Simulation of Deflection Coil Winding
Theory and verification of SWING

Ruud Voncken
Simulation of
Deflection Coil Winding

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SWING

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Preface

History of the project

This thesis is the result of a Philips project carried out by the author. The project was carried out at Philips CFT (Centre For manufacturing Technology) at the request of Philips-ITC (Innovation and Technology Centre), the development department of TV tubes and colour monitor tubes.

In 1989, a small group of people of Philips-ITC, Marti Leemans, Hein Lenders and Henk v.d. Berg, concluded that the traditional way of developing winding jigs for the winding of deflection coils would no longer be permissible in the near future.

The reason for this conclusion was that the traditional development was mainly based on the experience of a small group of skilled men and was very time consuming, resulting in long throughput times. As a result, they decided to ask the Mechanical Analysis group of the Philips CFT for assistance in finding a new way to develop winding jigs.

Consequently, the author, as a member of this mechanical analysis group, became involved in the project. After an explorative study, a project was proposed with the final target of developing a simulation program of the winding process. This simulation program should make it possible to develop winding jigs in a much faster way and should lead to objective design rules.

Thanks to the belief of the small group of people in the project proposal and thanks to their efforts to convince the management, the project was started in early 1989. After many alternating periods of progress and almost unsolvable problems, the first version of the program SWING (Simulation of WINding Geometry) was released in 1992, enabling the first verification tests.

After some years of simultaneous usage, verification and further development and several versions of the program, SWING is currently an accepted and even indispensable design tool for winding jigs. This design tool makes it possible to develop winding jigs in a much faster way and according to design rules, without relying purely on the skill of the people developing winding jigs.
Summary

The winding process of the deflection coils is an important process in the manufacture of TVs. During this process, a wire is wound by a rotating mechanism onto a winding jig, resulting in a deflection coil. Because there is a need to simulate this winding process, a simulation tool (the SWING program) has been developed. The purpose of simulating the winding process is both to speed up the development of winding jigs and to make the design skills transferable.

The main focus on the development of this tool was on the fast and accurate simulation of the three main factors affecting the winding process:

- The dynamics of the wire
- The contact description between the wire and the winding jig
- The friction description of the wire-to-jig contact

The theory needed to describe the dynamics of the wire is partly based on known multibody dynamics and further extended to be able to describe the contact and friction boundary conditions in an efficient manner. This is achieved by using local absolute coordinates, which results in an absence of Lagrange multipliers. It is shown that the proper choice of the local coordinate directions even results in an order $n$ method. At the moment, this approach seems limited to mechanisms with the same characteristic as the wire: the "single chain" character.

An algorithm has been developed for the surface contact description that enables the development of a fast and accurate algorithm to detect contact between wire and winding jig, including narrow segment wings. For this purpose, four different types of nodes have been used. Two of these types are non-material nodes, which have no fixed position with respect to the wire. The surface description can be generated from within a CAD package such as, in our case, UNIGRAPHICS.
A friction description needed to simulate the friction in the wire-to-jig contact has been developed. The special characteristics of the contact, such as, for instance, the large difference in wire stiffness in the longitudinal and tangential directions and the large angle of the wire around some surfaces with a small radius of curvature place special requirements on this friction description.

Some examples of the simulation results of the SWING program are shown and the functionality of the SWING program is verified by comparing simulation results to practical experiments.
Acknowledgements

Although taking a great risk in forgetting someone, I would like to thank a number of people who contributed most to the realisation of this thesis.

First of all, I want to thank my wife and daughter. My wife Maria I thank for her almost never-ending patience when I had to work even during our spare free moments. My four-year-old daughter Lieke I want to thank for saying "ga maar fijn werken Papa". I feel guilty to both of them because the moments that we have missed together will never come back.

Furthermore, I would like to thank Marti Leemans, Hein Lenders and Henk van de Berg. The work would never have been started without their initiative, and their great contribution to the continuation and success of the work by their never-ending enthusiastic belief and support.

I would also like to thank Edward Maesen and Antoine Grubben for performing the first tests with SWING and, in this way, giving enough reason for carrying on, as well as giving many directions for improvement without giving up during the many fatal crashes of SWING, especially with the first versions.

I would like to thank Alex Schipper for his enthusiastic usage of SWING. He contributed much to the further improvement of the program by chasing the program beyond its limits without becoming defeated when the program produced fuzzy error numbers.

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Last but not least, I give my gratitude and appreciation to Chris van Wintershoven. As my group leader during most the time that I worked on SWING, he provided me with extra time and facilities needed to write this thesis. He also gave me a lot of mental support by his never ending requests for my latest time schedule for completing my thesis.
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Chapter 1

Introduction

1.1 Scope of this thesis

During the past years, an increasing need has evolved within the development departments of industrial enterprises for tools that can predict processes and the operation of products in an early stage.

This need is mainly caused by competitive considerations to have fast product and process innovation, whereby the quality is very important. For a long time now, the extensive testing of prototypes and of new production processes has no longer fitted in this trend.

One illustration of this trend towards prediction and simulation is the progress that large commercially available simulation programs have made.

As far as simulation of multibody dynamics is concerned, we have, for example, the packages ADAMS, DADS, NEWEUL and DCAP-5. Here, we must note that the fast rise of these packages is not only the result of this modern trend, but also of the greatly increased power of modern computers.

A large diversity of the (thermo-)mechanical phenomena of products and processes can be simulated with the above-mentioned packages, which are therefore general purpose within their field of application. Besides this, there is still a large field of phenomena of which the simulation is technically possible, but for which the use of a general purpose program is not obvious. This is because simulation can be done much more efficiently or accurately with a simulation model that is specifically designed for the process or product behaviour.

A good example of a specific program is the subject of this thesis: the simulation of the winding of electron deflection coils for picture tubes.
Because of the heavy requirement to be able to accurately simulate this winding process, a special simulation software tool (which is described in this thesis) has been developed. The main focus on the development of this tool was on both a fast and accurate simulation of the three main factors affecting the winding process:

- The dynamics of the wire
- The contact description between the wire and the winding jig
- The friction description of the wire-to-jig contact

The theory needed to describe the dynamics of the wire is partly based on already known multibody dynamics and further extended to be able to describe the contact and friction boundary conditions in an efficient manner. Use has been made of the special characteristics of the wire, for instance, the "single chain" character.

For the description of the contact boundary condition, a contact description has been developed that enables the development of a fast and accurate algorithm to detect contact between wire and winding jig. This surface description can be generated from within a CAD package such as, in our case, UNIGRAPHICS.

Also, a special friction description is needed to simulate the friction in the wire-to-jig contact. This is because of the special characteristics of the contact, such as, for instance, the large difference in wire stiffness in the longitudinal and tangential directions and the large angle of the wire around some surfaces with a small radius of curvature.

1.2 Organisation of this thesis

The winding process of electron deflection coils is described in section 1.3, together with a description of the demands that the process puts on the simulation of this process.

In section 1.4, the need for a simulation of this process is explained. In section 1.5, the modelling of the winding process is discussed in general, whereby such aspects as the assumptions that were made with the modelling are explained. In section 1.7 the characteristics of describing the dynamics of the wire are explained and their relation with multibody dynamics.

In chapter 2, the way of defining the motion of the wire with respect to local coordinate systems is described. The use of local coordinate systems makes it
possible to account in a very easy way for local contact conditions and even more important: in chapter 4, it is shown that choosing the appropriate local coordinate directions leads to a very efficient solution procedure for the equations of motion. Chapter 3 shows the derivation of the equations of motion for the wire using the principle of virtual power as well as the calculation of local coordinate directions that lead to a set of equations of motion that can be solved in a very fast way.

In chapter 4 the numerical time integration method of the equations of motion and the implementation in a fast and efficient algorithm is described.

Chapter 5 contains the geometry description of the winding jig and the associated efficient contact algorithm.

Chapter 6 describes how the actual friction between wire and jig is described in the model.

Chapter 7 describes how the bending stiffness of the wire is accounted for by external forces acting on the wire. Furthermore, the dynamic model for the so-called flyer and air dereeler is described.

In chapter 8, a verification is given of the simulation program SWING (Simulation of WINding Geometry). This program is based on the theory as described in this thesis. The agreement between the results of SWING and experimental results are discussed.
1.3 Description of the winding process of deflection coils

The deflection unit

One important component of the picture tube of TVs and monitors is the deflection unit. This is mounted on the neck of a picture tube and basically has the purpose of deflecting the electron beam in the picture tube. When this electron beam hits the phosphors on the picture screen, they light up and produce the picture that we see.

Figure 1.1 shows an assembly of picture tube and deflection unit.

Fig. 1.1: The deflection of the electron beam in a picture tube
1.3 Description of the winding process of deflection coils

The deflection coils

Figure 1.2 shows an exploded view of a deflection unit. The only components that we indicate here are the cap (which forms the support for the deflection unit), the yoke ring (which provides the magnetic screening) and the four deflection coils. The two coils that are on the inside of the cap, i.e. the line coils, provide the horizontal deflection of the electron beam. The two deflection coils that are on the outside of the cap, the picture coils, provide the vertical deflection of the electron beam.

![Exploded view of the deflection unit](image)

Fig. 1.2: Exploded view of the deflection unit

The deflection coils consist of many windings of a single or multiple winding wire, which is made from a copper core with a few layers around it. From inside to outside, these layers are the insulation layer, the thermal adhesive layer and the lubrication layer, mostly consisting of paraffine wax.
The winding jig

Because the shape of the deflection coils is not completely convex, it is necessary to wind the deflection coil in a "winding jig", which mainly consists of an upper and a lower jig.

Figure 1.3 shows such a winding jig. Between the two jigs, there is a jig gap (or winding gap), whose shape corresponds with the constraining inner and outer surfaces of the coils. These surfaces are rotationally symmetric, whereby the cross-sections through the axis of rotation consist of straight lines and circle segments.

To guide the wire in the gap during the winding process, "segment wings" are fitted to the upper and lower jigs. These segment wings, which can clearly be seen in Figure 1.3, are relatively narrow, mostly circular discs, over which the wire slides during the winding process. A "torpedo" is also fixed to the lower jig to guide the wire. Figure 1.3 also shows the "nose" on the upper jig and the plate on the lower jig. These components continue the profile of the upper and lower jigs, respectively, and are required to improve the winding process.
1.3 Description of the Winding Process of Deflection Coils

Besides being determined by the shape of the recess between upper and lower jigs, the shape of the coil is also determined by the position of the "pins". These pins are positioned in the winding gap to prevent the winding wire from penetrating the winding gap to deep.

The winding

During the winding, a rotating winding motion is made by the "flyer", which is also shown in Figure 1.3. The flyer consists of a hollow shaft with an arm fixed to it, on which there are three wheels, called the flyer wheels.

The wire (possibly single, but mostly multiple) is fed through the hollow shaft and then over the three wheels. The flyer rotates at a speed of more than 500 rpm. The wire is guided from the winding reel by the "air dereeler" to the hollow flyer axis whereby, if the wire is multiple, a separate air dereeler is used for each separate wire. The function of the air dereeler is to give the winding wire the correct tension. By using air pressure, and by avoiding mechanical moving parts to tense the wire, an attempt is made to keep the wire tension virtually constant, even with large variations in the feed rate of the winding wire.

Figure 1.3 gives a somewhat simplified representation of the combination of flyer, air dereeler and winding jig.

The winding process is a complex process in which many factors are involved. These include the contact and the friction that occur between the wire and the segment wings or other surfaces with which the wire comes into contact, and the dynamic effects of the wire and the flyer wheels over which the wire runs at a strongly varying speed. All these factors have a large effect on how the wire is wound on to a coil and therefore also greatly influence the quality of the coil, which is determined by such aspects as:

- the degree of filling of the jig gap
- the symmetry of the deflection coil
- the internal stresses in the deflection coil, and
- whether the winding wire is damaged or marked as a result of contact with sharp corners
1.4 The need for simulation of the winding process

With a new type of coil to be developed, a new winding jig geometry is needed. The shape of the winding gap between the upper and the lower jigs and the position of the pins follow from the desired coil shape, which in turn is determined from the required electron-optical characteristics and the shape of the outside of the glass tube. To determine the electron-optical characteristics belonging to a certain coil shape, the simulation program DUCAD based on the PhD thesis of A. Osseyran ([12]) is used. This program calculates the electron-optical effects of the electron beam that passes through the magnetic field generated by the deflection coils. This offers the designer of the deflection coils a powerful tool to optimise the shape of the deflection coils.

After that, the number of segment wings and their position and the shape of the segment wings must be determined in such a way that a good windability of the coil is obtained at a high winding speed. By good windability, we mean that the winding process runs in such a manner that a qualitatively good coil is obtained.

This placing of the segment wings is done by carrying out winding tests in which the segment wing geometry is modified until a segment wing geometry is obtained that gives a satisfactory winding result. This process of trial and error is largely based on experience of the segment wing designer. However, this experience is very difficult to transfer to other people, because it is so closely related to one small group of people. Furthermore, this process is very time consuming because of the necessity of physically exchanging the segment wings on the winding jig and of carrying out the winding tests.

As stated in the Preface, developing winding jigs is an essential part of the process development of deflection units, which are in turn essential parts of TVs and monitors. Therefore, in a time where fast product and process innovation are a necessity to remain competitive, long throughput times for developing winding jigs are no longer permissible. The only way to speed up the development of winding jigs and to make it less dependent on experience appears to be simulation of the winding process with a software simulation tool.
1.4 The need for simulation of the winding process

The purposes of such a simulation tool are that:

- the program can be used to achieve a (drastic) reduction of the present design time required for a suitable segment wing geometry.
- the design skills that the segment wing designer has, but which are mainly intuitive and based on experience, can be described in a form that can be understood by people without special winding experience and thus be made transferable.

The above objectives can be realised with a software simulation tool because:

- The number of iterations is reduced as a result of the greater insight that the simulation offers in the winding process.
- The execution of an iteration by using simulation takes less time than with physical winding.
- The segment wing development skills can be described by using design rules based on quantitative simulation results.

It is essential that the simulation program cannot make the trial and error process superfluous, but that this process can be executed more quickly and effectively by simulating the winding process.

One problem with the simulation of the winding process is that it is not (yet) possible to make a simulation model of the winding process that directly gives the quality of the coil as a simulation result. We will therefore have to limit ourselves to results such as wire motion (position and speed as a function of time), wire tension, contact forces and the like.

Therefore, before a statement can be made about the windability of a simulated segment wing geometry, a conclusion must be drawn about the windability of the segment wing geometry from the calculated results. Because the relation between the simulation results (for example the wire motion) and the quality of the coil is not yet known, further investigation will be required to derive this relation. This investigation will not be a part of the project described in this thesis.

Figure 1.4 shows a schematic representation of the old segment wing development process. Figure 1.5 shows the future development process, whereby the possibility of simulating the winding process is used.
Fig. 1.4: The old development process

Fig. 1.5: The new development process
1.5 Assumptions made in modelling the winding process

To simplify the simulation of the winding process, a number of assumptions have been made with the modelling. These assumptions are now mentioned and justified.

The bending stiffness of the wire is not an important factor for the winding behaviour

If we assume an often used copper winding wire with a diameter \(d\) of 0.334 \(mm\) and a modulus of elasticity \(E\) of copper of 120000 \(N/mm^2\), we find the bending stiffness \(EI\) for this wire according to:

\[
EI = E \frac{\pi d^4}{64} = 73.3 N mm^2
\]  

(1.1)

The bending moment \(M^b\) required to bend this wire around a segment wing with a radius of curvature \(R = 5 mm\) is therefore approximately equal to:

\[
M^b = \frac{EI}{R} = 14.7 N mm
\]  

(1.2)

If we assume a value of 150 \(N/mm^2\) for the plastic yield stress \(\sigma_y\) of a copper wire with a diameter \(d\) of 0.334 \(mm\), and we assume that the tensile force in the wire results in a stress between one tenth of the yield stress and the yield stress itself, then we find a tensile force varying between 1.3 and 13 \(N\). This means that if we bend a copper wire over a segment with this tensile force, the shape that this copper wire assumes will differ by a maximum roughly between 1 and 11 \(mm\) from the shape that a wire without bending stiffness would assume. Figure 1.6 shows the shapes of both a wire with and without bending stiffness.

If we see this 1 to 11 \(mm\) in relation to the global dimensions of the winding jig (which is 200 to 300 \(mm\)), then the assumption that the bending stiffness is not an important factor appears to be justified.

Up to now, we have even ignored the fact that the copper wire will deform plastically with the bending around a segment wing so that the deviation will be less than calculated.
 INTRODUCTION

[Diagram: Wire with bending stiffness, tensile force, difference, Wire without bending stiffness]

Fig. 1.6: The maximum difference between a wire with and without bending stiffness

The maximum stress occurring in a wire that is bent can be found from:

$$\sigma_{\text{max}} = \frac{32M^b}{\pi d^3}$$  \hspace{1cm} (1.3)

For the assumed value of 150 \(N/mm^2\) for the plastic yield stress \(\sigma_y\) of a copper wire with a diameter \(d\) of 0.334 \(mm\), a value of 0.54 \(N/mm\) can be found from equation 1.3 for the moment where the wire will start showing plastic behaviour. Therefore, the moment of 14.7 \(N/mm\) that was calculated earlier will never be reached.

Because the bending stiffness of the wire has no influence on the global behaviour of the wire, the bending stiffness will be modelled in a strongly simplified way. The only purpose of the modelled bending stiffness will be to avoid local sharp corners in the wire.

**The torsional stiffness of the wire can be neglected**

For the earlier assumed copper wire with a diameter \(d\) of 0.334 \(mm\) and using a shear modulus \(G\) of 46150 \(N/mm^2\), we find the torsional stiffness \(GI\) from the following equation:

$$GI = \frac{G\pi d^4}{32} = 56.4 N/mm^2$$  \hspace{1cm} (1.4)

The pure elastic twisting moment \(M^t\) needed to twist the wire follows from the following equation, defining \(\omega\) as the twist angle in radians per length:

$$M^t = GI\omega$$  \hspace{1cm} (1.5)
When assuming a twisting of the wire of $2\pi$ radians per revolution of the flyer, then a maximum twist angle $\omega$ of 0.05 rad/mm will be found during the first windings of the deflection coil when the circumference is minimal and equals about 120 mm.

Then, a twisting moment of 3 Nmm will occur when assuming elastic behaviour. With the earlier mentioned tensile force between 1.3 and 13 N, only a torsion arm between 0.23 and 2.3 mm is needed, indicating that the torsional stiffness can be neglected even when assuming (stiffer) elastic behaviour. Therefore, the assumption of absence of torsional stiffness also appears to be justified.

The wire is single

Because the separate wires of a multiple wire are not joined together, the motion that the wire makes over the winding jig does not differ from the motion that a single wire makes. After all, the tensile force, stiffness and mass per wire are equal to those of a single wire. Therefore, to describe the motion of the multiple wire, we can suffice with the simulation of the motion of a single wire.

Whether the wire is single or multiple, account has to be taken of the calculation of the angular accelerations of the flyer wheels because, the total frictional force between wire and wheel depends on the number of wires.

The wire is infinitely stiff in the longitudinal direction

The tensile force between 1.3 and 13 N in the above copper wire with a diameter of 0.344 mm will result in a stretch between 0.012 and 0.12 percent in the longitudinal direction. For a wire with a length of 500 mm, this means a change in length between 0.06 and 0.6 mm. The assumption of infinite stiffness in the longitudinal direction appears, at least statically, to be an admissible assumption. The reason for the assumption is, however, mainly dynamic in character and is explained in more detail in chapter 3.

Coulomb friction with direction-dependent coefficient of friction

Frictional forces that occur in contact points of the wire with the winding jig satisfy Coulomb's law of friction, whereby the coefficient of friction may depend on the slip direction. Measurements have shown, at least for contact between wire and upper or lower jig, that this assumption approximates reality reasonably
well. The explanation for this must probably be looked for in the fact that the layer of paraffine wax on the wire has both a solid and a viscous character.

The same measurements showed a slight temperature and speed dependency on the coefficient of friction. Because of its small influence, this dependency is not included in the modelling.

No rolling is possible in the contact point

Besides normal forces, frictional forces can also occur in a contact point between wire and surface. These frictional forces exert a moment on the wire around the axis of the wire, except when they act parallel to the direction of the wire. In principle, the moment of these frictional forces can twist the wire, which makes rolling of the wire over a surface possible.

We will now calculate the degree of rolling for the wire with an assumed diameter of 0.334 mm and with a tensile force between 1.3 and 13 N. If we take twice the value of 13 N as an indication of the maximum possible value of the normal force and an (over) estimated value of 0.5 for the coefficient of friction, we find a maximum twisting moment by the friction force on the wire of 2.1 Nmm.

Assuming a wire of a total length of 200 mm clamped in at both ends and in the middle in contact with a surface, we find a maximum twisting angle of 1.8 rad by using the earlier calculated torsional stiffness of 56.4 Nmm². In this case, rolling is therefore only possible along a maximum distance of 0.3 mm. The assumption that the wire will never roll therefore very certainly appears justified.

The wire is assumed to be weightless

For the assumed copper wire with a specific mass of 8.97 \(10^3\) Kg/m³, we find a gravitational force of 7.7 \(10^{-3}\) N per meter length. Using the tensile force between 1.3 and 13 N, we find a characteristic length between 169 and 1688 m, which is very large in comparison to the dimensions of the winding jig.

From this, it appears to be a reasonable assumption that the wire is not exposed to the force of gravity.
1.6 Major aspects of spatial discretisation

The wire is modelled using the spatial discretisation approach, in which the continuous wire is described through discrete positions in space. This approach is very common in finite element practice and is widely described, including in a standard work in this field, such as [14].

A characteristic aspect of this approach is that the structures to be described are discretised with "nodal points" or "nodes". This was concretely executed for the winding wire in the following manner (see also Figure 1.7):

- The wire position is defined by the position of nodes on the wire. Between the nodes (which are indicated by a node number \(i\)), the wire is assumed to be straight. The number of nodes over the length of the wire is one of the factors that determine the accuracy with which the position of the wire is described.

- Nodes are mostly material points, which means that they have a fixed wire position, but they can sometimes also be non-material. This in turn means that they can glide along the wire, in which case the mass will be non-constant. To indicate the position with respect to the wire, each node has a wire coordinate \(l_i\). This is the length of wire between node \(i\) and a reference point on the wire.

- The mass of the wire is concentrated in the nodes, which have no rotational moment of inertia. The mass in node \(i\) is indicated by mass \(m_i\), which will be chosen equal to the mass of half the wire lengths to the left and to the right of the node. When the mass per length of the wire equals \(\rho A\), where \(A\) is the cross-sectional area, we find:

\[
m_i = \left(\frac{l_{i+1} - l_i}{2} + \frac{l_i - l_{i-1}}{2}\right) \rho A
\] (1.6)

The approach to concentrate the mass in the nodes has the same effect as so-called mass lumping (see for instance [14]). This approach will be justified in chapter 4.

- The wire can make contact with surfaces, whereby contact occurs each time in a node. If necessary, a new node is generated at the point of contact.

- At the point of contact, contact forces can be exerted on the wire. These can be both normal and frictional forces.

- There is a tensile force in the wire which varies in the longitudinal direction and which is assumed to be constant between two adjacent nodes. From
now on, we will call the tensile force between nodes $i - 1$ and $i$: $T_i$.

- The total wire to be described is positioned between two end points. On the one hand, this is the "flyer point" with node number $n$. This point has a position that corresponds with the hole in the flyer through which the wire is led. On the other hand, this is a point that is in the winding jig and that coincides with a point in that part of the wire that has come to rest in the winding jig. We will call this the fixed point. As the simulation progresses, new points can be selected for this point because more and more wire has come to rest. As a result, the length of wire that is included in the simulation can be minimised.

- The tensile force in the wire behind the flyer point, $T_{n+1}$, is calculated using a separate dynamic model of the flyer. In this dynamic model, account is taken of the inertia of the wire in the flyer, the inertia of the flyer wheels and the frictional forces between the flyer wheels and the wire, etc.

![Fig. 1.7: Spatial discretisation of the wire](image)
1.7 The use of multibody methods for modelling the winding process

When modelling the spatial motion of the wire as the motion of a mechanism consisting of multiple coupled nodes in space (as described in the previous section), use can be made of techniques for calculating the motion developed in multibody dynamics. Several different solution procedures have been described in literature for both mechanisms with rigid bodies and for mechanisms with flexible bodies. Most of the publications deal with space systems e.g. large manipulator arms consisting of several links.

The main characteristic of multibody dynamics is the fact that the coupling of the bodies leads to a coupling of the degrees of freedom and the deformation modes of the bodies, which in turn leads to a quite complex description of the motion and the deformation of the bodies.

Because of the assumed infinite stiffness of the coupling between the nodes, we can restrict ourselves in the overview of solution techniques to a mechanism consisting of rigid bodies. When considering the wire as an assembly of truss elements, the approach chosen in this thesis becomes a finite element approach, as described by van der Werff [13] and Jonker [7]. In this approach, the bodies and their links are replaced by finite elements which are coupled to each other at their nodes. The motion of the system is fully described by the coordinates of the nodes.

Because we have assumed that the nodes have no rotational moment of inertia, the rotation of the nodes is of no importance. Hence, to describe the motion of the wire, only three degrees of freedom per node are required.

These degrees of freedom are, however, not independent because of the coupling between the nodal points. Therefore, the equations of motion should be completed with one constraint equation per element, expressing that the distance between the nodal points is prescribed. The motion of the wire can then be calculated from three second order differential equations of motion and one constraint equation per node. Using the Lagrange’s multiplier method, the unknowns are three degrees of freedom per node and one Lagrange multiplier per node, whose physical meaning will be the tensile force in the wire.
The resulting set of Differential Algebraic Equations (DAE) is very time consuming to solve, because of the large number of unknowns. Furthermore, special measurements have to been taken to avoid stability problems (see for instance [5]). To reduce the computation time, several other methods for solving the dynamics of such systems have been developed which can be found in literature.

Van der Werff [13] used a method of eliminating the dependent degrees of freedom from the equations of motion for a mechanism without deformation modes. Jonker [7] also made this approach applicable for flexible bodies. The disadvantage of this method was the fact that the mass matrix in the resulting equations of motions was a full matrix. This therefore still leads to time-consuming solution times, although it is less time consuming than the solution of the earlier equation set, consisting of the equations of motion for all degrees of freedom and the constraint equations. When having a full mass matrix with \( n \) degrees of freedom, the computing time needed to solve the equations of motion will be proportional to \( n^3 \). Such an approach is therefore referred to in literature as an order \( n^3 \) method.

When making use of the special topology for an open-loop mechanism (consisting of a chain as we have for the wire where each body is only coupled to two other bodies), it is possible to derive a method leading to a computing time required to solve the equations of motion that is proportional to \( n \). Such a method is called an order \( n \) method in literature. The first methods described were only applicable for rigid bodies. Later on, the method was extended even to flexible bodies (see for instance [8]).

These methods are mainly based on defining the position of the bodies with respect to coordinate frames attached to their neighbours (called relative coordinates). These relative coordinates make it possible to eliminate, starting with the last body, the upward bodies in the loop from the equations of motion (see for instance [6]). These methods are particularly efficient for open loop mechanisms. This approach also appears possible for closed loop systems (see for instance [9]), or when extra constraints are present. However, the benefit then reduces because Lagrange multipliers have to be defined to fulfil these constraints.

In our particular case, a winding wire contacting surfaces in many of its nodes, the benefit of relative coordinates even will completely vanish because of the large number of Lagrange multipliers required. Therefore these relative coordinates are not chosen to describe the wire motion.
In this thesis, a method is presented that is far more efficient for the calculation of the motion of the wire than the use of relative coordinates. This method is completely based on absolute coordinates and yet results in an order $n$ method. It therefore combines the benefit of the order $n$ methods mentioned, based on relative coordinates and the benefit of using absolute coordinates, the absence of Lagrange multipliers. In this approach, the Lagrange multiplier satisfying the condition that a node remains on the contacting surface, is replaced by prescribing one of the absolute coordinates that is chosen perpendicular to the surface.

In this stage, the use of the presented method seems to be limited to open loop mechanisms.
Part I

Kinetics of the wire
Chapter 2

Kinematics

2.1 Introduction

Before the equations of motion for the wire can be set up, coordinates have to be chosen which define the shape of the wire. The choice of these coordinates has a substantial influence on the resulting equations of motion and on the calculation time required to solve these equations.

As explained in chapter 1, we shall use an approach where the positions of the nodes will be described directly with respect to an inertial coordinate system. Every node, however, will have its own local coordinate system that has a fixed position. This approach has the advantage that the boundary conditions arising from the nodal contact with surfaces can be directly accounted for by prescribing a local coordinate when choosing this coordinate perpendicular to the surface. The possible disadvantage that the resulting mass matrix is a full matrix can be avoided by choosing the directions of the local coordinate systems in a proper manner. In chapter 4, these directions are calculated and it is shown that these directions do not conflict with the demand to choose local coordinates perpendicular to contact surfaces.

To avoid a set of dependent degrees of freedom caused by the coupling between the nodes, an approach as described by van der Werff [13] for rigid bodies will be followed. Jonker [7] has extended this approach to flexible bodies. In this approach, the degrees of freedom are split into:

- Independent coordinates (called generalised)
- Coordinates depending on the generalised coordinates (called dependent)
- Prescribed coordinates
In contrast to Jonker, we use the prescribed coordinates not only for coordinates that are prescribed constant, but for all coordinates that are independent of a virtual displacement in the generalised coordinates. For instance, a coordinate chosen instantaneously perpendicular to a surface will be independent of virtual displacements along the surface. However, when the surface is not flat, this coordinate does not have to be constant.

The local coordinates of a node will depend on the type of node. We distinguish four types of nodes:

- **Free nodes:**
  These nodes are material nodes (fixed wire position) that are not in contact with any surface and will therefore have two generalised (independent) coordinates. The third coordinate is a dependent coordinate, because it depends on the generalised coordinates and on the position of the adjacent node.

- **Surface contact nodes:**
  These nodes are also material nodes but are in contact with a continuous surface. The coordinate perpendicular to this surface will be prescribed and these nodes will therefore only have one generalised coordinate. The remaining coordinate is again a dependent coordinate.

- **Line contact nodes:**
  These nodes are non-material nodes (i.e. without a fixed wire position) and they are in contact with a surface consisting of a line. These nodes will also have a prescribed coordinate and only one generalised coordinate. Again, the third coordinate is a dependent coordinate.

- **Combined surface-line contact nodes:**
  These nodes are non-material nodes that are in simultaneous contact with a line and a continuous surface that is intersected by the line almost perpendicular to it. Because these nodes will have two prescribed coordinates and one dependent coordinate, they have no generalised coordinates.
Figure 2.1 gives examples of these four types of nodes in a schematic representation. The different types of contact nodes and the way the wire contacts surfaces are explained further in chapter 5.

Fig. 2.1: Four types of nodes

In the following part of this chapter, the approach of expressing the dependent coordinates in the generalised coordinates and the application to the various node types are discussed in detail.
2.2 Splitting of the coordinates

The approach followed in this section to define the coordinates is the approach as described in [13] and [7]. We adapt this approach for use with local coordinate systems.

The position of nodes will be defined by local coordinates with respect to their own local rectangular Cartesian coordinate systems. The column vector containing the local coordinates will be called $\mathbf{x}_L$.

In node $i$, we define a local coordinate frame with the orthonormal base vectors $\mathbf{g}_i^1$, $\mathbf{g}_i^2$, and $\mathbf{g}_i^3$. The origin of these local coordinate frames will be chosen to coincide with the global origin.

The position vector $\mathbf{x}_i$ of node $i$ can be defined with respect to these base vectors by using its local coordinates:

$$\mathbf{x}_i = x_{L,i}^1 \mathbf{g}_i^1 + x_{L,i}^2 \mathbf{g}_i^2 + x_{L,i}^3 \mathbf{g}_i^3$$  \hspace{1cm} (2.1)

Where: $x_{L,i}^j$ : the coordinates of node $i$ with respect to the local coordinate frame of node $i$

When writing equation 2.1 in component form, we find the following expression for the components $x_{L,i}^j$ of $\mathbf{x}_i$:

$$\mathbf{x}_i = x_{L,i}^1 \mathbf{g}_i^1 + x_{L,i}^2 \mathbf{g}_i^2 + x_{L,i}^3 \mathbf{g}_i^3$$  \hspace{1cm} (2.2)

This expression can also be written as:

$$\mathbf{x}_i = \begin{bmatrix} x_{i}^1 \\ x_{i}^2 \\ x_{i}^3 \end{bmatrix} = \mathbf{G}_i \mathbf{x}_L$$  \hspace{1cm} (2.3)

Here, the matrix $\mathbf{G}_i$ is the orthonormal transformation matrix:

$$\mathbf{G}_i = \begin{bmatrix} \mathbf{g}_i^1 & \mathbf{g}_i^2 & \mathbf{g}_i^3 \end{bmatrix}$$  \hspace{1cm} (2.4)

Because the matrix is orthonormal, we have:

$$\mathbf{G}_i^{-1} = \mathbf{G}_i^T$$  \hspace{1cm} (2.5)
2.2 Splitting of the Coordinates

and therefore:

\[ x_{Li} = G^T_i x_i \] (2.6)

Because the local coordinate frames will be chosen as fixed in space, the velocities are given by:

\[ \dot{x}_i = \dot{x}_{Li}^1 g_i^1 + \dot{x}_{Li}^2 g_i^2 + \dot{x}_{Li}^3 g_i^3 \] (2.7)

The accelerations are given by:

\[ \ddot{x}_i = \ddot{x}_{Li}^1 g_i^1 + \ddot{x}_{Li}^2 g_i^2 + \ddot{x}_{Li}^3 g_i^3 \] (2.8)

The coordinates will now be split into:

- The generalised local coordinates \( x_{Lm} \): local coordinates that are independent of all other coordinates
- The prescribed local coordinates \( x_{L0} \): local coordinates that are independent of virtual displacements of the generalised coordinates
- The dependent local coordinates \( x_{Lc} \): local coordinates that are fully determined by the generalised coordinates.

If the local coordinate directions are chosen properly, the shape of the wire is completely defined by the generalised coordinates \( x_{Lm} \). Therefore, a function \( F^x \) will exist such that:

\[ x_L = F^x(x_{Lm}) \] (2.9)

Function \( F^x \) has subfunctions \( F^{x0}, F^{xc} \) and \( F^{xm} \) such that:

\[
x_L = \begin{bmatrix} x_{L0}^m \\ x_{Lc}^m \\ x_{Lm}^m \end{bmatrix} = \begin{bmatrix} F^{x0}(x_{Lm}^m) \\ F^{xc}(x_{Lm}^m) \\ F^{xm}(x_{Lm}^m) \end{bmatrix}
\] (2.10)
2.3 Calculation of the velocities

2.3.1 Definition of the Jacobian matrix

The kinematically admissible velocities $\dot{x}_L$ can be obtained by differentiation of equation 2.9 with respect to time:

$$\dot{x}_L = [D\mathcal{F}^x]\dot{x}_L^m \quad (2.11)$$

Here, the symbol $D$ means differentiation of a function with respect to its argument; in this case, differentiation of $\mathcal{F}^x$ with respect to $x_L^m$:

$$[D\mathcal{F}^x] = \left[ \frac{\partial \mathcal{F}^x(x_L^m)}{\partial x_L^m} \right] \quad (2.12)$$

The matrix $[D\mathcal{F}^x]$ represents the Jacobian matrix of the function $\mathcal{F}^x$. In this section, the components of the Jacobian matrix $[D\mathcal{F}^x]$ are calculated. First, analogous to the subfunctions of $\mathcal{F}^x$ in equation 2.10, we define the submatrices $[D\mathcal{F}^{x0}]$, $[D\mathcal{F}^{xc}]$ and $[D\mathcal{F}^{xm}]$ of $[D\mathcal{F}^x]$:

$$[D\mathcal{F}^x] = \begin{bmatrix} D\mathcal{F}^{x0} \\ D\mathcal{F}^{xc} \\ D\mathcal{F}^{xm} \end{bmatrix} \quad (2.13)$$

Such that:

$$\dot{x}_L = \begin{bmatrix} \dot{x}_L^0 \\ \dot{x}_L^c \\ \dot{x}_L^m \end{bmatrix} = \begin{bmatrix} D\mathcal{F}^{x0} \\ D\mathcal{F}^{xc} \\ D\mathcal{F}^{xm} \end{bmatrix} \begin{bmatrix} \dot{x}_L^m \end{bmatrix} \quad (2.14)$$

Because of the choice of the coordinates, it will be obvious that:

$$[D\mathcal{F}^{x0}] = [0] \quad (2.15)$$

Later on, when the components of the Jacobian matrix are calculated for the different node types, and when the prescribed coordinates are chosen, equation 2.15 will be formally proved for these prescribed coordinates.
Furthermore we have:

$$[DF^x_m] = [I]$$  \hspace{1cm} (2.16)

The components \((\partial x^c_{Li}/\partial x^m_{Li})\) of \([DF^x_c]\) can be calculated by demanding that the wire length remains constant. For this calculation, we have to distinguish between the different types of nodes.

### 2.3.2 The Jacobian matrix for free nodes

**Definition of local coordinates**

For all free nodes \(i\), we choose the coordinate \(x^3_{Li}\) as the one and only dependent coordinate. Because of the choice for \(x^3_{Li}\) as dependent coordinate for free nodes, the coordinates \(x^1_{Li}\) and \(x^2_{Li}\) will be generalised.

When the equations of motion are derived, \(\bar{g}_i^2\) is chosen to be instantaneously perpendicular to \(\bar{r}_i\) and \(\bar{r}_{i+1}\). The unit direction vectors from nodes \(i - 1\) to \(i\) and nodes \(i\) to \(i + 1\), respectively. \(\bar{g}_i^2\) will be calculated by:

$$\bar{g}_i^2 = \frac{\bar{r}_i \times \bar{r}_{i+1}}{|| \bar{r}_i \times \bar{r}_{i+1} ||}$$  \hspace{1cm} (2.17)

where \(\bar{r}_i \times \bar{r}_{i+1}\) indicates the vector product of the vectors \(\bar{r}_i\) and \(\bar{r}_{i+1}\).

Therefore:

$$\bar{g}_i^2 \cdot \bar{r}_i = \bar{g}_i^2 \cdot \bar{r}_{i+1} = 0$$  \hspace{1cm} (2.18)

The angle of the wire in node \(i\) defined by the angle between \(\bar{r}_i\) and \(\bar{r}_{i+1}\) will be called \(\phi_i\). This angle \(\phi_i\) \((0 \leq \phi_i \leq \pi)\) is defined by:

$$\phi_i = \cos^{-1} (\bar{r}_i \cdot \bar{r}_{i+1})$$  \hspace{1cm} (2.19)

The angle between \(\bar{g}_i^1\) and the vector \(\bar{r}_{i+1}\) will be called \(\psi_i\). Figure 2.2 shows the angles \(\phi_i\) and \(\psi_i\). The directions of \(\bar{g}_i^1\) and \(\bar{g}_i^3\) will always be chosen such that \(\bar{g}_i^3 \cdot \bar{r}_{i+1} \geq 0\). The directions of the local coordinates are then completely determined when defining this angle \(\psi_i\) \((0 \leq \psi_i \leq \pi)\) by:

$$\psi_i = \cos^{-1} (\bar{g}_i^1 \cdot \bar{r}_{i+1})$$  \hspace{1cm} (2.20)
From the definitions of the angles \( \phi_i \) and \( \psi_i \) and from Figure 2.2, it can be seen that:

\[
\mathbf{g}_i^3 \cdot \mathbf{r}_{i+1} = \sin(\psi_i) \quad (2.21)
\]

and:

\[
\begin{align*}
\mathbf{g}_i^3 \cdot \mathbf{r}_i &= \sin(\phi_i + \psi_i) \\
\mathbf{g}_i^1 \cdot \mathbf{r}_i &= \cos(\phi_i + \psi_i)
\end{align*} \quad (2.22)
\]

**The submatrix \( [D\mathcal{F}^\text{xc}] \)**

Using the local coordinates defined in the preceding, we will now calculate the components of the Jacobian submatrix \( [D\mathcal{F}^\text{xc}] \) that can be notated as \( \left( \partial x_{Li}^3 / \partial x_{Lr}^i \right) \).

If the wire is assumed to be infinitely stiff in the longitudinal direction (see section 1.5), then the difference in wire coordinate between nodes \( i \) and \( i - 1 \) can be calculated as:

\[
l_i - l_{i-1} = \sqrt{\left( \mathbf{x}_i - \mathbf{x}_{i-1} \right) \cdot \left( \mathbf{x}_i - \mathbf{x}_{i-1} \right)} \quad (2.23)
\]
By using equation 2.1, the following expression is found for the partial derivative \( \left( \frac{\partial (l_i - l_{i-1})}{\partial x_{L,i}^j} \right) \) with \( (j = 1, 3) \):

\[
\frac{\partial (l_i - l_{i-1})}{\partial x_{L,i}^j} = \frac{(\bar{x}_i - \bar{x}_{i-1}) \cdot \bar{g}_i^j}{\sqrt{(\bar{x}_i - \bar{x}_{i-1}) \cdot (\bar{x}_i - \bar{x}_{i-1})}} = \bar{r}_i \cdot \bar{g}_i^j
\]

(2.24)

Where: \( \bar{r}_i \) is the direction vector from node \( i - 1 \) to \( i \) with length 1.

For the partial derivative \( \left( \frac{\partial (l_i - l_{i-1})}{\partial x_{L(i-1)}^j} \right) \), we find the following with \( (j = 1, 3) \):

\[
\frac{\partial (l_i - l_{i-1})}{\partial x_{L(i-1)}^j} = -\frac{(\bar{x}_i - \bar{x}_{i-1}) \cdot \bar{g}_i^j}{\sqrt{(\bar{x}_i - \bar{x}_{i-1}) \cdot (\bar{x}_i - \bar{x}_{i-1})}} = -\bar{r}_i \cdot \bar{g}_i^j
\]

(2.25)

If we demand that the nodes have fixed wire coordinates, then we find:

\[
- (\bar{r}_i \cdot \bar{g}_{i-1}^1) \delta x_{L(i-1)}^1 - (\bar{r}_i \cdot \bar{g}_{i-1}^2) \delta x_{L(i-1)}^2 - (\bar{r}_i \cdot \bar{g}_{i-1}^3) \delta x_{L(i-1)}^3
\]

\[
+ (\bar{r}_i \cdot \bar{g}_i^1) \delta x_{L}^1 + (\bar{r}_i \cdot \bar{g}_i^2) \delta x_{L}^2 + (\bar{r}_i \cdot \bar{g}_i^3) \delta x_{L}^3 = 0
\]

(2.26)

Similar to equation 2.26, \( n \) relations can be found in this way for a wire with only material nodes numbered as 0 up to and including \( n \), with node 0 as fixed node.

