.1) in the form
\begin{equation}
\int \left( (v - b \eta(v))' - b v - \eta(v - 1) \right) dv = 0.
\end{equation}

It is in the form (2.1) that the equation for $\eta$ will arise in a natural way in our proof.

It is not difficult to derive from (i), (ii), (iii) that $\eta(u) - b^{-1}\eta(u)$ if $u > 0$, where $\theta_0(u)$ is the function occurring in (1.2) (it is characterized by $\theta_0(u) - u^0$ for $u > 0$).

3. The functional equation for $A_a(x, y)$

If $v > 1$, $y > 1$, then we have by (1.4),
\begin{equation}
A_a(y^n, y^p) - A_a(y^n, y) = \sum_{u < v} \lambda(n) u^m,
\end{equation}
where the dash indicates that only those $u$ are admitted whose largest prime factor $p$ satisfies $y < p < g^n$. For such a prime factor we have $p^m > g^n$, whence $p^m$ does not divide $u$ if $i > u$. Therefore the right-hand side of (3.1) equals
\begin{equation}
\sum_{u < v} \sum_{i < u} \lambda(n) p^m \sum_{u < v} \lambda(n) m^o,
\end{equation}
whence
\begin{equation}
A_a(y^n, y^p) - A_a(y^n, y) = \sum_{i < u} \sum_{u < v} \lambda(n) p^m A_a(y^n, p - 1)
\end{equation}
for all $u$, $v$ with $u > 0$, $y > 1$, $v > 1$.

In our proof of the main theorem it will turn out that the terms with $i > 1$ are negligible.

4. Some lemmas

Our first lemma deals with uniform Riemann integrability. We consider a function $f_a(x)$ defined for $\xi < x < \eta$, depending on the parameter $u$ $(a < u < \beta)$. If we have a dissection of the interval $[\xi, \eta]$, given by
\begin{equation}
\xi = x_0 < x_1 < \ldots < x_k = \eta,
\end{equation}
then we define the lower step-function $s_{ul}$ by $s_{ul}(x) = \inf_{x_i < x < x_{i+1}} f_a(y)$ $(x_{i+1} < x < x_{i+1})$.

and the upper step-function $s_{uL}$ similarly, with sup instead of inf.

We shall say that $f_a$ is uniformly Riemann integrable over $\xi < x < \eta$ if for every $e > 0$ there is a $\delta > 0$ such that for every dissection of $[\xi, \eta]$ with maximal interval length less than $\delta$ and for all $u$ in $(a, \beta)$ we have
\begin{equation}
\int \left( s_{ul}(x) - s_{uL}(x) \right) dx < e.
\end{equation}

The latter formula implies that the so-called upper and lower sums differ less than $e$ from the integral of $f_a$, uniformly with respect to $u$.

**Lemma 1.** Assume $0 < \xi < \eta$, $0 < \beta < \beta$, $b > 0$. Let $f(u)$ be defined and $\gamma > 0$ for all primes, and assume B. Let $f_a(x)$ be Riemann integrable over $[\xi, \eta]$, uniformly with respect to $u$ $(a < u < \beta)$. Put
\begin{equation}
\sum_{u < v} \lambda(p) f_a(x) \frac{\log p}{\log y} = S[f_a].
\end{equation}
Then we have
\begin{equation}
\lim S[f_a] = b \int_{\xi}^{\eta} \frac{f_a(x)}{x} dx,
\end{equation}
uniformly with respect to $u$ $(a < u < \beta)$.
Proof. If the dissection (4.1) is fixed (not depending on $u$ or $y$), we easily derive from $B$ that for $y \to \infty$
\begin{equation}
\lim_{y \to \infty} S[n_\alpha] - b \int x^1 S[n_\alpha(x)] x^1 dx < \epsilon
\end{equation}
uniformly with respect to $u$, since $s_n$ is uniformly bounded. Needless to say, we have a similar result for $s_m$.

Let $\epsilon > 0$ be given. By virtue of the uniform integrability we can take the dissection (4.1) such that
\begin{equation}
b \int (s_n(x) - s_m(x)) x^1 dx < \epsilon
\end{equation}
for all $u$ simultaneously ($s < u < \beta$). (Note that the factor $x^1$ is at most $\xi^{-1}$.) Next take $y_0$ such that for all $y > y_0$ the difference between $S[n_\alpha]$ and the right-hand side of (4.3) is less than $\epsilon/4$, for all $u$ simultaneously, and such that the analogous statement is true for the upper sum $S[n_\alpha]$. As $A(y) > 0$ for all $p$, we have
\begin{equation}
S[n_\alpha] < S[n_\alpha(u)] < S[n_\alpha]
\end{equation}
and it follows that
\begin{equation}
|S[n_\alpha(u)] - b \int x^1 S[n_\alpha(x)] x^1 dx| < \epsilon,
\end{equation}
uniformly for $s < u < \beta$. This proves the lemma.

Lemma 2. Assume A, B, E. Let the number $\beta$ satisfy $0 < \beta < \beta$ if $0 < \beta < 1$, and $1 < \beta < \beta$ if $b > 1$. Put $\gamma - \beta$ in the first case, $\gamma - \beta - 1$ in the second case. (So always $\gamma > 0$.) Then there is a positive constant $C = C(\beta)$ such that for all $y > 1$ and for all $s (0 < s < 1)$ we have
\begin{equation}
\sum_{y < p < y} \lambda(p) \left( \frac{\log (y/p)}{\log y} \right)^{1-\epsilon} < Cy^\epsilon
\end{equation}
Proof. a) If $1 < \beta < b$, the terms are at most $\lambda(p) y^{\beta-1}$, so for $0 < \epsilon \leq 1$ (4.4) follows from the fact that
\begin{equation}
\sum_{y < p < y} \lambda(p)
\end{equation}
is bounded (by $B$ it has a finite limit).

b) Assume $0 < \beta < 1$, $y > 1$, $0 < s < 1$, and let $N$ be the smallest integer such that $2^N > y$. We enlarge the sum in (4.4) by replacing the interval $y < p < y$ by $2^{-k} y < p < 2^{k} y$. Next we split this one into the intervals $2^{-k} y < p < 2^{k-1} y (k = 2, ..., N)$. On each one of these intervals we have
\begin{equation}
\log (y/p)^{1-\epsilon} < (k \log 2)^{1-\epsilon},
\end{equation}
whence, by $E$,
\begin{equation}
\sum_{y < p < y} \lambda(p) \log (y/p)^{1-\epsilon} < \sum_{k=2}^{N} (k \log 2)^{1-\epsilon} C_1 (\log (2^{-k} y)).
\end{equation}
If $y$ is large enough we have $2^{-k} y > y$. As $\beta > 0$ we have $\sum_{k=2}^{N} (k \log 2)^{1-\epsilon} = O(N^\epsilon)$. Finally $\b(N - 1) \log 2 < \log y$, by the definition of $N$. It follows that the right-hand side of (4.5) is less than a constant times $\log (y/p)^{1-\epsilon}$, and (4.4) follows.

Lemma 3. Let $L$ be a continuously slowly oscillating function defined for $x > 1$. Then for any $\delta > 0$ there exists a positive number $C = C(\delta)$ such that for all $x_1, x_2$ with $\frac{1}{2} < x_1 < x_2$ we have
\begin{equation}
|L(x_1)/L(x_2)| < C(\delta) (x_1/x_2)^\delta.
\end{equation}
For a proof we refer to [5], [6].

Our main theorem will be proved in sec. 5 by induction. The first step of this induction is the following lemma.

Lemma 4. Assume A, B, C, E. Let $M$ be any number $> 1$. Then as $y \to \infty$ we have, uniformly for $1 < u < M$,
\begin{equation}
\sum_{y < p < y} \lambda(p) \log (y/p) y^{\beta-1} \log (\log y)^{1-\epsilon} < Cy^\epsilon.
\end{equation}
Proof. We fix a number $\beta$ satisfying the conditions mentioned in lemma 2, and we take $\delta = b - \beta$, so $\delta > 0$. With this $\delta$ we apply lemma 3. If $2 < y^{\beta-1} < y^{\beta-1}$, we can take $x_1 = y^{\beta-1}/2, x_2 = y$, whence
\begin{equation}
\log (y^{\beta-1}/2) \log (\log y)^{1-\epsilon} < C_0 \log (y^{\beta-1}/2) y^\beta.
\end{equation}
It follows by $E$ that if $2 < y^{\beta-1} < y$, we have the following rough estimate: there is a constant $C$ with
\begin{equation}
\sum_{y < p < y} \lambda(n) y^{\beta-1} < C (\log (y^{\beta-1}/2) y^\beta.
\end{equation}
If $1 < y^{\beta-1} < 2$ this estimate is not efficient; in that case we just use that the left-hand side of (4.9) equals unity.