Because node 0 is the fixed node, we can consider the dependent coordinate \( x_{L1}^3 \) of node 1 as a function \( f_1 \) of its own generalised coordinates \( x_{L1}^j \), defined by equation 2.26 with \( i = 1 \). Furthermore, the dependent coordinate \( x_{Li}^3 \) of node \( i \) can be considered as a function \( f_i \) of its own generalised coordinates \( x_{Li}^j \), the coordinates \( x_{L(i-1)}^j \) and the dependent coordinate \( x_{L(i-1)}^3 \) of its adjacent node \( (j = 1, 2) \).

\[
x_{Li}^3 = f_i \left( x_{Li}^j, x_{L(i-1)}^j, x_{L(i-1)}^3 \right)
\]

(2.27)

The partial derivatives of the functions \( f_i \) can easily be found from equation 2.26:

\[
\frac{\partial f_i}{\partial x_{Li}^j} = -\frac{(\bar{r}_i \cdot \bar{g}_i^j)}{(\bar{r}_i \cdot \bar{g}_i^3)}
\]

(2.28)
The components \((\partial x_{Li}^3 / \partial x_{Lr}^i)\) of the Jacobian matrix \([DF^x]\) of the function \(\mathcal{F}^x\) (see equation 2.12), can now be calculated from relations 2.28 to 2.30 by using the chain rule if necessary:

For all \(i > 0\) we have:

\[
\frac{\partial x_{Li}^3}{\partial x_{Li}^j} = \frac{\partial f_i}{\partial x_{Li}^j} = -\frac{(\mathbf{r}_i \cdot \mathbf{g}_i^j)}{(\mathbf{r}_i \cdot \mathbf{g}_i^3)} \quad (2.31)
\]

and for all \(i > 1\):

\[
\frac{\partial x_{Li}^3}{\partial x_{L(i-1)}^j} = \frac{\partial f_i}{\partial x_{L(i-1)}^j} + \frac{\partial f_i}{\partial x_{L(i-1)}^q} \cdot \frac{\partial x_{L(i-1)}^j}{\partial x_{L(i-1)}^q}
\]

\[
= \left( (\mathbf{r}_i \cdot \mathbf{g}_{i-1}^j) - (\mathbf{r}_i \cdot \mathbf{g}_i^3) \frac{(\mathbf{r}_{i-1} \cdot \mathbf{g}_{i-1}^j)}{(\mathbf{r}_{i-1} \cdot \mathbf{g}_i^3)} \right) \frac{1}{(\mathbf{r}_i \cdot \mathbf{g}_i^3)} \quad (2.32)
\]

And for all \(i > k + 1\) with \(k \geq 1\):

\[
\frac{\partial x_{Li}^3}{\partial x_{L(i-1-k)}^j} = \left( \prod_{p=i-k+1}^{i} \left( \frac{\partial f_p}{\partial x_{L(p-1)}^j} \right) \right) \frac{\partial f_{i-k}}{\partial x_{L(i-1-k)}^j}
\]

\[
= \left( (\mathbf{r}_{i-k} \cdot \mathbf{g}_{i-1-k}^j) - (\mathbf{r}_{i-k} \cdot \mathbf{g}_{i-1-k}^3) \frac{(\mathbf{r}_{i-1-k} \cdot \mathbf{g}_{i-1-k}^j)}{(\mathbf{r}_{i-1-k} \cdot \mathbf{g}_{i-1-k}^3)} \right) \prod_{p=i-k+1}^{i} \left( \frac{\mathbf{r}_p \cdot \mathbf{g}_{p-1}^3}{(\mathbf{r}_{p-1} \cdot \mathbf{g}_{p-1}^3)} \right) \frac{1}{(\mathbf{r}_i \cdot \mathbf{g}_i^3)} \quad (2.33)
\]
2.3 Calculation of the velocities

The newly introduced symbol is the multiplication symbol with:

\[
\prod_{p=i-k}^{i} (\cdot)_p = (\cdot)_{i-k}(\cdot)_{i-k+1} \cdots (\cdot)_{i-1}(\cdot)_i
\]  

(2.34)

with the following for \( k < 0 \):

\[
\prod_{p=i-k}^{i} (\cdot)_p = 1
\]  

(2.35)

It is obvious that for all \( k \geq 1 \):

\[
\left( \frac{\partial x_{L_i}^3}{\partial x_{L_{i+k}}^j} \right) = 0
\]  

(2.36)

For convenience of notation, the following coefficients are now introduced:

- The coefficient \( S_i^j \):

\[
S_i^j = \left( \bar{r}_{i+1} \cdot \bar{g}_i^j \right) - \left( \bar{r}_{i+1} \cdot \bar{g}_i^3 \right) \frac{\left( \bar{r}_i \cdot \bar{g}_i^j \right)}{\left( \bar{r}_i \cdot \bar{g}_i^3 \right)}
\]  

(2.37)

The physical meaning of \( S_i^j \) is the ratio between the virtual displacement of node \( i \) in the direction of node \( i+1 \) and the virtual displacement \( \delta x_{L_i}^j \) of node \( i \). \( S_i^j \) can be considered as the sending coefficient of coordinate \( j \) of node \( i \).

- The coefficient \( T_{i+1} \):

\[
T_i^p = \left( \frac{\bar{r}_{i+1} \cdot \bar{g}_i^3}{\bar{r}_i \cdot \bar{g}_i^3} \right)
\]  

(2.38)

The physical meaning of \( T_{i+1} \) is the ratio between the virtual displacement of node \( p \) in the direction of node \( p+1 \) and the virtual displacement of node \( p-1 \) in the direction of node \( p \) when the virtual displacements direction of node \( p \) when the virtual displacements in the generalised coordinate directions of node \( p \) are equal to zero. \( T_{i+1} \) can be considered as the transmission coefficient of node \( i \).
For the multiplication of the transmission coefficients of node \( i - k \) to \( i \), we introduce:

\[
TM_{(i-k)(i)} = \prod_{p=i-k}^{i} T_p
\]  

(2.39)

- The coefficient \( R_i^3 \):

\[
R_i^3 = \frac{1}{(r_i \cdot \hat{g}_i^3)}
\]  

(2.40)

The physical meaning of \( R_i^3 \) is the ratio between the virtual displacement \( \partial x_{Li}^3 \) of node \( i \) and the virtual displacement of node \( i - 1 \) in the direction of node \( i \) when the virtual displacements in the generalised coordinate directions of node \( p \) are equal to zero. \( R_i^3 \) can be considered as the receiving coefficient of node \( i \).

- The coefficient \( S_{i,3j} \):

\[
S_{i,3j}^3 = -\frac{(r_i \cdot \hat{g}_i^j)}{(r_i \cdot \hat{g}_i^3)}
\]  

(2.41)

The physical meaning of \( S_{i,3j}^3 \) is the ratio between the virtual displacement \( \partial x_{Li}^3 \) and the virtual displacement \( \partial x_{Li}^j \) of node \( i \). \( S_{i,3j}^3 \) can be considered as a combined sending and receiving coefficient of node \( i \).

By making use of these coefficients and the following definition,

\[
TM_{(i)(i-1)} = \prod_{p=i}^{i-1} T_p = 1
\]  

(2.42)

the components of the Jacobian submatrix \([DF^{xc}]\) can be written in the following notation that will also be used for the other node types with different values for the coefficients:

\[
\frac{\partial x_{Li}^3}{\partial x_{Li}^j} = S_{i,3j}^3
\]  

(2.43)
For $k \geq 1$:

$$\frac{\partial x_{L(i-k)}^3}{\partial x_{L(i-k)}^j} = S_{(i-k)}^j T_{M(i-k+1)(i-1)} R_i^3$$

(2.44)

and:

$$\left( \frac{\partial x_{L(i+k)}^3}{\partial x_{L(i+k)}^j} \right) = 0$$

(2.45)

When using definitions 2.18 to 2.22, we find the following instantaneous values of the coefficients for free nodes:

$$S_i^1 = \cos (\psi_i) - \sin (\psi_i) \frac{\cos (\phi_i + \psi_i)}{\sin (\phi_i + \psi_i)}$$

(2.46)

$$S_i^2 = 0$$

(2.47)

$$T_i = \frac{\sin (\psi_i)}{\sin (\phi_i + \psi_i)}$$

(2.48)

$$R_i^3 = \frac{1}{\sin (\phi_i + \psi_i)}$$

(2.49)

$$S_{i31} = -\frac{\cos (\phi_i + \psi_i)}{\sin (\phi_i + \psi_i)} = - \cot (\phi_i + \psi_i)$$

(2.50)

$$S_{i32}^2 = 0$$

(2.51)

### 2.3.3 The Jacobian matrix for surface contact nodes

**Definition of local coordinates**

For surface contact nodes $i$, the coordinate $x_{Li}^3$ is again chosen as the one and only dependent coordinate. The coordinate $x_{Li}^2$ is chosen perpendicular to the surface in node $i$, in the direction of the outward normal. The remaining coordinate $x_{Li}^1$ will be the generalised coordinate.

The coordinate $x_{Li}^2$ will be a function of the nodal position on the surface, determined by the function $f_{i}^{\text{sur}}$ of $x_{Li}^1$ and $x_{Li}^3$:

$$x_{Li}^2 = f_{i}^{\text{sur}} (x_{Li}^1, x_{Li}^3)$$

(2.52)
Because the coordinate direction of $x_{Li}^2$ has been chosen perpendicular to the surface, we have:

$$
\frac{\partial f_{i+}^{\text{sur}}}{\partial x_{Li}^2} = \frac{\partial f_{i-}^{\text{sur}}}{\partial x_{Li}^2} = 0 \quad (2.53)
$$

Some assumptions with respect to surface contact

Within the surface contact model, it is assumed that the wire makes contact with a surface only in the node and that the wire remains straight to both sides of the node. When using these assumptions to calculate the dynamics of the wire, this will result in some unrealistic effects. The most obvious effect is that the wire tension at the left-hand and right-hand sides of the node will generally differ because the normal direction on the surface makes different angles with the wire at these two sides of the node. These angles of the normal direction with the vectors $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ will be called $\alpha_i^{\text{sur}} + \frac{\pi}{2}$ and $\alpha_i^{\text{sur}} + \frac{\pi}{2}$, respectively. The angle $\alpha_i^{\text{sur}} (-\frac{\pi}{2} \leq \alpha_i^{\text{sur}} \leq \frac{\pi}{2})$ is defined by:

$$
\alpha_i^{\text{sur}} = \sin^{-1} \left( \mathbf{g}_i^2 \cdot \mathbf{r}_i \right) \quad (2.54)
$$

The angle $\alpha_i^{\text{sur}}$ is defined by $(-\frac{\pi}{2} \leq \alpha_i^{\text{sur}} \leq \frac{\pi}{2})$:

$$
\alpha_i^{\text{sur}} = \sin^{-1} \left( -\mathbf{g}_i^2 \cdot \mathbf{r}_{i+1} \right) \quad (2.55)
$$

Figure 2.3 shows two different angles $\alpha_i^{\text{sur}}$ and $\alpha_i^{\text{sur}}$ in a strongly exaggerated way.

All the unrealistic effects are caused by the fact that in reality the wire lies over the surface instead of intersecting it. This means that the wire direction in all contact points of the wire is tangential to the normal direction.

To avoid unrealistic effects in a surface contact node, the following assumptions will be made:

- The normal direction in surface contact node $i$ represents the normal direction over the total wrapped angle $\alpha_i^{\text{sur}} + \alpha_i^{\text{sur}}$. Nevertheless, we will treat the direction $\mathbf{g}_i^2$, making the angles $\alpha_i^{\text{sur}} + \frac{\pi}{2}$ and $\alpha_i^{\text{sur}} + \frac{\pi}{2}$ (according to equations 2.54 and 2.55) with the wire, as the normal direction.
- It will be assumed that equations 2.20 to 2.22 as derived for free nodes remain valid, where the angle $\phi_i$ will be calculated as the angle between
2.3 Calculation of the velocities

Fig. 2.3: The contact geometry in surface contact node $i$

the unit projection vectors $\mathbf{F}_i^t$ and $\mathbf{F}_{i+1}^t$ of $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$, respectively, on the surface.

\[
\mathbf{F}_i^t = \frac{\mathbf{r}_i - \mathbf{g}_i^2 (\mathbf{r}_i \cdot \mathbf{g}_i^2)}{\| (\mathbf{r}_i - \mathbf{g}_i^2 (\mathbf{r}_i \cdot \mathbf{g}_i^2)) \|}
\]

\[
\mathbf{F}_{i+1}^t = \frac{\mathbf{r}_{i+1} - \mathbf{g}_i^2 (\mathbf{r}_{i+1} \cdot \mathbf{g}_i^2)}{\| (\mathbf{r}_{i+1} - \mathbf{g}_i^2 (\mathbf{r}_{i+1} \cdot \mathbf{g}_i^2)) \|}
\]

\[
\phi_i = \cos^{-1} (\mathbf{F}_i^t \cdot \mathbf{F}_{i+1}^t)
\]

Because the previous assumptions belong to a different definition of wire length between the nodes than will be used in the model, a correction on the node position will be necessary after each displacement increment, to ensure that the wire length remains constant. Generally, however, the use of surface contact nodes will be limited to situations where the curvature of the surface is large with respect to the nodal distances. Therefore, the directions of the projection vectors $\mathbf{F}_i^t$ and $\mathbf{F}_{i+1}^t$ will only differ slightly from the directions of $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ and thus the corrections required will also be small.
The submatrix \([DF^{x0}]\)

The components of the Jacobian submatrix \([DF^{x0}]\) for the surface contact node \(i\) are \(\left( \partial x_{Li}^2 / \partial x_{Lr}^i \right)\). When calculating these components using equation 2.53, we find the proof for equation 2.15:

\[
\frac{\partial x_{Li}^2}{\partial x_{Li}^1} = \frac{\partial \mathcal{H}_{i}^{sur}}{\partial x_{Li}^1} + \frac{\partial \mathcal{H}_{i}^{sur}}{\partial x_{Li}^3} \frac{\partial \mathcal{H}_{i}}{\partial x_{Li}^1} = 0 \tag{2.57}
\]

For \(k \geq 1\):

\[
\frac{\partial x_{Li}^2}{\partial x_{L(i-k)}^j} = \frac{\partial \mathcal{H}_{i}^{sur}}{\partial x_{Li}^3} \frac{\partial x_{Li}^3}{\partial x_{L(i-k)}^j} = 0 \tag{2.58}
\]

\[
\frac{\partial x_{Li}^2}{\partial x_{L(i+k)}^j} = 0 \tag{2.59}
\]

The submatrix \([DF^{xc}]\)

The functions \(f_i\) (as defined in equation 2.27) and the partial derivatives (as given by equations 2.28 to 2.30) will again be used to calculate the components of the Jacobian matrix \([DF^{xc}]\).

When node \(i\) is a surface contact node, these components \(\left( \partial x_{Li}^3 / \partial x_{Lr}^i \right)\) (for free nodes \(r: j = 1, 2\); for contact nodes \(r: j = 1\)) of this submatrix can now be calculated from relations 2.28 to 2.30, 2.52 and 2.53 by using the chain rule if necessary:

For all \(i > 0\), with node \(i\) being a surface contact node, we have:

\[
\frac{\partial x_{Li}^3}{\partial x_{Li}^1} = \frac{\partial \mathcal{H}_{i}}{\partial x_{Li}^1} + \frac{\partial \mathcal{H}_{i}}{\partial x_{Li}^2} \left( \frac{\partial \mathcal{H}_{i}^{sur}}{\partial x_{Li}^1} + \frac{\partial \mathcal{H}_{i}^{sur}}{\partial x_{Li}^3} \frac{\partial \mathcal{H}_{i}}{\partial x_{Li}^1} \right) = \frac{- \left( \bar{r}_i \cdot \bar{g}_{i}^1 \right)}{\left( \bar{r}_i \cdot \bar{g}_{i}^3 \right)} \tag{2.60}
\]

Similarly for all \(i - 1 > 1\), with node \(i - 1\) being a surface contact node, we have:

\[
\frac{\partial x_{Li}^3}{\partial x_{L(i-1)}^1} = \frac{\partial \mathcal{H}_{i-1}}{\partial x_{L(i-1)}^1} + \frac{\partial \mathcal{H}_{i}}{\partial x_{L(i-1)}^2} \left( \frac{\partial \mathcal{H}_{i-1}^{sur}}{\partial x_{L(i-1)}^1} + \frac{\partial \mathcal{H}_{i-1}^{sur}}{\partial x_{L(i-1)}^3} \frac{\partial \mathcal{H}_{i-1}}{\partial x_{L(i-1)}^1} \right) + \frac{\partial \mathcal{H}_{i-1}}{\partial x_{L(i-1)}^3} \frac{\partial \mathcal{H}_{i-1}}{\partial x_{L(i-1)}^1} \tag{2.61}
\]
2.3 Calculation of the velocities

For all $i - 1 > 1$, with node $i$ being a surface contact node, we have:

\[
\frac{\partial x_i^3}{\partial x_{L(i-1)}^j} = \frac{\partial^3 f_i}{\partial x_{L(i-1)}^j \partial x_{L(i-1)}^j} + \frac{\partial^3 f_i}{\partial x_{L(i-1)}^j \partial x_{L(i-1)}^j} \left( \frac{\partial^3 f_{i-1}^{sur}}{\partial x_{L(i-1)}^j \partial x_{L(i-1)}^j} + \frac{\partial^3 f_{i-1}^{sur}}{\partial x_{L(i-1)}^j \partial x_{L(i-1)}^j} \right)
\]

\[
= \left( \left( \frac{\mathbf{r}_i \cdot \mathbf{g}_{i-1}^1}{|\mathbf{r}_i \cdot \mathbf{g}_{i-1}^3|} - \left( \frac{\mathbf{r}_i \cdot \mathbf{g}_{i-1}^3}{|\mathbf{r}_i \cdot \mathbf{g}_{i-1}^3|} \right) \right) \frac{1}{|\mathbf{r}_i \cdot \mathbf{g}_{i}^3|} \right)
\]

The results of 2.60 to 2.63 are the same as the results of equations 2.31 to 2.33 (equation 2.33 with $k = 1$).

Using the same coefficients as introduced for free nodes, with only $S_i^1$ and $S_i^{31}$ instead of $S_i^1$ and $S_i^{3j}$, the components of the Jacobian submatrix $[D \mathcal{F}xc]$, as given by equations 2.43 to 2.50 (without equation 2.47 and only for $j = 1$) also remain valid for surface contact node $i$. The only difference with free nodes is that equation 2.56 should be used to calculate the angle $\phi_i$ for surface contact nodes instead of equation 2.19.
2.3.4 The Jacobian matrix for line contact nodes

Definition of local coordinates

Nodes in contact with a line will be non-material. For non-material nodes, the laws of Newton are no longer valid. Therefore, to allow the derivation of the equations of motion, we will add two material nodes to every non-material node. The position of these material nodes will be chosen on the wire at an infinitesimally small distance to the non-material node at both sides. Figure 2.4 shows the location of these material nodes. For practical reasons, the infinitesimally small distances between the nodes \( i^- \), \( i \) and \( i^+ \) are shown in Figure 2.4 as finite distances. These nodes are purely fictitious and are only needed during the derivation of the equations of motion. They also have no function in defining the geometry of the wire, because the wire always remains straight in these nodes. These extra nodes added to node \( i \) will be called \( i^- \) and \( i^+ \).

Fig. 2.4: The position of the nodes \( i^- \) and \( i^+ \) at both sides of node \( i \).

The mass of node \( i \) will be divided over both these material nodes. The mass in node \( i^- \) will be called \( m_{i^-} \) and the mass in node \( i^+ \) will be called \( m_{i^+} \). These masses are given by:

\[
\begin{align*}
m_{i^-} &= \left( \frac{l_i - l_{i-1}}{2} \right) \rho A \\
m_{i^+} &= \left( \frac{l_{i+1} - l_i}{2} \right) \rho A
\end{align*}
\]  

(2.64)
As can be seen from equation 2.64, the masses \( m_{i-} \) and \( m_{i+} \) depend on the wire length belonging to the nodes and because these are non-constant, the mass in the material nodes will be non-constant.

In node \( i- \), a local coordinate system will be defined with an orthonormal base \((\bar{g}_1^i, \bar{g}_2^i, \bar{g}_3^i)\). This coordinate system will also be used as the coordinate system of node \( i \). Therefore, its base will also be indicated as \((\bar{g}_1^i, \bar{g}_2^i, \bar{g}_3^i)\) and, for all local components with respect to this base, we will omit the minus sign.

When the equations of motion are set up, we instantly choose \( \bar{g}_1^i \) perpendicular both to \( \bar{r}_i \) and the contact line in node \( i \), such that the vector \( \bar{g}_1^i \) is pointing out of the contact surface. Therefore, \( \bar{g}_1^i \) can be treated as the normal vector on the line surface in node \( i- \).

The unit direction vector of the contact line in node \( i \) will be called \( \bar{t}_i \), and will be defined tangent to the contact line in such a direction that:

\[
\bar{g}_2^i = \frac{\bar{t}_i \times \bar{r}_i}{\| \bar{t}_i \times \bar{r}_i \|} \quad (2.65)
\]

Then, we have:

\[
\bar{g}_2^i \cdot \bar{r}_i = \bar{g}_2^i \cdot \bar{t}_i = 0 \quad (2.66)
\]

Figure 2.5 shows the geometrical situation when rotating the vector \( \bar{r}_{i+1} \) (and the local coordinate directions in node \( i+ \)) around the vector \( \bar{t}_i \) into the same plane as the vector \( \bar{r}_i \). To define the angle \( \phi_i \) for line contact nodes, we first introduce the angles \( \beta_{i-} \) and \( \beta_{i+} \). These angles are the angles of the wire with the circumferential direction of the line contact. The angle \( \beta_{i-} \) will be defined by \((-\frac{\pi}{2} \leq \beta_{i-} \leq \frac{\pi}{2})\):

\[
\beta_{i-} = \sin^{-1} (\bar{r}_i \cdot \bar{t}_i) \quad (2.67)
\]

The angle \( \beta_{i+} \) will be defined by \((-\frac{\pi}{2} \leq \beta_{i+} \leq \frac{\pi}{2})\):

\[
\beta_{i+} = \sin^{-1} (\bar{r}_{i+1} \cdot \bar{t}_i) \quad (2.68)
\]

The angle \( \phi_i \) for line contact nodes, is defined by \((-\pi \leq \phi_i \leq \pi)\):

\[
\phi_i = \beta_{i-} - \beta_{i+} \quad (2.69)
\]

Figure 2.5 shows that this angle \( \phi_i \) can be considered as the angle between \( \bar{r}_i \) and \( \bar{r}_{i+1} \) after rotating both around the vector \( \bar{t}_i \) into the same plane.
The angle between $\mathbf{t}_i$ and $\mathbf{r}_{i+1}$ will be called $\kappa_i$. This angle $\kappa_i$ ($0 \leq \kappa_i \leq \pi$) is defined by:

$$\kappa_i = \cos^{-1} (\mathbf{r}_{i+1} \cdot \mathbf{t}_i) \quad (2.70)$$

Furthermore, the angle $\psi_i$ ($-\frac{\pi}{2} \leq \psi_i \leq \frac{\pi}{2}$) is defined by:

$$\psi_i = \sin^{-1} (\mathbf{g}_i^3 \cdot \mathbf{r}_i) - \phi_i \quad (2.71)$$

Then, from Figure 2.5, it can be found that:

$$\begin{align*}
\mathbf{g}_i^1 \cdot \mathbf{r}_i &= \cos (\phi_i + \psi_i) \\
\mathbf{g}_i^1 \cdot \mathbf{t}_i &= \cos (\kappa_i + \psi_i) \\
\mathbf{g}_i^3 \cdot \mathbf{t}_i &= \sin (\kappa_i + \psi_i) \quad (2.72)
\end{align*}$$

In node $i^+$, a local coordinate system will be defined with an orthonormal base $(\mathbf{g}_{i+1}^1, \mathbf{g}_{i+1}^2, \mathbf{g}_{i+1}^3)$. When the equations of motion are set up, the direction of
\( \vec{g}_{i+}^3 \) is chosen instantaneously in the same direction as \( \vec{r}_{i+1} \), and \( \vec{g}_{i+}^2 \) is chosen perpendicular to both \( \vec{t}_i \) and \( \vec{r}_{i+1} \) by defining \( \vec{g}_{i+}^2 \) as:

\[
\vec{g}_{i+}^2 = \vec{t}_i \times \vec{r}_{i+1} \tag{2.73}
\]

Because \( \vec{g}_{i+}^1 \) is perpendicular to \( \vec{g}_{i+}^3 \), it is also perpendicular to \( \vec{r}_{i+1} \). Therefore we have:

\[
\vec{g}_{i+}^1 \cdot \vec{r}_{i+1} = \vec{g}_{i+}^2 \cdot \vec{r}_{i+1} = \vec{g}_{i+}^3 \cdot \vec{t}_i = 0 \tag{2.74}
\]

\[
\vec{g}_{i+}^3 \cdot \vec{r}_{i+1} = 1 \tag{2.75}
\]

\[
\vec{t}_i \cdot \vec{g}_{i+}^3 = \cos (\kappa_i) \tag{2.76}
\]

\[
\vec{t}_i \cdot \vec{g}_{i+}^1 = -\sin (\kappa_i) \tag{2.77}
\]

Furthermore, the angle \( \alpha_{lin}^i \) will be defined as the angle of the wire around the contact line. This angle \( \alpha_{lin}^i \) is defined by the following expressions:

If \( \vec{r}_i \cdot \vec{g}_{i+}^2 > 0 \) (\( 0 \leq \alpha_{lin}^i \leq \pi \)):

\[
\alpha_{lin}^i = \cos^{-1} (\vec{g}_i^2 \cdot \vec{g}_{i+}^2) \tag{2.77}
\]

If \( \vec{r}_i \cdot \vec{g}_{i+}^2 < 0 \) \( (\neg \pi \leq \alpha_{lin}^i \leq 0) \):

\[
\alpha_{lin}^i = -\cos^{-1} (\vec{g}_i^2 \cdot \vec{g}_{i+}^2) \tag{2.78}
\]

The practical meaning of this is that the angle \( \alpha_{lin}^i \) is defined as positive if the wire is wrapped around the contact line, otherwise as negative.

Because the local coordinates \( x_{Li}^2 \) and \( x_{Li+}^2 \) are instantaneously chosen perpendicular to the contact line, they are prescribed coordinates. Furthermore, we define the coordinate \( x_{Li}^3 \) in node \( i- \) as the generalised coordinate and \( x_{Li}^1 \) as the dependent coordinate. Because of the continuity of the wire, coordinates \( x_{Li+}^1 \) and \( x_{Li+}^3 \) are also dependent on both the coordinates of node \( i- \). Appendix A contains the derivation of the relations that express this dependency. Because of this dependency and for convenience of notation later on, we will consider node \( i \) as having six coordinates, one prescribed \( (x_{Li}^1) \), two prescribed coordinates \( (x_{Li}^2 \) and \( x_{Li+}^2 \)) and three dependent coordinates \( (x_{Li}^3, x_{Li+}^1 \) and \( x_{Li+}^3) \). The relations
found in appendix A are:

$$\begin{bmatrix}
\dot{x}_{Li+}^1 \\
\dot{x}_{Li+}^2 \\
\dot{x}_{Li+}^3 \\
\dot{x}_{Li}^3
\end{bmatrix} =
\begin{bmatrix}
A_{i+}^{11} & A_{i+}^{13} \\
A_{i+}^{21} & A_{i+}^{23} \\
A_{i+}^{31} & A_{i+}^{33} \\
A_{i}^{21} & A_{i}^{23}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_{Li}^1 \\
\dot{x}_{Li}^3
\end{bmatrix}$$  (2.79)

The submatrix $[DFx^0]$

The components of the Jacobian submatrix $[DFx^0]$ for the line contact node $i$ are $(\partial x_{Li}^2/\partial x_{Lr}^j)$ and $(\partial x_{Li+}^2/\partial x_{Lr}^j)$. These components can be found from relations 2.79 and A.13, as derived in appendix A.1.1. From these equations, we find the proof for equation 2.15 for a line contact node $i$:

$$\frac{\partial x_{Li}^2}{\partial x_{Li}^1} = A_{i}^{21} + A_{i}^{23} \mathcal{S}_i^{31} = 0$$  (2.80)

$$\frac{\partial x_{Li+}^2}{\partial x_{Li}^1} = A_{i+}^{21} + A_{i+}^{23} \mathcal{S}_i^{31} = 0$$  (2.81)

For $k \geq 1$:

$$\frac{\partial x_{Li}^j}{\partial x_{L(i-k)}^j} = A_{i}^{j3} \frac{\partial x_{Li}^3}{\partial x_{L(i-k)}^3} = 0$$  (2.82)

$$\frac{\partial x_{Li+}^j}{\partial x_{L(i-k)}^j} = A_{i+}^{j3} \frac{\partial x_{Li}^3}{\partial x_{L(i-k)}^3} = 0$$  (2.83)

$$\frac{\partial x_{Li}^2}{\partial x_{L(i+k)}^j} = 0$$  (2.84)

$$\frac{\partial x_{Li+}^2}{\partial x_{L(i+k)}^j} = 0$$  (2.85)

The submatrix $[DFxc]$

Analogous to section 2.3.2, where this has been done for free nodes, the components of the Jacobian submatrix $[DFxc]$ will now be calculated when node $i$ is a line contact node. These components are $(\partial x_{Li}^3/\partial x_{Lr}^j)$, $(\partial x_{Li+}^3/\partial x_{Lr}^j)$ and
When node $i$ is a line contact node, the function $f_{i+1}$ defined in section 2.3.2, expressing $x^3_{L(i+1)}$ as a function of $x^j_{L(i+1)}$, $x^i_{Li}$ and $x^3_{Li}$, will be redefined as $x^3_{L(i+1)}$ as a function of $x^j_{L(i+1)}$, $x^i_{Li+}$ and $x^3_{Li+}$:

$$x^3_{L(i+1)} = f_{i+1} \left( x^j_{L(i+1)}, x^i_{Li+}, x^3_{Li+} \right) \quad (2.86)$$

The partial derivatives of the function $f_{i+1}$ are:

$$\frac{\partial f_{i+1}}{\partial x^j_{L(i+1)}} = -\frac{\left( F_{i+1} \cdot g^j_{i+1} \right)}{\left( F_{i+1} \cdot g^3_{i+1} \right)} \quad (2.87)$$

$$\frac{\partial f_{i+1}}{\partial x^i_{Li+}} = \frac{\left( F_{i+1} \cdot g^i_{i+} \right)}{\left( F_{i+1} \cdot g^3_{i+1} \right)} \quad (2.88)$$

$$\frac{\partial f_{i+1}}{\partial x^3_{Li+}} = \frac{\left( F_{i+1} \cdot g^3_{i+} \right)}{\left( F_{i+1} \cdot g^3_{i+1} \right)} \quad (2.89)$$

The components $\left( \partial x^3_{L(i+1)} / \partial x^i_{Li} \right)$ of the Jacobian submatrix $[DF^{xc}]$ can again be calculated from relations 2.79 and 2.86 using the chain rule if necessary. Because the function $f_i$ for a line contact node $i$ is equal to this function for a free node with only $j = 1$, similar to equation 2.31, the following can be found for the coefficient $S_{3i}^j$ for all $i > 0$:

$$\frac{\partial x^3_{Li}}{\partial x^j_{Li}} = \frac{\partial f_i}{\partial x^j_{Li}} = -\frac{\left( F_i \cdot g^j_i \right)}{\left( F_i \cdot g^3_i \right)} \quad (2.90)$$

Also, similar to equation 2.32, it can be found that for all $i > 0$, with node $i$ being a line contact node, we have:

$$\frac{\partial x^3_{L(i+1)}}{\partial x^1_{Li}} = \frac{\partial f_{i+1}}{\partial x^1_{Li+}} \left( A_{i+}^{11} + A_{i+}^{13} \frac{\partial f_i}{\partial x^1_{Li}} \right) + \frac{\partial f_{i+1}}{\partial x^2_{Li+}} \left( A_{i+}^{21} + A_{i+}^{23} \frac{\partial f_i}{\partial x^2_{Li}} \right)$$

$$+ \frac{\partial f_{i+1}}{\partial x^3_{Li+}} \left( A_{i+}^{31} + A_{i+}^{33} \frac{\partial f_i}{\partial x^3_{Li}} \right) \quad (2.91)$$
Making use of equations 2.87 to 2.89 results in:

\[
\frac{\partial x_{L(i+1)}}{\partial x_{L(i-1)}} = (\vec{r}_{i+1} \cdot \vec{g}_{i+1}) \left( A^{11}_{i+1} - A^{13}_{i+1} \left( \frac{\vec{r}_i \cdot \vec{g}_i}{\vec{r}_i \cdot \vec{g}_i^3} \right) \right) \frac{1}{(\vec{r}_{i+1} \cdot \vec{g}_i^3)} \\
+ (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^2) \left( A^{21}_{i+1} - A^{23}_{i+1} \left( \frac{\vec{r}_i \cdot \vec{g}_i}{\vec{r}_i \cdot \vec{g}_i^3} \right) \right) \frac{1}{(\vec{r}_{i+1} \cdot \vec{g}_i^3)} \\
+ (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^3) \left( A^{31}_{i+1} - A^{33}_{i+1} \left( \frac{\vec{r}_i \cdot \vec{g}_i}{\vec{r}_i \cdot \vec{g}_i^3} \right) \right) \frac{1}{(\vec{r}_{i+1} \cdot \vec{g}_i^3)} (2.92)
\]

From equation 2.92, the coefficient $S_i^1$ for a line contact node can be found by dividing the right-hand term by the coefficient $R_{i+1}^3$ of node $i + 1$, given by equation 2.40, and making use of equations 2.41, 2.42 and 2.43, we obtain:

\[
S_i^1 = (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^1) \left(A^{11}_{i+1} + A^{13}_{i+1} \vec{g}_{i+1}^3\right) + (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^2) \left(A^{21}_{i+1} + A^{23}_{i+1} \vec{g}_{i+1}^3\right) + (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^3) \left(A^{31}_{i+1} + A^{33}_{i+1} \vec{g}_{i+1}^3\right) (2.93)
\]

Similarly, the component $\left(\partial x_{L(i+1)}^3 / \partial x_{L(i-1)}^j\right)$ of the Jacobian submatrix $[DF_{xc}]$ can be calculated:

\[
\frac{\partial x_{L(i+1)}^3}{\partial x_{L(i-1)}^j} = \left( \frac{\partial f_{i+1}}{\partial x_{L(i+1)}^j} A^{13}_{i+1} + \frac{\partial f_{i+1}}{\partial x_{L(i+1)}^j} A^{23}_{i+1} + \frac{\partial f_{i+1}}{\partial x_{L(i+1)}^j} A^{33}_{i+1} \right) \frac{1}{(\vec{r}_{i+1} \cdot \vec{g}_i^3)} \frac{\partial x_{L(i-1)}^3}{\partial x_{L(i-1)}^j} (2.94)
\]

Again, using equations 2.32 and 2.87 to 2.89 results in:

\[
\frac{\partial x_{L(i+1)}^3}{\partial x_{L(i-1)}^j} = \frac{1}{(\vec{r}_{i+1} \cdot \vec{g}_i^3)} \left( \frac{\vec{r}_i \cdot \vec{g}_{i-1}^j}{(\vec{r}_i \cdot \vec{g}_i^3)} - \frac{(\vec{r}_i \cdot \vec{g}_i^2)}{(\vec{r}_i \cdot \vec{g}_i^3)} \right) \left( \vec{r}_{i+1} \cdot \vec{g}_{i+1}^1 + (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^2) A^{23}_{i+1} + (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^3) A^{33}_{i+1} \right) (2.95)
\]

From equation 2.95, the coefficient $D_i^j$ for a line contact node can be found by dividing the right-hand term by the coefficient $S_{i-1}^j$ of node $i - 1$, given by equation 2.37, and the coefficient $R_{i+1}^3$ of node $i + 1$ given by equation 2.40.

\[
D_i^j = (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^1) \frac{A^{13}_{i+1}}{(\vec{r}_i \cdot \vec{g}_i^3)} + (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^2) \frac{A^{23}_{i+1}}{(\vec{r}_i \cdot \vec{g}_i^3)} + (\vec{r}_{i+1} \cdot \vec{g}_{i+1}^3) \frac{A^{33}_{i+1}}{(\vec{r}_i \cdot \vec{g}_i^3)} (2.96)
\]
Because of the choice of the local coordinates when setting up the equations of motion (the consequences of which are expressed by equations 2.71 to 2.76), the following can be found for the instantaneous values of the coefficients $S_i^j$ and $\tilde{T}_i$:

$$S_i^1 = (A_{i+}^{31} + A_{i+}^{33}S_i^{31})$$

and:

$$\tilde{T}_i = \frac{A_{i+}^{33}}{\sin (\phi_i + \psi_i)}$$

Expressions 2.49 and 2.50 remain valid for the coefficients $R_i^3$ and $S_i^{31}$ for a line contact node needed to calculate the components of the Jacobian submatrix $[DF^{xc}]$ with the dependent coordinate $x_{Li}^2$ (see equations 2.43 and 2.44). Before giving the components of the Jacobian submatrix $[DF^{xc}]$ with the remaining dependent coordinates $x_{Li}^1$ and $x_{Li}^3$, we define:

$$\frac{\partial x_{Li+}^1}{\partial x_{Li}^1} = S_{i+}^{11}$$

and for $k \geq 1$:

$$\frac{\partial x_{Li+}^1}{\partial x_{Li(i-k)}^1} = S_{(i-k)}^j T M_{(i-k+1)(i-1)} R_{i+}^1$$

$$\frac{\partial x_{Li+}^3}{\partial x_{Li(i-k)}^3} = S_{(i-k)}^j T M_{(i-k+1)(i-1)} R_{i+}^3$$

Using equation 2.79 and the chain rule, this results in:

$$S_{i+}^{11} = A_{i+}^{11} + A_{i+}^{13} S_i^{31}$$

$$S_{i+}^{31} = A_{i+}^{31} + A_{i+}^{33} S_i^{31}$$

$$R_{i+}^1 = A_{i+}^{13} R_{i}^3$$

$$R_{i+}^3 = A_{i+}^{33} R_{i}^3$$
2.3.5 The Jacobian matrix for combined surface-line contact nodes

A combined surface-line contact node that makes simultaneous contact with a line and a continuous surface almost perpendicular to this line will have no generalised coordinates. Similarly to the approach for line contact nodes, two material nodes $i-$ and $i+$ will be defined close to node $i$ with the same local directions as chosen for a line contact node (see section 2.3.4), with the only exception that the coordinate $x_{Li}$ will now also be chosen perpendicular to the wire (perpendicular to $\mathbf{F}_i$). In this way, coordinates $x_{Li}$ and $x_{Li+}$ are both prescribed coordinates, following from the definition of prescribed coordinates. Therefore, a combined surface-line contact node will have four prescribed coordinates and two dependent coordinates. The prescribed coordinates are $x_{Li}^1$, $x_{Li+}^1$, $x_{Li}^2$ and $x_{Li+}^2$. The dependent coordinates are $x_{Li}^3$ and $x_{Li+}^3$. The following now applies for the angle $\psi_i$:

$$
\psi_i = \frac{\pi}{2} - \phi_i
$$

(2.107)

The submatrix $[DFx^0]$

The components of the matrix $[DFx^0]$ for the combined surface-line contact node $i$ are $\left(\partial x_{Li}^2/\partial x_{Lr}^j\right)$, $\left(\partial x_{Li+}^2/\partial x_{Lr}^j\right)$, $\left(\partial x_{Li}^1/\partial x_{Lr}^j\right)$, and $\left(\partial x_{Li+}^1/\partial x_{Lr}^j\right)$. The first two components are again given by the relations 2.80 to 2.85, where the values for combined surface-line contact nodes, as calculated in appendix A.2.1, have to be used for the coefficients $A_{(i+)}^{kl}$. By using equation A.35, the components $\left(\partial x_{Li}^1/\partial x_{Lr}^j\right)$ and $\left(\partial x_{Li+}^1/\partial x_{Lr}^j\right)$ can be found in the same way as relations 2.82 to 2.85 have been calculated, again showing the reason why $x_{Li}^1$ and $x_{Li+}^1$ are also prescribed coordinates.

For $k \geq 1$ we have:

$$
\frac{\partial x_{Li}^1}{\partial x_{L(i-k)}^j} = A_{i-}^{13} \frac{\partial x_{Li}^3}{\partial x_{L(i-k)}^j} = 0
$$

(2.108)

$$
\frac{\partial x_{Li+}^1}{\partial x_{L(i-k)}^j} = A_{i+}^{13} \frac{\partial x_{Li}^3}{\partial x_{L(i-k)}^j} = 0
$$

(2.109)

$$
\frac{\partial x_{Li}^1}{\partial x_{L(i+k)}^j} = 0
$$

(2.110)
The submatrix $[DF^{xc}]$

The components of the Jacobian submatrix $[DF^{xc}]$ for the combined surface-line contact node $i$ can be calculated in a similar way to those for line contact nodes by calculating the new values for the coefficients $T_i, R_i^3$ and $R_{i+}^3$. Because, unlike line contact nodes, we have no generalised coordinate $x_{Li}^1$ in the combined surface-line contact node $i$, and because $x_{Li}^1$ has become a prescribed coordinate, the coefficients $S_i^1, S_i^{k1}$ and $R_{i+}^1$ do not exist in node $i$.

The coefficient $T_i$ can be calculated in a similar way as given by equation 2.96, using the coefficients $A_{i(i+)}^{kl}$ for combined surface-line contact nodes, as calculated in appendix A.2.1.

Because of the choice of the local coordinates (the consequences of which are expressed by equations 2.71 to 2.76), the following can be found for the instantaneous value of $T_i$ from 2.96, 2.107 and A.35:

$$T_i = 1 \quad (2.112)$$

The coefficient $R_i^3$ can be found from equation 2.49 by using 2.107:

$$R_i^3 = 1 \quad (2.113)$$

The components of the Jacobian submatrix $[DF^{xc}]$ with the dependent coordinate $x_{Li+}^3$ can again be found from equations 2.102 and 2.106 by using the values for $A_{i(i+)}^{kl}$, as calculated in appendix A.2.1.
2.4 Calculation of the accelerations

2.4.1 The time derivative of the Jacobian matrix

In the same way as the velocities in section 2.3, the accelerations \( \ddot{x}_L \) can be found by further differentiation of equation 2.11:

\[
\ddot{x}_L = [D\dot{F}_x] \dot{x}_L^m + [DF_x] \ddot{x}_L^m \quad (2.114)
\]

where \([D\dot{F}_x]\) is the time derivative of \([DF_x]\).

The components of the Jacobian matrix \([DF_x]\) are calculated in section 2.3. The components of \([D\dot{F}_x]\) are calculated in this section by differentiation of the components of the Jacobian with respect to time.

Again, using the submatrices \([DF_x^0]\), \([DF_x^c]\) and \([DF_x^m]\) of \([DF_x]\), as defined in equations 2.13 and 2.14, it will be obvious that:

\[
[DF_x^m] = [0] 
\]

(2.115)

As already mentioned in section 2.2, the prescribed coordinates do not have to be constant in time and therefore the time derivative \([D\dot{F}_x^0]\) does not have to be equal to the zero matrix \([0]\). This matrix \([D\dot{F}_x^0]\) is, however, not needed when setting up the equations of motion (see chapter 3). The accelerations in the prescribed coordinate directions are only needed to calculate the contact forces between the wire and any surface. When calculating these contact forces, all other accelerations and velocities have already been calculated (see chapter 3). Therefore, the accelerations in the prescribed coordinate directions in a node can be directly calculated from the other velocities in this node instead of using equation 2.114. This is shown in the next section.

The matrix \([D\dot{F}_x^c]\) is calculated in section 2.4.3.

2.4.2 The accelerations in the prescribed coordinate directions

Surface contact nodes

The accelerations in the prescribed coordinate direction of surface contact nodes (as defined in section 2.3.3) can be calculated from equation 2.52, using 2.53:

\[
\dddot{x}_{Li}^2 = \frac{\partial^2 f_{i\text{sur}}}{{\partial (x_{Li}^1)}^2} \ddot{x}_{Li}^1 \dddot{x}_{Li}^1 + 2 \frac{\partial^2 f_{i\text{sur}}}{{\partial x_{Li}^1 \partial x_{Li}^3}} \ddot{x}_{Li}^1 \dddot{x}_{Li}^3 + \frac{\partial^2 f_{i\text{sur}}}{{\partial (x_{Li}^3)}^2} \ddot{x}_{Li}^3 \dddot{x}_{Li}^3 
\]

(2.116)
2.4 Calculation of the accelerations

The second order derivatives of the function $f^\text{sur}_i$ in equation 2.116 are fully determined by the curvatures of the surface with which the node is in contact. Let us define these curvatures as:

\[
C_{11}^{\text{sur}} = -\frac{\partial^2 f^\text{sur}_i}{\partial x_{L_i}^1 \partial x_{L_i}^2}
\]
\[
C_{13}^{\text{sur}} = -\frac{\partial^2 f^\text{sur}_i}{\partial x_{L_i}^1 \partial x_{L_i}^3}
\]
\[
C_{33}^{\text{sur}} = -\frac{\partial^2 f^\text{sur}_i}{\partial x_{L_i}^3 \partial x_{L_i}^3}
\]

(2.117)

Using these curvatures, equation 2.116 leads to:

\[
\ddot{x}_{L_i}^2 = -C_{11}^{\text{sur}} (\dot{x}_{L_i}^1)^2 - 2C_{13}^{\text{sur}} \dot{x}_{L_i}^1 \dot{x}_{L_i}^3 - C_{33}^{\text{sur}} (\dot{x}_{L_i}^3)^2
\]

(2.118)

**Line contact nodes**

Before calculating the accelerations in the prescribed coordinate directions of a line contact node (as defined in section 2.3.4), we first define the normal vector $\vec{n}_i$ of the contact line, the angles $\alpha_{i-}$ and $\alpha_{i+}$, and the curvature $C_{\text{lin}}^{\text{sur}}$ of the contact line in node $i$.

The unit outward normal vector in node $i$ will be called $\vec{n}_i$. For a curved line, this vector $\vec{n}_i$ is defined by the following expressions. If $\partial \vec{t}_i / \partial t_i$ is pointing out of the material:

\[
\vec{n}_i = \frac{(\partial \vec{t}_i / \partial t_i)}{\| (\partial \vec{t}_i / \partial t_i) \|}
\]

(2.119)

If $\partial \vec{t}_i / \partial t_i$ is pointing into the material:

\[
\vec{n}_i = -\frac{(\partial \vec{t}_i / \partial t_i)}{\| (\partial \vec{t}_i / \partial t_i) \|}
\]

(2.120)

Where $t_i$ is the coordinate in the direction of $\vec{t}_i$.

The direction of the normal vector $\vec{n}_i$ is chosen such that this vector $\vec{n}_i$ is pointing out of the material.
For a straight line (with $\partial \vec{x}_i / \partial t_i = \vec{0}$), this vector $\vec{n}_i$ is defined by:

$$\vec{n}_i = \pm \frac{(\vec{g}_i^2 + \vec{g}_{i+}^2)}{\| (\vec{g}_i^2 + \vec{g}_{i+}^2) \|}$$  \hspace{1cm} (2.121)$$

Furthermore, we define the curvature $C_{lin}^{urv}$ of the line by (definition valid both for curved and straight lines):

$$C_{lin}^{urv} = - \left( \frac{\partial \vec{x}_i}{\partial t_i} \right) \cdot \vec{n}_i$$  \hspace{1cm} (2.122)$$

Therefore we have:

$$\dot{\vec{t}}_i = -\dot{t}_i C_{lin}^{urv} \vec{n}_i$$  \hspace{1cm} (2.123)$$

We find that $C_{lin}^{urv}$ will be positive for a convex contact line and negative for a concave contact line.

The angle between $\vec{g}_i$ and $\vec{n}_i$ will be called $\alpha_{i-}^{lin}$. This angle is defined by the following expressions.

If $\vec{r}_i \cdot \vec{n}_i \geq 0 \ (0 \leq \alpha_{i-}^{lin} \leq \pi)$:

$$\alpha_{i-}^{lin} = \cos^{-1} \left( \vec{g}_i^2 \cdot \vec{n}_i \right)$$  \hspace{1cm} (2.124)$$

And if $\vec{r}_i \cdot \vec{n}_i < 0 \ (-\pi \leq \alpha_{i-}^{lin} \leq 0)$:

$$\alpha_{i-}^{lin} = -\cos^{-1} \left( \vec{g}_i^2 \cdot \vec{n}_i \right)$$  \hspace{1cm} (2.125)$$

In the same way, the angle between $\vec{g}_{i+}^2$ and $\vec{n}_i$ will be called $\alpha_{i+}^{lin}$. This angle is defined by the following expressions.

If $\vec{r}_{i+1} \cdot \vec{n}_i \geq 0 \ (-\pi \leq \alpha_{i+}^{lin} \leq 0)$:

$$\alpha_{i+}^{lin} = -\cos^{-1} \left( \vec{g}_{i+}^2 \cdot \vec{n}_i \right)$$  \hspace{1cm} (2.126)$$

And if $\vec{r}_{i+1} \cdot \vec{n}_i < 0 \ (0 \leq \alpha_{i+}^{lin} \leq \pi)$:

$$\alpha_{i+}^{lin} = \cos^{-1} \left( \vec{g}_{i+}^2 \cdot \vec{n}_i \right)$$  \hspace{1cm} (2.127)$$

Figure 2.6 shows the vector $\vec{n}_i$, and the angles $\alpha_{i-}^{lin}$ and $\alpha_{i+}^{lin}$.

From the definition of the angle $\alpha_{i}^{lin}$ (see equations 2.77 and 2.78), we find:

$$\alpha_{i}^{lin} = \alpha_{i-}^{lin} + \alpha_{i+}^{lin}$$  \hspace{1cm} (2.128)$$
Fig. 2.6: The vector $\vec{n}_i$ and the angles $\alpha_{i-}^{lin}$ and $\alpha_{i+}^{lin}$

The velocity vector $\vec{v}_i$ of the non-material node $i$ equals:

$$\vec{v}_i = \dot{t}_i \vec{t}_i \quad (2.129)$$

For the acceleration vector $\vec{a}_i$ of the non-material node $i$, we find:

$$\vec{a}_i = \ddot{t}_i \vec{t}_i + \dot{t}_i \ddot{\vec{t}}_i \quad (2.130)$$

Using equations 2.66, 2.74, 2.123 to 2.127 and 2.130, we can find the following expressions for the accelerations in the prescribed coordinate directions:

$$\ddot{x}_{Li} = - (\dot{t}_i)^2 C_{lin}^{uv} \cos (\alpha_{i-}^{lin}) \quad (2.131)$$

and:

$$\ddot{x}_{Li+} = - (\dot{t}_i)^2 C_{lin}^{uv} \cos (\alpha_{i+}^{lin}) \quad (2.132)$$

**Combined surface-line contact nodes**

Because the velocity $\dot{t}_i$ equals zero for combined surface-line contact nodes, the accelerations in the prescribed coordinate directions are also equal to zero (see equations 2.131 and 2.132).
2.4.3 The time derivative of the Jacobian matrix for dependent coordinates

The components of the time derivative $[\dot{D}\dot{x}^c]$ of the Jacobian matrix $[D\dot{x}^c]$ for all node types can be found by taking the time derivative of equations 2.43, 2.44 and 2.45.

\[
\frac{d}{dt} \left( \frac{\partial x^2_{Li}}{\partial x^2_{Li}} \right) = \dot{R}_i^{3j} \quad (2.133)
\]

For $k \geq 1$, using definition 2.42, we have:

\[
\frac{d}{dt} \left( \frac{\partial x^3_{Li}}{\partial x^3_{L(i-k)}} \right) = \dot{S}^j_{(i-k)} \left( T^m_{(i-k+1)(i-1)} \dot{T}_p \right) R^3_i
\]

\[
+ S^j_{(i-k)} T^m_{(i-k+1)(i-1)} \dot{R}_i^3 \quad (2.134)
\]

\[
T^m_{(i-k+1)(i-1)} = \sum_{m=i-k+1}^{i-1} \left( \prod_{p=i-k+1}^{m-1} \dot{r}_p \right) \dot{r}_m \left( \prod_{p=m+1}^{i-1} \dot{T}_p \right) \quad (2.135)
\]

For $k \geq 1$ we have:

\[
\frac{d}{dt} \left( \frac{\partial x^3_{Li}}{\partial x^3_{L(i+k)}} \right) = 0 \quad (2.136)
\]

The time derivatives of the coefficients (as used in equations 2.134 to 2.136) are calculated in the following sections.

2.4.4 The rotational velocities

In this section, some relations are derived that make it possible to express the time derivatives $\dot{S}_i^j$, $\dot{T}_p$, $\dot{R}_i^3$ and $\dot{R}_i^{3j}$ in the rotational velocities of the wire in the next section. These rotational velocities will be fully determined by the time
2.4 Calculation of the accelerations

Derivatives of two angles per node, being $\psi_i$ and $\xi_i$. The angle $\xi_i$ is defined by:

$$\xi_i = \cos^{-1}\left(\frac{\mathbf{g}_i^{2(+)}}{\mathbf{r}_{i+1}}\right)$$

In equation 2.137 and in all subsequent equations, the plus sign between brackets (+), as in $\mathbf{g}_i^{2(+)i}$, indicates that (+) should be substituted by + if node $i$ is a line contact node. Hence, $\mathbf{g}_i^{2(+)i}$ should be substituted by $\mathbf{g}_i^{2i}$. Because of the choice of the directions of the local coordinates resulting in equations 2.18 for free nodes and surface contact nodes (and equation 2.74 for line contact nodes), $\xi_i$ equals instantaneous $\frac{\pi}{2}$. Therefore, the following can be found from equation 2.137:

$$\mathbf{g}_i^{2(+)i} \cdot \mathbf{r}_{i+1} = -\dot{\xi}_i$$

(2.138)

Using equation 2.20, we find the following results for free nodes and surface contact nodes:

$$\mathbf{g}_i^1 \cdot \mathbf{r}_{i+1} = -\dot{\psi}_i \sin (\psi_i)$$

(2.139)

$$\mathbf{g}_i^3 \cdot \mathbf{r}_{i+1} = \dot{\psi}_i \cos (\psi_i)$$

(2.140)

Before defining the twisting angle $\gamma_i$ and calculating more relations such as equations 2.138, 2.139 and 2.140, we first define the intermediate vectors $\mathbf{h}_i^1$ and $\mathbf{h}_i^1$:

$$\mathbf{h}_i^1 = \mathbf{g}_i^{2(+)i} \times \mathbf{r}_{i+1}$$

(2.141)

and:

$$\mathbf{h}_i^1 = \mathbf{g}_i^2 \times \mathbf{r}_i$$

(2.142)

$\mathbf{h}_i^1$ can also be calculated as:

$$\mathbf{h}_i^1 = \sin (\psi_i(+)) \mathbf{g}_i^{1(+)i} \mathbf{g}_i^{2(+)i} \mathbf{g}_i^3$$

(2.143)

The angle $\psi_i(+)$ used in this equation is equal to the angle $\psi_i$, except for a line contact node. If node $i$ is a line contact node, then, analogous to equation 2.20, $\psi_i$ is defined as: (0 $\leq \psi_i$ $\leq \pi$)

$$\psi_i = \cos^{-1}\left(\mathbf{g}_i^{1(+)i} \cdot \mathbf{r}_{i+1}\right) = \sin^{-1}\left(\mathbf{g}_i^{3(+)i} \cdot \mathbf{r}_{i+1}\right)$$

(2.144)
Because of the choice of the directions of $\mathbf{g}^1_{i+}$ and $\mathbf{g}^3_{i+}$, resulting in equations 2.74 and 2.75, the following time derivatives can be found for line contact nodes:

\[
\mathbf{g}^1_{i+} \cdot \dot{\mathbf{r}}_{i+1} = -\dot{\psi}_{i+} \tag{2.145}
\]

\[
\mathbf{g}^3_{i+} \cdot \dot{\mathbf{r}}_{i+1} = 0 \tag{2.146}
\]

$\mathbf{h}^1_i$ can also be calculated as:

\[
\mathbf{h}^1_i = \sin (\phi_i + \psi_i) \mathbf{g}^1_i - \cos (\phi_i + \psi_i) \mathbf{g}^3_i \tag{2.147}
\]

The angle $\gamma_i$ is defined as an imaginary twist angle of the wire between node $i-1$ and $i$:

\[
\gamma_i = \cos^{-1} \left( \mathbf{g}^2_{(i-1)(+)} \cdot \mathbf{g}^2_i \right) = -\sin^{-1} \left( \mathbf{h}^1_{(i-1)(+)} \cdot \mathbf{g}^2_i \right) \tag{2.148}
\]

This angle $\gamma_i$ is shown in Figure 2.7, from which it can be seen that the following also holds:

\[
\gamma_i = \cos^{-1} \left( \mathbf{h}^1_{(i-1)(+)} \cdot \mathbf{h}^1_i \right) = \sin^{-1} \left( \mathbf{g}^2_{(i-1)(+)} \cdot \mathbf{h}^1_i \right) \tag{2.149}
\]
From equations 2.20, 2.143, 2.144, 2.145 and 2.147, the following can now be found:

\[
\begin{align*}
\mathbf{\dot{r}}_i \cdot \mathbf{h}_{(i-1)(+)}^1 &= \mathbf{\dot{r}}_i \cdot \left( \mathbf{g}_{(i-1)(+)}^1 \sin \left( \psi_{(i-1)(+)} \right) - \mathbf{g}_{(i-1)(+)}^3 \cos \left( \psi_{(i-1)(+)} \right) \right) \\
&= -\dot{\psi}_{(i-1)(+)} \\
&= (2.150)
\end{align*}
\]

From equations 2.138, 2.148 and 2.150, we can find (see also Figure 2.7):

\[
\begin{align*}
\mathbf{\dot{r}}_i \cdot \mathbf{g}_i^2 &= \mathbf{\dot{r}}_i \cdot \left( \mathbf{g}_{(i-1)(+)}^1 \cos \left( \gamma_i \right) - \mathbf{h}_{(i-1)(+)}^1 \sin \left( \gamma_i \right) \right) \\
&= -\cos \left( \gamma_i \right) \dot{\xi}_{i-1} + \sin \left( \gamma_i \right) \dot{\psi}_{(i-1)(+)} \\
&= (2.151)
\end{align*}
\]

Similar to equation 2.150, the following can be found from equations 2.22 and 2.147:

\[
\begin{align*}
\mathbf{\dot{r}}_i \cdot \mathbf{h}_i &= \mathbf{\dot{r}}_i \cdot \left( \mathbf{g}_i^1 \sin \left( \phi_i + \psi_i \right) - \mathbf{g}_i^3 \cos \left( \phi_i + \psi_i \right) \right) = -\left( \dot{\phi}_i + \dot{\psi}_i \right) \\
&= (2.152)
\end{align*}
\]

From equations 2.138, 2.149, 2.150 and 2.152, we can find:

\[
\begin{align*}
\dot{\phi}_i + \dot{\psi}_i &= -\mathbf{\dot{r}}_i \cdot \mathbf{h}_i^1 \\
&= -\mathbf{\dot{r}}_i \cdot \left( \cos \left( \gamma_i \right) \mathbf{h}_{(i-1)(+)}^1 + \sin \left( \gamma_i \right) \mathbf{g}_{(i-1)(+)}^2 \right) \\
&= \left( \cos \left( \gamma_i \right) \dot{\psi}_{(i-1)(+)} + \sin \left( \gamma_i \right) \dot{\xi}_{i-1} \right) \\
&= (2.153)
\end{align*}
\]

For convenience of notation, equation 2.153 is used to define \( H_i^a \) as:

\[
H_i^a = \dot{\phi}_i + \dot{\psi}_i = \cos \left( \gamma_i \right) \dot{\psi}_{(i-1)(+)} + \sin \left( \gamma_i \right) \dot{\xi}_{i-1} \\
= (2.154)
\]

and \( H_i^b \) as:

\[
H_i^b = \cos \left( \gamma_i \right) \dot{\xi}_{i-1} - \sin \left( \gamma_i \right) \dot{\psi}_{(i-1)(+)} \\
= (2.155)
\]

Using equations 2.72, 2.153 and 2.154, we find:

\[
\begin{align*}
\mathbf{\dot{r}}_i \cdot \mathbf{g}_i^1 &= -\sin \left( \phi_i + \psi_i \right) H_i^a \\
\mathbf{\dot{r}}_i \cdot \mathbf{g}_i^3 &= \cos \left( \phi_i + \psi_i \right) H_i^a \\
&= (2.156)
\end{align*}
\]
2.4.5 The time derivatives of the coefficients

To calculate the time derivatives $\dot{R}_i^{3j}$, $\dot{R}_{i+1}^{31}$, $\dot{R}_{i+1}^{31}$, $\dot{R}_i^3$, $\dot{S}_i^j$ and $\dot{T}_p$, we again have to distinguish between the different types of node.

Free nodes

The time derivative $\dot{R}_i^{31}$ for free nodes can be found by differentiation of equation 2.41 with $j = 1$, using equations 2.49, 2.71, 2.72, 2.153, 2.154 and 2.156:

$$\dot{R}_i^{31} = H_i^a (R_i^3)^2 \quad (2.157)$$

The time derivative $\dot{R}_i^{32}$ for free nodes can be found by differentiation of equation 2.41 with $j = 2$, using equations 2.18, 2.49, 2.151 and 2.155:

$$\dot{R}_i^{32} = H_i^b R_i^3 \quad (2.158)$$

The time derivative $\dot{R}_i^3$ can be found by differentiation of equation 2.40 using equations 2.49, 2.50, 2.71 and 2.156:

$$\dot{R}_i^3 = S_i^{31} H_i^a R_i^3 \quad (2.159)$$

The time derivative $\dot{S}_i^1$ for free nodes can be found by differentiation of equation 2.37 with $j = 1$, using equations 2.20, 2.41, 2.49, 2.139, 2.140 and 2.157:

$$\dot{S}_i^1 = -\dot{\psi}_i \cos(\phi_i) R_i^3 + \sin(\psi_i) \dot{R}_i^{31} \quad (2.160)$$

The time derivative $\dot{S}_i^2$ can be found by differentiation of equation 2.37 with $j = 2$, using equations 2.20, 2.41, 2.49, 2.138, and 2.158:

$$\dot{S}_i^2 = -\ddot{\psi}_i + \sin(\psi_i) H_i^b R_i^3 \quad (2.161)$$

The time derivative $\dot{T}_i$ can be found by differentiation of equation 2.38, using equations 2.20, 2.49, 2.50, 2.140, 2.153, 2.154 and 2.156:

$$\dot{T}_i = \dot{\psi}_i \cos(\psi_i) R_i^3 + \sin(\psi_i) H_i^a S_i^{31} R_i^3 \quad (2.162)$$
Surface contact nodes

The time derivative $\dot{S}^{31}_i$ for surface contact nodes can be found by differentiation of equation 2.60 using equations 2.22, 2.28, 2.41, 2.49, 2.53, 2.54, 2.117 and the result given by equation 2.157:

$$\dot{S}^{31}_i = H^a_i (R^3_i)^2$$

$$+ \sin (\alpha_{i-}^\text{sur}) R^3_i (C_{11}^{\text{curv}} \dot{x}_L^1 + C_{13}^{\text{curv}} \dot{x}_L^3 + (C_{13}^{\text{curv}} \dot{x}_L^1 + C_{33}^{\text{curv}} \dot{x}_L^3) \dot{S}^{31}_i)$$

Equation 2.62 with node $i$ being a surface contact node can be written as:

$$\frac{\partial x^3_{Li}}{\partial x^L_{(i-1)}} = \left(1 + \frac{\partial f_i}{\partial x^2_L} \frac{\partial f_i}{\partial x^3_L} \right) \left( \frac{\partial f_i}{\partial x^L_{(i-1)}} + \frac{\partial f_i}{\partial x^3_{Li}} \right)$$

Analogous to the derivation of equation 2.163, using the result given by equation 2.159, from equation 2.164 the following expression can be found for the time derivative $\dot{R}^3_i$:

$$\dot{R}^3_i = S^{31}_i H^a_i R^3_i + \sin (\alpha_{i-}^\text{sur}) R^3_i (C_{11}^{\text{curv}} \dot{x}_L^1 + C_{13}^{\text{curv}} \dot{x}_L^3) R^3_i$$

Equation 2.61 for node $i-1$ being a surface contact node can be written as:

$$\frac{\partial x^3_{Li}}{\partial x^L_{(i-1)}} = \frac{\partial f_i}{\partial x^2_L} \frac{\partial f_i}{\partial x^3_L} + \frac{\partial f_i}{\partial x^L_{(i-1)}} \frac{\partial f_i}{\partial x^1_L} \frac{\partial f_i}{\partial x^3_L}$$

Analogous to the derivation of equations 2.163 and 2.165, and using the result given by 2.160, from equation 2.166 the following expression can be found for the time derivative $\dot{S}^1_i$:

$$\dot{S}^1_i = -\dot{\psi}_i \cos (\phi_i) R^3_i + \sin (\psi_i) \dot{S}^{31}_i + (\sin (\alpha_{i+}^\text{sur}) + \sin (\psi_i) \sin (\alpha_{i-}^\text{sur}) R^3_i) (C_{11}^{\text{curv}} \dot{x}_L^1 + C_{13}^{\text{curv}} \dot{x}_L^3 + (C_{13}^{\text{curv}} \dot{x}_L^1 + C_{33}^{\text{curv}} \dot{x}_L^3) \dot{S}^{31}_i)$$
Equation 2.63 for node $i$ being a surface contact node can be written as:

$$\frac{\partial x_{L(i+1)}^3}{\partial x_{L(i-1)}^3} = \left( \frac{\partial f_{i+1}^3}{\partial x_{L(i+1)}^3} + \frac{\partial f_i^3}{\partial x_{L(i)}^3} + \frac{\partial f_{i-1}^3}{\partial x_{L(i-1)}^3} \right) \left( \frac{\partial f^3_{\text{sur}}}{\partial x_{L(i)}^3} \right)$$

(2.168)

Analogous to the derivation of equations 2.163, 2.165 and 2.167, and using the result given by 2.162, from equation 2.168 the following expression can be found for the time derivative $\dot{f}_i$:

$$\dot{f}_i = \dot{\psi}_i \cos(\psi_i) R_i^3 + \sin(\psi_i) R_i^3 \mathcal{S}_{i}^{31} R_i^3$$

(2.169)

$$+ \left( \sin(\psi_i) \sin(\alpha_i^3_{\text{sur}}) R_i^3 + \sin(\alpha_{i+1}^3_{\text{sur}}) \right) \left( C_{13}^{\text{curv}} \dot{x}_L^1 + C_{33}^{\text{curv}} \dot{x}_L^3 \right) R_i^3$$

### Line contact nodes

The coefficients $\mathcal{S}_{i}^{31}$ and $R_i^3$ for line contact nodes have been calculated in the same way as for free nodes. Therefore, for line contact nodes, the values for $\mathcal{S}_{i}^{31}$ and $R_i^3$ given in equations 2.157 and 2.159, respectively, remain valid.

The time derivatives $\dot{\mathcal{S}}_{i+1}^{11}$ and $\dot{\mathcal{S}}_{i+1}^{31}$ can be found by differentiation of equations 2.103 and 2.104:

$$\dot{\mathcal{S}}_{i+1}^{11} = \dot{A}_{i+1}^{11} + \dot{A}_{i+1}^{13} \mathcal{S}_{i}^{31} + A_{i+1}^{13} \dot{\mathcal{S}}_{i}^{31}$$

(2.170)

$$\dot{\mathcal{S}}_{i+1}^{31} = \dot{A}_{i+1}^{31} + \dot{A}_{i+1}^{33} \mathcal{S}_{i}^{31} + A_{i+1}^{33} \dot{\mathcal{S}}_{i}^{31}$$

(2.171)

The coefficients $\dot{A}_{i+1}^{11}$, $\dot{A}_{i+1}^{13}$, $\dot{A}_{i+1}^{31}$ and $\dot{A}_{i+1}^{33}$ are calculated in appendix A.1.2.

The time derivatives $\dot{R}_{i+1}^1$ and $\dot{R}_{i+1}^3$ can be found by differentiation of equations 2.105 and 2.106:

$$\dot{R}_{i+1}^1 = \dot{A}_{i+1}^{13} R_i^3 + A_{i+1}^{13} \dot{R}_i^3$$

(2.172)

$$\dot{R}_{i+1}^3 = \dot{A}_{i+1}^{33} R_i^3 + A_{i+1}^{33} \dot{R}_i^3$$

(2.173)
2.4 Calculation of the accelerations

By using equations 2.74, 2.75, 2.138, 2.145, 2.146 and A.13, the time derivative $\dot{S}_i^1$ for line contact nodes can be found by differentiation of equation 2.93:

$$
\dot{S}_i^1 = -\dot{\psi}_{i+} (A_{i+}^{11} + A_{i+}^{13} \dot{R}_i^3) + \left( \dot{A}_{i+}^{31} + \dot{A}_{i+}^{33} \dot{R}_i^3 + A_{i+}^{33} \dot{R}_i^3 \right)
$$

By using equations 2.49, 2.50, 2.71, 2.74, 2.75, 2.138, 2.145, 2.146, 2.153, 2.154 and A.13, the time derivative $\dot{T}_i$ for line contact nodes can be found by differentiation of equation 2.96:

$$
\dot{T}_i = -\dot{\psi}_{i+} A_{i+}^{13} R_i^3 + \dot{A}_{i+}^{33} R_i^3 + A_{i+}^{33} \dot{R}_i^3 H_i R_i^3
$$

**Combined surface-line contact nodes**

By using equations 2.40, 2.71 and 2.107, we can find the time derivative $\dot{R}_i^3$ for combined surface-line contact nodes:

$$
\dot{R}_i^3 = 0
$$

Using equations 2.106, 2.113, 2.176, and A.38, we can also find:

$$
\dot{R}_{i+}^3 = 0
$$

The time derivative $\dot{T}_i$ for combined-line contact nodes can be found in an analogous manner to $\dot{T}_i$ for line contact nodes (see equation 2.175):

$$
\dot{T}_i = -\dot{\psi}_{i+} A_{i+}^{13} R_i^3 + \dot{A}_{i+}^{33} R_i^3 + A_{i+}^{33} \dot{R}_i^3 H_i R_i^3
$$

Using equations 2.107, A.13 and A.38, we find:

$$
\dot{T}_i = 0
$$
Chapter 3

Dynamics

3.1 Introduction

In this chapter, the equations of motion for the wire are set up. As already stated in chapter 2, we will use an approach leading to a set of equations consisting of ordinary second order differential equations only, containing the generalised coordinates, without algebraic equations describing constraints. No constraints are needed because local absolute coordinates have been used and one of these coordinates is chosen perpendicular to the contact surface in contact nodes.

As already stated in section 1.7, most methods using absolute coordinates that can be found in literature result in order \( n^2 \) methods (see for instance [13]). The minimum required calculation time to solve the equations of motion for such methods strongly depends (by the third power) on the number of nodes (or better on the degrees of freedom).

The methods resulting in an order \( n \) method, where the minimum required calculation time depends only linearly on the degrees of freedom, are mainly based on relative coordinates (see for instance [6]).

In this chapter a method for calculating the dynamics of the wire is described, that results in an order \( n \) method although it uses (local) absolute coordinates. In this way, the advantages of an order \( n \) method (lesser calculation time) and absolute coordinates (absence of constraints resulting in Lagrange multipliers) are combined. The method described in this chapter seems limited to systems with a "single chain" character, such as we have for the wire.
3.2 External forces

In section 3.4, the equations of motion are derived, using the principle of virtual work. For this derivation, an expression is needed for the external forces acting on the wire.

The external forces $f_L$ acting on the wire will be divided into the velocity dependent forces $f^{v_d}_L$ that depend linearly on the velocity of the wire, and the velocity independent forces $f^{v_i}_L$, consisting of contact forces, friction forces and convective inertia forces (see section 3.5). Here, generally the friction forces are treated as velocity independent because they depend only on the direction of the velocity and are independent of the magnitude of the velocity. In section 3.3 however is shown that some of the friction forces are treated as velocity dependent for numerical reasons. Therefore the velocity dependent forces $f^{v_d}_L$ are divided into the velocity dependent friction forces $f^{f_d}_L$ and all other velocity dependent forces $f^{d}_L$, called damping forces.

Then we have for the external forces $f_L$:

$$f_L = f^{v_i}_L + f^{d}_L + f^f_L \tag{3.1}$$

The damping force vector corresponding to the local coordinates $x_L$ can be expressed in the velocities $\dot{x}_L$ by:

$$f^{d}_L = -K^{d}_L \dot{x}_L \tag{3.2}$$

The matrix $K^{d}_L$ will be called the damping matrix. We will only assume damping that can be described by a diagonal damping matrix $K^{d}_L$. For free nodes and surface contact nodes, the damping matrix $K^{d}_{Li}$ at nodal level is given by:

$$K^{d}_{Li} = D_M \begin{bmatrix} k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} \\ k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} \\ k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} \end{bmatrix} \tag{3.3}$$

Here, $D_M[\,]$, indicates a diagonal matrix with its diagonal terms $k^{d,1}_{Li}$, $k^{d,2}_{Li}$ and $k^{d,3}_{Li}$. These diagonal terms are the damping coefficients belonging to the $x^1_{Li}$, $x^2_{Li}$ and $x^3_{Li}$ coordinate direction, respectively. For line contact and combined surface-line contact nodes, the nodal damping matrix $K^{d}_{Li}$ is given by:

$$K^{d}_{Li} = D_M \begin{bmatrix} k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} & k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} \\ k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} & k^{d,1}_{Li} & k^{d,2}_{Li} & k^{d,3}_{Li} \end{bmatrix} \tag{3.4}$$

Because line contact nodes and combined surface-line contact nodes have six coordinates, $K^{d}_{Li}$ now has six damping coefficients corresponding to the three coordinates in node $i(-)$ and $i+$. 
3.3 Treatment of damping and friction forces

3.3.1 Velocity dependency in explicit time integration

In chapter 4, a method is presented to solve the equations of motion using an explicit numerical time integration, in which the only unknowns during solving the equations of motion are the accelerations. The positions and velocities are calculated from these accelerations after solving the equations of motion (see chapter 4).

It is common practice in explicit time integration to base the velocity dependent forces in the equations of motion on the velocities at the end of the previous increment. Jonker ([7]), for instance, replaced the current velocities $\dot{x}_L^m$ in equation 2.114 for the second order inertia terms by $\dot{x}_L^{m,\text{old}}$, the velocities at the end of the previous increment. This results in the following equation for $\ddot{x}_L$ using equation 2.114:

$$\ddot{x}_L = [D \dot{F}_x] \dot{x}_L^{m,\text{old}} + [D F_x] \ddot{x}_L^m$$ (3.5)

The same could be done for the velocity dependent external forces, the damping and the friction forces. However, when the damping forces or the friction forces are based on the old velocities, this can result in instabilities. Suppose the wire is exposed to damping or friction forces based on the velocity at the end of the previous increment, with a magnitude that can change the velocity direction of the wire within one increment, resulting in a greater velocity in the opposite direction. Then, an unstable situation will develop where the velocity changes direction every increment because the damping or friction force also changes direction. This only applies for forces in the direction perpendicular to the wire because the stiffness of the wire in this direction is very low. This is in contrast to the average wire direction, in which direction the wire is very stiff.

To avoid such kinds of instabilities, we will implicitly create a dependency of the damping and some of the friction forces on the current velocity by making them dependent on a linear combination of (current) accelerations and old velocities.

This dependency is explained further in chapter 4. In this chapter, we only calculate the consequences of this approach on the equations of motion.
3.3.2 Acceleration dependency of damping forces

The approach to make the damping forces dependent on the accelerations is based on expressing the velocities $\dot{x}_L$ in equation 3.2 in the old velocities and the current accelerations:

$$\dot{x}_L = \dot{x}_L^{\text{old}} + D_d \ddot{x}_L$$

(3.6)

Eliminating $\ddot{x}_L$ from equation 3.6 using equation 3.5 results in:

$$\dot{x}_L = \dot{x}_L^{\text{old}} + D_d (\left[ D \ddot{x}_x \right] \ddot{x}_L^{m,\text{old}} + [D \ddot{x}_x] \dddot{x}_L^m)$$

(3.7)

The value for $D_d$ is calculated in chapter 4. Elimination of $\ddot{x}_L$ from equation 3.2 using equation 3.7 results in the following expression for the damping forces $f_L^d$:

$$f_L^d = -K_L^d (\dot{x}_L^{\text{old}} + D_d (\left[ D \ddot{x}_x \right] \ddot{x}_L^{m,\text{old}} + [D \ddot{x}_x] \dddot{x}_L^m))$$

(3.8)

3.3.3 Acceleration dependency of friction forces

As already stated, the friction forces are dependent on the direction of the velocity and independent on the magnitude of the velocity.

However, to avoid an instability as described earlier, it will be supposed that, for contact nodes, the friction force perpendicular to the wire depends linearly on the current displacement perpendicular to the wire divided by the time increment $\Delta t$. This is discussed in detail in section 6.6. Here, we only mention that we have a contribution to the equations of motion in the $\bar{g}_1^3$ and $\bar{g}_2^3$ direction linearly depending on the displacement $\Delta x_L$.

Analogous to equation 3.2 we find for these velocity dependent friction forces $f_L^f$:

$$f_L^f = -K_L^f \frac{\Delta x_L}{\Delta t}$$

(3.9)

The matrix $K_L^f$ can be called the friction damping matrix, analogous to the damping matrix $K_L^d$. Just as the matrix $K_L^d$, it is a diagonal matrix. Most of the diagonal terms however are equal to zero. Only the diagonal terms belonging to the first generalised and the dependent coordinate directions of contact nodes will be unequal to zero. For free nodes the friction damping matrix $K_L^{f,\text{free}}$ at nodal level is equal to zero.
3.3 TREATMENT OF DAMPING AND FRICTION FORCES

For surface contact nodes $K^f_{Li}$ is given by:

$$K^f_{Li} = DM \left[ k^{f,1}_{Li}, 0, k^{f,3}_{Li} \right]$$  \hspace{1cm} (3.10)

For line contact and combined surface-line contact nodes, it is supposed that the friction forces are only acting on node $i(-)$. Therefore $K^f_{Li}$ is given by:

$$K^f_{Li} = DM \left[ k^{f,1}_{Li}, 0, k^{f,3}_{Li}, 0, 0 \right]$$  \hspace{1cm} (3.11)

The diagonal terms $k^{f,1}_{Li}$ and $k^{f,3}_{Li}$ are calculated in section 6.6. We replace the displacement $\Delta x_L$ in equation 3.9 by the old velocities and the current accelerations:

$$\frac{\Delta x_L}{\Delta t} = \dot{x}_L^{\text{old}} + D_f \ddot{x}_L$$  \hspace{1cm} (3.12)

The value for $D_f$ is calculated in chapter 4. Elimination of $\Delta x_L$ from equation 3.9 using equations 3.5 and 3.12 results in:

$$f^f_L = -K_L \left( \dot{x}_L^{\text{old}} + D_f \left[ D_F \ddot{x}_L^{\text{old}} + [D_F \dddot{x}_L^{\text{old}}] \right] \right)$$  \hspace{1cm} (3.13)
3.4 Principle of virtual work

The equations describing the wire motion can be found from the principle of virtual work. Before using this principle, we first define the (diagonal) matrix $M_{Li}$ as the mass matrix of the wire containing the nodal masses corresponding to the local coordinates $x_L$. For free nodes and surface contact nodes, the mass matrix $M_{Li}$ at nodal level is given by:

$$M_{Li} = \mathcal{D}M [m_i, m_i, m_i] \quad (3.14)$$

The nodal mass $m_i$ has been defined by equation 1.6. For line contact and combined surface-line contact nodes, the nodal mass matrix $M_{Li}$ is given by:

$$M_{Li} = \mathcal{D}M [m_{i-}, m_{i-}, m_{i+}, m_{i+}, m_{i+}] \quad (3.15)$$

The nodal masses $m_{i-}$ and $m_{i+}$ have already been defined by equation 2.64.

In our case, where it has been assumed that the wire length remains constant, the principle of virtual work states that for all kinematically admissible virtual displacements $\delta x_L$, the virtual work of the external forces $f_L$, plus inertia forces, acting on the nodes, is equal to zero. Using equation 3.1, we find:

$$\delta x_L^T \left[ f_L^y + f_L^d + f_L^i - M_L \ddot{x}_L \right] = 0 \quad (3.16)$$

In this case, kinematically admissible virtual displacements are displacements that do not violate the requirement of constant wire length. These virtual displacements can be calculated from the same expression 2.11 as formulated for the velocities:

$$\delta x_L = [DF^x] \delta x_L^m \quad (3.17)$$

Substitution of equations 3.5, 3.8, 3.13 and 3.17 into equation 3.16 and using the fact that 3.16 holds for all $\delta x_L^m$ results in:

$$[DF^x]^T \left( M_L + D_d K_L^d + D_f K_L^f \right) [DF^x] \ddot{x}_L^m =$$

$$[DF^x]^T \left[ f_L^y - \left( K_L^d + K_L^f \right) \dot{x}_L^{old} - f_L^{in} \right] \quad (3.18)$$
3.4 Principle of virtual work

With:

\[
f_{L}^{\text{in}} = \left( M_{L} + D_{d}K_{L}^{d} + D_{f}K_{L}^{f} \right) \left[ D \hat{\mathbf{x}}^x \right] \hat{x}_{L}^{m,\text{old}}
\]  

(3.19)

The term \( f_{L}^{\text{in}} \) can be considered as second order inertia (plus damping) forces, mainly resulting from (local or global) rotations of the wire. The reason for this is that in section 2.4 it has been shown that these second order inertia forces are closely related to the rotational speeds \( \psi_i \) and \( \xi_i \). The second order accelerations, resulting from the curvature of surfaces that the wire is contacting, are an exception to this.

It should be noticed that the resulting mass matrix in equation 3.18 will generally not be a diagonal matrix, although the matrices \( M_{L}, K_{L}^{d} \) and \( K_{L}^{f} \) are diagonal (see equations 3.3, 3.4, 3.10, 3.11, 3.14 and 3.15).
3.5 Convective terms

Because the wire position of non-material nodes is not constant, the mass in the added nodes \( i- \) and \( i+ \) and even in the nodes \( i-1 \) and \( i+1 \) will generally be non-constant. This is because the mass of these nodes will change when the wire length represented by such nodes changes (see equation 2.64). Therefore, a convective transfer of mass takes place between these nodes. Formally, it should be required that three laws have to be satisfied per two nodes exchanging mass separately:

- Conservation of mass
- Conservation of momentum
- Conservation of angular momentum

The conservation of mass will always be satisfied when calculating the nodal masses using equations 1.6 and 2.64, because the total wire length remains constant.

In a wire without bending stiffness, a consequence of requiring conservation of angular momentum for two nodes with different velocities approaching each other will be that the difference in velocity increases. Figure 3.1 shows such a situation. Material node \( i-1 \) with a velocity component pointing downwards approaches non-material line contact node \( i \) (without vertical velocity). The mass of node \( i-1 \) decreases according to equation 1.6. To satisfy the law of conservation of angular momentum with respect to node \( i \), at least a vertical force will be acting downwards on node \( i-1 \). This force acting on node \( i-1 \) will result in an increase of the vertical velocity component of node \( i-1 \).

![Figure 3.1: Increase in velocity of node \( i-1 \) approaching line contact node \( i \)](image)

The increase of velocity of node \( i-1 \) is very unrealistic and is caused by the assumption that the wire has no bending stiffness. In reality, the bending
stiffness (although small) will become increasingly important when two nodes are approaching each other. Therefore, we will assume that the angular momentum conservation law will be satisfied when including moments in the wire. These moments and the angular momentum conservation law will not be considered further.

In reality, the velocities of nodes that approach each other will tend to synchronise as a result of bending moments in the wire. To describe this effect without (optional) bending stiffness of the wire as well, we define a force $f_{L(i-1)}^{\text{conv}}$ acting on node $i-1$ when approaching non-material node $i$. The value of this force will be chosen such that it will have synchronised the velocities of node $i-1$ and node $i$ at the time node $i-1$ reaches the position of node $i$. Using equation 2.3, the following expression can be found for the force $f_{L(i-1)}^{\text{conv}}$ (only valid for $\dot{i}_i < 0$!):

$$f_{L(i-1)}^{\text{conv}} = \frac{\rho A_i \dot{i}_i}{2} \left( \dot{x}_{L(i-1)} - G_{i-1}^{-1} G_i \dot{x}_{Li} \right)$$ (3.20)

When node $i+1$ is approaching node $i$, we can find the following expression for the force $f_{L(i+1)}^{\text{conv}}$ (only valid for $\dot{i}_i > 0$!):

$$f_{L(i+1)}^{\text{conv}} = -\frac{\rho A_i \dot{i}_i}{2} \left( \dot{x}_{L(i+1)} - G_{i+1}^{-1} G_{i+} \dot{x}_{Li+} \right)$$ (3.21)

In order to satisfy the momentum conservation law, we assume that forces are also acting on nodes $i-$ and $i+$. Assuming that $f_{L(i+1)}^{\text{conv}}$ is equal to zero when $\dot{i}_i < 0$ and that $f_{L(i-1)}^{\text{conv}}$ is equal to zero when $\dot{i}_i > 0$, the following convective forces satisfy the momentum conservation law. The force $f_{L(i)}^{\text{conv}}$ acting on node $i-$ equals:

$$f_{L(i)}^{\text{conv}} = \frac{\rho A_i \dot{i}_i}{2} \left( \dot{x}_{Li} - G_{i-1}^{-1} G_{i-1} \dot{x}_{L(i-1)} + G_{i}^{-1} G_{i} \dot{x}_{Li} + \dot{x}_{Li} \right)$$ (3.22)

The force $f_{Li+}^{\text{conv}}$ acting on node $i+$ equals:

$$f_{Li+}^{\text{conv}} = \frac{\rho A_i \dot{i}_i}{2} \left( \dot{x}_{Li+} - G_{i+1}^{-1} G_i \dot{x}_{Li} + G_{i+1}^{-1} G_{i+1} \dot{x}_{L(i+1)} - \dot{x}_{Li+} \right)$$ (3.23)
The forces $\mathbf{f}_{L(i-1)}^{\text{conv}}$, $\mathbf{f}_{L(i+1)}^{\text{conv}}$, $\mathbf{f}_{L(i)}^{\text{conv}}$ and $\mathbf{f}_{L(i+1)}^{\text{conv}}$ are included in the external force vector $\mathbf{f}_L^{\text{vi}}$ in equation 3.18.
3.6 Splitting the equations of motion

In this section, the equations of motion given by equation 3.18 are split into two separate parts. One part only contains the first generalised coordinates and the other part only contains the second generalised coordinates. Because these two parts of the equations of motion are fully uncoupled, they can be solved successively.