The total contribution to the left-hand side of (4.7) produced by those $p$ for which both $y < p < y^{\beta-1}$ and $1 < y^{\beta-1} < 2$ hold, is relatively small. This contribution is at most
\begin{equation}
\sum_{y < p < y} \lambda(p) y^{\beta-1},
\end{equation}
and by $E$ this is less than $C_1 y^{\beta-1} \log (\log y)^{1-\epsilon}$ when $y > 2$. By lemma 3 we have $L(\log y)^{1-\epsilon} = O(\log y)^{1-\epsilon}$, since $\delta$ is positive. It follows that the contribution of the $p$ with $y < p < y^{\beta-1}, 1 < y^{\beta-1} < 2$ is $O(\Phi(y, u))$, uniformly with respect to $u$.

Next choose an $\epsilon, 0 < \epsilon < M^\beta$, and consider the total contribution of
those $p$ for which both $y < p < y^*$ and $y^{1 - \alpha n'_{p}} < p < \frac{1}{2} y^*$ hold. For these terms we use (4.9), producing at most
\[ C \Phi(y, u) \sum_{\nu = y^{1 - \alpha n'_{p}} < p < \frac{1}{2} y^*} \phi(p) \left( \frac{\log (\nu/p)}{\log y} \right)^{\nu - 1} \]
and this is at most $C' \Phi(y, u)$ according to lemma 2, with a new constant $C - C'$. Finally we take the terms for which simultaneously
\[ y < p < y^*, \quad p < y^{1 - \alpha n'_{p}}. \]
We remark that $C$ now gives
\[ \sum_{n'_{p} \leq y} \lambda(n) n^s \sim \left( \frac{\log y/p}{\log y} \right)^{y - 1} L(\log y), \]
if $y \to \infty$, uniformly with respect to $p$ and $\nu$ ($p$ restricted by (4.10), $\nu$ by $1 < \nu < M$). Note that $L(\log y) \sim L(\log (y^* / p))$, since (by (4.10) and $1 < \nu < M$)
\[ \log y \sim \log (y^* / p) < M \log y. \]

It does not do any harm to replace in (4.7) the expression on the left-hand side of (4.11) by the one on the right-hand side of (4.11). We then obtain as the contribution of the terms restricted by (4.10):
\[ \sum_{\nu = y^{1 - \alpha n'_{p}} < p < \frac{1}{2} y^*} \lambda(p) f_{\nu} \left( \frac{\log \nu}{\log y} \right), \]
where $f_{\nu}(x)$ is defined for $1 < x < M$, $1 < \nu < M$ by
\[ f_{\nu}(x) = \begin{cases} \frac{(x - \nu)^{-1}}{(1 - \nu)^{-1}} & \text{if } 1 < x \leq (1 - \nu) \nu, \\ 0 & \text{if } x > (1 - \nu) \nu. \end{cases} \]
(Note that for $1 < \nu < (1 - \epsilon) < 1$ we have $f_{\nu}(x) = 0$ for all $x$, and, accordingly, the sum (4.12) is empty in that case.)

Now lemma 1 provides the asymptotic behaviour of (4.12). It results that the left-hand side of (4.12) is
\[ \Phi(y, u) \left[ a^{-1} b \int f_{\nu}(x) x^{-1} dx + B \right], \]
where $\lim \sup_{\nu < y} |B| < C e^{2}$, uniformly with respect to $u$ ($1 < u < M$).

As finally
\[ \lim_{\nu \to \infty} \int f_{\nu}(x) x^{-1} dx = \int (u - \nu)^{-1} x^{-1} dx, \]
uniformly with respect to $u$ ($1 < u < M$), the lemma follows.