In order to derive these splitted equations of motion, firstly the generalised coordinates are split into the first and second generalised coordinates:

$$x_L^m = \begin{bmatrix} x_L^{m,1} \\ x_L^{m,2} \end{bmatrix}$$ (3.24)

We define the splitting of the Jacobian matrix $[DF^x]$ in a consistent manner to the splitting of the coordinates in prescribed, dependent and generalised coordinates (see equation 2.14) and the splitting of the generalised coordinates into the coordinates $x_L^{m,1}$ and $x_L^{m,2}$ as given before. When using equations 2.15 and 2.16 and using the fact that the dependent coordinates are independent of the generalised coordinates $x_L^{m,2}$ (see chapter 2), we find:

$$[DF^x] = \begin{bmatrix} 0 & 0 \\ Df^{xc} & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ [DF^{xc,1}] & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$$ (3.25)

In the same way, using equation 2.115 and using the fact that the second order accelerations of the prescribed coordinates are independent of the coordinates $x_L^{m,2}$, we find:

$$[D\ddot{f}^x] = \begin{bmatrix} D\ddot{f}^{x0,1} & 0 \\ D\ddot{f}^{xc,1} & D\ddot{f}^{xc,2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ (3.26)

Because all nodal mass matrices will be diagonal, splitting the mass matrix in the same way leads to the following mass matrix:

$$[M_L] = DM \begin{bmatrix} M_L^{0,0} & M_L^{0,c} & M_L^{1,1} & M_L^{2,2} \end{bmatrix}$$ (3.27)
Splitting up the damping matrix leads to:

\[
[K_L^d] = \mathcal{D}\mathcal{M} \begin{bmatrix}
K_{L}^{d0,0}, K_{L}^{dc,c}, K_{L}^{d1,1}, K_{L}^{d2,2}
\end{bmatrix} \tag{3.28}
\]

Splitting up the friction damping matrix and using the fact that only the matrix components belonging to the first generalised and the dependent coordinates are non-zero, leads to:

\[
[K_L^f] = \mathcal{D}\mathcal{M} \begin{bmatrix}
0, K_{L}^{fc,c}, K_{L}^{f1,1}, 0
\end{bmatrix} \tag{3.29}
\]

Defining the same splitting for the forces \(f_L^{vi}\) and \(f_L^{in}\) yields:

\[
f_L^{vi} = \begin{bmatrix}
    f_L^{vi,0} \\
    f_L^{vi,c} \\
    f_L^{vi,1} \\
    f_L^{vi,2}
\end{bmatrix}, \quad f_L^{in} = \begin{bmatrix}
    f_L^{in,0} \\
    f_L^{in,c} \\
    f_L^{in,1} \\
    f_L^{in,2}
\end{bmatrix} \tag{3.30}
\]

Using equations 3.25 to 3.30, the following equations of motion are found from the equation of motion 3.18:

\[
[D_f^{xc,1}]^T \left( [M_L^{c,c} + D_d K_{L}^{dc,c} + D_f K_{L}^{fc,c}] \right) [D_f^{xc,1}] \hat{x}_L^{m,1} + \left( K_{L}^{d1,1} + K_{L}^{f1,1} \right) \hat{x}_L^{m,1,old} + [D_f^{xc,1}]^T \left[ f_L^{vi,c} - \left( K_{L}^{dc,c} + K_{L}^{fc,c} \right) \hat{x}_L^{c,old} - f_L^{in,c} \right]
\]

with:

\[
f_L^{in,c} = \left( [M_L^{c,c} + D_d K_{L}^{dc,c} + D_f K_{L}^{fc,c}] \right) \left( [D_f^{xc,1}] \hat{x}_L^{m,1,old} + [D_f^{xc,2}] \hat{x}_L^{m,2,old} \right) \tag{3.32}
\]

and:

\[
[M_L^{2,2} + D_d K_{L}^{d2,2}] \hat{x}_L^{m,2} = f_L^{vi,2} - K_{L}^{d2,2} \hat{x}_L^{m,2,old} \tag{3.33}
\]
3.7 Towards a diagonal system mass matrix

The system mass matrix $M_{L}^{2,2} + D_{d}K_{L}^{d,2,2}$ in equation 3.33 is a diagonal matrix with the masses $m_{i}$, $m_{i-}$ and $m_{i+}$ and the damping terms $D_{d}k_{L_{i}}^{d,2}$ as diagonal terms. This fact can be used to reduce the calculation time when calculating the accelerations $\ddot{x}_{L_{i}}^{m,2}$ from equation 3.33.

The system mass matrix (belonging to the accelerations $\ddot{x}_{L_{i}}^{m,1}$) in equation 3.31 will generally be a full matrix. In this section, however, local coordinate directions defined by the angles $\psi_{i}$ for all nodes are calculated that result in a diagonal system mass matrix in equation 3.31. The solution process for a set of equations of motion with a diagonal mass matrix will take less time than a set with a full mass matrix.

To calculate the system matrix corresponding to the accelerations $\ddot{x}_{L_{i}}^{m,1}$ in equation 3.31, the components of this matrix will be indicated by the indices $r$ and $k$, where $r$ indicates the row number and $k$ the column number. For convenience of notation, we assume that the row number $r$ and column number $k$ also indicate the node number to which the indicated row or column belongs. The consequence of this assumption is that row $i$ and column $i$ do not exist when node $i$ is a combined surface-line contact node. This is because these nodes have no generalised coordinate $x_{1}^{1}$.

For convenience of notation, we define the following for a free or surface contact node $i$:

\begin{align*}
m_{i} = m_{i} & \quad mk_{L_{i}}^{1} = m_{i} + D_{d}k_{L_{i}}^{d,1} + D_{f}k_{L_{i}}^{f,1} & \quad mk_{L_{i}}^{1,1} = 0 \\
m_{i+} = 0 & \quad mk_{L_{i}}^{3} = m_{i} + D_{d}k_{L_{i}}^{d,3} + D_{f}k_{L_{i}}^{f,3} & \quad mk_{L_{i}}^{3} = 0
\end{align*}

(3.34)

and for a combined surface-line or line contact node $i$:

\begin{align*}
m_{i} = m_{i-} & \quad mk_{L_{i}}^{1} = m_{i} + D_{d}k_{L_{i}}^{d,1} + D_{f}k_{L_{i}}^{f,1} & \quad mk_{L_{i}}^{1} = m_{i+} + D_{d}k_{L_{i}}^{d,1} \\
m_{i+} = m_{i+} & \quad mk_{L_{i}}^{3} = m_{i} + D_{d}k_{L_{i}}^{d,3} + D_{f}k_{L_{i}}^{f,3} & \quad mk_{L_{i}}^{3} = m_{i+} + D_{d}k_{L_{i}}^{d,3}
\end{align*}

(3.35)
For the component \((r, k)\) of the system mass matrix corresponding to the accelerations \(\ddot{x}^m_L\) in equation 3.31, the following expression then can be found:

\[
\left(\left[D\mathcal{F}^{xc.1}\right]^T \left(M_L^{c,c} + D_dK_L^{dc,c} + D_fK_L^{fc,c}\right) \left[D\mathcal{F}^{xc.1}\right]\right)_{r,k} \\
+ \left(M_L^{1,1} + D_dK_L^{d1,1} + D_fK_L^{f1,1}\right)_{r,k} = \\
\delta^r_k m^1_{Lr} + \sum_{i=1}^n \left(\frac{\partial x^3_i}{\partial x_{Lr}} m^3_{Li} \frac{\partial x^3_i}{\partial x_{Lk}}\right) \\
+ \sum_{i=1}^n \left(\frac{\partial x^3_{Li+}}{\partial x_{Lr}} m^3_{Li+} \frac{\partial x^3_{Li+}}{\partial x_{Lk}} + \frac{\partial x^3_{Li+}}{\partial x_{Lr}} m^3_{Li+} \frac{\partial x^3_{Li+}}{\partial x_{Lk}}\right)
\]

Where \(\delta^r_k\) is the kronecker delta: 
\[
\delta^r_k = 1 \text{ for } r = k \\
\delta^r_k = 0 \text{ for } r \neq k
\]

For convenience of notation in the following sections, we define the following much shorter notation of expression 3.36:

\[
\left(\left[D\mathcal{F}^{xc.1}\right]^T \left(M_L^{c,c} + D_dK_L^{dc,c} + D_fK_L^{fc,c}\right) \left[D\mathcal{F}^{xc.1}\right]\right)_{r,k} \\
+ \left(M_L^{1,1} + D_dK_L^{d1,1} + D_fK_L^{f1,1}\right)_{r,k} = \\
\delta^r_k m^1_{Lr} + \sum_{i=1}^n \left(\sum_{j=1,3} \left(\frac{\partial x^j_i}{\partial x_{Lr}} m^j_{Li} \frac{\partial x^j_i}{\partial x_{Lk}}\right)\right)
\]

The newly introduced summation symbol means a summation over the following combinations of \(i\) and \(j\) if node \(i\) is a non-material node: \(i\) with \(j = 3\), \(i+\) with \(j = 1\) and \(i+\) with \(j = 3\). If node \(i\) is a material node, the summation symbol represents only \(i\) with \(j = 3\).

The non-diagonal components of the system matrix in equation 3.31 can be found from equation 3.37 using \(r < k\) and the results of chapter 2:

\[
\left(\left[D\mathcal{F}^{xc.1}\right]^T \left(M_L^{c,c} + D_dK_L^{dc,c} + D_fK_L^{fc,c}\right) \left[D\mathcal{F}^{xc.1}\right]\right)_{r,k}
\]
Towards a Diagonal System Mass Matrix

\[ M_{L}^{1,1} + D_{d}K_{L}^{d1,1} + D_{f}K_{L}^{fi,1} \] \( r, k \) =

\[ \left( S_{r}^{1}TM(k+1) \right) \left( \sum_{j=1,3}^{j} \sum_{k=k+1}^{k} \left( R_{i}^{j}m_{k}^{i}L_{k}^{j}S_{k}^{j} \right) \right) \]

\[ + \sum_{i=k+1}^{n} \left( S_{r}^{1}TM(r+1)(i-1) \right) \left( \sum_{j=1,3}^{j} \sum_{i=i,i+}^{i} \left( R_{i}^{j}m_{k}^{i}L_{i}^{j}R_{i}^{j} \right) \right) S_{k}^{1}TM(k+1)(i-1) \]

From the results of chapter 2, the following can be found for \( i > k > r \) if node \( k \) is a material node:

\[ S_{r}^{1}TM(r+1)(i-1)R_{i}^{j} = S_{r}^{1}TM(r+1)(k-1)R_{k}^{3} \left( \vec{r}_{k+1} \cdot \vec{F}_{k}^{2} \right) TM(k+1)(i-1)R_{i}^{j} \] (3.38)

Also, the following can be found for \( i > k > r \) if node \( k \) is a non-material node:

\[ S_{r}^{1}TM(r+1)(i-1)R_{i}^{j} = S_{r}^{1}TM(r+1)(k-1)R_{k}^{3}A_{k+1}^{33}TM(k+1)(i-1)R_{i}^{j} \] (3.39)

Furthermore, we define the coefficient \( C_{m}^{nk} \) as the contribution of the nodes with higher node numbers to the mass of node \( k \) in the equation of motion:

\[ C_{m}^{nk} = \sum_{i=k+1}^{n} \left( TM(k+1)(i-1) \right) \left( \sum_{j=1,3}^{j} \sum_{i=i,i+}^{i} \left( R_{i}^{j}m_{k}^{i}L_{i}^{j}R_{i}^{j} \right) \right) \left( TM(k+1)(i-1) \right) \] (3.40)

The time derivative \( \dot{C}_{m}^{nk} \) can be found from equation 3.40:

\[ \dot{C}_{m}^{nk} = 2 \sum_{i=k+1}^{n} TM(k+1)(i-1) \left( \sum_{j=1,3}^{j} \sum_{i=i,i+}^{i} \left( R_{i}^{j}m_{k}^{i}L_{i}^{j}R_{i}^{j} \right) \right) TM(k+1)(i-1) \] (3.41)

\[ + 2 \sum_{i=k+1}^{n} \left( TM(k+1)(i-1) \right) \left( \sum_{j=1,3}^{j} \sum_{i=i,i+}^{i} \left( R_{i}^{j}m_{k}^{i}L_{i}^{j}R_{i}^{j} \right) \right) TM(k+1)(i-1) \]
We define:

\[ \frac{\dot{C}_{nk}}{k} = \frac{\dot{C}_{nk}}{2} \]  

(3.42)

Because of equations 3.38 and 3.39, the non-diagonal terms of the resulting mass matrix (defined by expression 3.38) are equal to zero if for all \( r \) and \( k \):

\[ S^{r}_{k}M_{(r+1)(k-1)}R^{3}_{k} = 0 \]  

(3.43)

If we have for any \( r \) or \( k \):

\[ S^{r}_{k}M_{(r+1)(k-1)}R^{3}_{k} \neq 0 \]  

(3.44)

from equations 3.38 to 3.41, we find that the requirement that the non-diagonal terms given by equation 3.38 are equal to zero, for a material node \( k \) can very briefly be written as:

\[ \left( m_k + D_{k}k_{Lk}^{d,3} + D_{f}k_{Lk}^{f,3} \right) S^{31}_{k} + (\tilde{r}_{k+1} \cdot \tilde{g}_{k}) S^{1}_{k}C^{nk}_{k} = 0 \]  

(3.45)

Similarly for a line contact node \( k \) we have:

\[ \left( m_k + D_{k}k_{Lk}^{d,3} + D_{f}k_{Lk}^{f,3} \right) S^{31}_{k} + A_{k+}^{13} \left( m_{k+} + D_{k}k_{Lk+}^{d,1} \right) S^{11}_{k+} + A_{k+}^{33} \left( m_{k+} + D_{k}k_{Lk+}^{d,3} \right) S^{31}_{k+} + A_{k+}^{33} S^{1}_{k}C^{nk}_{k} = 0 \]  

(3.46)

Expressions 3.45 and 3.46 implicitly contain only the angles \( \psi_{i} \) with \( i \geq k \). Therefore, the angles \( \psi_{i} \) can be calculated by using expression 3.45 or 3.46 (depending on the node type) for all nodes, successively starting with the expression 3.45 or 3.46 for node number \( n - 1 \), containing only \( \psi_{n-1} \) and \( \psi_{n} \). The angle \( \psi_{n} \) of the flyer node is found by treating the flyer node in the same way as all combined surface-line contact nodes, hence \( \psi_{n} = \frac{\pi}{2} \) (see equation 2.107 with \( \phi_{i} = 0 \)). In appendix B, the values for the angles \( \psi_{i} \) are calculated for material nodes and line contact nodes from equations 3.45 and 3.46, respectively.

The physical interpretation of the generalised coordinate directions \( \tilde{g}_{i}^{l} \) leading to a diagonal mass matrix is shown in Figure 3.2.

The generalised coordinate directions as calculated in this section are such that when a force is acting on node \( i \) in this direction, the resulting acceleration of node \( i \) caused by this force is in a direction perpendicular to \( \tilde{r}_{i} \). Therefore, this force has no influence on the wire tension between nodes \( i - 1 \) and \( i \) and
Fig. 3.2: Forces acting in the generalised coordinate directions on the accelerations of the nodes $k$ with $k < i$. This also means that, when calculating the accelerations of node $k$, the forces in the generalised directions acting on the nodes $i$ with $i > k$ are not yet needed.

On the other hand, when a force is acting on node $i$ in the direction of $\vec{r}_i$, the resulting acceleration of node $i$ caused by this force is in a direction perpendicular to the generalised direction. And this means that the accelerations of the nodes $k$ with $k < i$ are not needed when calculating the accelerations of node $i$.

Figure 3.2 shows two forces acting in the generalised coordinate directions and the resulting accelerations in a direction perpendicular to $\vec{r}_i$. 
3.8 The equations of motion

From equation 3.33, we find the following equations for the accelerations $\ddot{x}^{\text{in,2}}_L$ for all free $k$ nodes:

$$M_k^2 \ddot{x}_k = F_k^2 \quad (3.47)$$

With:

$$M_k^2 = m_k + D_k d,2 L_k \quad (3.48)$$

$$F_k^2 = f_{v,2}^{\text{in,2}} - k^{d,2 \text{old}}_L \ddot{x}_L$$

For convenience of notation of the equations of motion and to allow an efficient calculation of these equations (see chapter 4), we first define some additional coefficients:

- The coefficient $C^d_k$, which can be interpreted as the contribution of the (external and damping) forces acting on the higher nodes (nodes with a higher node number) to the total force on node $k$ in the equations of motion.
- The coefficient $C^n_k$, which can be interpreted as the contribution of the lower nodes (nodes with a lower node number) to the first order acceleration of node $k$ in the dependent coordinate direction.
- The coefficient $C^v_k$, which can be interpreted as the contribution of the lower nodes to the (old) velocity of node $k$.
- The coefficient $C^2_k$, which can be interpreted as the contribution of the lower nodes to the second order acceleration of node $k$.
- The coefficient $C^m_k$, which can be interpreted as the contribution of the second order inertia (plus damping) forces on the higher nodes, resulting from the old generalised velocities of these nodes, to the force acting on node $k$. 
The expressions for the above-defined coefficients are:

\[ C^f_k = \sum_{i=k+1}^{n} \left( TM_{(k+1)(i-1)} \right) \sum_{i=i,i+1}^{j=1,3} \left( (f_{Li}^{vi,j} - (k_{Li}^d + k_{Li}^f) \dot{x}_{Li}^{j,old}) R_i^i \right) \]  \hspace{1cm} (3.49)

\[ C^d_k = \sum_{i=1}^{k-1} \left( S_i^1 TM_{(i+1)(k-1)} \right) \ddot{x}_{Li}^1 \]  \hspace{1cm} (3.50)

\[ C^u_k = \sum_{i=1}^{k-1} \left( S_i^1 x_{Li}^{1,old} TM_{(i+1)(k-1)} \right) \]  \hspace{1cm} (3.51)

\[ C^{\alpha 2}_k = \sum_{i=1}^{k-1} \left( \left( \dot{S}_i^1 x_{Li}^{1,old} + \dot{S}_i^2 x_{Li}^{2,old} \right) TM_{(i+1)(k-1)} + S_i^1 x_{Li}^{1,old} \dot{M}_{(i+1)(k-1)} \right) \]  \hspace{1cm} (3.52)

\[ C^{\alpha n}_k = - \sum_{i=k+1}^{n} \left( TM_{(k+1)(i)} \left( \left( \dot{S}_i^1 x_{Li}^{1,old} + \dot{S}_i^2 x_{Li}^{2,old} \right) C_{i}^{n,i} + \left( S_i^1 x_{Li}^{1,old} \right) \dot{C}_{i}^{n,i} \right) \right) \]  \hspace{1cm} (3.53)
When calculating the accelerations $\ddot{x}_{Lk}^1$ from the equations of motion, starting with node 1 up to $n$, we can define the following short notation of the equation of motion for $\ddot{x}_{Lk}^1$ in an analogous manner to equation 3.47:

$$ M_k^1 \ddot{x}_{Lk}^1 = F_k^1 \tag{3.54} $$

Using the results from the previous section (choosing appropriate values for all $\psi_i$ leading to a diagonal mass matrix), and using the coefficients defined by equations 3.49 to 3.53, the following expressions can be found from equation 3.31 for the total mass $M_k^1$ and total force $F_k^1$:

$$ M_k^1 = m_k^1 L_k + \sum_{k=k,k+1}^{j=1,3} \left[ \left( S_{k}^{j1} \right)^2 m_{k_j}^j \right] + (S_k^1)^2 C_{rk}^{mk} \tag{3.55} $$

$$ F_k^1 = \sum_{j=1,3}^{j=1,3} \left( f_{Lk}^{vi,j} - \left( k_{Lk}^{d,j} + k_{Lk}^{f,j} \right) \dot{x}_{Lk}^{1,old} + S_k^1 C_k^j \right) + \sum_{k=k,k+1}^{j=1,3} \left( S_{k}^{j1} \left( f_{Lk}^{vi,j} - \left( k_{Lk}^{d,j} + k_{Lk}^{f,j} \right) \dot{x}_{Lk}^{1,old} \right) \right) \tag{3.56} $$

$$ -S_k^1 \left( C_{k}^{\psi} F_{k}^{\psi} C_{k}^{\psi k} + C_{k}^{\psi} F_{k}^{\psi} C_{k}^{\psi k} + C_{k}^{\psi} F_{k}^{\psi} C_{k}^{\psi k} \right) $$
3.9 The wire tension

The equations of motion 3.54 do not explicitly contain the wire tension. Therefore, after solving the equations of motion, the wire tension has to be calculated separately. This section gives the derivation of the expression needed to calculate this wire tension.

The wire tension $T_p$ between node $p - 1$ and node $p$ can be calculated by applying the principle of virtual work for a virtual wire elongation $\delta e_p$ between node $p - 1$ and node $p$.

We first introduce the elongation $e_p$ of the wire between nodes $p - 1$ and $p$ as additional independent coordinate in the function $F^x$, as first defined in equation 2.9:

$$x_L = F^x(x^m_L, e_p) \quad (3.57)$$

The kinematically admissible virtual displacements $x_L$ can now be obtained by differentiating equation 3.57:

$$\delta x_L = [DF^x] \delta x^m_L + [D_{e_p}F^x] \delta e_p \quad (3.58)$$

In equation 3.58, the symbol $D_{e_p}$ means differentiation of the function $F^x$ with respect to $e_p$:

$$[D_{e_p}F^x] = \left[ \frac{\partial F^x(x^m_L, e_p)}{\partial e_p} \right] \quad (3.59)$$

Using the principle of virtual work for all kinematically admissible virtual displacements $x_L$ yields:

$$\delta x_L^T \left[ f^{vi}_L + f^d_L + f^i_L - M_L \ddot{x}_L \right] = T_p \delta e_p \quad (3.60)$$

Equation 3.60 is valid for all $\delta x^m_L$ and $\delta e_p$. We choose $\delta x^m_L = 0$ and therefore obtain from equation 3.58:

$$\delta x_L = [D_{e_p}F^x] \delta e_p \quad (3.61)$$

Substitution of $\delta x_L$ from equation 3.61 into 3.60 leads to:

$$T_p = [D_{e_p}F^x]^T \left[ f^{vi}_L + f^d_L + f^i_L - M_L \ddot{x}_L \right] \quad (3.62)$$
If \( i < p \), the dependent coordinates of node \( i \) will be independent of the elongation \( e_p \). Therefore, we have for \( i < p \):

\[
\frac{\partial x_{L_i}^3}{\partial e_p} = 0, \quad \frac{\partial x_{L_{i+}}^1}{\partial e_p} = 0, \quad \frac{\partial x_{L_{i+}}^3}{\partial e_p} = 0
\]  

(3.63)

Using the results of chapter 2, plus the fact that the non-diagonal components (according to equation 3.37) have been chosen as equal to zero, and using equation 3.63, we find the following expression for the wire tension \( T_p \) from equation 3.62:

\[
T_p = \sum_{i=p}^{n} \sum_{i=i,i+}^{j=1,3} \left( \frac{\partial x_{L_i}^j}{\partial e_p} \left( f_{L_i}^{vi,j} - \left( k_{L_i}^{d,j} + k_{L_i}^{f,j} \right) \dot{x}_{L_i}^{i,old} - f_{L_i}^{in,j} \right) \right)
\]  

- \sum_{k=1}^{n} \sum_{i=p}^{j=1,3} \left( \frac{\partial x_{L_i}^j}{\partial e_p} \left( m_i + D_d k_{L_i}^{d,j} + D_f k_{L_i}^{f,j} \right) \frac{\partial x_{L_i}^j}{\partial x_{L_k}^1} \right) \dot{x}_{L_k}^1
\]  

(3.64)

If the local directions have been chosen such that the mass matrix has become diagonal, expression 3.37 is equal to zero for \( r < k \). Furthermore, we use the following equation that is valid for all \( i \geq p \):

\[
\frac{\partial x_{L_i}^i}{\partial e_p} = C_p \frac{\partial x_{L_i}^1}{\partial x_{L(p-1)}^1}
\]  

(3.65)

In equation 3.65 the constant \( C_p \) depends on \( p \) but is independent of \( i \) and \( j \). In this case, we find the following from equation 3.64:

\[
T_p = \sum_{i=p}^{n} \sum_{i=i,i+}^{j=1,3} \left( \frac{\partial x_{L_i}^j}{\partial e_p} \left( f_{L_i}^{vi,j} - \left( k_{L_i}^{d,j} + k_{L_i}^{f,j} \right) \dot{x}_{L_i}^{i,old} - f_{L_i}^{in,j} \right) \right)
\]  

- \sum_{k=1}^{p-1} \sum_{i=p}^{j=1,3} \left( \frac{\partial x_{L_i}^j}{\partial e_p} \left( m_i + D_d k_{L_i}^{d,j} + D_f k_{L_i}^{f,j} \right) \frac{\partial x_{L_i}^j}{\partial x_{L_k}^1} \right) \dot{x}_{L_k}^1
\]  

(3.66)
We have:

\[
\frac{\partial x_{Li}}{\partial e_p} = T M_{(p)(i-1)} R_i^j \tag{3.67}
\]

Using the results of chapter 2, the coefficients as defined in equations 3.40, 3.41 and 3.49 to 3.53, and using equation 3.67, we find the following from equation 3.66:

\[
T_p = C_{p-1}^f - C_{p}^{ul} C_{p-1}^{nk} - C_{p}^{v} C_{p-1}^{njk/2} - C_{p}^{u2} C_{p-1}^{nk} + C_{p-1}^{n} \tag{3.68}
\]
3.10 A wire tension dependent friction force

The equations of motion 3.18 are solved using an explicit time integration scheme, as discussed in detail in chapter 4. When using an explicit time integration, the external forces will generally depend on the state variables of the wire in the previous increment. This also holds for the friction force, which will depend on the contact forces and wire tension in the previous increment.

Because of large non-linearities in the tensioning of the wire, large fluctuations can occur in the wire tension between subsequent increments. Therefore, basing the friction forces on state variables in the previous increment can lead to instabilities. These instabilities, which are caused by large fluctuations in wire tension, can only be avoided by making the friction forces dependent on the current wire tension. In this section, a method is developed that makes it possible to introduce a friction force in the equations of motion that depends on the current wire tension.

Suppose a friction force is acting on node $p$ with the components $f_{Lp}^{ric,1}$ and $f_{Lp}^{ric,3}$ in the coordinate directions $x_{L1}^1$ and $x_{Li}^3$, respectively, where both the components $f_{Lp}^{ric,1}$ and $f_{Lp}^{ric,3}$ depend on the wire tension $T_p$. The velocity dependent friction forces $f_{Lp}^v$ as introduced in section 3.3.3 (see equation 3.9), are not part of $f_{Lp}^{ric,3}$ and are supposed to be independent on the current wire tension.

In chapter 4, it is shown that the equations of motion can be solved starting with node 1 up to node $n$. From equation 3.66 it can be seen that the tension $T_p$ cannot yet be calculated before calculation of $\ddot{x}_{L1}^1$ of node 1 to $p - 1$. The component $f_{Lp}^{ric,1}$ is not needed before $\ddot{x}_{L1}^1$ has been calculated (see equations 3.54 and 3.56). At that time, $T_p$ can be calculated from equation 3.66. However, the components $f_{Lp}^{ric,3}$ are needed for the calculation of $\ddot{x}_{Li}^3$ of node 1 to $p - 1$.

Let us define the friction factor $\lambda_p$ in such a way that:

$$f_{Lp}^{ric,3} = \left(\frac{\lambda_p - 1}{\lambda_p}\right) \frac{T_p}{\frac{\partial x_{Li}^3}{\partial \varepsilon_p}}$$

(3.69)

The friction force component $f_{Lp}^{ric,3}$ will be part of the external force $f_{Lp}^{vi,3}$. We define:

$$f_{Lp}^{vi,3} = f_{Lp}^{vi*,3} + f_{Lp}^{ric,3}$$

(3.70)
3.10 A wire tension dependent friction force

Equation 3.68 can be written as:

\[
T_p = \sum_{j=1,3}^{j=1,3} \left( R_p^i \left( f_{Lp}^{vi,j} - (k_{Lp}^{d,j} + k_{Lp}^{f,j}) \dot{x}_{Lp}^{j,old} \right) \right) + Tr_p C_p^{j} - C_p^{ul} \dot{C}_{p-1}^{mk} - C_p^{v} \dot{C}_{p-1}^{mk/2} - C_p^{a} C_p^{mk} - C_p^{\in} + C_{p-1}^{\in}
\]

(3.71)

Elimination of \( T_p \) from equations 3.69 and 3.71 leads to an expression for \( f_{Lp}^{ric,3} \) without the wire tension \( T_p \):

\[
f_{Lp}^{ric,3} = (\lambda_p - 1) \left( R_p C_p^{j} - C_p^{ul} \dot{C}_{p-1}^{mk} - C_p^{v} \dot{C}_{p-1}^{mk/2} - C_p^{a} C_p^{mk} - C_p^{\in} + C_{p-1}^{\in} \right) \]

\[
+ (\lambda_p - 1) \left( f_{Lp}^{v,i,3} + f_{Lp}^{v,i,1} + f_{Lp}^{v,i,3} - \sum_{p=p, p+}^{j=1,3} \left( \left( k_{Lp}^{d,j} + k_{Lp}^{f,j} \right) \dot{x}_{Lp}^{j,old} + f_{Lp}^{in,j} \right) \right)
\]

(3.72)

The tension dependent friction force \( f_{Lp}^{ric,3} \) is only needed in the equations of motion 3.54 for \( k \leq p \). When solving the equation of motion for \( k = p \), the tension force \( T_p \) can be calculated from equation 3.68. From this tension force, we can now calculate the friction force.

To enable substitution of the expression 3.72 for the tension dependent friction force \( f_{Lp}^{ric,3} \) in the equations of motion 3.54 with \( k < p \), we change the notation of the term \( S_k^{1,C}_k^{d} \) in the right-hand term containing \( f_{Lp}^{ric,3} \). Using 3.70 results in:

\[
S_k^{1,C}_k^{d} = S_k^{1} \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i, i+}^{j=1,3} \left( \left( f_{Lp}^{v,i,j} - \left( k_{Lp}^{d,j} + k_{Lp}^{f,j} \right) \dot{x}_{Lp}^{j,old} \right) R_{i}^{j} \right) \right) + S_k^{1} TM_{(k+1)(p-1)} \left( f_{Lp}^{v,i,3} R_{p}^{3} + f_{Lp}^{ric,3} R_{p}^{3} + f_{Lp}^{v,i,1} R_{p+}^{1} + f_{Lp}^{v,i,3} R_{p+}^{3} \right)
\]

\[
- S_k^{1} TM_{(k+1)(p-1)} \sum_{p=p, p+}^{j=1,3} \left( \left( k_{Lp}^{d,j} + k_{Lp}^{f,j} \right) \dot{x}_{Lp}^{j,old} + f_{Lp}^{in,j} \right) R_{i}^{j}
\]

(3.73)
\begin{equation}
\phi_k = \sum_{i=p+1}^{n} \left( TM_{(p+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( f^{vi,j}_{Li} - \left( k^{d,j}_{L_i} + k^{f,j}_{L_i} \right) \dot{x}^{old}_{Li} \right) R^i_{L_i}
\end{equation}

When substituting expression 3.72 in the equations of motion 3.54 using equation 3.73, the resulting equations of motion remain unchanged when redefining $C^{nk}_k$, $C^{nk/2}_k$, $C^f_k$ and $C^i_k$ (for $k < p$) as:

\begin{align}
C^{nk}_k &= \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&= \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&+ \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&+ \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&+ \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right)
\end{align}

\begin{align}
C^{nk/2}_k &= \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&= \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&+ \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&+ \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right) \\
&+ \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( R^i_{L_i} \bar{R}^i_{L_i} \right) \right) \left( TM_{(k+1)(i-1)} \right)
\end{align}

\begin{align}
C^f_k &= \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+1}^{j=1,3} \left( f^{vi,j}_{Li} - \left( k^{d,j}_{L_i} + k^{f,j}_{L_i} \right) \dot{x}^{old}_{Li} \right) \right) R^i_{L_i}
\end{align}
\[ + \lambda_p TM_{(k+1)(p-1)} \left( f_{Lp}^{vi,3} R_p^3 + f_{Lp}^{vi,1} R_p^1 + f_{Lp}^{vi,3} R_p^3 \right) \]
\[ - \lambda_p TM_{(k+1)(p-1)} \left( \sum_{j=1,3} \left( \left( k_{Lp}^{d,j} + k_{Lp}^{f,j} \right) \dot{x}_j + f_{Lp}^{in,j} \right) R_p^i \right) \]
\[ + \lambda_p \sum_{i=p+1}^{n} \left( TM_{(k+1)(i-1)} \left( \sum_{j=1,3} \left( \left( f_{Li}^{vi,j} - \left( k_{Li}^{d,j} + k_{Li}^{f,j} \right) \dot{x}_j + f_{Li}^{in,j} \right) R_i^j \right) \right) \right) \]

\[ C_{in}^k = - \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \left( \left( \hat{S}_i^1 \dot{x}_L^1 + \hat{S}_i^2 \dot{x}_L^2 \right) C_i^{nk} + \left( S_i^1 \dot{x}_L^1 \right) C_i^{mk/2} \right) \right) \]
\[ - \sum_{i=k+1}^{p-1} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+}^{j=1,3} \left( R_i^j m k_{Li}^{j1} \left( \hat{S}^{j1}_{Li} \dot{x}_L^{1,old} + \hat{S}^{j2}_{Li} \dot{x}_L^{2,old} \right) \right) \right) \]
\[ - \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \left( \left( S_i^1 \dot{x}_L^1 + S_i^2 \dot{x}_L^2 \right) C_i^{nk} + \left( S_i^1 \dot{x}_L^1 \right) C_i^{mk/2} \right) \right) \]
\[ - \lambda_p \sum_{i=p}^{n} \left( TM_{(k+1)(i-1)} \sum_{i=i,i+}^{j=1,3} \left( R_i^j m k_{Li}^{j1} \left( \hat{S}^{j1}_{Li} \dot{x}_L^{1,old} + \hat{S}^{j2}_{Li} \dot{x}_L^{2,old} \right) \right) \right) \]

Expressions 3.74 to 3.77 can be generalised for friction forces acting on any node \( i \), when defining in agreement with the definition of \( \lambda_i \) in equation 3.69: \( \lambda_i = 1 \) if no tension dependent friction force is present.

When we define:

\[ f_{Lp}^{vi,1} = f_{Lp}^{vi,1} \]
\[ f_{Lp}^{vi,3} = f_{Lp}^{vi,3} \]

then, instead of the definitions for \( C_{nk}^k, C_{nk}^{mk/2}, C_j^f \) and \( C_{in}^k \) in the equations 3.74 to 3.77, we find:
\[ C^{\text{mk}}_k = \left( \sum_{i=k+1}^{n} \left( \prod_{p=k+1}^{i} \lambda_p \right) (TM_{(k+1)}(i-1))^{2} \sum_{i=i,i+1}^{j=1,3} \left( R^j_i m k^j_{Li} R^j_i \right) \right) \]  

\[ \dot{C}^{\text{mk}/2}_k = \sum_{i=k+1}^{n} \left( \prod_{p=k+1}^{i} \lambda_p \right) (TM_{(k+1)}(i-1))^{2} \sum_{i=i,i+1}^{j=1,3} \left( R^j_i m k^j_{Li} \dot{R}^j_i \right) \]  

\[ C^i_k = \sum_{i=k+1}^{n} \left( \prod_{p=k+1}^{i} \lambda_p \right) \]  

\[ \left( \sum_{i=i,i+1}^{j=1,3} \left( f_{Li}^{vi,j} - (k^{d,j}_{Li} + k^{f,j}_{Li}) \dot{x}^{i,old}_{Li} \right) R^j_i \right) \]  

\[ \dot{C}^{\text{in}}_k = - \sum_{i=k+1}^{n} \left( \prod_{p=k+1}^{i} \lambda_p \right) \left( TM_{(k+1)}(i-1) \left( (S^1_i \dot{x}^{1,old}_{Li} + S^2_i \dot{x}^{2,old}_{Li}) C^\text{mk}_i + (S^1_i \dot{x}^{1,old}_{Li}) \dot{C}^{\text{mk}/2}_i \right) \right) \]  

\[ \sum_{i=k+1}^{n} \left( \prod_{p=k+1}^{i} \lambda_p \right) \left( TM_{(k+1)}(i-1) \sum_{i=i,i+1}^{j=1,3} \left( R^j_i m k^j_{Li} \left( \dot{S}^j_i \dot{x}^{1,old}_{Li} + \dot{S}^j_i \dot{x}^{2,old}_{Li} \right) \right) \right) \]
3.11 Calculation of normal contact forces

To enable the detection of end of contact between the wire and a contact surface (or contact line) caused by a negative normal force between wire and surface (see chapter 5), expressions for the normal contact forces are needed.

In this section the expressions for these normal contact forces are derived. These expressions are derived for surface contact nodes and line contact nodes. Combined surface-line contact nodes can end their contact with the contact surface or contact line separately. The sign of both these normal forces can be calculated separately using the expressions for surface contact nodes and line contact nodes.

3.11.1 The normal contact force in surface contact nodes

To determine whether the contact in a surface contact node \( i \) should be ended, an expression for the normal force in this node is needed. The contact will be ended whenever this normal force becomes negative (see chapter 5). Because the normal direction in a surface contact node is not constant over the finite contact angle (as already mentioned in section 2.3.3), no strict distinction can be made between the normal forces and the tangential contact forces. Nevertheless, we will consider the force in the direction of \( \mathbf{g}_i^2 \) as a normal contact force. This is because when the normal force becomes negative, the tangential (friction) force becomes zero (see chapter 6), so that its contribution to the calculated contact force vanishes.

We will indicate the normal force acting on node \( i \) as \( f_{Li}^n \) and define \( f_{Li}^{ui,2} \) as external forces, excluding normal force and damping force. The normal force \( f_{Li}^n \) can be calculated using the principle of virtual work. Assuming a virtual displacement \( \delta x_{Li}^2 \) and two virtual wire elongations \( \delta e_i \) and \( \delta e_{i+1} \), we find:

\[
\delta x_{Li}^2 \left( f_{Li}^n + f_{Li}^{ui,2} - m_i x_{Li}^2 \right) = \delta e_i T_i + \delta e_{i+1} T_{i+1}
\]  

(3.83)

The virtual wire elongations \( \delta e_i \) and \( \delta e_{i+1} \) can be expressed in the virtual displacement \( \delta x_{Li}^2 \) by using equations 2.54 and 2.55:

\[
\delta e_i = -\cos (\alpha_{i-}^\text{sur}) \delta x_{Li}^2
\]

(3.84)

\[
\delta e_{i+1} = -\cos (\alpha_{i+}^\text{sur}) \delta x_{Li}^2
\]
Using the expression 2.118 for the acceleration \( \ddot{x}_{Li}^2 \), the normal contact force can be found from equations 3.83 and 3.84:

\[
\begin{align*}
\tau_{Li}^n &= -\tau_{Li}^{v_{1,2}} - \cos (\alpha_i^{sur}) T_i - \cos (\alpha_i^{sur}) T_{i+1} \\
&\quad - m_i \left( C_{11}^{sur} (\dot{x}_{Li}^1)^2 + 2 C_{13}^{sur} \dot{x}_{Li}^1 \ddot{x}_{Li}^3 + C_{33}^{sur} (\ddot{x}_{Li}^3)^2 \right)
\end{align*}
\] (3.85)

3.11.2 The normal contact force in line contact nodes

Even more than in a surface contact node, no strict distinction can be made between normal forces and tangential forces in a line contact node \( i \) because of the usually large angle of the wire around the contact line.

Similarly to the surface contact, the only reason why the normal force is needed is to determine when the contact between the wire and the contact line has to be ended.

We will assume that the normal force can be calculated by using the principle of virtual work when assuming that the virtual displacement \( \Delta x_{Li}^2 \) equals the virtual displacement \( \Delta x_{Li+1}^2 \) and when using virtual wire elongations \( \delta e_i \) and \( \delta e_{i+1} \). We will indicate the normal force in node \( i \) as \( \tau_{Li}^n \).

The forces \( \tau_{Li}^{v_{1,2}} \) and \( \tau_{Li+1}^{v_{1,2}} \) will be defined as external forces, excluding normal and damping forces.

From the principle of virtual work, we find:

\[
\begin{align*}
\delta x_{Li}^2 \left( \tau_{Li}^n + \tau_{Li}^{v_{1,2}} - m_i \ddot{x}_{Li}^2 + f_{Li+1}^{v_{1,2}} - m_{i+1} \ddot{x}_{Li+1}^2 \right) &= \delta e_i T_i + \delta e_{i+1} T_{i+1}
\end{align*}
\] (3.86)

For the virtual wire elongations \( \delta e_i \) and \( \delta e_{i+1} \) we assume:

\[
\begin{align*}
\delta e_i &= \frac{\alpha_i^{lin}}{2} \sin (\kappa_i - \phi_i) \delta x_{Li+1}^2 \\
\delta e_{i+1} &= \frac{\alpha_i^{lin}}{2} \sin (\kappa_i) \delta x_{Li}^2
\end{align*}
\] (3.87)
3.11 Calculation of normal contact forces

Using equations 3.86, 3.87 and the expressions 2.131 and 2.132, we find the following for the normal contact force:

\[
\begin{align*}
J_{Li}^n &= -J_{Li}^{v_i,2} + \frac{\alpha_{i}^{lin}}{2} \sin (\kappa_i) T_{i+1} - m_i (i_i)^2 C_{lin}^{uv} \cos (\alpha_{i-}^{lin}) \\
&\quad -J_{Li+}^{v_i,2} + \frac{\alpha_{i}^{lin}}{2} \sin (\kappa_i - \phi_i) T_i - m_i + (i_i)^2 C_{lin}^{uv} \cos (\alpha_{i+}^{lin})
\end{align*}
\]
Chapter 4

Solution procedure

4.1 Introduction

This chapter describes the solution procedure for the equations of motion, which have been derived in the previous chapter. Firstly, the most suitable numerical integration method is chosen. Subsequently, an efficient implementation of this numerical integration method is described, resulting in a minimum calculation time being required.

4.2 Choice of time integration method

4.2.1 Implicit versus explicit time integration

For the numerical integration of the equations of motion described in the previous chapter, there are basically two different methods. These are the implicit and explicit methods. A description of both methods can be found in [1]. An overview of the advantages and disadvantages of both methods is given in [4].

A brief survey of the main advantages and disadvantages is given below.

- The main advantage of an implicit method compared to an explicit method is that an implicit method is more accurate and that it is unconditionally stable, whereas an explicit method is only conditionally stable. The time step $\Delta t$ for an explicit method has to be smaller than the critical time step $\Delta t_c$. This critical time step $\Delta t_c$ depends on the time period $T_{min}$ of the
largest eigen-frequency of the system:

\[ \Delta t_c = \frac{T_{min}}{\pi} \] (4.1)

- The main advantage of an explicit method compared to an implicit method is the greater robustness. This means that it is less sensitive to large non-linearities in the equations of motion. When using an implicit time integration method, large non-linearities in the equations of motion can result in a non-convergence of this method, making it useless.
- A possible advantage or disadvantage of both methods is the possible large difference in required calculation time. There are two aspects of interest.

  - Firstly, an implicit method generally requires more calculation time to solve the equations of motion than an explicit method. This is because an explicit method often leads to a diagonal set of equations as a result of the diagonal mass matrix, and an implicit method often leads to a non-diagonal set of equations as a result of the non-diagonal stiffness matrix.
  - Secondly, when using an explicit method, the time step is limited by the critical time step. For an explicit method, the total time to be simulated is therefore very important in proportion to the critical time step because of the calculation time.

Depending on the situation, the calculation time required for the two methods can differ strongly.

4.2.2 Motivation of choice for explicit time integration

Because of the requirement that the time integration method has to be robust and the fact that the equations of motion are highly non-linear, the central difference method (an explicit method) has been chosen. This method is described in section 4.3. The reasons why the equations of motion are highly non-linear are:

- The geometrical non-linearity. This is caused by the continuously changing shape of the wire, the changing discretisation of the wire caused by the generation and deletion of nodes (see Section 5), plus the continuously changing contact situation.
• The highly non-linear friction description. As described in section 6, this is caused by stick-slip and the dependency of both the magnitude and the direction of the friction force on the slip direction.
• The non-linearity of the tension force acting on the wire, caused by the air dereeler.

4.2.3 Consequence of the choice for explicit time integration

As mentioned in Section 4.2.1, when using an explicit method, the time step will depend on the largest eigen-frequency of the system. In our case, the largest eigen-frequency is determined by the discretisation of the wire.