**Lemma 5.** Assume $A, C, D, E$. Let $i$ be a fixed integer $> 1$ and let $M$ be any number $> 1$. Then we have
\[ \sum_{\nu < y^*} \lambda(p) p^s \sum_{\nu < \nu' < y^*} \lambda(n) n^s = o(\Phi(y, u)) \]
uniformly for $1 < u < M$.

**Proof.** We shall use the letter $q$ as a summation index running through all numbers $y^* < p < \frac{1}{2} y^*$.

The inner sum in (4.14) is certainly zero if $y^{1 - \alpha n'_{p}} < 1$, so the left-hand side of (4.14) equals
\[ \sum_{\nu < y^*} \lambda(q) q^s \sum_{\nu < \nu' < y^*} \lambda(n) n^s, \]
(so this is zero for $u < i$).

Next we remark that if $\xi, \eta, u, \beta, f_{\nu}$ satisfy the conditions of lemma 1, then
\[ \lim_{\nu \to \infty} \sum_{\nu < y^*} \lambda(q) f_{\nu} \left( \frac{\log q}{\log y} \right) = 0, \]
uniformly for $1 < u < M$. The fact that $q$ are not prime is of no concern in the proof of that lemma: the lemma can still be used to show that our assumption $D$, i.e.
\[ \lim_{\nu \to \infty} \sum_{\nu < y^*} \lambda(q) = 0 \]
(for every fixed $u > 1$) leads to (4.16). (This means specializing $b$ in lemma 1 to $b = 0$, but this is not the same $b$ we have in our present lemma 5: the $b$ occurring in assumption $C$ is positive according to $A$.)

A further preparatory remark is that lemma 2 and its proof remain true if we replace $p$ by $q$, provided that $\sum_{\nu < \nu' < y^*} \lambda(q)$ is bounded, and this is certainly the case because it has limit $0$, by $D$.

We can now prove lemma 5 by repetition of the proof of lemma 4, replacing $p$'s by $q$'s. There are two minor differences:

(i) The summation in (4.15) runs from $y^*$ onward instead of from $y$ onward. This gives no trouble, we can first show that the sum with $y^* < q < y^*$ is $o(\Phi(y, u))$, and then remark that (4.15) is even less.

(ii) In (4.13) we have to replace $a^{-1} b \int f_{\nu}(x) x^{-1} dx$ by $\sum_{\nu < y^*} \lambda(q) q^s$.
Applying \( C \) to \( A_a(y, y) \) (see (5.1)) and then lemma 4, to the double sum, we obtain

\[
A_a(y, y)\Phi(y, y) = a^{-1} b \left\{ (u-x)^{-1} \frac{1}{x} \sum_{n \leq x} \frac{1}{n} \right\} 
\]

uniformly for \( 1 < u < 2 \). Since (2.1) (with \( a = 1 \)) gives

\[
w^{-1} b \left\{ (u-x)^{-1} \frac{1}{x} \sum_{n \leq x} \frac{1}{n} \right\} = \eta(u) \quad (1 < u < 2),
\]

we have now proved the theorem for \( M < 2 \).

We proceed by induction. Assuming that the theorem has been proved for a certain \( M = 2 \), we show that it is correct for \( M \) replaced by \( M' = M + \frac{1}{2} \), i.e., we show that (1.10) holds uniformly for \( 1 < u < M + \frac{1}{2} \).

We apply (3.3) with \( v = 1 \):

\[
(5.2)\quad A_a(y, y) = \sum_{i = 1}^{\infty} \sum_{x < n \leq x+i} \eta(x) \frac{1}{n},
\]

We have

\[
A_d(y^p, p^{-1}) = \sum_{y < x < z} \eta(x) n^p,
\]

and so, by lemma 3, the contribution of each fixed \( i > 1 \) to the right-hand side is \( o(\Phi(y, u)) \), uniformly for \( 1 < u < M + \frac{1}{2} \). We have to consider at most \( M - \frac{1}{2} \) different values of \( i \), so their total contribution is \( o(\Phi(y, u)) \), and we can restrict ourselves to the remaining terms with \( i = 1 \).