The critical time step (determined by the smallest time period in the transversal direction of a wire modelled by nodes with a nodal distance $l^\Delta$ and with a tension force $T$) is given by:

$$\Delta t^{\text{trans}} = \sqrt{\frac{m_i l^\Delta}{T}} \quad (4.2)$$

The critical time step (determined by the smallest time period in the longitudinal direction of the same wire with modulus of elasticity $E$ and cross-sectional area $A$) is given by:

$$\Delta t^{\text{long}} = \sqrt{\frac{m_i l^\Delta}{EA}} \quad (4.3)$$

Strictly speaking, equations 4.2 and 4.3 are only valid for linear systems. In [2], it can be found that the critical time steps according to equations 4.2 and 4.3 can be used as an estimate for non-linear systems.

For a wire that can vibrate in both the transversal and longitudinal directions, the critical time step will be determined by the smallest value of $\Delta t^{\text{long}}_c$ or $\Delta t^{\text{trans}}_c$. The ratio of both values can be calculated from equations 4.2 and 4.3:

$$\frac{\Delta t^{\text{trans}}_c}{\Delta t^{\text{long}}_c} = \sqrt{\frac{EA}{T}} \quad (4.4)$$

Using practical values for the ratio of the tensile force $T$ and the cross-sectional area $A$ ($T/A = 56.8 \text{ N/mm}^2$) and the modulus of elasticity $E$ ($E = 1.2 \times 10^5 \text{ N/mm}^2$) for an often used copper wire with a diameter $d$ of 0.334 mm, we find that the critical time step $\Delta t^{\text{trans}}_c$ caused by transversal vibrations is about 50 times larger than the critical time step $\Delta t^{\text{long}}_c$ caused by longitudinal vibrations.
This ratio of 50 shows why it is advantageous from a dynamic point of view to assume that the wire is infinitely stiff in the longitudinal direction (see section 1.5).

4.2.4 Central difference and mass lumping

As described in chapter 1, the mass of the wire has been concentrated in the nodes and these nodes have no rotational moment of inertia. This approach is called mass lumping in literature (see [14]). In [11], it is shown that the errors caused by the central difference method and mass lumping are compensatory. Therefore, the approach of mass lumping can be used without introducing an additional error.
4.3 The time integration scheme

The central difference method has been chosen as the explicit time integration method. Small differences can be found between the central difference methods as given in literature (see, for instance, [1] and [2]). We use the method as described in [1].

Within the central difference method, the acceleration $\ddot{x}_L^t$ is given by:

$$\ddot{x}_L^t = \frac{1}{\Delta t^2} \left( x_L^{t+\Delta t} - 2x_L^t + x_L^{t-\Delta t} \right) + O\left(\Delta t^2\right) \tag{4.5}$$

where $O\left(\Delta t^2\right)$ is an error of order $\Delta t^2$.

The velocity $\dot{x}_L^t$ is given by:

$$\dot{x}_L^t = \frac{1}{2\Delta t} \left( x_L^{t+\Delta t} - x_L^{t-\Delta t} \right) + O\left(\Delta t^2\right) \tag{4.6}$$

Based on this method, a scheme has been used for which a non-constant time step $\Delta t$ can be used.

We define:

$$\Delta x_L^t = x_L^{t+\Delta t} - x_L^t \tag{4.7}$$

Using equation 4.7, equations 4.5 and 4.6 can be written as:

$$\ddot{x}_L^t = \frac{1}{\Delta t^2} \left( \Delta x_L^t - \Delta x_L^{t-\Delta t} \right) + O\left(\Delta t^2\right) \tag{4.8}$$

and:

$$\dot{x}_L^t = \frac{1}{2\Delta t} \left( \Delta x_L^t + \Delta x_L^{t-\Delta t} \right) + O\left(\Delta t^2\right) \tag{4.9}$$

From these expressions it can be concluded that:

$$\Delta x_L^t = \frac{\Delta t^2}{2} \ddot{x}_L^t + \Delta t \dot{x}_L^t \tag{4.10}$$

and:

$$\Delta x_L^{t-\Delta t} = -\frac{\Delta t^2}{2} \ddot{x}_L^t + \Delta t \dot{x}_L^t \tag{4.11}$$
The equations of motion 3.47 and 3.54 can now be rewritten as:

\[
\left( \frac{M^j_k}{\Delta t^2} \right) \Delta x_{Lk}^{j,t} = \left( \frac{M^j_k}{\Delta t^2} \right) \Delta x_{Lk}^{j,t-\Delta t} + F^j_k \tag{4.12}
\]

To find all \( x_L^{t_0+n\Delta t} \) with \( n > 0 \), the scheme according to Figure 4.1 is used. The advantage of this scheme is the possibility of changing the time step \( \Delta t \). Changing the time step is necessary because the critical time step is related to the minimum nodal distance according to equation 4.2 and this nodal distance changes continuously. Basing the time step on the smallest possible nodal distance would result in an unnecessary high number of time steps and thus also in an unnecessary amount of calculation time. The scheme in Figure 4.1 is based on the continuity of \( \ddot{x}_L, \dot{x}_L \) and, of course, \( x_L \) whenever the time step \( \Delta t \) is changed.
4.3 The time integration scheme

\[ t = t_0 \]

\[ x_L^t = x_L^{t_0} \]
\[ \dot{x}_L^t = \dot{x}_L^{t_0} \]
\[ \ddot{x}_L^t = \ddot{x}_L^{t_0} \]

\[ \Delta x_L^t = \frac{\Delta t^2}{2} \ddot{x}_L^t + \Delta t \dot{x}_L^t \]

\[ x_L^{t+\Delta t} = x_L^t + \Delta x_L^t \]

\[ t = t + \Delta t \]

Calculate and solve for \( k \) from 1 up to \( n \) (\( j = 1, 2 \)):

\[ \left( \frac{M_k^j}{\Delta t^2} \right) \Delta x_{L_k}^{j,t} = \left( \frac{M_k^j}{\Delta t^2} \right) \Delta x_{L_k}^{j,t-\Delta t} + F_k^j \]

\[ \ddot{x}_L^t = \frac{1}{\Delta t^2} \left( \Delta x_L^t - \Delta x_L^{t-\Delta t} \right) \]
\[ \dot{x}_L^t = \frac{1}{2\Delta t} \left( \Delta x_L^t + \Delta x_L^{t-\Delta t} \right) \]

Calculate new time step \( \Delta t \) based on nodal distances

\[ \text{time step } \Delta t \text{ unchanged?} \]

yes

no

Fig. 4.1: The time integration scheme with variable time step
4.4 Calculation of the time step

The time step used to integrate the equations of motion has to be smaller than the critical time step.

To calculate the critical time step at any time \( t \), equation 4.2 is not suitable because this equation is based on a constant nodal distance \( l^\Delta \) between all nodes. In practice these distances will differ. To calculate the critical time for that situation, we first define the transversal stiffness \( C^\text{trans}_i \) in node \( i \) as:

\[
C^\text{trans}_i = \frac{2T_i}{l_i^\Delta} + \frac{2T_{i+1}}{l_{i+1}^\Delta}
\]

(4.13)

with:

\[
l_i^\Delta = l_i - l_{i-1}
\]

(4.14)

We now assume that the critical time step for the wire is determined by the smallest value of the following expression for any node \( i \):

\[
\Delta t^\text{prac\ min} = \sqrt{\frac{m_i}{C^\text{trans}_i}}
\]

(4.15)

Using equations 1.6 and 4.13, we find the following for the critical time step:

\[
\Delta t^\text{prac\ min} = \sqrt{\frac{\rho A \left(l_i^\Delta + l_{i+1}^\Delta\right) l_i^\Delta l_{i+1}^\Delta}{T_i l_{i+1}^\Delta + T_{i+1} l_i^\Delta}}
\]

(4.16)

Because the calculation of the critical time step in the equation above is not exact, the practical time step \( \Delta t \) will be chosen as 0.5 times the critical time step \( \Delta t^\text{prac\ min} \) as given in the equation above.
4.5 The description of damping and friction forces

In chapter 3, the actual velocities in the expressions 3.2 for the damping forces and 3.9 for the friction forces are substituted by the old velocities and the current accelerations (see equation 3.6). Using equations 4.8 and 4.9, we find:

\[
\dot{x}_L^t = \frac{1}{2\Delta t} \left( \Delta x_L^t + \Delta x_L^{t-\Delta t} \right) \\
= \frac{\Delta t}{2} \left( \frac{\Delta x_L^t - \Delta x_L^{t-\Delta t}}{\Delta t^2} \right) + \frac{1}{\Delta t} \left( \Delta x_L^{t-\Delta t} \right)
\]

When comparing equation 4.17 to equation 3.6, we find the value for \( D_d \):

\[
D_d = \frac{\Delta t}{2}
\]

When comparing equation 4.17 to equation 3.12, we find the value for \( D_f \):

\[
D_f = \Delta t
\]

Furthermore, we find that the old velocity \( \dot{x}_L^{\text{old}} \) in the equations 3.6, 3.7 and 3.12 should be calculated by:

\[
\dot{x}_L^{\text{old}} = \frac{\Delta x_L^{t-\Delta t}}{\Delta t}
\]
4.6 Efficient calculation of the coefficients in the equations of motion

In this section, it is shown that the coefficients $M_k^1$ and $F_k^1$ in the equation of motion 3.54 can be calculated in an efficient manner.

First, the following relations can be found from the definitions of the coefficients as given in equations 3.50 to 3.52 and 3.79 to 3.82:

\[
C_{k-1}^{nk} = \lambda_k \left( T_k^{nk} C_k^{nk} T_k^* + \sum_{k=k, k+}^{j=1,3} \left( R_k^j m_k^j L_k R_k^j \right) \right) \tag{4.21}
\]

\[
\dot{C}_{k-1}^{nk/2} = \lambda_k \left( T_k^{nk} \dot{C}_k^{nk} T_k^* + \dot{C}_k^{nk/2} T_k^* \right) + \sum_{k=k, k+}^{j=1,3} \left( R_k^j m_k^j L_k \dot{R}_k^j \right) \tag{4.22}
\]

\[
C_{k-1}^f = \lambda_k \left( T_k^{f} C_f^k T_k^* + \sum_{k=k, k+}^{j=1,3} \left( R_k^j f^{ui,j}_{Lk} \left( k_{Lk}^i + k_{Lk}^{f,j} \dot{x}_{Lk}^{i,old} \right) \right) \right) \tag{4.23}
\]

\[
C_{k-1}^n = \lambda_k T_k^{C_k} \left( C_k^{in} - C_k^{nk} \left( \dot{S}_{k, Lk}^1 \dot{x}_{Lk}^{1,old} + \dot{S}_{k, Lk}^2 \dot{x}_{Lk}^{2,old} \right) - \dot{C}_k^{nk/2} S_k^1 \dot{x}_{Lk}^{1,old} \right) \tag{4.24}
\]

\[
-\lambda_k \sum_{k=k, k+}^{j=1,3} \left( R_k^j m_k^j L_k \left( \dot{S}_{k, Lk}^1 \dot{x}_{Lk}^{1,old} + \dot{S}_{k, Lk}^2 \dot{x}_{Lk}^{2,old} \right) \right) \tag{4.24}
\]

\[
C_{k+1}^{ud} = T_k^{C_k} C_k^{ud} + S_k^1 \dot{x}_{Lk}^{1,old} \tag{4.25}
\]

\[
C_{k+1}^{u} = T_k^{C_k} C_k^u + S_k^1 \dot{x}_{Lk}^{1,old} \tag{4.26}
\]

\[
C_{k+1}^{\omega} = T_k^{C_k} C_k^{\omega} + \dot{S}_{k, Lk}^1 \dot{x}_{Lk}^{1,old} + \dot{S}_{k, Lk}^2 \dot{x}_{Lk}^{2,old} \tag{4.27}
\]
Coefficients $M^1_k$ and $F^1_k$ in equation 4.12 now can be calculated very efficiently, using the following four loops over all nodes 1 to $n$:

- **Loop 1**: node $k$ from 1 up to $n$.
  - Calculation of $\bar{g}^2_k$ direction
  - Calculation of twist angle $\gamma_k$
  - Calculation of rotational velocities $\dot{\psi}_k$ and $\dot{\xi}_k$
  - Calculation of (old) velocities $\dot{t}_i$ or $\dot{x}^1_{L,k}$ and $\dot{x}^3_{L,k}$ for contact nodes

- **Loop 2**: node $k$ from $n$ down to 1.
  - Calculation of $C^m_{nk}$ using equation 4.21
  - Calculation of $\dot{C}^m_{nk/2}$ using equation 4.22
  - Calculation of angle $\psi_k$
  - Calculation of directions $\bar{g}_i^1$ and $\bar{g}_i^3$
  - Calculation of the coefficients $S^j_k$, $\bar{R}^1_k$, $\bar{S}^i_k$, $\dot{R}^i_k$, $\dot{S}^i_k$, $\bar{S}^i_k$, $\bar{R}^i_k$, $\bar{S}^i_k$, $\dot{A}^{ij}_{k(±)}$ and $\dot{A}^{i+1,j}_{k(±)}$
  - Calculation of friction factor $\lambda_k$
  - Calculation of local external forces

- **Loop 3**: node $k$ from 1 up to $n$.
  - Calculation of local old displacements $\Delta x_L^{t-\Delta t}$
  - Calculation of damping forces based on old velocities
  - Calculation of convective forces

- **Loop 4**: node $k$ from $n$ down to 1.
  - Calculation of $C^f_k$ using equation 4.23
  - Calculation of $C^n_k$ using equation 4.24
4.7 Efficient solution of the equations of motion

Because the resulting mass matrix in the equations of motion in chapter 3 has been chosen as diagonal, the displacement increments $\Delta x_{Lk}^{j,t}$ can easily be calculated from equation 4.12 in a loop for node $k$ with $k$ from 1 up to $n$:

- Calculation of wire tension $F_k$ using equation 3.68
- Calculation of $M^j_k$ using equations 3.48 and 3.55
- Calculation of tension dependent friction force using equation 3.69
- Calculation of $F^j_k$ using equations 3.48 and 3.56
- Calculation of $\Delta x_{Lk}^{j,t}$ for $j = 1, 2$ using equation 4.12
- Calculation of $\ddot{x}_{Lk}^j$ for $j = 1, 2$ using equation 4.8
- Calculation of $\dot{x}_{Lk}^2$ and $\ddot{x}_{Lk}^2$ using:

$$
\begin{align*}
\dot{x}_{Lk}^3 &= C_k^v R_k^3 + \mathbf{R}_{k}^{31} \dot{x}_{Lk}^1 \\
\ddot{x}_{Lk}^3 &= C_k^{11} R_k^3 + C_k^v \dot{R}_k^3 + C_k^{22} R_k^3 \\
&\quad + \mathbf{R}_{k}^{31} \ddot{x}_{Lk}^{1,old} + \mathbf{R}_{k}^{32} \ddot{x}_{Lk}^{2,old}
\end{align*}
$$

(4.28)

- Calculation of $\Delta x_{Lk}^{j,t}$ using equation 2.27 based on constant wire length
- Calculation of global displacement increments
- Calculation of global accelerations and velocities
- Calculation of $C_{k+1}^{1l}$ using equation 4.25
- Calculation of $C_{k+1}^{v1}$ using equation 4.26
- Calculation of $C_{k+1}^{v2}$ using equation 4.27

As mentioned earlier, the method to calculate the motion of the wire is indeed an order $n$ method. This is because no other loops exist within the loops over all nodes needed to calculate the motion of the wire.
Part II

Contact and friction
Chapter 5

Description of the contact between wire and winding jig

5.1 Introduction

For a realistic simulation of the winding process, the surfaces of the winding jig (including segment wings, see Figure 1.3) which the wire may contact have to be described in a form that can function as contact surface for the wire. Therefore, to satisfy the objectives defined in section 1.4, the following requirements must be placed on the description of the surfaces:

- The description should be sufficiently accurate. In the first instance, this means that a smooth surface in the direction in which the wire moves must be modelled smoothly. This is because of the sensitivity of the winding process for the shape of the surfaces over which the wire slides. Therefore, a model of a curved surface with the aid of a piece-by-piece straight surface is not allowed because it is practically impossible to have a smooth wire motion over such a surface. In the second instance, sufficiently accurate means that a continuous surface has to be described continuously. This is because the wire has no thickness in the simulation, so that the wire can in principle catch in even the smallest gap between two surfaces.
- The description should be sufficiently flexible to ensure that all essential surfaces of a winding jig can be described.
- The description should allow for an effective manner of contact detection between surface and wire. In other words, it should be possible, with a
minimum amount of processor time, to determine whether and where the wire makes contact with a surface.

- It should be possible to automatically generate the surface description by using a description of the winding jig within the UNIGRAPHICS CAD program.

To satisfy the above requirements, a special method for surface modelling has been chosen. This surface modelling is described in section 5.2.

The detection and other characteristics of contact with the modelled surfaces are described in sections 5.3 and 5.4. In section 5.5 the implementation of the contact description in a contact algorithm is described. In section 5.6 an example of the modelling of a complete jig with segment wings is given.

## 5.2 Method of surface modelling

To satisfy the requirements defined in the previous section, a surface description has been chosen in two different ways. (There are three different types of contact that correspond with the contact with both surfaces separately or simultaneously.) The two different ways are:

- Surface modelling with the aid of straight or curved lines.
- Surface modelling with the aid of a (mostly) rotationally symmetric continuous surface.

### 5.2.1 Surface modelling with the aid of contact lines

The lines used to describe a surface (which we will call contact lines in the remainder of this thesis) are in principle the intersecting lines of the original surface with intersecting surfaces. These lines can be straight or a segment of a circle. This modelling is excellently suited for surfaces with a strong curvature in one direction when the wire is mainly in the direction of this strong curvature. With such a surface, the intersections are chosen perpendicular to this minimum direction of curvature and therefore almost perpendicular to the wire direction, and a new node is generated in the contact point. There is therefore no danger that the wire will penetrate the surface. The type of surfaces for which this method of modelling is excellently suited are the segment wings. In reality these segment wings also are straight or are circle segments. Figure 5.1 shows
a circular segment wing which is modelled with five contact lines that form the intersecting lines of five parallel surfaces with the surface of the segment wing.

Fig. 5.1: The modelling of a segment wing with contact lines

In principle, a contact line is capable of describing the contact with a large wrapped angle of the wire. Because the radius of curvature in the contact of the wire with a contact line is zero, the finite radius of the segment wing preferably will be modelled with several contact lines, for instance five, as shown in Figure 5.1.

The contact between the wire and contact lines is called line contact in the remainder of this thesis.

5.2.2 Surface modelling with the aid of a continuous surface

The method of surface modelling with the aid of a (mostly) rotationally symmetric continuous surface is chosen when the wire contacting these surfaces can be in an arbitrary direction. For these surfaces, modelling with the aid of lines is unsuitable because of the requirement of a smooth surface. Because mainly the upper and lower jigs form such a surface and they are both rotationally symmetric, a model has been chosen for these jigs with the aid of continuously rotationally symmetric surfaces with which only existing nodes of the wire can make contact.

Because only existing nodes can contact these surfaces, this method of modelling is unsuited for such items as the relatively narrow segment wings. The reason is that these segment wings will often be narrower than the distance between the nodes, so that the wire can intersect these wings without making
contact. Therefore, this method of surface modelling is only suited for slightly curved surfaces. By slightly curved surfaces, we mean surfaces whose radius of curvature is much larger than the distance between the nodes on the wire. Figure 5.2 shows a rotationally symmetric surface and the wire contacting this surface.

To avoid penetration of the wire at the edges of a surface, these edges of a surface will always be modelled separately with contact lines. Figure 5.2, showing the edges of a rotationally symmetric surface, can therefore also be considered as showing the contact lines at the edges of this surface.

The contact between the wire and a (mostly) rotationally symmetric surface is called surface contact in the remainder of this thesis.

Fig. 5.2: The wire in contact with the model of a rotationally symmetric surface
5.3 The line contact

5.3.1 Method of modelling with the aid of contact lines

As already stated, the contact lines are composed of line segments that are either straight or a segment of a circle. These line segments are defined in the first instance by a defining plane $S^{def}_j$ in which the line segment is located. The defining plane is chosen in such a way that the line segment is the intersecting line of this defining plane and the surface to be modelled.

The defining plane $S^{def}_j$ is defined by a normal vector $\mathbf{n}^{def}_j$ and a constant $c^{def}_j$ in such a way that the following expression holds for each node $i$ with position vector $\mathbf{x}_i$ that is located in this plane:

$$S^{def}_j (\mathbf{x}_i) = \mathbf{n}^{def}_j \cdot \mathbf{x}_i - c^{def}_j = 0$$ (5.1)

Where $\mathbf{n}^{def}_j$ : the normal vector of the defining plane $S^{def}_j$

$c^{def}_j$ : the constant belonging to the defining plane $S^{def}_j$

The function $S^{def}_j$ will be used later on for contact detection.

The described method of surface modelling enables an efficient method of detecting new line contact. This method is be described in the following subsections.

5.3.2 Main principle of detecting new line contact

The detection of new contact between the wire and the segment lines is carried out by testing the possible contact between the wire and all segment lines successively for a number of conditions. Because all conditions must be satisfied for the occurrence of contact between the wire and a segment line, the remaining conditions will no longer need to be tested when one of the conditions is not satisfied. The most important of these conditions are explained in the following text.

Condition 1: The existence of a point of intersection of the wire with the defining plane of the line segment

Contact between the wire and a line segment in the defining plane $S^{def}_j$ is only possible when there is a point of intersection of the wire with the defining plane.
Figure 5.3 shows such a point of intersection with position vector $\mathbf{x}^s$ between the nodes $i$ and $i + 1$.

The existence of such a point of intersection in the wire between two nodes $i$ and $i + 1$ can easily be checked because such a point only exists when the functional values $S_j^{\text{def}}(\mathbf{x}_i)$ and $S_j^{\text{def}}(\mathbf{x}_{i+1})$ satisfy the following condition:

$$S_j^{\text{def}}(\mathbf{x}_i) \cdot S_j^{\text{def}}(\mathbf{x}_{i+1}) \leq 0 \quad (5.2)$$

If the above condition is satisfied, the position vector $\mathbf{x}^s$ of the point of intersection easily can be calculated by a linear interpolation between the position vectors $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ according to the following equation:

$$\mathbf{x}^s = \frac{\mathbf{x}_i |S_j^{\text{def}}(\mathbf{x}_{i+1})| + \mathbf{x}_{i+1} |S_j^{\text{def}}(\mathbf{x}_i)|}{|S_j^{\text{def}}(\mathbf{x}_i)| + |S_j^{\text{def}}(\mathbf{x}_{i+1})|} \quad (5.3)$$
Condition 2: The existence of a point of intersection of the motion path with the line segment

Another important condition for contact creation between the wire and a line segment is the existence of a point of intersection of the line segment with the so-called motion path. This motion path is supposed to be the straight line between the positions of the points of intersection at the previous and the current contact checks. Figure 5.4 shows such a point of intersection between the motion path and line segment 1.

![Diagram showing the point of intersection of the motion path with the line segment](image)

Fig. 5.4: The point of intersection of the motion path with the line segment

Figure 5.4 also shows why this condition has to be placed on a new contact instead of the less stringent condition that the current point of intersection has to lie at the material side of the line segment. Although the current point of intersection is also at the material side of line segment 2, where line segments 1 and 2 together form a sharp angled surface, contact is only created with line segment 1 because of this second condition.

The testing on this condition is carried out in the following way. The distance to the line segment is calculated for both the previous and the current points of intersection. For the previous point of intersection, this distance is called $D^{old}$ and for the current point $D^{cur}$. This distance is defined as negative when the
point of intersection is at the material side of the line segment and as positive otherwise.

Furthermore, the distance to the end of the line segment is calculated. For the previous point of intersection this distance is called $D_{end}^{old}$ and for the current point $D_{end}^{cur}$. This distance to the end of the line segment is defined as negative when the point of intersection is between the end points of the line segment and as positive otherwise. The calculation of these distances $D_{cur}^{cur}$, $D_{old}^{old}$, $D_{cur}^{cur}$ and $D_{end}^{old}$ for straight line segments, and also for circular line segments, is outlined in appendix C.

Finally, the interpolated distance $D_{end}^{int}$ to the end of the line segment is defined by:

$$D_{end}^{int} = \frac{D_{end}^{cur} \cdot |D_{end}^{old}| + D_{end}^{old} \cdot |D_{end}^{cur}|}{|D_{end}^{cur}| + |D_{end}^{old}|}$$

(5.4)

Figure 5.5 shows these distances for a straight line segment.

---

**Fig. 5.5:** The interpolated position of the point of intersection
For the contact check, it will be supposed that a point of intersection of the motion path with the defining plane only exists when:

\[ D^{old} > 0 \]  
and:

\[ D^{cur} \leq 0 \]  
and:

\[ D^{int}_{end} < 0 \]

It should be noticed that the previous point of intersection does not necessarily need to lie between the same two wire nodes as in the current contact check. Therefore, this method of contact detection also allows for a change in nodal distribution in the wire between the previous and the current contact checks. However, it has to be checked that the previous and current points of intersection belong to the same piece of wire because the wire can have more than one point of intersection with one defining plane. How this check is carried out is not explained here further.
**Exception to condition 2**

Sometimes, an exception is made to the requirement that a point of intersection of the motion path with the line segment has to exist for a new contact. This exception will only be made when the point of intersection of the wire with a defining plane has the same wire position as a node that made contact with another line segment in the previous increment. When the tangent vectors to both line segments have the same directions, it will be assumed that both line segments are neighbouring line segments that together form a smooth surface. In such a situation, a new contact can also be created when the point of intersection of the wire with the defining plane has a positive distance to the line segment. Figure 5.6 shows such a situation where the wire is sliding from a straight line segment to a circular line segment. In such a situation, the small positive distance is completely caused by the linearisation of the displacement increment.

![Diagram](image)

**Fig. 5.6: Change of contact between two line segments**

When the tangents to both line segments have different directions, no new contact is created until the wire penetrates the line segment, according to the original requirement of existence of a point of intersection. Figure 5.7 shows such a situation. In this situation, the creation of new contact without penetration would result in a discontinuity in the motion direction of the contact node. This is very unrealistic since the mass of the wire is concentrated in these nodes.
5.3.3 Alternative method of detecting new line contact

Detecting a new contact for a piece of wire between the nodes \( i \) and \( i + 1 \) as described in the previous section may fail in situations where that piece of wire is parallel or almost parallel to the defining plane. This is because no clear motion path of the intersection point of the wire with the defining plane can be discerned in such situations.

Therefore, in these situations where the wire between two nodes is (almost) parallel to the defining plane, line contact also will be detected with an alternative method. This alternative method is based on checking whether the following conditions are both satisfied:
Condition 1: Existence of a point of intersection of the wire with the normal plane in the previous and current contact checks

For the alternative method of detecting line contact, a point of intersection of the wire with the normal plane has to exist in the previous and the current contact checks. The plane called the normal plane is the plane that can be created by sweeping the line in the direction normal to the defining plane. When defining this plane, an endless line (straight line or circle) has to be taken. Figure 5.8 shows a part of such a normal plane for a circular line segment and the point of intersection of the wire with this plane, with position vector $\overline{x}^s$. Because the line segment shown is a circular line segment, the total normal plane will be a full cylinder.

![Diagram](image.png)

Fig. 5.8: The normal plane and the point of intersection of the wire with the normal plane

The existence of a point of intersection of the wire with the normal plane can be checked by first projecting the wire on the defining plane of the line segment, and then searching for a point of intersection between the projected wire and the line segment. This calculation is outlined in appendix C. It should be noticed that this calculation requires more computing effort than checking on condition 1 in section 5.3.2. This is the main reason why this alternative method of checking for contact is only carried out in situations where the wire is almost parallel to the defining plane.
Condition 2: Existence of a point of intersection of the normal motion path with the line segment

The second condition for the alternative method of detecting contact, is the existence of a point of intersection of the line segment with the normal motion path. Analogous to the motion path defined in section 5.3.2, the normal motion path is the (curved) line between the points of intersection of the wire with the normal plane in the previous and current contact checks. Instead of calculating this motion path, we suppose that the existence of such an intersection point of the normal motion path with the line segment can be checked in the following way. We first calculate the functional values $S_{j}^{def}(\mathbf{x}^{s})$ of the point of intersection of the wire with the normal plane in the previous and the current contact checks, which we will call $D_{n}^{old}$ and $D_{n}^{cur}$, respectively.

Furthermore, we define (analogous to section 5.3.2) the distances to the end of the line segment of the projections on the defining plane of the intersection points in the previous and the current contact checks. These distances are called $D_{n, end}^{old}$ and $D_{n, end}^{cur}$, respectively, and are calculated in the same way as described in appendix C for the distances $D_{end}^{old}$ and $D_{end}^{cur}$. Figure 5.9 shows these distances.
Also (analogous to section 5.3.2), we define the interpolated distance $D_{n,\text{end}}^{\text{int}}$ to the end of the line segment as:

$$D_{n,\text{end}}^{\text{int}} = D_{n,\text{cur}}^{\text{end}} \frac{|D_{n}^{\text{old}}| + D_{n,\text{end}}^{\text{old}} |D_{n}^{\text{cur}}|}{|D_{n}^{\text{cur}}| + |D_{n}^{\text{old}}|} \quad (5.8)$$

It then will be supposed that this condition 2 is satisfied when:

$$D_{n}^{\text{old}} \cdot D_{n}^{\text{cur}} \leq 0 \quad (5.9)$$

and:

$$D_{n,\text{end}}^{\text{int}} \leq 0 \quad (5.10)$$

### 5.3.4 The creation of new line contact

When a new line contact has been detected as described in the previous sub-sections, a new contact node will be created. For reasons of efficiency, the position of this contact node is determined by a projection of the current point of intersection of the wire with the defining plane, normal to the line segment, when the contact has been detected in the way as described in subsection 5.3.2. Figure 5.10 shows the projection normal to a straight line segment.

![Fig. 5.10: The position of a new contact node](image-url)
Generally, this position is different from the position of the point of intersection of the line segment and the motion path as shown in Figure 5.10. This point of intersection would be a more realistic contact position. Because the displacement increments of the wire are kept small, the position error, however, will be very small.

For the alternative method of detecting line contact, the position of the new contact node is determined by a projection normal to the defining plane of the intersection point of the wire with the normal plane.

As already outlined in chapter 2, the new created line contact node is a non-material node without a fixed wire coordinate. The position of the node will be chosen such that it is always located in the defining plane of the line segment.
5.3.5 **Ending line contact**

There are three possible reasons for ending the contact between the wire and a line segment:

- The normal force between the wire and the line segment becomes negative. The calculation of this normal force is outlined in section 3.11.2.
- The contact node passes beyond the end of the line segment (see for instance Figure 5.6).
- The angle between the wire and the line segment becomes too small (smaller than 5 degrees). This requirement follows from the fact that the line contact is only intended for contact between the wire and a line segment making a sufficiently large angle. To avoid penetration of the wire in a surface modelled with contact lines, surfaces have to be modelled in two directions with contact lines when the direction of the wire contacting this surfaces is unknown or not constant in time. An example of this is the modelling of the torpedo (see Figure 1.3), as shown in Figure 5.11.

![Fig. 5.11: The modelling of the torpedo with contact lines in two directions](image)
5.4 The surface contact

5.4.1 Method of modelling with rotationally symmetric surfaces

The rotationally symmetric surface of the upper and the lower jigs is defined in a local coordinate system where the z-axis is the axis of rotation of the surface. In principle, only those surfaces are used that occur by rotating a curve (which will further be called the profile and which is defined in the local y-z plane) around the local z-axis. Figure 5.12 shows the rotationally symmetric surface of the upper jig that has been defined in this way. The lines drawn on the surface of the jig only serve for the visualisation of the surface and are therefore of no importance in the realisation of contact between the wire and the surface.

![Diagram of rotationally symmetric surface]

Fig. 5.12: The rotationally symmetric surface of the upper jig

Only that part of the surface is used that contains a local x-coordinate which
is larger than a minimum x-coordinate belonging to that surface. The profile is defined by a succession of straight and circular line segments in an identical manner to the contact lines as described in the previous section, whereby the local y-z plane now serves as a defining plane for the curve. The material side follows from the jig that is described; for the upper jig, the material side is on the outside, for the lower jig on the inside.

The only non-rotationally symmetric surfaces that are also described with this description method are the surface of the nose on the upper jig and the surface of the plate on the lower jig (see Figure 1.3). These surfaces can be described by extending the profile of upper or lower jig in the negative x-direction. If there is such a surface, the connection to the rotationally symmetric surface must occur at the local \( x = 0 \) position.

The surface described in the above manner is called the main turning profile. In principle, the description of the surface can be extended by combining the main turning profile with extra turning profiles. These are surfaces that occur by rotating another profile around the z-axis of a coordinate system shifted with respect to the original local coordinate system in the x-direction. For the upper jig, the composite surface is defined by the largest distance to the z-axis of all original coordinate systems and, for the lower jig, by the smallest distance. In practice, the physical background of this is that such a surface is also manufactured by carrying out several turning operations on a surface, whereby new material is removed by a shift of the axis of rotation.

5.4.2 The detection of new surface contact

A surface contact can only be realised by contact between existing nodes and the surface. During the contact, these nodes have a fixed wire position. This is in contrast to the line contact, whereby new nodes are generated at the contact point, and where the position of these nodes can vary with respect to the wire.

The condition for new contact of a node with a surface (upper or lower jig), can be formulated by one main condition:

**Main condition: Node is within the material of the jig**

Checking on the condition whether a node is within a jig is carried out by testing on several subconditions. For instance, to contact the upper jig, the z-coordinate of a node has to be larger than the minimum z-coordinate and smaller
than the maximum z-coordinate of the upper jig. Another subcondition is that the node has to lie at the material side of the turning profile.

The same holds as for line contact. When one of the subconditions is not satisfied, the remaining conditions will no longer needed to be tested on.

Testing on the requirement that the node has to lie on the material side of the turning profile is carried out by first transforming the global coordinates of the node to the coordinates \( r, \phi, z \) in a local cylindrical coordinate system. The \( z \)-axis of this local coordinate system coincides with the axis of rotation of the jig. If a surface arises from a continuation of the main profile in the negative \( x \)-direction and the \( x \)-coordinate of the node in the local Cartesian coordinate system is smaller than zero, then, after the transformation, the \( r \)-coordinate is chosen to be equal to the absolute value of the \( y \)-coordinate of the node.

After this transformation, the three-dimensional contact problem is reduced to a two-dimensional problem in \( r \) and \( z \). The contact algorithm to be further applied strongly resembles the algorithm described in section 5.3 for determining the contact of a wire with a surface described with lines. The node with coordinates \( r \) and \( z \) corresponds with the point of intersection of the wire with the defining plane determined in section 5.3. The line segments of which the profile of the jig is constructed correspond with the line segments of which the lines are composed.

Contrary to the contact checking for line contact, the node position at the previous contact check is not of any importance when checking for surface contact. This is because the turning profiles have no sharp corners. This is in contrast to the surfaces modelled with contact lines (see Figure 5.4).

Just as in section 5.3, there are two possibilities for a node that can come into contact with the surface of a profile:

- The node that comes into contact previously made contact with a piece of jig wall belonging to another of the line segments of the profile of the same jig (upper or lower), and has just slid from it in the current time step. In this case, no requirements are placed on the distance of the node to the surface.
- The node that can come into contact was not in contact with a jig wall in the previous time step. In this case, the distance of the node to the surface has to be negative, which means that the node has to lie at the material side of the surface.
The above resembles to a large extent the description in section 5.3. For surface contact, however, the distance of the node to the surface in the previous contact check is of no importance.

5.4.3 The creation of new surface contact

When contact of a node with a surface has been detected, the new contact position will be found by a projection of the node normal to the surface. Similarly to the creation of line contact, only a small error will be made in contact position because the displacements between two contact checks will be kept very small.

5.4.4 Ending surface contact

The contact between a node and the piece of surface belonging to a line segment of the profile can be ended for two reasons:

- The contact force between the node and the surface becomes negative. The calculation of this normal force is outlined in section 3.11.1.
- The node slides beyond the end points of the current line segment of the profile or the x-coordinate of the node becomes smaller than the minimum x-coordinate of the surface. Sliding beyond the end of the line segment is determined in a corresponding manner to that described in the previous section for line contact.
5.5 The implementation of the contact description

Sections 5.3 and 5.4 contain a brief description of how contact of the wire with a surface can be determined in an efficient manner. In this way, if implemented in a computer program, a check can be made with relatively little processor time after an incremental displacement of the wire to see whether the wire has made any contact with a surface. Because an explicit time integration (see chapter 3) has been chosen for the dynamics description of the wire, so many time steps are required for a simulation that the total processor time required for the contact algorithm becomes very large (at least if a check were made for a new contact after each time step).

To prevent this, the contact algorithm is implemented as described in the following two sections:

5.5.1 The subdivision of the increments into subincrements

The simulation is subdivided into a number of increments, whereby the size of an increment is determined by a maximum permissible incremental displacement (the displacement of the wire within an increment). After each increment, a check is made on whether the wire has made contact with a surface. If this appears to be the case, the increment is restarted. Each increment consists of a number of subincrements, whereby the size of a subincrement is determined by stability and accuracy requirements following from the explicit time integration (see chapter 3). After each subincrement, only the existing contact of the wire with surfaces is modified to the changed geometry of the wire (modify direction of normal, remove contact etc.). In principle, no check is made on whether a new contact occurs. This is only checked in the following cases:

- If the wire has slid from a surface in this subincrement.
- If the instantaneous increment is restarted because a new contact appeared to occur in this increment.
- If a new node is just created. In this case, the only check is for a new surface contact.

    Because the number of checks for a new contact can be much lower than the number of subincrements in the above manner, a significant saving in processor time can be achieved. The condition for this is that the increments are so large that the number of increments is much lower than the number of subin-
ments. Here, however, the increments may not be chosen so large that too many increments must be restarted.

5.5.2 The estimate for the minimum needed displacement

Contact checking for line contact is only carried out when the estimate for the minimum displacement for the wire needed to come into contact is smaller than twice the maximum displacement between two contact checks.

The estimate for the minimum displacement of the wire needed to come into contact with a particular line segment can be found by taking the smallest value of the following two distances:

- The first distance is the estimate for the smallest distance of all nodes to the line segment. This estimate is taken as the largest value of (see Figure 5.13):

  - The distance of a node to the defining plane of the line segment ($D_1$ in Figure 5.13).
  - The distance of the projection of the node in the defining plane to the line segment ($D_2$ in Figure 5.13).
  - The distance of this projection point to the end of the segment, which is only meaningful when the projection point is lying outside the endpoints of the segment ($D_3$ in Figure 5.13).

- The second distance is only meaningful if there is an intersection point between the wire and the defining plane. Then, this distance is the maximum of the distance of this intersection point to the line segment times the cosine of the angle $\alpha_s$, and the distance to the end of the line segment times the cosine of $\alpha_s$ (again only meaningful when the intersection point is lying outside the endpoints of the segment). The angle $\alpha_s$ is the angle between the wire and the normal $\bar{n}_{j}^{\text{def}}$ of the defining plane. Multiplying by $\cos(\alpha_s)$ accounts for the effect that the point of intersection of the wire with a surface can have a displacement exceeding the wire displacement. This effect increases when the angle $\alpha_s$ increases.

After each contact check in which no contact checking is carried out for a particular line segment because the estimate for the minimal distance is larger than twice the maximum displacement between two contact checks for the wire,
the estimate is decreased by this maximum displacement. In this way, contact checking will not be carried out earlier for a particular line segment than the earliest time that contact can be expected.
5.6 Example of the modelling of the winding jig

Figure 5.14 shows an example of the model of a complete jig with segment wings. A total of 912 line segments has been used to describe the surfaces which the wire can contact. The rotationally symmetric surfaces of the upper and lower jig have been defined using seven line segments in the profile of the upper jig and six lines in the profile of the lower jig. The nose on the upper jig has been modelled with contact lines instead of with an extension of the profile of the upper jig.

The contours of the block of the upper jig have also been modelled with contact lines, just as the top part of the lower jig. This top part of the lower jig has been modelled with contact lines because it has sharp corners that will make a surface contact approach inadmissible. When the profile of the jig has a small radius at any place, contact lines are also defined in this radius to prevent excessive penetration of the wire in the surface.
Fig. 5.14: The model of a complete jig with segment wings
DESCRIPTION OF THE CONTACT BETWEEN WIRE AND WINDING JIG
Chapter 6

Description of friction between wire and winding jig

6.1 Introduction

An important factor of influence with the winding process is the friction between the wire and the surface with which it makes contact. Because of the great sensitivity of the wire motion for the size and direction of the frictional forces, an accurate description of these forces is essential for the simulation of the winding process.

The chosen modelling of the friction is as follows:

- The friction satisfies the Coulomb law of friction, i.e. that the maximum frictional force that occurs with slip is proportional to the normal force and has the opposite direction to the slip direction.
- The coefficient of friction is dependent on the direction in which the wire slips. The motivation for this dependency of the slip direction is that experiments have shown that the coefficient of friction is larger when the wire slips in the longitudinal direction than when it slips in the direction perpendicular to the wire. The directions of the slip and the frictional force remain opposite.

Furthermore, the following requirements are placed on the modelling:

- The description must be capable of properly describing the friction that occurs in a contact where the wire forms a large wrapped angle around a surface, e.g. around a segment. In such a contact, there is a large inter-
action between the frictional forces and the (non-constant) tensile force in the wire over the wrapped angle.

- The description must be suitable for application in the dynamics description as described in chapter 3. This means that it must be applicable for both a material node, i.e. for a surface contact, and for a non-material node, i.e. for a line contact. Furthermore, it must also be applicable in an explicit time integration.

This chapter contains the friction description. In sections 6.2 and 6.3, the friction is described for a non-material node. In section 6.4, the friction is described for a material node. The friction for a combined surface-line contact node is described in section 6.5. In section 6.6, the modification of the friction perpendicular to the wire is described. This modification is needed to prevent a situation where the slip direction and the frictional force have the same direction, a situation that would become possible when basing the frictional force direction on the slip direction in the previous increment as usual in explicit time integration.

6.2 Description of friction for a line contact

6.2.1 Quasi-static friction model for a slipping node

Line contact should in principle be usable for large wrapped angles. This is in spite of the fact that the most common line contact (the contact of the wire with segment wings) is preferably modelled with several contact lines (see Figure 5.1) and therefore the wrapped angle per contact line remains relatively small. There will also always be sharp corners in the model (see for instance Figure 5.14) that have to be modelled with a single contact line, leading to possible large wrapped angles in one line contact. Therefore, for line contact, a friction description has to be used that remains valid for large wrapped angles.

In [10] a friction description can be found that is valid for large wrapped angles but limited to the quasi-static contact situation. The next section presents a friction description that is based on this model, that remains valid for large wrapped angles and that is usable in a (dynamic) explicit time integration algorithm.

In the following, the friction description for a quasi-static line contact, as given in [10], is described.
For the description of the friction in the contact point of the wire with a contact line at node $i$, the contact in the contact point is regarded as the contact between a wire and a cylinder over which this wire is wrapped. This is shown in Figure 6.1. Here, the angle $\alpha_i^{\text{lin}}$ is the wrapped angle of the wire around the cylinder.

The axis of the cylinder is in the bearing surface $S_j^{\text{def}}$ of the contact line (see section 5.3), and is parallel to the tangent at the contact line in the contact point. Because the radius of the cylinder does not appear to occur in the friction description, it can remain undetermined.

If a segment is modelled with several contact lines, as shown in Figure 5.1, the contact of the wire with each contact line can be separately regarded as the contact of the wire with a cylinder, whereby the wrapped angle corresponds to the wrapped angle of the wire on the relevant contact line. Per cylinder, we have the friction description, as described in the following.

The contact geometry drawn in Figure 6.1 can also be drawn in the flat plane (as shown in Figure 6.2) by drawing the surface of the cylinder in a flat plane. Here, the horizontal direction in Figure 6.2 corresponds with the circumferential direction in Figure 6.1 and the vertical direction with the axial direction. We define the angles $\beta^+$ and $\beta^-$ as the angles of the wire between nodes $i$ and $i + 1$. 

Fig. 6.1: The cylinder as contact geometry
and between nodes $i$ and $i - 1$, respectively, with the circumferential direction of the cylinder. These angles are defined as positive in the situation given in Figure 6.2.

![Diagram of contact geometry and forces](image)

**Fig. 6.2:** The contact geometry and the interplay of forces drawn in the flat plane

In the following description, we assume that the wire in Figure 6.2 has the same slip direction in each point of the contact with the cylinder belonging to node $i$, where the direction is such that it makes an angle $\alpha_{\text{slip}}$ with the circumferential direction of the cylinder. This assumption is only completely correct if the wire does not move with respect to the contact point with the cylinder, i.e. if the contact point forms a material point of the wire, or if the angles $\beta^-$ and $\beta^+$ are equal. In all other cases, the wire motion in its longitudinal direction has part of the slip direction and therefore does not have the same direction in the whole contact.

The above assumption is permitted because in practical situations the longitudinal wire motion will be relatively small and because, for practical values of the coefficients of friction, the difference between $\beta^-$ and $\beta^+$ in a node will be small.

We will define the tensile force in the wire in the contact node $i$ as $T_\theta$, whereby this is a function of the position on the cylinder, indicated by the angle $\theta$ (see
6.2 Description of friction for a line contact

In the contact point that is closest to node \( i - 1 \), we define \( \theta = 0 \). In the contact point that is closest to node \( i + 1 \) we define \( \theta = \alpha_{i}^{\text{lin}} \). The angle of the wire with the circumferential direction at position \( \theta \) is called \( \beta \), see also Figure 6.2.

For the increase of the component of the tensile force in the circumferential direction, we find using equilibrium of forces:

\[
\frac{dT_{\theta} \cos(\beta)}{T_{\theta} \cos(\beta)} = f_{\text{lin}}^{\text{ric}} P_{\text{norm}} R_{\text{cyl}} \cos(\alpha_{\text{slip}}) \, d\theta
\]  

(6.1)

Where \( P_{\text{norm}} \): the normal force per wire length

\( f_{\text{lin}}^{\text{ric}} \): the coefficient of friction for line contact

\( R_{\text{cyl}} \): the radius of the cylinder

For the normal force \( P_{\text{norm}} \) per wire length between the wire and the cylinder, we can write:

\[
P_{\text{norm}} = \frac{T_{\theta} \cos(\beta)}{R_{\text{cyl}}}
\]  

(6.2)

Substitution of equation 6.2 in equation 6.1 and separation of variables gives the following differential equation:

\[
\frac{d \left( T_{\theta} \cos(\beta) \right)}{\left( T_{\theta} \cos(\beta) \right)} = f_{\text{lin}}^{\text{ric}} \cos(\alpha_{\text{slip}}) \, d\theta
\]  

(6.3)

Integration of equation 6.3 between the limits \( \beta^- \) and \( \beta \) for the left part and the limits 0 and \( \theta \) for the right part gives:

\[
T_{\theta} \cos(\beta) = T_i \cos(\beta^-) \exp f_{\text{lin}}^{\text{ric}} \cos(\alpha_{\text{slip}})
\]  

(6.4)

Integration of equation 6.3 between the limits \( \beta^- \) and \( \beta^+ \) for the left part and the limits 0 and \( \alpha_{i}^{\text{lin}} \) for the right part gives:

\[
T_{i+1} \cos(\beta^+) = T_i \cos(\beta^-) \exp f_{\text{lin}}^{\text{ric}} \alpha_{i}^{\text{lin}} \cos(\alpha_{\text{slip}})
\]  

(6.5)

For the increase of the component of the tensile force in the axial direction, we can now write the following in connection with equilibrium of forces in a corresponding manner:

\[
\frac{dT_{\theta} \sin(\beta)}{T_{\theta} \sin(\beta)} = f_{\text{lin}}^{\text{ric}} T_{\theta} \cos(\beta) \sin(\alpha_{\text{slip}}) \, d\theta
\]  

(6.6)
Substitution of equation 6.4 into 6.6 results in the following equation:

\[ dT_\theta (\sin (\beta)) = f_{lin}^{ric}T_i \cos (\beta^-) \exp f_{lin}^{ric} \cos (\alpha_{slip}) \sin (\alpha_{slip}) d\theta \] (6.7)

Integration of equation 6.7 between the limits \( \beta^- \) and \( \beta^+ \) for the left part and the limits 0 and \( \alpha_{lin}^i \) for the right part gives:

\[ T_{i+1} \sin (\beta^+) = T_i \sin (\beta^-) \]
\[ + T_i \cos (\beta^-) \tan (\alpha_{slip}) \left( \exp f_{lin}^{ric} \alpha_{lin}^i \cos (\alpha_{slip}) - 1 \right) \] (6.8)

Equations 6.5 and 6.8 are the equations as derived in a slightly different form in [10]. These equations describe the quasi-static balance of a wire that is in contact and whereby the forces of inertia are of no importance. If, for example, the tensile force \( T_{i+1} \) and the angle \( \beta^+ \) are known, the tensile force \( T_i \) and the angle \( \beta^- \) can be calculated with the aid of these two equations.

With the aid of these equations, the frictional force components for the quasi-static situation can be calculated.

### 6.2.2 The explicit-dynamics friction model for a slipping node

The friction description as given in the previous section is unsuited for application in an explicit time integration as described in chapter 3. This is because equations 6.5 and 6.8 describe a quasi-static force balance that is not applicable with a dynamics description in which accelerations are important. Even if we consider it permissible to calculate the frictional forces acting on node \( i \) based on a quasi-static balance for node \( i \), then we must ascertain that these equations cannot be used. This is because, with the application of an explicit time integration, the geometry and therefore the angles \( \beta^- \) and \( \beta^+ \) are known at the moment of calculation of the frictional forces. As a result, there are only two unknown factors \( T_i \) and \( T_{i+1} \) for the two equations mentioned, which are determined by the geometry and the angles. The latter would mean that the friction algorithm itself (i.e. without the dynamic algorithm) would completely determine the tensile forces left and right of the node, which cannot be combined with the dynamic algorithm. Apart from this, the two equations mentioned in general only allow the solution \( T_i = T_{i+1} = 0 \).

If we want to make the above friction model suited for application in combination with an explicit time integration, we will have to add the forces of inertia.
The most important influence of the forces of inertia on the friction model is the influence of the inertia of the wire in a direction perpendicular to the average wire direction. We can prove this as follows. To start with, we assume that the mass of the wire that was in contact with the cylinder in the previously-described situation is negligible. This assumption is allowed because of the slight mass belonging to the short length of the piece of wire in contact.

If we take the tensile forces between nodes $i-$ and $i$ and between nodes $i+$ and $i$ for the tensile forces $T_i$ and $T_{i+1}$, respectively, then we may assume that these tensile forces are the tensile forces that correspond with the actual dynamic tensile forces working on node $i$. This is because the mass of the wire to the left and to the right of node $i$ is concentrated in the nodes $i-$ and $i+$, respectively.

The only remaining influence of the forces of inertia on the friction algorithm is the fact that the angles $\beta-$ and $\beta+$ are not the actual angles because they are based on the assumption of a straight wire between the nodes, whereas in reality this does not need to be straight. This bending of the wire especially occurs with an acceleration of the wire in a direction perpendicular to its longitudinal direction. This is shown in Figure 6.3, in which the shape of the wire is shown dotted as if it were straight between the nodes.

![Diagram](image)

**Fig. 6.3:** A wire in contact with a line segment, whereby the wire accelerates in a direction perpendicular to the wire
This therefore shows that the angles $\beta^-$ and $\beta^+$, which are important in the given friction model, can have a significantly different size to that assumed in the model. A change of these angles mainly results in a change of the force balance in the direction perpendicular to the average wire direction. Instead of calculating the correct angles $\beta^-$ and $\beta^+$, which means that we have to relinquish the assumption of a straight wire between the nodes, we choose to add a force of inertia to the force balance in the direction mentioned. We try to select the size of this force of inertia such that it is equal to the size of the extra force component resulting from the change of the angles $\beta^-$ and $\beta^+$.

We do this by letting forces of inertia apply to the left and right of the node $i$. Here, the size of these forces of inertia is proportional to the masses of the nodes $i-$ and $i+$, which we have already called $m_{i-}$ and $m_{i+}$ (see section 2.3.4). Furthermore, we assume that these forces of inertia are also proportional to a fictive acceleration $A_{ic}^i$ of the wire around node $i$ in a direction perpendicular to the average wire direction. This acceleration $A_{ic}^i$ is fictive because it is not calculated directly by the dynamic algorithm and because it is meant to introduce an extra unknown to the friction equations.

This acceleration $A_{ic}^i$ is the additional unknown factor which, besides the only unknown factor $T_{i+1}$ or $T_i$ per node up to now, forms the second unknown factor belonging to two new friction equations that can be derived from equations 6.5 and 6.8. This is done by including the forces of inertia in these equations.

After the mentioned forces of inertia are added, equation 6.5 becomes as follows:

\[
T_{i+1} \cos (\beta^+) + m_{i+} A_{ic}^i \sin (\beta^{lon}) =
\]
\[
( T_i \cos (\beta^-) - m_{i-} A_{ic}^i \sin (\beta^{lon}) ) \exp \int_{lin}^i \alpha_i^{lin} \cos (\alpha_{slip})
\]

Where the angle $\beta^{lon}$ defines the average wire direction according to:

\[
\beta^{lon} = \frac{\beta^- + \beta^+}{2}
\]

and where it is assumed that the acceleration $A_{ic}^i$ is in the perpendicular direction defined by the angle $\beta^{lon} + \frac{\pi}{2}$.
Equation 6.8 becomes the following equation:

\[
T_{i+1} \sin (\beta^+) - m_{i+} A_i^{cc} \cos (\beta^{lon}) = T_i \sin (\beta^-) + m_{i-} A_i^{cc} \cos (\beta^{lon}) \quad (6.11)
\]

\[
+ \left( T_i \cos (\beta^-) - m_{i-} A_i^{cc} \sin (\beta^{lon}) \right) \tan (\alpha_{\text{slip}}) \left( \exp^{f_{\text{fric}} i \alpha_{\text{lin}} \cos (\alpha_{\text{slip}})} - 1 \right)
\]

From equations 6.9 and 6.11, the fictive acceleration \( A_i^{cc} \) and the tensile force \( T_{i+1} \) can be calculated as a function of the tensile force \( T_i \). Therefore, from these equations also the friction forces can be calculated as a function of \( T_i \).

The friction component in the circumferential direction of the imaginary contact cylinder is called \( F_{\text{fric}}^{\text{circ}} \). The friction component in the axial direction is called \( F_{\text{fric}}^{\text{axi}} \). These components are defined by:

\[
F_{\text{fric}}^{\text{circ}} = T_i \cos (\beta^-) - T_{i+1} \cos (\beta^+) - (m_{i-} + m_{i+}) A_i^{cc} \sin (\beta^{lon}) \quad (6.12)
\]

and:

\[
F_{\text{fric}}^{\text{axi}} = T_i \sin (\beta^-) - T_{i+1} \sin (\beta^+) + (m_{i-} + m_{i+}) A_i^{cc} \cos (\beta^{lon}) \quad (6.13)
\]

Components \( F_{\text{fric}}^{\text{circ}} \) and \( F_{\text{fric}}^{\text{axi}} \) are calculated from equations 6.9 and 6.11 in Appendix D as a function of \( T_i \). In the explicit time integration, the slip angle \( \alpha_{\text{slip}} \) is based on the incremental displacement direction of the node in the previous (sub)increment. To prevent a situation occurring whereby the frictional force and the slip direction have the same sign in the following increment, the frictional force is corrected if this situation threatens to occur. This correction is extensively described in section 6.6.

The incremental displacement direction is composed of the incremental displacement \( dl_i \) of the wire with respect to the node and the incremental displacement \( dx_i \) of the node itself.

The axial component \( \Delta S_{\text{axi}}^{\text{lip}, t} \) of the slip at time \( t \) is defined by:

\[
\Delta S_{\text{axi}}^{\text{lip}, t} = \mathbf{t}_i \cdot \Delta \mathbf{x}_{i}^{t-\Delta t} - \Delta l_{i}^{t-\Delta t} \sin (\beta^{lon}) \quad (6.14)
\]

Here, \( \mathbf{t}_i \) is the tangent vector in the node \( i \) to the contact line (see section 2.3.4, equation 2.65). The component \( \Delta S_{\text{circ}}^{\text{lip}, t} \) of the slip in the circumferential direction at time \( t \) is:
\[ \Delta S_{\text{circip}, t} = -\Delta l_i^{t-\Delta t} \cos (\beta_{\text{lon}}) \]  

(6.15)

The angle \( \alpha_{\text{slip}} \) follows from:

For \( \Delta S_{\text{circip}, t} = 0 \):

\[ \alpha_{\text{slip}} = \text{Sign} \left( \Delta S_{\text{axi}, t} \right) \frac{\pi}{2} \]  

(6.16)

For \( \Delta S_{\text{circip}, t} > 0 \) (with \(-\frac{\pi}{2} \leq \tan^{-1}() \leq \frac{\pi}{2}\)):

\[ \alpha_{\text{slip}} = \tan^{-1} \left( \frac{\Delta S_{\text{axi}, t}}{\Delta S_{\text{circip}, t}} \right) \]  

(6.17)

For \( \Delta S_{\text{circip}, t} < 0 \):

\[ \alpha_{\text{slip}} = \pi + \tan^{-1} \left( \frac{\Delta S_{\text{axi}, t}}{\Delta S_{\text{circip}, t}} \right) \]  

(6.18)

### 6.2.3 The transition from slip to stick

The previous section contains a description of how the frictional forces acting on that node are calculated for a slipping node. To determine whether a node continues to slip, the frictional forces are calculated that would be required if the node had not slipped any more since the previous node. Therefore, with the calculation of the required frictional force working on a node with complete stick, we take account of the true stiffness of the wire. If a node in the previous increment was subject to sticking, then the calculated incremental wire displacement \( \Delta l_i^{t-\Delta t} \) in the node would have resulted in a tensile force change \(dT_i\) according to:

\[ \Delta T_i^{t-\Delta t} = S_{i}^{\text{tiffs}} \times \Delta l_i^{t-\Delta t} \]  

(6.19)

Here, \( S_{i}^{\text{tiffs}} \) is the stiffness of the wire between node \( i \) and the fixed point or the nearest node \( j \) to node \( i \) that also sticks with \( j < i \).

The tensile force \( T_i \) belonging to stick follows by increasing the tensile force in the previous increment with the change according to equation 6.19:

\[ T_i^{\text{stick}} = T_i + \Delta T_i \]  

(6.20)
If the node sticks, the slip angle $\alpha_{slip}$ (which then can perhaps better be called sticking angle) can be calculated from the ratio of the required frictional forces in axial and circumferential direction. This can be found in an identical manner to equations 6.16 to 6.18:

For \( T_{stick}^i \sin (\beta^-) - T_{i+1} \sin (\beta^+) \) = 0:

$$\alpha_{slip} = \text{Sign} \left( T_{stick}^i \cos (\beta^-) - T_{i+1} \cos (\beta^+) \right) \frac{\pi}{2} \quad (6.21)$$

For \( T_{stick}^i \sin (\beta^-) - T_{i+1} \sin (\beta^+) \) > 0:

$$\alpha_{slip} = \tan^{-1} \left( \frac{T_{stick}^i \cos (\beta^-) - T_{i+1} \cos (\beta^+)}{T_{stick}^i \sin (\beta^-) - T_{i+1} \sin (\beta^+)} \right) \quad (6.22)$$

For \( T_{stick}^i \sin (\beta^-) - T_{i+1} \sin (\beta^+) \) < 0:

$$\alpha_{slip} = \pi + \tan^{-1} \left( \frac{T_{stick}^i \cos (\beta^-) - T_{i+1} \cos (\beta^+)}{T_{stick}^i \sin (\beta^-) - T_{i+1} \sin (\beta^+)} \right) \quad (6.23)$$

This result is identical to the result that can be found by combination of equations 6.5 and 6.8. Using equation 6.5 gives the following equation for the required coefficient of friction for stick:

$$f_{rim} = \frac{1}{\alpha_{rim} \cos (\alpha_{slip})} \ln \left( \frac{T_{i+1} \cos (\beta^+)}{T_{stick}^i \cos (\beta^-)} \right) \quad (6.24)$$

For sticking, the required coefficient of friction according to equation 6.24 must be less than or equal to the actual coefficient of friction that is calculated in section 6.6. In that case, the node has actually stopped slipping. Then, we demand that $dx_i = 0$ in the dynamic algorithm and the frictional force components $F_{a \text{rim}}^{ric}$ and $F_{c \text{rim}}^{ric}$ follow from the force balance. The frictional force component $F_{c \text{rim}}^{ric}$ is calculated as follows from the assumed tensile forces resulting from stick:

$$F_{c \text{rim}}^{ric} = T_{stick}^i \cos (\beta^-) - T_{i+1} \cos (\beta^+) \quad (6.25)$$

If the required coefficient of friction according to equation 6.24 is greater than the actual coefficient of friction, then the assumption that the node has stopped slipping is incorrect and it must be assumed that the node continues to slip. In this case, the frictional forces working on the node are calculated according to equations D.16 and D.17, whereby the slip angle is calculated according to equations 6.16 to 6.18.
6.2.4 Friction model for a sticking node

For a sticking node, we assume that the node displacement $\Delta x_{Li}^t$ is equal to zero. For the wire displacement $\Delta l_i^t$, however, we do not require this to be equal to zero. As explained in the previous section, this is because of the assumed infinite stiffness of the wire. However, a wire displacement $\Delta l_i^t-\Delta t$ does result in a change of the tensile force $\Delta T_i^t-\Delta t$ corresponding with equation 6.19. If the node continues to stick, the required coefficient of friction follows from equation 6.24, whereby equation 6.21 to 6.23 applies for the slip angle. The required coefficient of friction must be less than or equal to the actual coefficient of friction.

6.2.5 The transition from stick to slip

If the required coefficient of friction is greater than the actual coefficient of friction, then the node starts to slip and the frictional forces will be equal to the maximum that follows from equations D.16 and D.17. Here, the value according to equation 6.21 to 6.23 must again be used in the first increment after the beginning of slip for the slip angle $\alpha_{\text{slip}}$ because there is no slip direction known from the previous increment. For the following increments, the slip angle according to equation 6.16 to 6.18 can again be used.

6.3 The coefficient of friction for line contact

The coefficient of friction for a line contact is made dependent on the current slip direction of the wire.

The motivation for this dependency of the slip direction is that experiments have shown that the coefficient of friction is greater when the wire slips in the longitudinal direction than when it slips in the direction perpendicular to the wire (transversal direction). It will be assumed that the coefficient of friction for the slip directions lying between the longitudinal and the transversal slip direction, can be calculated using an ellipse. Then, the actual coefficient of friction $f_{\text{lin}}^{\text{ric}}$ is given as the following function of both the coefficient of friction $f_{\text{long}}^{\text{ric}}$ for a wire slipping in its longitudinal direction and the coefficient of friction $f_{\text{tran}}^{\text{ric}}$ for a wire slipping in the transversal direction:
6.3 The coefficient of friction for line contact

\[ f_{\text{lin}} = \sqrt{\left(f_{\text{fric long}} \frac{\Delta S_{\text{lon}}^{\text{lip},t}}{\Delta S_{\text{tot}}^{\text{lip},t}}\right)^2 + \left(f_{\text{fric tran}} \frac{\Delta S_{\text{tra}}^{\text{lip},t}}{\Delta S_{\text{tot}}^{\text{lip},t}}\right)^2} \]  

(6.26)

Here, \( \Delta S_{\text{lon}}^{\text{lip},t} \) is the slip in the longitudinal direction of the wire defined by:

\[ \Delta S_{\text{lon}}^{\text{lip},t} = -\Delta l_{i}^{t} - \Delta t + \sin \left(\beta_{\text{lon}}^{t}\right) \overline{t}_{i} \times \Delta \overline{x}_{i}^{t} - \Delta t \]  

(6.27)

Furthermore, \( \Delta S_{\text{tra}}^{\text{lip},t} \) is the slip in the transversal direction of the wire defined by:

\[ \Delta S_{\text{tra}}^{\text{lip},t} = \cos \left(\beta_{\text{lon}}^{t}\right) \overline{t}_{i} \times \Delta \overline{x}_{i}^{t} - \Delta t \]  

(6.28)

And \( \Delta S_{\text{tot}}^{\text{lip},t} \) is the total slip of the wire defined by:

\[ \Delta S_{\text{tot}}^{\text{lip},t} = \sqrt{\left(\Delta S_{\text{lon}}^{\text{lip},t}\right)^2 + \left(\Delta S_{\text{tra}}^{\text{lip},t}\right)^2} \]  

(6.29)

Figure 6.4 shows the ellipse for the actual coefficient of friction in the directions lying between the longitudinal and transversal direction.

![Fig. 6.4: Ellipse used for coefficient of friction](image-url)
6.4 Description of friction for a surface contact

6.4.1 General

The friction description for a node in surface contact is based on the fact that the wrapped angle around the surface in such a node will always be small when choosing the nodal distance small with respect to the radius of curvature of the surfaces, which are modelled as a continuous surface (see section 5.4.) This is in contrast to the friction description given for line contact in section 6.2, where the wrapped angle will be independent of the nodal distance. Because the frictional forces with respect to the tensile forces will therefore be small, the frictional force according to a Coulomb friction model can be calculated from a normal force based on only one of two tensile forces because these forces will only slightly differ. We choose to base the normal force on the tensile force $T_i$.

For the friction description, we define the unit direction vector $\mathbf{r}_{i}^{\text{lon}}$ for the average (longitudinal) wire direction touching the surface:

$$\mathbf{r}_{i}^{\text{lon}} = \frac{\mathbf{F}_{i}^{*} + \mathbf{F}_{i+1}^{*}}{\| (\mathbf{F}_{i}^{*} + \mathbf{F}_{i+1}^{*}) \|} \quad (6.30)$$

Vectors $\mathbf{F}_{i}^{*}$ and $\mathbf{F}_{i+1}^{*}$ are the projections of the vectors $\mathbf{F}_{i}$ and $\mathbf{F}_{i+1}$ respectively on the surface and are defined in section 2.3.3 by equation 2.56. Furthermore, we define the unit direction vector $\mathbf{r}_{i}^{\text{tra}}$ perpendicular both to $\mathbf{r}_{i}^{\text{lon}}$ and the outward normal vector $\mathbf{g}_{i}^{2}$, (transversal direction) according to:

$$\mathbf{r}_{i}^{\text{tra}} = \mathbf{g}_{i}^{2} \times \mathbf{r}_{i}^{\text{lon}} \quad (6.31)$$

Using the angles $\alpha_{i-}^{\text{sur}}$ and $\alpha_{i+}^{\text{sur}}$ as defined in section 2.3.3, the total wrapped angle $\alpha_{i}^{\text{sur}}$ in node $i$ is equal to:

$$\alpha_{i}^{\text{sur}} = \alpha_{i-}^{\text{sur}} + \alpha_{i+}^{\text{sur}} \quad (6.32)$$

The normal force in node $i$ is equal to:

$$F_{i}^{\text{norm}} = \alpha_{i}^{\text{sur}} T_i \quad (6.33)$$

We motivate equation 6.33 as follows. The contact between the wire and surface is modelled as a point contact, whereby the wire is straight on both sides of the node and does not make contact with the surface. This is shown dotted in
6.4 Description of friction for a surface contact

Figure 6.5, where the curvature of the surface relative to the nodal distance is strongly exaggerated.

Figure 6.5 also shows the situation that will be more in agreement with reality, namely that the wire will lie over a finite piece of wire on the surface over an angle of approximately $\alpha_{i}^{\text{sur}}$. In this case, the total normal force will follow from equation 6.33.

6.4.2 Friction model for a slipping node

Just as with the node in the line contact, the slip direction of the node $i$ is again based on the direction of the incremental displacement in the previous (sub)increment, whereby the appropriate frictional force is corrected if force and displacement have the same sign in the following increment (see section 6.6). The slip direction in the node $i$ at time $t$ is determined by the components of the incremental slip in the directions of $\vec{r}_{i}^{\text{lon}}$ and $\vec{r}_{i}^{\text{tra}}$ in the previous increment, $\Delta S_{\text{lon},i}^{\text{lip},t-\Delta t}$ and $\Delta S_{\text{tra},i}^{\text{lip},t-\Delta t}$, respectively:

$$\Delta S_{\text{lon},i}^{\text{lip},t-\Delta t} = \vec{r}_{i}^{\text{lon}} \cdot \Delta \vec{x}_{i}^{t-\Delta t}$$  \hspace{1cm} (6.34)
The angle $\alpha_{\text{slip}}$ between the slip direction of the wire over the surface and vector $\mathbf{r}_i^{\text{lon}}$ can be calculated in an identical manner to equation 6.16 to 6.18 as follows:

For $\Delta S_{\text{lon}}^{\text{lip},t-\Delta t} = 0$:

$$\alpha_{\text{slip}} = \text{Sign} \left( \Delta S_{\text{tra}}^{\text{lip},t-\Delta t} \right) \frac{\pi}{2}$$

(6.36)

For $\Delta S_{\text{lon}}^{\text{lip},t-\Delta t} > 0$:

$$\alpha_{\text{slip}} = \tan^{-1} \left( \frac{\Delta S_{\text{tra}}^{\text{lip},t-\Delta t}}{\Delta S_{\text{lon}}^{\text{lip},t-\Delta t}} \right)$$

(6.37)

For $\Delta S_{\text{lon}}^{\text{lip},t-\Delta t} < 0$:

$$\alpha_{\text{slip}} = \pi + \tan^{-1} \left( \frac{\Delta S_{\text{tra}}^{\text{lip},t-\Delta t}}{\Delta S_{\text{lon}}^{\text{lip},t-\Delta t}} \right)$$

(6.38)

The component $F_{\text{lon}}^{\text{ric}}$ of the maximum frictional force in the direction of $\mathbf{r}_i^{\text{lon}}$ is calculated as follows:

$$F_{\text{lon}}^{\text{ric}} = -\alpha_{\text{sur}}^{\text{sur}} T_i f_{\text{sur}}^{\text{ric}} \cos (\alpha_{\text{slip}})$$

(6.39)

Here, $f_{\text{sur}}^{\text{ric}}$ is the coefficient of friction for the wire on a surface modelled with a continuous surface.

Because this description method is only used for the upper or lower jig, we assume the coefficient of friction between wire and upper or lower jig for it. In contrast to the coefficient of friction for line contact, this coefficient of friction is chosen independent on the slip direction.

Similarly, we find the following for the component $F_{\text{tra}}^{\text{ric}}$ of the maximum frictional force in the direction of $\mathbf{r}_i^{\text{tra}}$:

$$F_{\text{tra}}^{\text{ric}} = -\alpha_{\text{sur}}^{\text{sur}} T_i f_{\text{sur}}^{\text{ric}} \sin (\alpha_{\text{slip}})$$

(6.40)

6.4.3 The transition from slip to stick

Just as with the friction description for a node in line contact, we must also determine when a node that is in surface contact stops slipping. For this purpose, the original stiffness of the wire is again taken into account and the tensile force...
change $\Delta T_i$ is calculated for the case that the node would no longer have slipped in the previous increment, similar to equation 6.19:

$$\Delta T_i^{t=\Delta t} = S_i^{t_g} \cdot \Delta S_{lon}^{\text{lip},t=\Delta t}$$  \hspace{1cm} (6.41)

For simplicity, we calculate the frictional forces required for sticking $F_{lon}^{\text{ric}}$ and $F_{tra}^{\text{ric}}$ as follows:

$$F_{lon}^{\text{ric}} = (T_i + \Delta T_i - T_{i+1}) \left( \bar{r}_i^t \cdot \bar{r}_{i}^{\text{lon}} \right)$$  \hspace{1cm} (6.42)

$$F_{tra}^{\text{ric}} = (T_i + \Delta T_i + T_{i+1}) \left( \bar{r}_i^t \cdot \bar{r}_{i}^{\text{tra}} \right)$$  \hspace{1cm} (6.43)

The friction angle $\alpha_{\text{slip}}$ belonging to stick is defined by:

For $F_{lon}^{\text{ric}} \neq 0$:

$$\alpha_{\text{slip}} = \text{Sign} \left( F_{tra}^{\text{ric}} \right) \frac{\pi}{2}$$  \hspace{1cm} (6.44)

For $F_{lon}^{\text{ric}} > 0$:

$$\alpha_{\text{slip}} = \tan^{-1} \left( \frac{F_{tra}^{\text{ric}}}{F_{lon}^{\text{ric}}} \right)$$  \hspace{1cm} (6.45)

For $F_{lon}^{\text{ric}} < 0$:

$$\alpha_{\text{slip}} = \pi + \tan^{-1} \left( \frac{F_{tra}^{\text{ric}}}{F_{lon}^{\text{ric}}} \right)$$  \hspace{1cm} (6.46)

The maximum frictional force component $F_{lon}^{\text{ric}}$ follows from:

$$F_{lon}^{\text{ric}} = -\alpha_{i}^{\text{sur}} T_i f_{\text{sur}}^{\text{ric}} \cos (\alpha_{\text{slip}})$$  \hspace{1cm} (6.47)

If the required frictional force component $F_{lon}^{\text{ric}}$ according to equation 6.42 is less than the value according to 6.47, the node stops slipping, otherwise it continues to slip.

### 6.4.4 Friction model for a sticking node

For a sticking surface contact node, we demand that the node displacement $\Delta S_{tra}^{\text{lip},t=\Delta t}$ is equal to zero. In a similar way to the line contact node, the wire displacement $\Delta S_{lon}^{\text{lip},t=\Delta t}$ results in a change of the tensile force $T_i$ according
to equation 6.41. If we assume that the node has continued to stick, then the appropriate required frictional forces can be calculated from equations 6.42 and 6.43. These frictional forces must be smaller than the maximum for which it is sufficient to demand that the value according to equation 6.42 is less than that according to equation 6.47.

6.4.5 The transition from stick to slip

If the frictional force according to equation 6.42 is larger than the value according to equation 6.47, then the node starts to slip. The appropriate frictional forces then follow from equations 6.39 and 6.40. Here, the value according to equations 6.44 to 6.46 must be used for the slip angle $\alpha_{slip}$ in the first (sub)increment after the start of slipping and, in the next increments, the value according to equations 6.36 to 6.38.
6.5 Description of friction for a combined surface-line contact

6.5.1 Description of friction for a slipping node

If the wire is wrapped around one of the pins in the jig gap, it is possible that a node of the wire not only makes contact with a pin, but also with the upper or lower jig. In section 2.2, such a node is called a combined surface-line contact node.

Because the centre lines of the pins are always perpendicular to the jig surface, if the wire also makes contact with a jig wall, the wire will be wrapped almost circumferentially around the pin. Also, the possible slip direction of the wire will be in the wire direction. Considering this, the friction description from section 6.2 can be used (after a small modification) to calculate the friction to which the wire is subjected at such a combined line-surface contact.

Because the wire is wrapped almost circumferentially to the pin, the following applies by approximation in Figure 6.2:

\[
\cos (\beta^-) = 1 \quad (6.48)
\]

and:

\[
\cos (\beta^+) = 1 \quad (6.49)
\]

The following also applies by approximation if the wire slips in the wire direction:

\[
\cos (\alpha_{\text{slip}}) = \text{Sign} \left( \Delta S_{\text{cir}}^{\text{lip},t} \right) \quad (6.50)
\]

If we consider that, apart from friction on the pin, the wire is also subjected to friction against the jig wall resulting from the normal force with it, and that the change of the angle \( \beta \) describes the curvature around the jig wall, then we find the following instead of equation 6.3:

\[
\frac{dT_\theta}{T_\theta} = \text{Sign} \left( \Delta S_{\text{cir}}^{\text{lip},t} \right) \left( f_{\text{lin}}^{\text{ric}} d\theta + f_{\text{sur}}^{\text{ric}} d\beta \right) \quad (6.51)
\]

Integration of equation 6.51 over the complete contact of wire with jig and pin in the node under consideration gives:

\[
T_{i+1} = T_i \exp \left( \text{Sign} \left( \Delta S_{\text{cir}}^{\text{lip},t} \right) \left( f_{\text{lin}}^{\text{ric}} \alpha_{\text{lin},i} + f_{\text{sur}}^{\text{ric}} \alpha_{\text{sur},i} \right) \right) \quad (6.52)
\]
Here, the angle $\alpha_{i}^{\text{sur}}$ is again the total angle of the wire around the surface, as calculated in equation 6.32.

We find the following equation for the size of the maximum frictional force:

$$F_{\text{surlin}}^{\text{ric}} = T_{i} - T_{i+1} = T_{i} \left(1 - \exp \left(\Delta S_{\text{circ}}^{\text{lip},i} \right) \left(f_{\text{lin}}^{\text{ric}} \alpha_{i}^{\text{lin}} + f_{\text{surlin}}^{\text{ric}} \alpha_{i}^{\text{sur}} \right)\right)$$  \hfill (6.53)

### 6.5.2 The transition from slip to stick

Determining whether the wire stops slipping in the node is done in an almost identical manner to that described in section 6.2. If the wire would no longer have slipped during the previous increment, the change of the tensile force $T_{i}$ is calculated with the aid of equation 6.19. The frictional force required for this follows from:

$$F_{\text{surlin}}^{\text{ric}} = T_{i} + \Delta T_{i} - T_{i+1}$$  \hfill (6.54)

The wire has only stopped slipping if the absolute value of this required frictional force is less than or equal to the absolute value of the maximum frictional force, according to equation 6.53.

### 6.5.3 Description of friction for a sticking node

For a node that is sticking, the frictional force is calculated according to equation 6.54, whereby $\Delta T_{i}^{1-\Delta t}$ is again calculated according to equation 6.19. If this frictional force becomes larger than the maximum value according to equation 6.53, then the wire will start to slip in the node.
6.6 Modification of the frictional force

As described in the previous sections, the frictional force in a node that is in line or surface contact is chosen such that it is directionally opposed to the displacement increment in the previous (sub)increment. If, under influence of the frictional force, the slip direction changes direction in the instantaneous increment, then an unstable situation may arise, whereby the slip and the frictional force continuously change direction. This danger is especially present with respect to the frictional forces in the direction perpendicular to the wire. This is because the motion of the wire in a direction perpendicular to the wire is relatively sensitive for the size of the frictional force in that direction. To prevent such instability, the frictional forces as calculated in the previous sections for a slipping contact node are modified in this section.

The main goal is to avoid a situation where the friction force in the transversal direction has the same direction as the slip in that direction, instead of being opposed.

Firstly, the friction forces in the transversal and the longitudinal direction of the wire are defined for line contact nodes. These friction forces have already been defined as $F_{irc}^{ric}$ and $F_{lon}^{ric}$, respectively, for surface contact nodes.

For a line contact node $i$, these friction forces are defined in the average wire (longitudinal) direction (defined by the angle $\beta^{lon}$) and in the perpendicular (transversal) direction (defined by the angle $\beta^{lon} + \frac{\pi}{2}$). Analogous to the surface contact, these forces will be called $F_{lon}^{ric}$ and $F_{tra}^{ric}$ respectively. These forces can be calculated from the friction forces $F_{cic}^{ric}$ and $F_{axi}^{ric}$ (see equations 6.12 and 6.13) using the following relations:

$$F_{lon}^{ric} = F_{cic}^{ric} \sin \left( \kappa_i - \frac{\phi_i}{2} \right) + F_{axi}^{ric} \cos \left( \kappa_i - \frac{\phi_i}{2} \right)$$ (6.55)

and:

$$F_{tra}^{ric} = -F_{cic}^{ric} \cos \left( \kappa_i - \frac{\phi_i}{2} \right) + F_{axi}^{ric} \sin \left( \kappa_i - \frac{\phi_i}{2} \right)$$ (6.56)

The approach to avoid instabilities is based on the assumption that friction forces $F_{lon}^{ric,act}$ and $F_{tra}^{ric,act}$ are working in the longitudinal and transversal direction and that these forces depend linearly on the incremental slip displacements $\Delta S_{lon}^{lip,t}$ and $\Delta S_{tra}^{lip,t}$, respectively, divided by the time increment $\Delta t$.

When expressing the incremental displacement $\Delta S_{lon}^{lip,t}$ in the old incremental displacement $\Delta S_{lon}^{lip,t-\Delta t}$ and the acceleration $A_{lon}^{cc,t}$ in the longitudinal direction
using equation 4.8, the following expression is found for the actual friction force $F_{\text{lon}}^{\text{ric}, \text{act}}$ in the longitudinal direction:

$$F_{\text{lon}}^{\text{ric}, \text{act}} = -K^{\text{fr}i} \Delta t^2 A_{\text{lon}}^{\text{cc}, t} - K^{\text{fr}i} \Delta S_{\text{lon}}^{\text{lip}, t-\Delta t}$$  \hspace{1cm} (6.57)

In an analogous manner, an expression for the actual friction force $F_{\text{tra}}^{\text{ric}, \text{act}}$ is found:

$$F_{\text{tra}}^{\text{ric}, \text{act}} = -K^{\text{fr}i} \Delta t^2 A_{\text{tra}}^{\text{cc}, t} - K^{\text{fr}i} \Delta S_{\text{tra}}^{\text{lip}, t-\Delta t}$$  \hspace{1cm} (6.58)

Symbol $K^{\text{fr}i}$ can be treated as the friction damping. When assuming that the actual friction forces are equal to the theoretical values when the accelerations are equal to zero, we find the following for the friction damping $K^{\text{fr}i}$ (using the fact that the friction force is opposite to the slip in the previous increment):

$$K^{\text{fr}i} = -\frac{F_{\text{tra}}^{\text{ric}}}{\Delta S_{\text{tra}}^{\text{lip}, t-\Delta t}} = -\frac{F_{\text{lon}}^{\text{ric}}}{\Delta S_{\text{lon}}^{\text{lip}, t-\Delta t}}$$  \hspace{1cm} (6.59)

When the actual friction forces $F_{\text{tra}}^{\text{ric}, \text{act}}$ and $F_{\text{lon}}^{\text{ric}, \text{act}}$ differ too much from the required values $F_{\text{tra}}^{\text{ric}}$ and $F_{\text{lon}}^{\text{ric}}$, one or more new iterations are made for the calculation of the accelerations. For these iterations the friction damping is calculated from the incremental slip $\Delta S_{\text{tra}}^{\text{lip}, \text{prev}}$ in the transversal direction in the previous iteration:

$$K^{\text{fr}i} = -\frac{F_{\text{tra}}^{\text{ric}}}{\Delta S_{\text{tra}}^{\text{lip}, \text{prev}}}$$  \hspace{1cm} (6.60)

However, when the slip $\Delta S_{\text{tra}}^{\text{lip}}$ in the transversal direction becomes too small, the value of the friction damping $K^{\text{fr}i}$ will be limited, thus allowing the actual friction forces to become smaller than the theoretical values. In that case, the current contact node probably will stop slipping in the next increment. Figure 6.6 shows this approach schematically.

Furthermore, during these iterations, correction forces are added in the longitudinal direction to compensate for the change in friction force due to the change in slip in this direction.

In the practical implementation damping is added, both to the generalised direction $g_{1i}$ and to the dependent direction $g_{3i}^j$. In section 3.3, these damping components are used in the derivation of the equations of motion and they are called $k_{Li}^{f,1}$ and $k_{Li}^{f,3}$ respectively.
Fig. 6.6: The friction force linearly depending on the slip displacement

We have:

$$k_{Li}^{f,1} = k_{Li}^{f,3} = K^{fri}$$  \hspace{1cm} (6.61)
DESCRIPTION OF FRICTION BETWEEN WIRE AND WINDING JIG
Chapter 7

External forces

7.1 Bending forces

In section 1.5, it is supposed that the bending stiffness of the wire has no influence of importance on the winding behaviour. It has been shown that the influence of the bending stiffness is relatively small compared to the influence of the usual tensile forces in the wire. Therefore, in the dynamic theory of chapter 3 and the friction theory in chapter 6, the bending stiffness of the wire has been neglected.

However, when totally neglecting the bending stiffness of the wire, this can result in a very unrealistic wire shape when the tensile force approaches zero or even becomes negative for a short time (due to inertia effects). To avoid this, the elastic bending stiffness of the wire has been accounted for by external forces acting on the wire nodes, where these forces will have a minor influence on the wire as long as the tensile forces have their nominal values.

Furthermore, the material damping has been accounted for by viscous bending forces to prevent high frequent (small wavelength) vibrations in the wire. This is because these vibrations are very unrealistic as a result of the occurring material damping in reality. These vibrations would especially occur in situations during the simulation where no (energy dissipating) friction forces are acting on the wire.

These elastic and viscous bending forces are described in this section.
7.1.1 Elastic bending moment

Firstly, the bending angle $\varphi_i$ is calculated for all nodes $i$. This is the angle of the wire in node $i$ defined by: $0 \leq \varphi_i \leq \pi$

$$\varphi_i = \cos^{-1}(\vec{r}_i \cdot \vec{r}_{i+1})$$ (7.1)

For material (free and surface contact) nodes, the angle $\varphi_i$ is equal to the angle $\phi_i$ (see equation 2.19).

It is assumed that the elastic bending moment $M_{i}^{el}$ in the wire at node $i$ follows from:

$$M_{i}^{el} = -c_i^b \varphi_i$$ (7.2)

where $c_i^b$ can be treated as the angle stiffness in node $i$ with:

$$c_i^b = \frac{2EI}{l_i^A + l_{i+1}^A}$$ (7.3)

In the above equation, use has been made of the distances $l_i^A$ and $l_{i+1}^A$ between the nodes $i$ and $i - 1$ and $i$ and $i + 1$, respectively, as defined in equation 4.14.

Furthermore, it is assumed that the bending moment $M_{i}^{el}$ can never exceed the plastic bending moment of the wire:

$$M_{i}^{el} \leq \frac{\pi d^3 \sigma_y}{32}$$ (7.4)

Here, $\sigma_y$ is the yield stress of the wire material.

The value of the maximum bending moment according to equation 7.4 is only a rough estimate of the maximum bending moment in the wire, which also depends on the current tensile force. Because the bending stiffness of the wire has only a small influence on the wire, it is allowed to use this rough estimate according to equation 7.4 as maximum.

7.1.2 Viscous bending moment

Before defining the bending rates of the wire, needed to calculate the viscous bending moment in the wire, the vector $\vec{k}_i$ with unit length and perpendicular to the direction vectors $\vec{r}_i$ and $\vec{r}_{i+1}$ is firstly defined by:

$$\vec{k}_i = \frac{(\vec{r}_i \times \vec{r}_{i+1})}{\| (\vec{r}_i \times \vec{r}_{i+1}) \|}$$ (7.5)
For material (free and surface contact) nodes the vector $\mathbf{K}_i$ is equal to the vector $\mathbf{g}_i^2$ (see equation 2.17).