For the values of \( v \) and \( \eta \) under consideration \( M < u < M + \frac{1}{2}, y < p < y^u \) we have

\[
\frac{1}{log(y^p/p)} < \frac{y}{log(p-1)} < (M-\frac{1}{2}) \frac{log p}{log (p-1)} < M
\]

for all \( y \) exceeding a certain constant \( C = C(M) \). Hence we may apply the induction hypothesis:

\[
A_a(y^p, p^{-1}) = (1 + o(1)) a^{-1} b \eta \left\{ \frac{log(y^p/p)}{log(p-1)} \right\} (y^p - p) \Phi(p, y) \left( \frac{log(p)}{p} \right)^{b-1} L(\log(p-1)) - 
\]

\[
= (1 + o(1)) a^{-1} b \eta \left\{ \frac{y log y}{p log(p)} \right\} \Phi(p, y) \left( \frac{log y}{p} \right)^{b-1},
\]

uniformly for \( M < u < M + \frac{1}{2} \). (Note that \( \eta \) is uniformly continuous and positive on \( \left( \frac{1}{2}, M \right) \); moreover \( log(p-1)/log y \) lies between \( \frac{1}{2} \) and \( 1M + \frac{1}{2} \), whence \( L(\log(p-1)) \) may be replaced by \( L(\log y) \).)

As (1.10) has already been proved for \( u = 2 \) we have

\[
A_a(y^p, y^q) \sim a^{-1} b \eta(2) (\frac{1}{2})^{b-1} \Phi(y, u),
\]

uniformly for \( M < u < M + \frac{1}{2} \). So it follows from (5.2) that

\[
A_a(y^p, y^q) \Phi(y, u) = a^{-1} b \eta(2) (\frac{1}{2})^{b-1} \left( \frac{y^{-1}}{u^{-1}} \right) \Phi(p, y) \left( \frac{log y}{p} \right)^{b-1} L(\log(p-1)),
\]

uniformly for \( M < u < M + \frac{1}{2} \).

We now apply lemma 1 with \( \xi = -1, \eta = \frac{1}{M + \frac{1}{2}}, \alpha = M, \beta = M + \frac{1}{2}, \) and

\[
I_d(x) = \left\{ \begin{array}{ll}
\eta(u-1) x^{-1} & \text{if } 1 < x < \frac{1}{M},
0 & \text{if } \frac{1}{M} < x < \frac{1}{M + \frac{1}{2}}.
\end{array} \right.
\]

This leads to

\[
A_a(y^p, y^q) \Phi(y, u) = \left( a^{-1} b \eta(2) (\frac{1}{2})^{b-1} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \Phi(y, u) \left( \frac{log y}{p} \right)^{b-1} L(\log(p-1)),
\]

uniformly for \( M < u < M + \frac{1}{2} \). By (2.1) (with \( a = 2 \)) the right-hand side is \( \alpha^{-1} b \eta(u) + o(1) \), and this completes the induction step.

6. Applications

6.1 If \( A\eta(n) \sim n^{-1} \) for all \( n \), and if \( a = 1, b = 1 \), the conditions of our theorem are satisfied, with \( L = 1 \). The result is that if \( \Psi(x, y) \) is the number of integers \( \leq x \) free of prime factors \( \leq y \), then \( \Psi(y, y) \sim \eta(y) y^u \) \( u \) fixed, \( y \rightarrow \infty \). This result was first obtained by A. A. BeICHTS [8], and extended to cases where \( u \rightarrow \infty \) in [1].

6.2 In Part I ([7]) we proved

\[
(6.1)\quad \sum_{y < n < 2^y} \mu(y) (\Psi(y))^2 \sim 0(y) log y.
\]

Inserting an extra factor \( d \), we now obtain from our present theorem (see (1.10))

\[
\sum_{y < n < 2^y} \mu(y) d \Psi(y)^2 \sim \eta(y) y^u
\]

if \( u > 0 \) is fixed, \( y \rightarrow \infty \). In this case we have \( \Psi(n) = \mu^2(n) \psi(n) \), \( a = 1, \) \( b = 1, \) \( L = 1 \). We omit a detailed verification of the conditions A, B, C, D, E; A and D are trivial, B and E depend on the facts that the expression

\[
\sum_{y < n < 2^y} \mu(y) \psi(n) \sim \eta(y) y^u
\]

has a limit if \( x \rightarrow \infty \); for C we need

\[
\sum_{y < n < 2^y} \mu(n) \psi(n) \sim y.
\]

The latter relation can be seen, for example, from the identity

\[
\sum_{y < n < 2^y} \mu(n) \psi(n) \sim \eta(y) \prod \left( \frac{1}{1 - \frac{1}{p}} - \frac{1}{p} \frac{1}{p} \frac{1}{p^2} \right).
\]
where the infinite product can be expanded into a Dirichlet series which converges absolutely for $s > \frac{1}{2}$ and has the value 1 at $s = 1$.