Furthermore, the bending rate $\dot{\varphi}_i$ is defined for all nodes $i$. This is the time derivative of the angle $\varphi_i$ (defined in equation 7.1). The bending rate $\dot{\varphi}_i$ is calculated using the incremental displacements in the nodes $i-1$, $i$ and $i+1$ in the previous increment:

$$
\dot{\varphi}_i = \frac{(\Delta \mathbf{x}_i^{t+\Delta t} - \Delta \mathbf{x}_i^{t-\Delta t}) \cdot (\mathbf{r}_i \times \mathbf{K}_i)}{(\Delta t) (l_i^\Delta)} + \frac{(\Delta \mathbf{x}_{i+1}^{t+\Delta t} - \Delta \mathbf{x}_{i+1}^{t-\Delta t}) \cdot (\mathbf{r}_{i+1} \times \mathbf{K}_i)}{(\Delta t) (l_{i+1}^\Delta)}
$$

(7.6)

In an analogous manner, the bending rate $\dot{\varphi}_i^P$ in the direction perpendicular both to $\mathbf{K}_i$ and the wire is calculated for all nodes $i$:

$$
\dot{\varphi}_i^P = \frac{(\Delta \mathbf{x}_i^{t+\Delta t} - \Delta \mathbf{x}_i^{t-\Delta t}) \cdot \mathbf{K}_i}{(\Delta t) (l_i^\Delta)} + \frac{(\Delta \mathbf{x}_{i+1}^{t+\Delta t} - \Delta \mathbf{x}_{i+1}^{t-\Delta t}) \cdot \mathbf{K}_i}{(\Delta t) (l_{i+1}^\Delta)}
$$

(7.7)

It is assumed that the viscous bending moment $M_{vi}^i$ in the wire at node $i$ follows from:

$$
M_{vi}^i = -k_i^b \dot{\varphi}_i
$$

(7.8)

and that the viscous bending moment $M_{vi,p}^i$ in the direction perpendicular to the above direction follows from:

$$
M_{vi,p}^i = -k_i^{b,p} \dot{\varphi}_i^P
$$

(7.9)

The bending damping $k_i^b$ is chosen such that the damping is 0.1 times the critical damping $k_i^{b,crit}$, where the critical damping $k_i^{b,crit}$ is defined as the smallest damping resulting in a non-periodic solution of the following differential equation for $\varphi_i$ (or $\varphi_i^P$):

$$
m_i \ddot{\varphi}_i(p) + k_i^b \dot{\varphi}_i(p) + c_i \ddot{\varphi}_i(p) = 0
$$

(7.10)

In the above differential equation, which we assume to be representative for the dynamic bending behaviour of the wire in node $i$, $m_i^b$ is the mass belonging to
the bending angle $\phi_i^{(p)}$ and is, consistently to the definition of $c_i^b$ in equation 7.3, given by:

$$m_i^b = \frac{(l_i^\Delta l_{i+1}^\Delta)^2}{4(l_i^\Delta + l_{i+1}^\Delta)^2} m_i$$  \hspace{1cm} (7.11)

The critical damping $k_i^b, \text{crit}$ is given by:

$$k_i^b, \text{crit} = 2 \sqrt{m_i^b c_i^b}$$  \hspace{1cm} (7.12)

For the bending damping $k_i^b$, the following is then found, using equations 7.3, 7.11 and 7.12:

$$k_i^b = 0.2 \frac{l_i^\Delta l_{i+1}^\Delta}{(l_i^\Delta + l_{i+1}^\Delta)} \sqrt{\frac{EIm_i}{2(l_i^\Delta + l_{i+1}^\Delta)}}$$  \hspace{1cm} (7.13)

### 7.1.3 Dynamical aspects of the bending stiffness

In chapter 4, it is shown that the time step for the time integration of the equations of motion is chosen smaller than the critical time step found for a wire without bending stiffness to obtain a stable time integration.

Because it is shown in the above that a small amount of bending stiffness is added to the wire, it has to be verified whether the time step chosen in chapter 4 is still small enough.

When assuming that the critical time step is determined by the undamped form of the differential equation 7.10, we find:

$$\Delta t_c^b = 2 \sqrt{\frac{m_i^b}{c_i^b}}$$  \hspace{1cm} (7.14)

When, for any node $i$, the critical time step $\Delta t_c^b$ appears to be smaller than the critical time step $\Delta t_c^{\text{prac}}$ (as calculated in equation 4.15), then the bending stiffness $c_i^b$ is decreased for that node as much as necessary to result in a critical time step $\Delta t_c^b$ equal to $\Delta t_c^{\text{prac}}$. 
7.1 BENDING FORCES

7.1.4 Calculation of bending forces

The elastic and viscous bending moments as calculated above act on the wire by means of external forces. These (external) bending forces are defined as force vectors \( \bar{f}_{i-1}^{bi}, \bar{f}_{i}^{bi} \) and \( \bar{f}_{i+1}^{bi} \) in the plane perpendicular to \( k_i \), and as force vectors \( \bar{f}_{i-1}^{bip}, \bar{f}_{i}^{bip} \) and \( \bar{f}_{i+1}^{bip} \) in the plane through the wire and \( k_i \). These forces act on the nodes \( i - 1, i \) and \( i + 1 \), respectively.

The force vectors acting on node \( i - 1 \) are defined by:

\[
\bar{f}_{i-1}^{bi} = \left( \bar{r}_i \times \bar{k}_i \right) \frac{M_i^{el} + M_i^{vi}}{|| (\bar{x}_i - \bar{x}_{i-1}) ||}
\]

(7.15)

The force vectors acting on node \( i \) are defined by:

\[
\bar{f}_{i}^{bi} = -\left( \bar{r}_i \times \bar{k}_i \right) \frac{M_i^{el} + M_i^{vi}}{|| (\bar{x}_i - \bar{x}_{i-1}) ||} - \left( \bar{r}_{i+1} \times \bar{k}_i \right) \frac{M_i^{el} + M_i^{vi}}{|| (\bar{x}_{i+1} - \bar{x}_i) ||}
\]

(7.16)

The force vectors acting on node \( i + 1 \) are defined by:

\[
\bar{f}_{i+1}^{bi} = \left( \bar{r}_{i+1} \times \bar{k}_i \right) \frac{M_i^{el} + M_i^{vi}}{|| (\bar{x}_{i+1} - \bar{x}_i) ||}
\]

(7.17)
Figure 7.1 shows the bending force vectors acting on the nodes $i - 1$, $i$ and $i + 1$ as a result of the bending angle $\varphi_i$ in node $i$.

The (global) force components of the external bending forces can be calculated from expressions 7.15 to 7.17. Using equation 2.6, these global components can be transformed to the local external force components needed in the equations of motion.
7.2 Wire tension at the flyer outlet

An important aspect in the winding process is the tensile force in the wire, which is strongly influenced by the dynamic behaviour of the air dereeler and the flyer (see Figure 1.3). However, the wire in the air dereeler and the flyer is not modelled using the same approach as described in the previous chapters (see Figure 1.7 in section 1.6). The approach is to calculate the motion of the wire between the fixed node and the flyer node with the dynamic theory as described in chapters 2 and 3, and to apply an external force in the flyer node that equals the wire tension at the outlet of the flyer. This external force in the flyer node separately will be calculated using a dynamic model of the flyer and the air dereeler. This dynamic model is described in this section.

7.2.1 Dynamic aspects in the flyer and air dereeler

The aspects that have an important influence on the dynamic behaviour of the flyer and the air dereeler are:

- The mass of the wire in the flyer and the air dereeler. In practice this mass is even larger than the mass of the wire moving in(to) the winding jig because of the larger length of the wire in the flyer and the air dereeler.
- The moment of inertia of the flyer wheels and the friction between these wheels and the wire. Experiments have shown that these wheels slip with respect to the wire most of the winding time.
- The dynamic air pressure in the air dereeler. Although the air dereeler is designed to keep the wire tension constant by using air pressure instead of mechanical moving parts, wire tension measurements at the outlet of the air dereeler show significant variations in this wire tension.
- The plastic elongation of the wire when the wire tension in the wire somewhere reaches the maximum elastic wire tension.
- The possible buckling of the wire when the wire tension decreases to zero.

It is almost impossible to accurately account for all aspects mentioned above. Therefore, a simplified dynamic model has been made of the flyer and the air dereeler in an attempt to calculate the main effects in the dynamic behaviour.
7.2.2 The simplified dynamic model of the flyer and air dereeler

It will be supposed that the wire tension at the flyer outlet can be calculated using a dynamic model of the flyer and the air dereeler based on the following assumptions (see Figure 7.2):

- The motion of the wire in the flyer and air dereeler is one-dimensional, in the longitudinal direction of the wire. The velocity in this direction will be indicated by \( \dot{x}^w \), the displacement by \( x^w \).
- The wire in the flyer and the air dereeler is infinitely stiff in the longitudinal direction.
- The air dereeler exerts a force on the wire consisting of two parts, the constant nominal (air pressure) force \( F^{air} \), and the velocity dependent force \( k^{air} \dot{x}^w \). This latter force represents the damping in the air dereeler.
- The only masses of the wire are the concentrated masses \( m_j^w \) (\( j = 1, 2, 3 \)), where \( m_j^w \) is the mass of the wire between wheel \( j \) and wheel \( j + 1 \). Here, the air dereeler should be considered as wheel 4. The mass \( m_j^w \) is defined by:

\[
m_j^w = \rho A L_j^w
\]  

where \( L_j^w \) is the wire length between wheel \( j \) and \( j + 1 \).
- Friction forces \( f_j^w \) act between the wire and the flyer wheels \( j \) (\( j = 1, 2, 3 \)), with moment of inertia \( I_j^w \), radius \( r_j^w \) and angular speed \( \dot{\theta}_j^w / r_j^w \). These friction forces satisfy Coulomb’s law of friction.
For a wheel sticking to the wire ($\dot{x}_{j}^{wh} = \dot{x}_{w}$), we have:

$$T_{j}^{w} \left(1 - \exp^{-\alpha_{j}^{w} f_{w}^{ric}}\right) \leq f_{j}^{w} \leq T_{j}^{w} \left(1 - \exp^{-\alpha_{j}^{w} f_{w}^{ric}}\right) \tag{7.19}$$

Where $\alpha_{j}^{w}$: wrapped angle of the wire at wheel $j$
$f_{w}^{ric}$: coefficient of friction between wire and flyerwheels
$T_{j}^{w}$: wire tension between wheel $j$ and mass $m_{j}^{w-1}$

For a slipping wheel ($\dot{x}_{j}^{wh} \neq \dot{x}_{w}$), we have:

$$f_{j}^{w} = T_{j}^{w} \left(1 - \exp^{-\alpha_{j}^{w} f_{w}^{ric}}\right) \text{ for: } \dot{x}_{j}^{wh} < \dot{x}_{w} \tag{7.20}$$

$$f_{j}^{w} = T_{j}^{w} \left(1 - \exp^{-\alpha_{j}^{w} f_{w}^{ric}}\right) \text{ for: } \dot{x}_{j}^{wh} > \dot{x}_{w}$$

- The wire will buckle when the tensile force decreases to zero. This buckling occurs in a buckling element. When defining $x^{b}$ as the wire displacement between the buckling element and the plastic element, we have:
  - for $T_{1}^{w} > 0$:
    $$x^{w} = x^{b} \tag{7.21}$$
  - and for $T_{1}^{w} = 0$:
    $$x^{b} \geq x^{w} \tag{7.22}$$
  - for $x^{b} > x^{w}$:
    $$T_{1}^{w} = 0 \tag{7.23}$$
- When the tensile force in the wire exceeds the maximum elastic tensile force $T^{y}$, plastic deformation occurs in the wire. This plastic deformation is called $e^{p}$ and is concentrated in the plastic element.
When defining \( x^p \) as the wire displacement between this plastic element and the spring, we have:

for \( T_1^\text{w} < T^y \):

\[
\dot{x}^p = \dot{x}^b \\
x^p = x^b - e^p
\]  
(7.24)

and for \( T_1^\text{w} = T^y \):

\[
\dot{e}^p = \dot{x}^b - \dot{x}^p
\]  
(7.25)

Here, \( \dot{e}^p \) is the plastic elongation rate of the wire in the plastic element. Both \( e^p \) and \( \dot{e}^p \) are always larger than or equal to zero. For \( \dot{e}^p > 0 \), we always have:

\[
T_1^\text{w} = T^y
\]  
(7.26)

For the maximum tensile force \( T^y \), we find:

\[
T^y = \frac{\pi d^2 \sigma_y}{4}
\]  
(7.27)

- The elasticity of the wire between the air dereeler and the fixed point is concentrated in one elastic spring with stiffness \( C^\text{w} \). This stiffness \( C^\text{w} \) is defined as:

\[
C^\text{w} = \frac{\pi d^2 E}{4 L_{\text{tot}}^\text{w}}
\]  
(7.28)

Here, \( L_{\text{tot}}^\text{w} \) is the total length of the wire in the flyer:

\[
L_{\text{tot}}^\text{w} = \sum_{j=1}^{3} L_j^\text{w}
\]  
(7.29)

For the tensile force acting on the wire in flyer node \( n \) as external force \( f_{Ln}^{\text{vi},3} \), we find:

\[
f_{Ln}^{\text{vi},3} = C^\text{w} (x^p - x_{Ln}^3)
\]  
(7.30)
The material damping of the wire between the air dereeler and the fixed point is concentrated in one dash pot with damping coefficient $k^w$. The velocity of the right side of this dash pot is chosen equal to $\dot{x}^3_{Ln}$, the damping force from this dash pot, however, is not acting on the winding wire, only on the wire in the flyer. The resulting force $T^w_i$ in the wire in the flyer is given by:

$$T^w_i = C^w (x^p - x^3_{Ln}) + k^w (\dot{x}^3_{Ln} - \dot{x}^w_i)$$  \hspace{1cm} (7.31)

The value for the damping $k^w$ is chosen as half the critical damping for the one-dimensional mass-spring system of the flyer:

$$k^w = \sqrt{C^w \rho A f^w_{tot}}$$  \hspace{1cm} (7.32)

### 7.2.3 Equation of motion for the wire in the flyer

The equation of motion for the degree of freedom $x^w$ of the wire in the flyer is:

$$M^w_{res} x^w = F^w_{res} - \dot{x}^w k^{res}$$  \hspace{1cm} (7.33)

Where $M^w_{res}$: resulting mass in the flyer

$F^w_{res}$: resulting force acting on the resulting mass in the flyer

$k^{res}$: resulting coefficient of damping in the flyer

The resulting mass $M^w_{res}$ is found in a similar manner to the theory described in chapter 3. It can be calculated by:

$$M^w_{res} = \sum_{j=1}^{3} ( \prod_{k=j+1}^{3} \lambda^w_k ) m^w_j$$  \hspace{1cm} (7.34)

Here, $m^w_j$ and $\lambda^w_j$ are defined by:

if wheel $j$ is sticking:

$$m^w_j = m^w_j + \frac{f^w_j}{(r^w_j)^2}$$  \hspace{1cm} (7.35)

$$\lambda^w_j = 1$$
if wheel $j$ is slipping:

$$m_j^{\omega} = m_j^w$$

$$\lambda_j^{\omega} = \lambda_j^w$$

(7.36)

Here, $\lambda_j^w$ is the friction factor for flyer wheels, which has the same function as the friction factor $\lambda_p$ defined in equation 3.69. We define $\lambda_j^w$ as:

$$\lambda_j^w = \frac{T_j^w - f_j^w}{T_j^w}$$

(7.37)

Using equation 7.20, we find:

$$\lambda_j^w = \exp^{-\alpha_j^w f_j^{ric}} \text{ for: } \dot{x}_j^{wh} < \dot{x}^w$$

(7.38)

$$\lambda_j^w = \exp^{\alpha_j^w f_j^{ric}} \text{ for: } \dot{x}_j^{wh} > \dot{x}^w$$

The resulting force $F_{res}^w$ is calculated by:

$$F_{res}^w = -F_{air} + k^w \ddot{x}_{Ln} + \left( \prod_{k=j+1}^{3} \lambda_j^w \right) T_1^w$$

(7.39)

The resulting damping $k_{res}$ is calculated by:

$$k_{res} = k_{air} + \left( \prod_{k=1}^{3} \lambda_k^w \right) k^w$$

(7.40)

For a slipping wheel, the following equation of motion is found:

$$I_j^w \dddot{x}_j^{wh} = -f_j^w \left( r_j^w \right)^2$$

(7.41)
7.2 Wire tension at the flyer outlet

7.2.4 Solution procedure

Because the external force $f_{LN}^{vt,3}$ is needed for the calculation of the coefficients in the equations of motion for the wire outside the flyer as described in chapter 3 (see equation 4.23), the equation of motion for the wire in the flyer has to be solved before calculating these coefficients.

Analogous to the solution procedure for the wire as described in chapter 4, the central difference method has been chosen to calculate the motion of the wire in the flyer. This method is described in section 4.3.

Choice of time step

Analogous to the time integration of the equations of motion for the winding wire, the explicit time integration will only be stable if the time step is smaller than the critical time step $\Delta t_c^w$ where equation 4.1 also holds for the value of $\Delta t_c^w$.

For the calculation of the critical time step $\Delta t_c^w$, we will use an estimate for the largest eigen-frequency for the one-dimensional damped mass-spring system of the flyer. This estimate is found when assuming: $\lambda_j^w = 1$ for all $j$, $0 < T_j^w < T_y$, and $k^{res} = 0$.

For the time period of this (estimated) largest eigen-frequency can be found:

$T_{min} = 2\pi \sqrt{m_1^w + m_2^w + m_3^w} / C^w$ \hspace{1cm} (7.42)

Using equations 4.1, 7.18, 7.28 and 7.29, we find the following for the critical time step $\Delta t_c^w$:

$\Delta t_c^w = 4\sqrt{\rho A L_{tot}^w / \pi d^2 E}$ \hspace{1cm} (7.43)

The time steps of the subincrements in the time integration of the wire are based on the critical time step according to equation 4.2. When according to equation 7.43, the critical time step $\Delta t_c^w$ for the flyer is equal to or larger than the critical time step $\Delta t_c$ for the wire, then the time integration for the flyer can be integrated in the same time stepping scheme as used for the wire.
When demanding $\Delta t^w \geq \Delta t_c$, assuming a constant nodal distance $l^\Delta$ and using equations 1.6 and 4.2, the following requirement is found for the wire length in the flyer:

$$L^w_{tot} \geq \frac{l^\Delta}{4} \sqrt{\frac{\pi d^2 E}{T}}$$

(7.44)

When using the following values for a copper wire with a diameter $d$ of 0.334 [mm] (see also section 1.5):

$$1.3 \, N \leq T \leq 13 \, N$$
$$E = 1.2 \times 10^5 \, N/mm^2$$

we find that requirement 7.44 is always satisfied if:

$$\sum_{j=1}^{3} L^w_j \geq 44l^\Delta$$

(7.45)

In practical situations where $l^\Delta$ will never be chosen larger than 20 mm, $L^w_{tot}$ will always be larger than 900 mm and equation 7.44 will therefore always be satisfied. The time steps for the integration of the motion of the wire in the flyer can be chosen equal to the time step for the integration of the winding wire.

**Explicit time integration**

The explicit time integration of the motion of the wire in the flyer is carried out by the following calculation steps every subincrement:

- **Step 1:**
  The new displacements $x^{w,t}$ and $x^{3,t}_{Ln}$ at time $t$ are calculated from the calculated incremental displacements $\Delta x^{w,t-\Delta t}$ and $\Delta x^{3,t-\Delta t}_{Ln}$. These incremental displacements have been calculated in the previous increment (see also chapter 4).

- **Step 2:**
  A first estimate for the new wire velocity $\dot{x}^{w,t}$, the wheel angular velocities $\dot{x}^{wh,t}_j$ and the winding wire velocity $\dot{x}^{3}_{Ln}$ is calculated from an equation
with an error $O(\Delta t)$:

\[
\dot{w}_w^{*,t} = \frac{\Delta x_{w,t}^{*,t} - \Delta t}{\Delta t} + \frac{\Delta t}{2} w_{w,t}^{*,t} + O(\Delta t)
\]

\[
\dot{x}_j^{wh,t} = \frac{\Delta x_{j}^{wh,t} - \Delta t}{\Delta t} + \frac{\Delta t}{2} x_{j}^{wh,t} + O(\Delta t)
\]

\[
\dot{x}_{Ln}^{3,t} = \frac{\Delta x_{Ln}^{3,t} - \Delta t}{\Delta t} + \frac{\Delta t}{2} x_{Ln}^{3,t} + O(\Delta t)
\]

- **Step 3:**
  The new states of slipping wheels, slipping or sticking, are determined using the first estimates $\dot{x}_w^{*,t}$ for the wire velocity and the first estimate $\dot{x}_j^{wh,t}$ for the wheel velocities. It is verified whether the slipping wheel is still slipping in the same direction with respect to the wire, as in the previous increment. If not, sticking will be assumed.

- **Step 4:**
  The new state of the buckling element is determined. There is no buckling if:

  \[
  x_{w,t}^{*,t} \leq x_{Ln}^{3,t} - e_{p,t}
  \]  
  (7.47)

  The buckling element is buckling if:

  \[
  x_{w,t}^{*,t} > x_{Ln}^{3,t} - e_{p,t}
  \]  
  (7.48)

- **Step 5:**
  If the buckling element is not buckling, we have:

  \[
  \dot{x}_b^{*,t} = \dot{x}_w^{*,t}
  \]  
  (7.49)

  \[
  x_b^{*,t} = x_w^{*,t}
  \]

The current wire tension $T_1^{w}$ is calculated using equations 7.24, 7.28, 7.30 and 7.49.
If the buckling element is buckling (see previous step), we have $T_1^{w} = 0$ because of equation 7.23.
Step 6:
If the current wire tension $T^w_1$ exceeds the maximum ($T^w_1 > T^y$), yielding of the plastic element is assumed. Then we have $T^w_1 = T^y$. In that case the following is found for the plastic elongation of the wire:

$$e_p, t = x_{Ln}^{3,t} - x_{w}^{w,t} \tag{7.50}$$

Step 7:
For wheels that remain slipping, the friction factors $\lambda_j^w$ are calculated using equation 7.38.

Step 8:
The resulting mass $M_{res}^w$ is calculated from equation 7.34 and the resulting force $F_{res}^w$ from equation 7.39.

Step 9:
The incremental displacement $\Delta x_{w,t}^w$ is calculated from the following equation that can be found from the equation of motion 7.33 using the central difference scheme:

$$\Delta x_{w,t}^w \left( \frac{M_{res}^w}{\Delta t^2} + \frac{k_{res}}{2\Delta t} \right) = \Delta x_{w,t}^w - \Delta t \left( \frac{M_{res}^w}{\Delta t^2} - \frac{k_{res}}{2\Delta t} \right) + F_{res}^w \tag{7.51}$$

Using the central difference scheme, from $\Delta x_{w,t}^w$ the acceleration $\ddot{x}_{w,t}^w$ and the velocity $\dot{x}_{w,t}^w$ is calculated.

Step 10:
For sticking wheels, the acceleration $\ddot{x}_{j,wh,t}^w$ is made equal to $\ddot{x}_{w,t}^w$ and the velocity $\dot{x}_{j,wh,t}^w$ equal to $\dot{x}_{w,t}^w$. The friction force $f_{j}^w$ acting on sticking wheels is calculated from:

$$f_{j}^w = -\frac{I_j^y \dot{x}_{j,wh,t}^w}{(r_j^w)^2} \tag{7.52}$$

If the friction force $f_{j}^w$ does not satisfy equation 7.19 (friction force greater than maximum), wheel j starts slipping. In the next subincrement wheel j is treated as a slipping wheel. For slipping wheels, the new velocity $\dot{x}_{j,wh,t}^w$ is calculated using the following relation (with the value for $f_{j}^w$ according to equation 7.20):

$$\dot{x}_{j,wh,t}^w = \dot{x}_{j,wh,t}^w - \Delta t + \frac{f_{j}^w (r_j^w)^2}{I_j^y} \frac{\Delta t}{2} \tag{7.53}$$
7.2 Wire tension at the flyer outlet

7.2.5 Practical values for the parameters in the dynamic model

The values of the geometrical parameters $L^w_j$, $r^w_j$ and $\alpha^w_j$, used in the dynamic model of the flyer and the air dereeler, can be determined directly because they are closely related to the geometry of the flyer and the flyer wheels.

The values for the parameters $m^w_j$, $I^w_j$, $C^w$ and $T^w$ follow directly from the geometrical parameters (including $A$ and $d$ of the wire) and the material properties $\rho$ and $\sigma_y$, using the relations 7.18, 7.27 and 7.28.

The practical value for the coefficient of friction $f^w_{vic}$ between the flyer wheels and the wire can be determined by a separate experiment where the wire is wrapped round a wheel and the tensile force at which the wire starts slipping is measured.

The nominal force $F^{air}$ can be determined from a static measurement of the wire tension before the air dereeler. In practice, it is related to the chosen air pressure in the air dereeler.

The determination of the remaining parameter, the damping $k^{air}$ in the air dereeler, is more difficult. This is discussed separately in the following section.

7.2.6 Measurement of the damping of the air dereeler

The practical value of the damping coefficient $k^{air}$ of the air dereeler has been determined by measuring the dynamic tensile force in the wire before the air dereeler ($T^w_4$ in Figure 7.2) during the winding process and by choosing $k^{air}$ in the simulation such that the calculated $T^w_4$ fits the measured value as well as possible.

The dynamic measurement of the tensile force in the wire before the air dereeler has been carried out by using a three-point support of the wire. Figure 7.3 shows the principle of this three-point support. The wire tension is measured by measuring the displacement of the inner support and by calculating the force from this displacement using the known stiffness of this inner support.

This three-point support has been designed for wire tension measurements at very high frequencies, up to $10^4\ [rad/sec]$. This is high enough to measure the wire tension during the winding process.
Both the wrapped angle $\alpha$ round the middle support and the distance $L^s$ between the outer supports (see Figure 7.3) appear to have a large influence on the measured wire tensions. For best results, it appears that the distance $L^s$ between the supports has to be increased for large wrapped angles and decreased for small wrapped angles.

However, the reproducibility of the measured wire tensions appears to be low. Further investigation into this manner of wire tension measurement will be necessary to improve the accuracy of the measurements. Here, we will only use the measured wire tension at two combinations of $\alpha$ and $L^s$ to fit the practical value of $k_{air}$.

Figure 7.4 shows the measured wire tension variations as a function of the flyer angle for the innermost winding of a TV tube line coil. The measurements have been carried out with a large wrapped angle ($\alpha = 36$ degrees and $L^s = 180$ mm) as well as with a very small wrapped angle ($\alpha = 3$ degrees and $L^s = 8$ mm). Some of the input parameters for the simulation of the winding of this coil are summarised in chapter 8, where the results of this simulation will be presented.

Fitting of the damping $k_{air}$ on this measured wire tension, results in a value of $10^{-4} \ N/\text{sec/mm}$. Figure 7.4 also shows the calculated wire tension using this best fit value of $k_{air}$. Since using a value for $k_{air}$ would result in a constant wire tension, the usefulness of the damping to describe the dynamic behaviour of the flyer is clearly demonstrated by Figure 7.4. The mentioned low accuracy of the measurements can be concluded from the difference between the measurements using the large and the small wrapped angle.
Fig. 7.4: Comparison between measured and calculated wire tension variations using fitted damping coefficient.
Part III

Results and verification
Chapter 8

Results and verification

8.1 Introduction

The theory described in the previous chapters has been implemented in the simulation program SWING (Simulation of WINding Geometry). This program is used to evaluate the windability of winding jigs in the design stage when the jig exists only as a CAD model (see chapter 1).

In this chapter, the functionality of this program is illustrated by presenting some simulation results and an attempt is made to verify the simulation program SWING on the following two aspects:

- Verification of the correctness of the implemented theory to result in a simulation model based on the assumptions described in section 1.5. This theory can be divided into three main parts, the dynamic theory, (chapters 2 to 4), the contact theory (chapter 5) and the friction theory (chapter 6).
- Verification of the usefulness of the simulation model to predict the windability of a realistic winding jig. In fact, this means that the assumptions defined in section 1.5 have to be tested on their practical applicability.

Finally, the CPU time required is compared between the implemented dynamic theory for a wire with infinite stiffness and the far simpler dynamic approach when assuming finite stiffness of the wire in its length direction.
8.2 Example of simulation results

8.2.1 Simulation results for the first winding

Firstly, the simulation results for the first winding of a typical TV tube line coil are presented. There now follows a list of the physical input parameters used:

- Nominal distance between nodes: 20 mm
- Ratio of initial/main angular speed of flyer: 0.6
- Specific mass of wire: $0.897 \times 10^{-2} \text{ g/mm}^3$
- Modulus of elasticity of wire: $1.2 \times 10^6 \text{ N/mm}^2$
- Yield stress of the wire: $150 \text{ N/mm}^2$
- Damping of wire per unit of length: $2 \times 10^{-7} \text{ Ns/mm}^2$
- Damping of air dereeler: $1 \times 10^{-4} \text{ Ns/mm}$

Figure 8.1 shows the calculated positions of the wire and the calculated wire tension variations in the wire during the first winding. The winding starts from the straight wire position in Figure 8.1 and turns to the left in the top view. The model of the winding jig in Figure 8.1 and all following figures in this chapter have been simplified for reasons of clarity. Only the most essential parts of the models have been plotted. For a more complete model, the reader is referred to Figure 5.14.

The results have been plotted using a constant time interval. As usual in practice, the angular speed during part of the first winding is chosen lower than the main angular speed to improve the catch-in of the wire. During the first winding, the wire is coming from totally outside the winding jig, which makes the catch-in more difficult. This is in contrast to the following windings, where the wire is already partially inside the winding jig.

Figure 8.1 clearly demonstrates that the wire tension decreases when the flyer needs to take wire back. This mainly occurs when the wire accelerates after almost having been stuck to one of the segment wings.

Furthermore, the wire tension increases when the take-back of the wire stops abruptly. This happens, for instance, when the wire has reached the torpedo and the wire gets tightly stretched.

From Figure 8.1, it can be concluded that the chosen winding jig geometry leads to an acceptable catch-in during the first winding of the wire with the initial angular speed used.
Fig. 8.1: Calculated positions and tension variations in $N$ (colours) during the first winding
8.2.2 Simulation results for the second winding

Figure 8.2 shows the simulation results for the second winding. The positions of the wire are again plotted with a constant time interval. The chosen time interval, however, is half the time interval used for the simulation results of the first winding in Figure 8.1. This second winding appears to be representative for a large part of the successive windings.

Figure 8.2 shows that the chosen main angular speed results in the wire almost sticking to the torpedo. At the moment the wire gets loose, the wire tension decreases strongly, which results in a buckling behaviour of the wire. During such buckling behaviour of the wire, the wire shape will be not very realistic. The small amount of bending stiffness used (see chapter 7), however, becomes important in such situations to prevent far more unrealistic wire shapes. The strong decrease in wire tension demonstrated by Figure 8.2 is undesired in practice and the simulation clearly indicates the situations where this occurs.

The above results demonstrate the capability of SWING for evaluating segment wings that have been positioned on a winding jig within UNIGRAPHICS. At present, the evaluation is mainly done by checking whether the segment wings result in a smooth motion of the wire into the winding jig.

Whether the quality of such a coil, which greatly depends on the winding behaviour, is satisfactory cannot be simulated at the moment with SWING. Further investigation will be necessary into the relationship between the winding behaviour and the resulting quality of the coil.
Fig. 8.2: Calculated positions and tension variations in $N$ (colours) during the second winding
8.3 Verification of the correctness of the implemented theory

8.3.1 Verification of the implemented dynamic theory

The implemented dynamic theory is quite complex. The accelerations and the wire tensions in the wire are calculated in a formal way that is difficult to check. Nevertheless, the accelerations and the wire tensions resulting from the implemented dynamic theory, can be quite simply verified on the important aspect of equilibrium. This is because the calculated accelerations $\ddot{x}_i$ and wire tensions $T_i$ and $T_{i-1}$ have to satisfy the equations of equilibrium for every node.

Checking these equations of equilibrium for every node results in a maximum error in the order of $10^{-18}$, which can totally be explained by the maximum computer accuracy caused by the chosen coding in FORTRAN-77 double precision.

It is also possible to check whether the calculated velocities are kinematically allowable. However, this check requires the same simple equations as used to calculate the velocities, and is therefore not meaningful.

To check whether the time step has been chosen sufficiently small (see section 4.4), six simulations have been carried out using three different time steps, both with and without using friction. Besides the nominal time step of 0.5 times the critical time step, time steps of 0.2 and 0.1 times the critical time step have been used.

Figure 8.3 shows the results of the three simulations using these three different time steps. The results plotted above are for the frictionless case. The results plotted below are for the friction case. The results for the nominal time step have been plotted in red. The results for the 0.2 time step in green and the results of the 0.1 time step in blue.

The calculated positions of the wire agree reasonably well for the frictionless simulation. There is only one position of the wire where substantial differences exist between the various time steps. This can be explained by the large sensitivity of the wire for small differences in the contact creation between the wire and surfaces.

For the simulation with friction, the differences are greater. From this, it can be concluded that the implemented friction theory causes small differences when using different time steps. However, the differences are sufficiently small, especially for the two smallest time steps.
Fig. 8.3: Comparison of results using different time steps. The upper results are without friction, the lower results with friction.
It can be concluded that there is no strong need to decrease the time step of 0.5 times the critical time step. The time step should only be chosen smaller than 0.5 times the critical time step when high accuracy is required in the friction case.

To check whether the calculation results converge when using different nodal distances, six calculations have been carried out using three different nodal distances both with and without using friction. The time steps for these calculations have been nominally chosen as 0.5 times the critical time step, where the critical time step depends on the nodal distance according to equation 4.16.

Figure 8.4 shows the results of these simulations both for the frictionless case and using practical values for the coefficients of friction. The results are plotted in red for a nodal distance of 20 mm, in green for a nodal distance of 10 mm and in blue for a nodal distance of 5 mm.
Fig. 8.4: Comparison of results using different nodal distances. Upper results without friction, lower results with friction
8.3.2 Verification of the implemented contact model

The purpose of the contact model as described in chapter 5 is mainly to prevent penetration of the wire into surfaces and to end contact when the wire is leaving a surface.

Whether the implemented contact model prevents penetration and ends contact correctly can be checked by visual inspection of the results as shown for instance in Figures 8.1 and 8.2. No penetration has been detected in these results.

8.3.3 Verification of the implemented friction model

Verification of the correctness of the implemented (modified quasi-static) friction theory is very difficult because of the following reasons:

- For quasi-static friction situations, the modified theory for quasi-static situations gives the same results as the quasi-static theory, and the quasi-static friction theory does not need to be verified.
- For non quasi-static situations, e.g. situations where the wire accelerates significantly in a direction perpendicular to its longitudinal direction, it is (almost) impossible to measure friction forces.

Therefore, no separate verification on the correctness of the implemented friction theory will be carried out. The only verification will be done by verification of the general results of the simulation program, as described in the following sections.
8.4 Verification of the usefulness of SWING

An accurate verification of the wire positions calculated by SWING is very difficult because the wire motion is difficult to record. When, for instance, filming the winding process with a high speed camera, the wire has a bad visibility because of its slenderness. High speed cameras that offer the possibility of recording the wire motion have only recently become available. Also, the verification of the wire tension in the wire during the winding process is (almost) impossible.

Therefore, only the simulation results of SWING with respect to the windability of a winding jig will be verified. Because the prediction of this windability is also the main purpose of SWING, this verification is very meaningful.

To verify the results of SWING with respect to the windability, the influence of the following parameters on the windability of a winding jig will be compared between SWING results and experiments:

- The influence of the positioning of the segment wings
- The influence of the rotational speed
- The influence of the nominal wire tension

The above verifications are given in the next subsections. In the last subsection, a few preliminary results of the comparison between SWING results and high speed films are given.
8.4.1 The influence of the positioning of the segment wings

To investigate the capability of SWING to predict the correctness of the position of a segment wing, a simulation has been made of the winding on a winding jig that in practice leads to sticking of the wire because of wrong positioning of one of the segment wings.

Figure 8.5 shows the simulation results. The (simplified) model has been plotted in black and the wire positions in red. The simulation clearly shows the position where the wire is sticking to the corner of the block. This position agrees perfectly with the position where the wire in practice is sticking to the block. This example demonstrates the agreement between the SWING results and practical results with respect to the positioning of segment wings.
Fig. 8.5: Sticking of the wire to the corner of the winding block
8.4.2 The influence of the rotational speed

To investigate the influence of the rotational speed of the flyer on the windability of a winding jig, the winding behaviour has been investigated in practice at several increased rotational speeds of the flyer during the winding on a specific winding jig. It was found that the wire starts sticking fully to the torpedo at the first winding at a rotational speed between 2 and 2.4 times the nominal rotational speed. The result is that the wire no longer penetrates the gap between upper and lower jig and winding of a deflection coil becomes impossible.

Figure 8.6 shows the simulation results for the rotational speeds of 2 and 2.4 times the nominal speed. The same effects as noticed in practice can be seen in the simulation. At the highest rotational speed of 2.4 times nominal, the wire is sticking fully to the torpedo and no longer penetrates the winding gap. At the lower rotational speed of 2 times nominal, the wire sticks to the torpedo for a short time.
Fig. 8.6: Sticking of the wire to the torpedo when increasing the rotational speed of the flyer. The upper results are for 2 times nominal, the lower results for 2.4 times nominal rotational speed.
8.4.3 The influence of the nominal wire tension

To investigate the influence of the wire tension of the air dereeler on the windability of a winding jig, the winding behaviour at several decreased wire tensions has been investigated in practice using the same winding jig as in the experiments described above.

It was found that, at a wire tension of about half the nominal value, the wire starts hitting the front of the upper jig before penetrating the winding gap in the first winding when using a rotational speed of 2 times the nominal value. The upper picture in Figure 8.6 shows the simulation results for this rotational speed using nominal wire tension.

Figure 8.7 shows the simulation results for this situation using the tensile force at the air dereeler of half the nominal value and for the rotational speed of 2 times nominal. The same effects as noticed in practice can be seen in the simulation: the wire is hitting the front of the upper jig. In the simulation, however, this front has not been modelled as surface and the wire is therefore free to penetrate the upper jig in this way.

It can be concluded that, for this experiment, SWING also gives results that are in agreement with experimental results.

8.4.4 Preliminary high speed results

As stated earlier, high speed video cameras have only recently become available that make it possible to record the wire motion at high angular speeds of the flyer. Preliminary verification results using this high speed cameras show good agreement between the filmed wire motion and the wire motion calculated by SWING.

More verification tests using these high speed films that will be carried out will result in more detailed results concerning the agreement between SWING results and the filmed wire motion.
Fig. 8.7: Hitting of the front of the upper jig when using tensile force at air dereeler of half the nominal value
8.5 Required CPU time for different wire stiffnesses

In section 4.2, it is shown that, in practical situations, the critical time step for an explicit time integration using a wire that is infinitely stiff will be larger than for a wire that has finite stiffness.

Because of this, a dynamic algorithm (described in chapters 2 to 4) has been specially developed for a wire that is infinitely stiff in the length direction. This algorithm requires more computing per time increment than a simple algorithm for a wire with finite stiffness. This is because of the relatively complex calculation of the coefficients in the equations of motion.

It is therefore very interesting to investigate how much faster the resulting dynamic algorithm is for an infinitely stiff wire than the simple algorithm for a wire with finite stiffness.

Therefore, the simple algorithm for a wire with finite stiffness has also been tested in the simulation program. The required calculation time of both algorithms has been compared for the simulation of a wire rotated by the flyer without contacting any surface. Enabling contact for the algorithm using finite wire stiffness would require far more programming effort and has therefore been omitted.

For the wire with finite stiffness, a transition from a stable time integration to an unstable time integration has been found between the time steps of $5.2 \times 10^{-3}$ ms and $5.9 \times 10^{-3}$ ms, using the simple dynamic algorithm and the physical parameters of section 4.2.3, a specific mass of $0.897 \times 10^{-2}$ g/mm$^3$, a nodal distance $l_\Delta$ of 20 mm and equation 1.6. This is in perfect agreement with the value of $5.4 \times 10^{-3}$ ms that can be found from equation 4.3.

The calculation time required for the simulation of a rotation of 58 degrees with a rotational speed of 60 rpm and using a time step of $5.2 \times 10^{-3}$ ms appeared to be 101 seconds of CPU time on a SUN-1000 computer.

For the wire with infinite stiffness, using the dynamic algorithm as described in this thesis, a transition from a stable time integration to an unstable time integration has been found between the time steps of $239 \times 10^{-3}$ ms and $248 \times 10^{-3}$ ms. This is in perfect agreement with the value $250 \times 10^{-3}$ ms that can be found from equation 4.2. For this wire, the calculation time required for the same simulation appeared to be 7 seconds of CPU time.
Comparing the calculation times for both algorithms in this way, it can be seen that the dynamic algorithm for the wire with infinite stiffness results in a 14 times faster algorithm than the algorithm for a wire with finite stiffness.

It can therefore be concluded that assuming that the wire has infinite stiffness in its length direction and developing a dynamic algorithm for such a wire is indeed meaningful.
Part IV

Conclusions and recommendations
Chapter 9

Conclusions and recommendations

The following conclusions can be drawn from the preceding chapters:

• An efficient dynamic algorithm can be used to calculate the motion of the wire during the deflection coil winding process when assuming that the wire is infinitely stiff in the longitudinal direction.
• No Lagrange multipliers are needed to describe the contact conditions of the wire when absolute coordinates are used to describe the position of the wire and when these coordinates are chosen perpendicular to the surfaces the wire is contacting.
• Although using absolute coordinates, a dynamic algorithm is possible, which results in an order $n$ method.
• Although using an explicit time integration, it is possible to define friction forces acting on the wire that depend on the current wire tension.
• When making use of four different node types and two different ways of surface modelling, a description of contact is possible that is both accurate, flexible and effective.
• It is possible to adapt the known friction theory for a wire wrapped around a cylinder for use in an explicit time integration. This adapted theory is suitable for describing the friction in large wrapped angles.
• Adding a velocity dependent term to the friction acting on the wire results in a more stable time integration with a minimum loss of accuracy.
• The low stiffness of the wire in its tangential direction, together with a non-optimum reproducability of the contact and (especially) the friction description result, prohibite the convergence of the results when decreasing the time step or the nodal distance.
From comparisons between experimental results and simulation results, it can be concluded that the results of the SWING simulation program are well suited to evaluate in the design stage the winding behaviour onto a winding jig.

With respect to further development of SWING the following recommendations are given:

- Further development is needed to improve the convergence behaviour of SWING, especially with respect to the friction description. Also, the reproducability of the contact behaviour when using different nodal distances and time steps can be improved.
- Improvement of the contact description will be necessary in the near future to handle all new types of winding jigs.
- More reliable measurements of the wire tension are required to improve the dynamic model of the flyer and the air dereeler and to determine the model parameters for this model.
- Further research is required to find the relationship between the winding behaviour and the quality of the resulting deflection coil. This will enable usage in the near future of SWING to optimise not only the winding process but also the deflection coil quality.
Appendix A

Relations between the coordinates of node \(i+\) and \(i−\)

This appendix contains calculations of the relations between the velocities and accelerations of node \(i−1\) and \(i+1\) for non-material nodes. These relations are needed in section 2 to calculate the velocities and accelerations of the wire. These relations are calculated for line contact nodes and combined surface-line contact nodes separately.

A.1 The relations for line contact nodes

A.1.1 The relations between the velocities

To calculate the relations between the velocities of the nodes \(i−\) and \(i+\) for line contact nodes, we first define the velocity \(i\_\_i\) of the non-material line contact node \(i\) in the direction of the vector \(t_i\), the direction vector of the contact line. The material velocities of the wire in the nodes \(i−\) and \(i+\) can be found by taking the vector sums of the velocity \(i\_\_\_i\) of node \(i\) in the direction of \(\bar{t}_i\) and the wire velocities relative to node \(i\).