6.3 If we define the multiplicative function $\lambda$ by $\lambda(n) = (n d(n))^{-1}$, where $d(n)$ stands for the number of divisors of $n$, then we have by (10)

$$\sum_{n \leq x} \lambda(n) n - \sum_{n \leq x} (d(n))^{-1} \sim e \pi(x) x^{-1},$$

with a certain positive constant $c$. The function $\lambda$ evidently satisfies conditions A, B, C, D, E with $a = 1$, $b = \frac{1}{2}$, $L = c$. Therefore by (1.10) we have

$$\sum_{n \leq x} \mu^2(n) (d(n))^{-1} \sim e \eta(n) y^0 \log y x^{-1},$$

where $\eta$ is the function defined in sec. 2 with $b = \frac{1}{2}$.

6.4 Another example with $b = \frac{1}{2}$ is found by defining

(i) $\lambda(p^i) = 0$ if $i = 1, 3, 5, \ldots, p = 3 \pmod{4},$

(ii) $\lambda(p^i) = p^{i-1}$ otherwise,

(iii) $\lambda$ multiplicative.

It is well-known that for $n > 1$ we have $\lambda(n) = 1$ if $n$ is the sum of two squares, $\lambda(n) = 0$ otherwise. Thus we have in this case

$$A(y, y) = \sum_{n \leq x} \lambda(n) n - \sum_{n \leq x} 1,$$

the dash indicates that $n$ is omitted if $n$ is not the sum of two squares.

For the partial sums we have

$$\sum_{n \leq x} \lambda(n) n \sim e \pi(x) x^{-1},$$

where $e = (2 \prod_{p \text{ prime}} (1 - p^{-2}))^{-1}$ (cf. [9], § 176), and the verification of A, B, C, D, E (with $a = 1$, $b = \frac{1}{2}$, $L = c$) is easy. So by (1.10) we have

$$A(y^0, y) \sim \eta(n) y^0 \log y x^{-1},$$

with the same function $\eta$ as in example 2.

6.5 In all previous examples the function $L$ occurring in our theorem was constant. It is not difficult to construct an example where this is not the case. We define

(i) $\lambda(p) = 1 + (\log \log p)^{-1}$ if $p > 2$,

(ii) $\lambda(p) = 0$ if $p = 2$ or $i > 2$,

(iii) $\lambda$ multiplicative.

By a theorem of Wirsing [11] we now have

$$\sum_{n \leq x} \lambda(n) n \sim e^{-x} (\log x)^{-1} \prod_{p \leq x} (1 + \lambda(p)) \sim 4n^{-2} x G(x),$$

where $G(x) = \prod_{p \text{ prime}} (1 + (p + 1)^{-1} (\log \log p)^{-1})$, and $y$ is Euler’s constant.

In order to prove that $G$ is a slowly oscillating function of $\log x$ we must show that

$$\lim_{x \to \infty} \prod_{n \leq x} (1 + (p + 1)^{-1} (\log \log p)^{-1}) = 1$$

for every $c > 1$, and to show this it is sufficient to show that

$$\lim_{x \to \infty} \frac{1}{x} \int_{\log x}^{\log x} \frac{dt}{t \log t \log t} = 0$$

for every $c > 1$. (Here the prime number theorem is applied in the familiar way.) This is verified by straightforward calculation. Also, it is easy to see that $L(x) \to \infty$ if $x \to \infty$.

Thus we have given an example of a multiplicative function $\lambda$, satisfying A, B, C, D, E with $a = 1$, $b = 0$ and $L$ is a slowly oscillating function which is not a constant (not even asymptotically). We omit the simple verification of A, B, C, D, E.

**REFERENCES**

(References 1, 2, 3 are the same as in part I, the others are in alphabetical order)