In this way, the material velocity in node \(i−\) is:

\[
\dot{\bar{x}}_i = \dot{i}\_i t_i - \dot{i}\_i \bar{r}_i \tag{A.1}
\]

The material velocity in node \(i+\) is:

\[
\dot{\bar{x}}_{i+} = \dot{i}\_i \bar{t}_i - \dot{i}\_i \bar{r}_{i+1} \tag{A.2}
\]
Equations A.1 and A.2 can also be written in terms of local coordinates with respect to the local coordinate frames of nodes \( i^- \) and \( i^+ \). Written as local coordinates in the coordinate frame of node \( i^- \), this gives:

\[
\mathbf{\dot{x}}_{Li} = \dot{t}_i \mathbf{t}_{Li} - \dot{r}_i \mathbf{r}_{Li}
\]

(A.3)

Here, \( \mathbf{r}_{Li} \) are local components of \( \mathbf{r}_i \) in \( i^- \)

Written as local coordinates in the coordinate frame of node \( i^+ \), this gives:

\[
\mathbf{\dot{x}}_{Li^+} = \dot{t}_i \mathbf{t}_{Li^+} - \dot{r}_i \mathbf{r}_{Li^+}
\]

(A.4)

Here, \( \mathbf{r}_{Li^+} \) are local components of \( \mathbf{r}_{i^+} \) in \( i^+ \)

Equation A.3 can also be written as:

\[
\dot{x}_L^1 = \dot{t}_L^1 - \dot{r}_L^1 i_L^1
\]

(A.5)

\[
\dot{x}_L^2 = \dot{t}_L^2 - \dot{r}_L^2 i_L^2
\]

(A.6)

\[
\dot{x}_L^3 = \dot{t}_L^3 - \dot{r}_L^3 i_L^3
\]

(A.7)

Since \( t_{Li}^2 \) and \( r_{Li}^2 \) will generally be equal to zero because of the choice of \( \mathbf{g}_i^2 \) perpendicular to \( \mathbf{t}_i \) and \( \mathbf{r}_i \), the unknowns \( \dot{t}_i \) and \( \dot{r}_i \) will be determined by \( \dot{x}_L^1 \) and \( \dot{x}_L^3 \). Calculating \( \dot{t}_i \) from equations A.5 and A.7 by eliminating \( \dot{r}_i \) results in:

\[
\dot{t}_i = \frac{r_{Li}^3 \dot{x}_L^3 - r_{Li}^1 \dot{x}_L^1}{t_{Li}^1 r_{Li}^3 - t_{Li}^3 r_{Li}^1}
\]

(A.8)

Calculating \( \dot{r}_i \) from equations A.5 and A.7 by eliminating \( \dot{t}_i \) results in:

\[
\dot{r}_i = \frac{t_{Li}^3 \dot{x}_L^3 - t_{Li}^1 \dot{x}_L^1}{t_{Li}^1 r_{Li}^3 - t_{Li}^3 r_{Li}^1}
\]

(A.9)

When substituting \( \dot{t}_i \) and \( \dot{r}_i \) given by equations A.8 and A.9 in equations A.4 and A.6, we find the following relations:

\[
\begin{bmatrix}
\dot{x}_L^1 \\
\dot{x}_L^2 \\
\dot{x}_L^3 \\
\end{bmatrix}
=
\begin{bmatrix}
A_{i^+}^{11} & A_{i^+}^{13} \\
A_{i^+}^{21} & A_{i^+}^{23} \\
A_{i^+}^{31} & A_{i^+}^{33} \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}_L^1 \\
\dot{x}_L^3 \\
\end{bmatrix}
\]

(A.10)
The components used in equation A.10 can be found from substituting equations A.8 and A.9 in equation A.4:

\[
\begin{align*}
A_{i+}^{11} &= \frac{(t_{Li+} r_{Li}^3 - r_{Li}^1 t_{Li}^3)}{(t_{Li+} r_{Li}^3 - t_{Li}^3 r_{Li}^1)} \\
A_{i+}^{21} &= \frac{(t_{Li+} r_{Li}^3 - r_{Li}^2 t_{Li}^3)}{(t_{Li+} r_{Li}^3 - t_{Li}^3 r_{Li}^2)} \\
A_{i+}^{31} &= \frac{(t_{Li+} r_{Li}^3 - r_{Li}^3 t_{Li}^3)}{(t_{Li+} r_{Li}^3 - t_{Li}^3 r_{Li}^3)} \\
A_{i+}^{22} &= \frac{(t_{Li+} r_{Li}^2 - r_{Li}^1)}{(t_{Li+} r_{Li}^2 - t_{Li}^1)} \\
A_{i+}^{33} &= \frac{(t_{Li+} r_{Li}^3 - r_{Li}^1)}{(t_{Li+} r_{Li}^3 - t_{Li}^1)} \\
A_{i+}^{13} &= \frac{(-t_{Li+} r_{Li}^1 + r_{Li+}^1)}{(t_{Li+} r_{Li}^3 - t_{Li}^3 r_{Li}^1)} \\
A_{i+}^{23} &= \frac{(-t_{Li+} r_{Li}^1 + r_{Li+}^2)}{(t_{Li+} r_{Li}^3 - t_{Li}^3 r_{Li}^2)} \\
A_{i+}^{33} &= \frac{(-t_{Li+} r_{Li}^1 + r_{Li+}^3)}{(t_{Li+} r_{Li}^3 - t_{Li}^3 r_{Li}^3)}
\end{align*}
\]

(A.11)

Making use of the instantaneous choice of directions of the local coordinate systems, we find the following from equations 2.66 to 2.76:

\[
\begin{align*}
r_{Li}^1 &= \cos (\phi_i + \psi_i) \\
r_{Li+}^1 &= 0 \\
r_{Li}^2 &= 0 \\
r_{Li+}^2 &= 0 \\
r_{Li}^3 &= \sin (\phi_i + \psi_i) \\
r_{Li+}^3 &= 1 \\
t_{Li}^1 &= \cos (\kappa_i + \psi_i) \\
t_{Li+}^1 &= -\sin (\kappa_i) \\
t_{Li}^2 &= 0 \\
t_{Li+}^2 &= 0 \\
t_{Li}^3 &= \sin (\kappa_i + \psi_i) \\
t_{Li+}^3 &= \cos (\kappa_i)
\end{align*}
\]

(A.12)

For the instantaneous values of the components \(A_{i(+)}^{kl}\) from A.11, can be found:

\[
\begin{align*}
A_{i+}^{11} &= -\sin (\kappa_i) \sin (\phi_i + \psi_i) \\
A_{i+}^{13} &= \frac{\cos (\phi_i + \psi_i) \sin (\kappa_i)}{\sin (\phi_i - \kappa_i)} \\
A_{i+}^{21} &= 0 \\
A_{i+}^{23} &= 0 \\
A_{i+}^{31} &= \frac{-\sin (\kappa_i + \psi_i) + \cos (\kappa_i) \sin (\phi_i + \psi_i)}{\sin (\phi_i - \kappa_i)} \\
A_{i+}^{33} &= \frac{-\sin (\kappa_i) \cos (\phi_i + \psi_i)}{\sin (\phi_i - \kappa_i)}
\end{align*}
\]

(A.13)
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\[
A_{i+}^{33} = -\frac{\cos(\phi_i + \psi_i) \cos(\kappa_i) + \cos(\kappa_i + \psi_i)}{\sin(\phi_i - \kappa_i)}
\]

\[
A_{i}^{21} = 0
\]

\[
A_{i}^{23} = 0
\]

A.1.2 The relation between the accelerations

To calculate the relations between the accelerations (and velocities) of nodes \( i- \) and \( i+ \) for line contact nodes from relation A.10, the time derivatives of the components \( A_{i(+)i}^{kl} \) are needed according to:

\[
\begin{bmatrix}
\ddot{x}_{Li+}^1 \\
\ddot{x}_{Li+}^2 \\
\ddot{x}_{Li+}^3
\end{bmatrix} = \begin{bmatrix}
\dot{A}_{i+}^{11} & \dot{A}_{i+}^{13} \\
\dot{A}_{i+}^{21} & \dot{A}_{i+}^{23} \\
\dot{A}_{i+}^{31} & \dot{A}_{i+}^{33}
\end{bmatrix} \begin{bmatrix}
\dot{x}_{Li}^1 \\
\dot{x}_{Li}^2 \\
\dot{x}_{Li}^3
\end{bmatrix} + \begin{bmatrix}
A_{i+}^{11} & A_{i+}^{13} \\
A_{i+}^{21} & A_{i+}^{23} \\
A_{i+}^{31} & A_{i+}^{33}
\end{bmatrix} \begin{bmatrix}
\ddot{x}_{Li}^1 \\
\ddot{x}_{Li}^2 \\
\ddot{x}_{Li}^3
\end{bmatrix}
\] (A.14)

To calculate these time derivatives \( A_{i(+)i}^{kl} \), the curvatures of the contact line in the directions of the local coordinate directions have first to be calculated. Therefore, the vectors \( \overline{h}_i^2 \) and \( \overline{h}_i^{2+} \) with unit length are first defined:

\[
\overline{h}_i^2 = \overline{g}_i^2 \times \overline{t}_i
\] (A.15)

\[
\overline{h}_i^{2+} = \overline{g}_i^{2+} \times \overline{t}_i
\] (A.16)

Figure A.1 shows the normal vector \( \overline{n}_i \), the angles \( \alpha_i^{lin} \), \( \alpha_i^{lin} \) and the vectors \( \overline{h}_i^2 \) and \( \overline{h}_i^{2+} \). From equations 2.124, 2.126 and 2.123 the following can be found:

\[
\dot{\overline{h}}_i \cdot \overline{g}_i^2 = -C_{lin}^{uv} \dot{t}_i \cos(\alpha_i^{lin})
\] (A.17)

\[
\dot{\overline{h}}_i \cdot \overline{g}_i^{2+} = -C_{lin}^{uv} \dot{t}_i \cos(\alpha_i^{lin})
\] (A.18)

And as can easily be seen from Figure 2.5:

\[
\dot{\overline{h}}_i \cdot \overline{h}_i^2 = +C_{lin}^{uv} \dot{t}_i \sin(\alpha_i^{lin})
\] (A.19)

\[
\dot{\overline{h}}_i \cdot \overline{h}_i^{2+} = -C_{lin}^{uv} \dot{t}_i \sin(\alpha_i^{lin})
\] (A.20)
Now, using \( \vec{t}_i \cdot \vec{t}_i = 0 \), we have:

\[
\begin{align*}
\dot{t}_i \cdot \vec{g}^1_i &= \dot{t}_i \cdot (\vec{t}_i \cos (\psi_i + \kappa_i) + \bar{h}_i^2 \sin (\psi_i + \kappa_i)) \\
&= +C_{lin}^{ur} i \sin (\alpha_{lin}^i) \sin (\psi_i + \kappa_i) & (A.21)
\end{align*}
\]

\[
\begin{align*}
\dot{t}_i \cdot \vec{g}^3_i &= \dot{t}_i \cdot (\vec{t}_i \sin (\psi_i + \kappa_i) - \bar{h}_i^2 \cos (\psi_i + \kappa_i)) \\
&= -C_{lin}^{ur} i \sin (\alpha_{lin}^i) \cos (\psi_i + \kappa_i) & (A.22)
\end{align*}
\]

\[
\begin{align*}
\dot{t}_i \cdot \vec{g}^4_i &= \dot{t}_i \cdot (-\vec{t}_i \sin (\kappa_i) + \bar{h}_i^2 \cos (\kappa_i)) \\
&= -C_{lin}^{ur} i \sin (\alpha_{lin}^i) \cos (\kappa_i) & (A.23)
\end{align*}
\]

\[
\begin{align*}
\dot{t}_i \cdot \vec{g}^3_{i+1} &= \dot{t}_i \cdot (\vec{t}_i \cos (\kappa_i) + \bar{h}_{i+1}^2 \sin (\kappa_i)) \\
&= -C_{lin}^{ur} i \sin (\alpha_{lin}^i) \sin (\kappa_i) & (A.24)
\end{align*}
\]

From this we find:
Relations between the coordinates of node $i+$ and $i-$

\[
i_{Li}^1 = C_{lin}^{ur}i_1 \sin (\alpha_{i-}^{lin}) \sin (\psi_i + \kappa_i) \\
i_{Li}^2 = -C_{lin}^{ur}i_1 \cos (\alpha_{i-}^{lin}) \\
i_{Li}^3 = -C_{lin}^{ur}i_1 \sin (\alpha_{i-}^{lin}) \cos (\psi_i + \kappa_i) \tag{A.25}
\]

\[
i_{Li+}^1 = -C_{lin}^{ur}i_1 \sin (\alpha_{i+}^{lin}) \cos (\kappa_i) \\
i_{Li+}^2 = -C_{lin}^{ur}i_1 \cos (\alpha_{i+}^{lin}) \\
i_{Li+}^3 = -C_{lin}^{ur}i_1 \sin (\alpha_{i+}^{lin}) \sin (\kappa_i) \tag{A.26}
\]

Using equations 2.154 and 2.155, from equations 2.138, 2.145, 2.146, 2.151, 2.156 and 2.156 it follows that:

\[
\begin{align*}
\dot{r}_{Li}^1 &= -\sin (\phi_i + \psi_i) H_i^a \\
\dot{r}_{Li}^2 &= -H_i^b \\
\dot{r}_{Li}^3 &= \cos (\phi_i + \psi_i) H_i^a \\
\dot{r}_{Li+}^1 &= -\dot{\psi}_i(+) \\
\dot{r}_{Li+}^2 &= -\dot{\xi}_i \\
\dot{r}_{Li+}^3 &= 0 \tag{A.27}
\end{align*}
\]

Then, from equations A.11, A.12, A.25 and A.27, the following time derivatives of the components are found:

\[
\begin{align*}
\dot{A}_{i+}^{11} &= -C_{lin}^{ur}i_1 \sin (\phi_i + \psi_i) \\
&\quad \left( \cos (\kappa_i) \sin (\alpha_{i+}^{lin}) \sin (\phi_i - \kappa_i) - \sin (\kappa_i) \sin (\alpha_{i-}^{lin}) \cos (\phi_i - \kappa_i) \right) \frac{\sin^2 (\phi_i - \kappa_i)}{\sin^2 (\kappa_i) \\
&\quad + \sin (\kappa_i) \sin (\psi_i + \kappa_i) H_i^a + \dot{\psi}_i(+) \sin (\kappa_i + \psi_i) \right) \\
&\quad \frac{\sin^2 (\phi_i - \kappa_i)}{\sin^2 (\kappa_i)} \\
\dot{A}_{i+}^{13} &= -C_{lin}^{ur}i_1 \cos (\phi_i + \psi_i) \\
&\quad \left( -\cos (\kappa_i) \sin (\alpha_{i+}^{lin}) \sin (\phi_i - \kappa_i) + \sin (\kappa_i) \sin (\alpha_{i-}^{lin}) \cos (\phi_i - \kappa_i) \right) \frac{\sin^2 (\phi_i - \kappa_i)}{\sin^2 (\kappa_i) \\
&\quad - \sin (\kappa_i) \cos (\psi_i + \kappa_i) H_i^a - \dot{\psi}_i(+) \cos (\kappa_i + \psi_i) \right) \\
&\quad \frac{\sin^2 (\phi_i - \kappa_i)}{\sin^2 (\kappa_i)} \\
\dot{A}_{i+}^{21} &= -C_{lin}^{ur}i_1 \sin (\phi_i + \psi_i) \cos (\alpha_{i+}^{lin}) + \dot{\xi}_i \sin (\kappa_i + \psi_i) \frac{\sin (\phi_i - \kappa_i)}{\sin (\phi_i - \kappa_i)} \tag{A.28}
\end{align*}
\]
A.1 THE RELATIONS FOR LINE CONTACT NODES

\[ \dot{A}_{i+}^{23} = \frac{C_{i+}^{uv} \cos (\phi_i + \psi_i) \cos (\alpha_{i+}^{\text{lin}}) - \dot{\xi}_i \cos (\kappa_i + \psi_i)}{\sin (\phi_i - \kappa_i)} \]

\[ \dot{A}_{i+}^{31} = -H^c_k \sin (\phi_i + \psi_i) + \frac{H^a_i \sin (\kappa_i + \psi_i) (\cos (\phi_i - \kappa_i) - \cos (\kappa_i))}{\sin^2 (\phi_i - \kappa_i)} \]

\[ \dot{A}_{i+}^{33} = H^c_k \cos (\phi_i + \psi_i) + \frac{H^a_i \cos (\kappa_i + \psi_i) (-\cos (\phi_i - \kappa_i) + \cos (\kappa_i))}{\sin^2 (\phi_i - \kappa_i)} \]

\[ \dot{A}_{i-}^{21} = \frac{-C_{i-}^{uv} \sin (\phi_i + \psi_i) \cos (\alpha_{i-}^{\text{lin}}) + H^b_i \sin (\kappa_i + \psi_i)}{\sin (\phi_i - \kappa_i)} \]

\[ \dot{A}_{i-}^{23} = \frac{C_{i-}^{uv} \cos (\phi_i + \psi_i) \cos (\alpha_{i-}^{\text{lin}}) - H^b_i \cos (\kappa_i + \psi_i)}{\sin (\phi_i - \kappa_i)} \]

With:

\[ H^c_k = \frac{-C_{i+}^{uv} \sin (\alpha_{i+}^{\text{lin}}) (1 - \cos (\kappa_i) \cos (\phi_i - \kappa_i)) - \sin (\alpha_{i+}^{\text{lin}}) \sin (\kappa_i) \sin (\phi_i - \kappa_i)}{\sin^2 (\phi_i - \kappa_i)} \] (A.29)
A.2 The relations for combined surface-line contact nodes

A.2.1 The relations between the velocities

Relations A.3 and A.4 can again be used to calculate the relations between the velocities of the nodes $i-$ and $i+$ for combined surface-line contact nodes. Because now node $i$ also contacts a surface, we have:

$$i_i = 0$$ (A.30)

From equation A.7, the following then can be found:

$$i_i = \frac{-\dot{x}_{Li}}{r_{Li}^3}$$ (A.31)

When substituting $i_i$ (given by equation A.31) in equations A.4, A.5 and A.6, the following relations are found:

$$\begin{bmatrix}
\dot{x}^{1}_{Li+} \\
\dot{x}^{2}_{Li+} \\
\dot{x}^{3}_{Li+} \\
\dot{x}^{1}_{Li} \\
\dot{x}^{2}_{Li}
\end{bmatrix}
= \begin{bmatrix}
A^{13}_{i+} \\
A^{23}_{i+} \\
A^{33}_{i+} \\
A^{13}_{i} \\
A^{23}_{i}
\end{bmatrix}
\begin{bmatrix}
\dot{x}^{3}_{Li}
\end{bmatrix}$$ (A.32)

The components used in equation A.32 can be found by substituting equations A.31 in equations A.4, A.5 and A.6:

$$\begin{align*}
A^{13}_{i+} &= \frac{r_{Li+}^1}{r_{Li}^3} \\
A^{23}_{i+} &= \frac{r_{Li+}^2}{r_{Li}^3} \\
A^{33}_{i+} &= \frac{r_{Li+}^3}{r_{Li}^3} \\
A^{13}_{i} &= \frac{r_{Li}^1}{r_{Li}^3} \\
A^{23}_{i} &= \frac{r_{Li}^2}{r_{Li}^3}
\end{align*}$$ (A.33)
A.2 The relations for combined surface-line contact nodes

By making use of the instantaneous choice of the directions of the local coordinate systems (see equations 2.66 to 2.76 and the extra condition 2.107), we find:

\[
\begin{align*}
    r_{Li}^1 &= 0 & r_{Li+}^1 &= 0 \\
    r_{Li}^2 &= 0 & r_{Li+}^2 &= 0 \\
    r_{Li}^3 &= 1 & r_{Li+}^3 &= 1 \\
\end{align*}
\]  \hspace{1cm} (A.34)

The following can be found for the instantaneous values of the components \(A_{i(+)}^{kl}\) in A.33:

\[
\begin{align*}
    A_{i+}^{13} &= 0 \\
    A_{i+}^{23} &= 0 \\
    A_{i+}^{33} &= 1 \\
    A_{i}^{13} &= 0 \\
    A_{i}^{23} &= 0 \\
\end{align*}
\]  \hspace{1cm} (A.35)

A.2.2 The relations between the accelerations

To calculate the relations between the accelerations (and velocities) of the nodes \(i-\) and \(i+\) for combined surface-line contact nodes from relation A.32, the time derivatives of the components \(A_{i(+)}^{kl}\) are needed according to:

\[
\begin{bmatrix}
    \dot{x}_{Li+}^1 \\
    \dot{x}_{Li+}^2 \\
    \dot{x}_{Li+}^3 \\
\end{bmatrix}
= \begin{bmatrix}
    \dot{A}_{i+}^{13} \\
    \dot{A}_{i+}^{23} \\
    \dot{A}_{i+}^{33} \\
\end{bmatrix}
\begin{bmatrix}
    x_{Li+}^3 \\
\end{bmatrix}
= \begin{bmatrix}
    \dot{x}_{Li+} \\
    \dot{x}_{Li} \\
\end{bmatrix}
\begin{bmatrix}
    A_{i+}^{13} \\
    A_{i+}^{23} \\
\end{bmatrix}
\begin{bmatrix}
    x_{Li}^3 \\
\end{bmatrix}
\]  \hspace{1cm} (A.36)

Using equations 2.154 and 2.155, from the equations 2.107, 2.138, 2.145, 2.146, 2.151 and 2.156 it follows that:

\[
\begin{align*}
    \dot{r}_{Li}^1 &= -H_i^p & \dot{r}_{Li+}^1 &= -\dot{\psi}_{i+} \\
    \dot{r}_{Li}^2 &= -H_i^p & \dot{r}_{Li+}^2 &= -\dot{\xi}_{i} \\
    \dot{r}_{Li}^3 &= 0 & \dot{r}_{Li+}^3 &= 0 \\
\end{align*}
\]  \hspace{1cm} (A.37)
Then, from equations A.33, A.34 and A.37, the following time derivatives of the components are found:

\[
\begin{align*}
\dot{A}_{i+}^{13} &= -\dot{\psi}_{i+} \\
\dot{A}_{i+}^{23} &= -\dot{\xi}_i \\
\dot{A}_{i+}^{33} &= 0 \\
\dot{A}_i^{13} &= -H_i^q \\
\dot{A}_i^{23} &= -H_i^p
\end{align*}
\] (A.38)
Appendix B

Calculation of the local directions

This appendix contains the calculations of the values for the angles $\psi_i$ from the requirement expressed by equations 3.45 or 3.46 that the resulting mass matrix in the equations of motion has to be diagonal.

The value of the angle $\psi_k$ following from equation 3.45 or 3.46 depends on the node type of node $k$. We will distinguish between material nodes (free and surface contact nodes) and non-material (line contact) nodes. As already stated in chapter 3, if node $k$ is a combined surface-line contact node, row and column $k$ do not exist in the resulting mass matrix and the angle $\psi_k$ has already been chosen as equal to $\frac{\pi}{2} - \phi_k$ (see equation 2.107).
B.1 Material nodes

Using the results of chapter 2 (see equations 2.46, 2.50, 2.157 and 2.160), equation 3.45 can be written as follows for a material node $k$:

$$
\frac{\cos(\phi_k + \psi_k)}{\sin(\phi_k + \psi_k)} \left( m_k + D_k d_{L,k} \right) = \sin(\psi_k) \left( \cos(\psi_k) - \sin(\psi_k) \frac{\cos(\phi_k + \psi_k)}{\sin(\phi_k + \psi_k)} \right) C^{\nu}_{nk} \tag{B.1}
$$

Now, we will further assume that $\sin(\phi_k + \psi_k) \neq 0$, which is reasonable because $\sin(\phi_k + \psi_k) = 0$ means that the generalised coordinate $x_{L,k}^1$ is chosen in the direction of $\bar{r}_k$. In this case, the position of node $k$ would remain undetermined after choosing the coordinate $x_{L,k}^1$.

When assuming $\sin(\phi_k + \psi_k) \neq 0$ and using:

$$
\cos(\phi_k + \psi_k) = \cos(\phi_k) \cos(\psi_k) - \sin(\phi_k) \sin(\psi_k)
$$

$$
\sin(\phi_k + \psi_k) = \sin(\phi_k) \cos(\psi_k) + \cos(\phi_k) \sin(\psi_k)
$$

After multiplying by $\sin(\phi_k + \psi_k)$, we find the following from equation B.1:

$$
\cos(\phi_k) \cos(\psi_k) \left( m_k + D_k d_{L,k} \right) = \sin(\psi_k) \sin(\phi_k) \left( m_k + D_k d_{L,k} \right) + C^{\nu}_{nk} \tag{B.2}
$$

From equation B.2, we find:

$$
\tan(\psi_k) = \frac{\cos(\phi_k) \left( m_k + D_k d_{L,k} \right)}{\sin(\phi_k) \left( m_k + D_k d_{L,k} \right) + C^{\nu}_{nk}} \tag{B.3}
$$

From equation B.4, we find:

$$
\psi_k = \tan^{-1} \left( \frac{\cos(\phi_k) \left( m_k + D_k d_{L,k} \right)}{\sin(\phi_k) \left( m_k + D_k d_{L,k} + C^{\nu}_{nk} \right)} \right) \tag{B.5}
$$
B.2 Non-material nodes

Using the results of chapter 2 (see equations 2.50, 2.97, 2.103, 2.104, 2.157 and 2.174), and again using \( \sin (\phi_k + \psi_k) \neq 0 \), equation 3.46 can be written as follows for a line contact node \( k \):

\[
\cos (\phi_k + \psi_k) \left( m_k + D_d k_L k_k + A_{k+}^{13} \left( m_{k+} + D_d k_{L+}^{d,1} \right) A_{k+}^{13} + A_{k+}^{33} \left( m_{k+} + D_d k_{L+}^{d,3} \right) A_{k+}^{33} \right) - \sin (\phi_k + \psi_k)
\]

\[
\left( A_{k+}^{13} \left( m_{k+} + D_d k_{L+}^{d,1} \right) A_{k+}^{11} + A_{k+}^{33} \left( m_{k+} + D_d k_{L+}^{d,3} \right) A_{k+}^{31} \right) =
\]

\[
A_{k+}^{33} \left( \sin (\phi_k + \psi_k) A_{k+}^{31} - \cos (\phi_k + \psi_k) A_{k+}^{33} \right) C_{nk}^k
\]

Using equation A.13, we find the following:

\[
\cos (\phi_k + \psi_k) A_{k+}^{33} - \sin (\phi_k + \psi_k) A_{k+}^{31} = \frac{\cos (\phi_k - \kappa_k) - \cos (\kappa_k)}{\sin (\phi_k - \kappa_k)}
\]

and:

\[
\cos (\phi_k + \psi_k) A_{k+}^{13} - \sin (\phi_k + \psi_k) A_{k+}^{11} = \frac{\sin (\kappa_k)}{\sin (\phi_k - \kappa_k)}
\]

Now we define:

\[
K = \frac{\cos (\phi_k - \kappa_k) - \cos (\kappa_k)}{\sin (\phi_k - \kappa_k)}
\]

Using equations B.7, B.8 and B.9, we find the following from equation B.6:

\[
\cos (\phi_k + \psi_k) \left( m_k + D_d k_L k_k + A_{k+}^{13} \left( m_{k+} + D_d k_{L+}^{d,1} \right) \frac{\sin (\kappa_k)}{\sin (\phi_k - \kappa_k)} \right) + A_{k+}^{33} \left( m_{k+} + D_d k_{L+}^{d,3} \right) K = -A_{k+}^{33} K C_{nk}^k
\]

Furthermore, the following well-known formulas will be used:

\[
\cos (\psi_k + \kappa_k) = \cos (\phi_k + \psi_k) \cos (\phi_k - \kappa_k) + \sin (\phi_k + \psi_k) \sin (\phi_k - \kappa_k)
\]

\[
\sin (\psi_k + \kappa_k) = \sin (\phi_k + \psi_k) \cos (\phi_k - \kappa_k) - \cos (\phi_k + \psi_k) \sin (\phi_k - \kappa_k)
\]
Using equations A.13 and A.28, the terms containing \((\phi_k + \psi_k)\) will be split off from the various coefficients \(A_{k+}^{ij}\) as follows:

\[
A_{k+}^{11} = -\sin(\phi_k + \psi_k) \frac{\sin(\kappa_k)}{\sin(\phi_k - \kappa_k)} \\
A_{k+}^{13} = \cos(\phi_k + \psi_k) \frac{\sin(\kappa_k)}{\sin(\phi_k - \kappa_k)} \\
A_{k+}^{33} = \cos(\phi_k + \psi_k) K + \sin(\phi_k + \psi_k) \\
A_{k+}^{31} = \sin(\phi_k + \psi_k) \left( H_k^a \frac{\cos(\phi_k - \kappa_k)}{\sin(\phi_k - \kappa_k)} K - H_k^c \right) - \cos(\phi_k + \psi_k) H_k^a K \\
A_{k+}^{33} = -\cos(\phi_k + \psi_k) \left( H_k^a \frac{\cos(\phi_k - \kappa_k)}{\sin(\phi_k - \kappa_k)} K - H_k^c \right) - \sin(\phi_k + \psi_k) H_k^a K
\]

Equation B.10 then becomes:

\[
\cos(\phi_k + \psi_k) \left( m_k + D_k L_k^d,3 + \left( m_{k+} + D_k L_k^d,1 \right) \frac{\sin^2(\kappa_k)}{\sin^2(\phi_k - \kappa_k)} \right) \\
+ \cos(\phi_k + \psi_k) K^2 \left( m_{k+} + D_k L_k^d,3 + C_k^{\text{mk}} \right) \\
+ \sin(\phi_k + \psi_k) K \left( m_{k+} + D_k L_k^d,1 + C_k^{\text{mk}} \right) = 0
\]

From equation B.11, we find:

\[
\tan(\phi_k + \psi_k) = \frac{C^b}{C^a} 
\]

With:

\[
C^a = K \left( m_{k+} + D_k L_k^d,1 + C_k^{\text{mk}} \right) \\
C^b = m_k + D_k L_k^d,3 + \left( m_{k+} + D_k L_k^d,1 \right) \frac{\sin^2(\kappa_k)}{\sin^2(\phi_k - \kappa_k)} \\
+ K^2 \left( m_{k+} + D_k L_k^d,3 \right) + C_k^{\text{mk}} 
\]

From equation B.12, we find:

\[
\psi_k = \tan^{-1} \left( -\frac{C^b}{C^a} \right) - \phi_k
\]
Appendix C

Some calculations used in the detection of line contact

This appendix describes the calculation of the distances $D_{\text{cur}}$ to a line segment and $D_{\text{end}}$ to the end of a line segment of a point of intersection of a wire with a defining plane as well as the calculation of the point of intersection of a wire with the normal plane. These distances and points of intersection are used in the contact check as described in sections 5.3.2 and 5.3.3.

C.1 Calculations for circular line segments

C.1.1 Definition of circular line segments

In the chosen method of modelling, a circular line segment is defined by its defining plane and the following parameters, shown in Figure C.1:

- $\mathbf{x}_{\text{cen}}$: position vector of the centre of the circle segment
- $\mathbf{n}_{\text{cen}}$: the vector from the centre to the middle of the circle segment
- $r$: the radius of the circle segment as well as the length of the vector $\mathbf{n}_{\text{cen}}$
- $C_{\alpha}^{\text{seg}}$: the half angle of the circle segment

Here, the centre $\mathbf{x}_{\text{cen}}$ must lie in the surface $S_i^{\text{def}}$. If the material side is located on the outside of the circle, then the radius is chosen to be negative. A negative radius therefore indicates that the surface is concave instead of convex.
C.1.2 Calculation of the distances for circular line segments

Before calculating the distances $D_{cur}$ and $D_{end}$, the normalised vector $\overline{n}_{cs}$ is defined as the vector from the centre pointing to the point of intersection between the wire and the defining plane:

$$\overline{n}_{cs} = \frac{\overline{x}^s - \overline{x}_{cen}}{|| (\overline{x}^s - \overline{x}_{cen}) ||} \quad (C.1)$$

The distance $D_{cur}$ of the point of intersection to the circular line segment is found by:

$$D_{cur} = || (\overline{x}^s - \overline{x}_{cen}) || - |r| \quad (C.2)$$

The distance $D_{end}$ of the point of intersection to the end of the circular line segment is defined by (negative when within the endpoints):

$$D_{end} = (C_{cc} \alpha - C_{seg} \alpha) (D_{cur} + |r|) \quad (C.3)$$

In equation C.3, the value $C_{cc} \alpha$ is the angle between the vectors $\overline{n}_{cen}$ and $\overline{n}_{cs}$:

$$C_{cc} \alpha = \cos^{-1} \left( \frac{\overline{n}_{cs} \cdot \overline{n}_{cen}}{|r|} \right) \quad (C.4)$$
C.1.3 Calculation of the point of intersection of the wire with the normal plane of circular line segments

The point of intersection of the wire between nodes \(i\) and \(i+1\) with the normal plane of a circular line is calculated by first calculating the projections of the nodes \(i\) and \(i+1\) on the defining plane. The position vectors \(\vec{x}_{i}^{\text{pro}}\) and \(\vec{x}_{i+1}^{\text{pro}}\) of these projections of the nodes \(i\) and \(i+1\), respectively, can be calculated by:

\[
\vec{x}_{i}^{\text{pro}} = \vec{x}_{i} - \vec{n}_{j}^{\text{def}} \cdot S_{j}^{\text{def}}(\vec{x}_{i})
\]
\[
\vec{x}_{i+1}^{\text{pro}} = \vec{x}_{i+1} - \vec{n}_{j}^{\text{def}} \cdot S_{j}^{\text{def}}(\vec{x}_{i+1})
\]

The projected wire can then be represented by:

\[
\vec{x}^{\text{pro}} = \vec{x}_{i}^{\text{pro}} + \lambda (\vec{x}_{i+1}^{\text{pro}} - \vec{x}_{i}^{\text{pro}})
\]

The position vector of the possible intersection points of the projected line with the circular line segment can be found from expression C.6 using the solution(s) \(\lambda_{\text{sol}}\) of the following equation:

\[
\| (\vec{x}_{i}^{\text{pro}} + \lambda (\vec{x}_{i+1}^{\text{pro}} - \vec{x}_{i}^{\text{pro}}) - \vec{x}_{\text{cen}}) \| = r^{2}
\]

One or two point(s) of intersection of the wire between nodes \(i\) and \(i+1\) is (are) found if equation C.7 has one or two solutions \(\lambda_{\text{sol}}\) respectively, lying in the interval \(0 \leq \lambda_{\text{sol}} \leq 0\). The position vector of the point(s) of intersection of the projected wire with the line is (are) then given by:

\[
\vec{x}^{\text{pro}}_{\text{s}} = \vec{x}_{i}^{\text{pro}} + \lambda_{\text{sol}} (\vec{x}_{i+1}^{\text{pro}} - \vec{x}_{i}^{\text{pro}})
\]

The position vector of the point(s) of intersection of the wire with the normal plane is (are) then given by:

\[
\vec{x}^{\text{n}} = \vec{x}_{i} + \lambda_{\text{sol}} (\vec{x}_{i+1} - \vec{x}_{i})
\]

C.2 Calculations for straight line segments

C.2.1 Definition of straight line segments

Straight line segments are defined with parameters corresponding with the parameters defined for circular line segments. For straight line segments, a point in
the surface $S^\text{def}_j$ lying on the perpendicular bisector of the line, at a distance of the line equal to the half line length, functions as centre. Figure C.2 shows these parameters for a straight line.

![Diagram of line segment and parameters](image)

**Fig. C.2: Parameters describing a straight line segment**

The value 0 for the radius $r$ indicates that the line is straight. The parameter $C^{\text{seg}}_\alpha$ is now defined as the half line length. The position of the point that functions as centre is chosen to be on the material side of the surface.

### C.2.2 Calculation of the distances for straight line segments

Before calculating the distances $D_{\text{cur}}$ and $D_{\text{end}}$, the vector $\mathbf{n}_{\text{cs}}$ is calculated in a corresponding manner for a circular line segment, but is not normalised:

$$\mathbf{n}_{\text{cs}} = \mathbf{x}_s - \mathbf{x}_{\text{cen}} \quad \text{(C.10)}$$

Also, the normalised direction vector $\mathbf{n}_{\text{rv}}$ of the line segment is calculated by:

$$\mathbf{n}_{\text{rv}} = \frac{\mathbf{n}_{\text{cen}} \times \mathbf{n}_{\text{def}}^{\text{seg}}}{C^{\text{seg}}_\alpha} \quad \text{(C.11)}$$

The distance $D_{\text{cur}}$ of the point of intersection to the straight line segment is found by:

$$D_{\text{cur}} = (\mathbf{n}_{\text{cs}} \cdot \mathbf{n}_{\text{cen}}) - C^{\text{seg}}_\alpha \quad \text{(C.12)}$$

The distance $D_{\text{end}}$ of the point of intersection to the end of the straight line segment is defined by (negative when within the end points):
C.2 Calculations for straight line segments

\[ D_{cur}^{end} = \| (\vec{n}_{cs} \cdot \vec{n}_{rv}) \| - C_{seg} \]  \hspace{1cm} (C.13)

C.2.3 Calculation of the point of intersection of the wire with the normal plane of straight line segments

The point of intersection of the wire between nodes \( i \) and \( i+1 \) with the normal plane of a straight line is calculated directly by using the following expression that holds for every position vector \( \vec{x} \) lying in the normal plane of a straight line:

\[ \vec{x} \cdot \vec{n}_{cen} = \vec{x}_{cen} \cdot \vec{n}_{cen} \]  \hspace{1cm} (C.14)

Again, the wire between the nodes \( i \) and \( i + 1 \) can be represented by:

\[ \vec{x} = \vec{x}_i + \lambda (\vec{x}_{i+1} - \vec{x}_i) \]  \hspace{1cm} (C.15)

The position vector of the possible intersection point of the wire with the normal plane is then given by expression C.15, using the solution \( \lambda^{sol} \) of the following equation:

\[(\vec{x}_i + \lambda (\vec{x}_{i+1} - \vec{x}_i)) \cdot \vec{n}_{cen} = \vec{x}_{cen} \cdot \vec{n}_{cen} \]  \hspace{1cm} (C.16)

A point of intersection of the wire between nodes \( i \) and \( i + 1 \) is found if equation C.16 has a solution \( \lambda^{sol} \), lying in the interval \( 0 \leq \lambda^{sol} \leq 0 \). The position vector of the point of intersection of the wire with the normal plane is then given by:

\[ \vec{x} = \vec{x}_i + \lambda^{sol} (\vec{x}_{i+1} - \vec{x}_i) \]  \hspace{1cm} (C.17)
SOME CALCULATIONS USED IN THE DETECTION OF LINE CONTACT
Appendix D

Calculation of the friction forces for line contact

This appendix describes the calculation of the friction forces $F_{\text{circ}}^{\text{ric}}$ and $F_{\text{axi}}^{\text{ric}}$ as defined in chapter 6. For this calculation, equations 6.9 and 6.11 are used. Equation 6.9 can be written as:

$$T_{i+1} \cos (\beta^+) = T_i \cos (\beta^-) \exp^{\text{fric} \alpha_{i}^{\text{lin}}} \cos(\alpha_{\text{slip}}) - C_1 A_i^{cc} \quad (D.1)$$

Where:

$$C_1 = \sin (\beta_{\text{lon}}) \left( m_{i+} + m_{i-} \exp^{\text{fric} \alpha_{i}^{\text{lin}}} \cos(\alpha_{\text{slip}}) \right) \quad (D.2)$$

Equation 6.11 can be written as:

$$T_{i+1} \sin (\beta^+) = T_i \sin (\beta^-) + \cos (\beta^-) C_0 + C_2 A_i^{ce} \quad (D.3)$$

Where:

$$C_2 = (m_{i-} + m_{i+}) \cos (\beta_{\text{lon}}) - m_{i-} \sin (\beta_{\text{lon}}) C_0 \quad (D.4)$$

And, if $\alpha_{\text{slip}} \neq \pm \frac{\pi}{2}$, then the following is valid for $C_0$:

$$C_0 = \tan (\alpha_{\text{slip}}) \left( \exp^{\text{fric} \alpha_{i}^{\text{lin}}} \cos(\alpha_{\text{slip}}) - 1 \right) \quad (D.5)$$

If $\alpha_{\text{slip}} = \pm \frac{\pi}{2}$, then the following is valid for $C_0$:

$$C_0 = \sin (\alpha_{\text{slip}}) f_{\text{lin}}^{\text{ric} \alpha_{i}^{\text{lin}}} \quad (D.6)$$
Elimination of $A_{i}^{cc}$ from equations D.1 and D.3 gives:

$$T_{i+1} = T_{i} \frac{C_{4}}{C_{3}}$$  \hspace{1cm} (D.7)

Where:

$$C_{3} = C_{1} \sin (\beta^{+}) + C_{2} \cos (\beta^{+})$$  \hspace{1cm} (D.8)

and:

$$C_{4} = C_{1} (\sin (\beta^{-}) + \cos (\beta^{-}) C_{0}) + C_{2} \cos (\beta^{-}) \exp^{f_{\text{fric}}^{\text{lin}} \alpha_{i}^{\text{lin}} \cos(\alpha_{\text{slip}})}$$  \hspace{1cm} (D.9)

If $C_{3} = 0$ and $T_{i} \neq 0$, then equations 6.9 and 6.11 conflict with each other. We will assume that this cannot occur in practical situations. If $| C_{1} | \geq | C_{2} |$, we use equations D.1 and D.7 to calculate $A_{i}^{cc}$. If $| C_{1} | < | C_{2} |$, we use equations D.3 and D.7.

In this way, the calculation of $A_{i}^{cc}$ remains valid, even if $C_{1}$ or $C_{2}$ is equal to zero. The case that both $C_{1}$ and $C_{2}$ are equal to zero can be excluded because $C_{1}$ is only equal to zero for $\sin (\beta^{\text{lon}}) = 0$. In that case, we have:

$$C_{2} = (m_{i-} + m_{i+}) \cos (\beta^{\text{lon}}) \neq 0$$  \hspace{1cm} (D.10)

We further define:

$$A_{i}^{cc} = T_{i} \frac{C_{5}}{C_{6}}$$  \hspace{1cm} (D.11)

For $| C_{1} | \geq | C_{2} |$, we find the following using equations D.1 and D.7:

$$C_{5} = - \frac{C_{4}}{C_{3}} \cos (\beta^{+}) + \cos (\beta^{-}) \exp^{f_{\text{fric}}^{\text{lin}} \alpha_{i}^{\text{lin}} \cos(\alpha_{\text{slip}})}$$  \hspace{1cm} (D.12)
and:

\[ C_6 = C_1 \]  
(D.13)

For \(| C_2 | > | C_1 |\), we find using equations D.3 and D.7:

\[ C_5 = \frac{C_4}{C_3} \sin (\beta^+) - \sin (\beta^-) - \cos (\beta^-) C_0 \]  
(D.14)

and:

\[ C_6 = C_2 \]  
(D.15)

Using equations 6.12 and 6.13, we find the following results for the components \( F^\text{ric}_{\text{cir}} \) and \( F^\text{ric}_{\text{axi}} \):

\[ F^\text{ric}_{\text{cir}} = T_i \left( \cos (\beta^-) - \cos (\beta^+) \frac{C_4}{C_3} - (m_i_- + m_i^+) \sin (\beta_{\text{lon}}) \frac{C_5}{C_6} \right) \]  
(D.16)

and:

\[ F^\text{ric}_{\text{axi}} = T_i \left( \sin (\beta^-) - \sin (\beta^+) \frac{C_4}{C_3} + (m_i_- + m_i^+) \cos (\beta_{\text{lon}}) \frac{C_5}{C_6} \right) \]  
(D.17)
Bibliography


[10] Koster M.P., *Constructieprincipes - voor het nauwkeurig bewegen en posi-
tioneren, (in Dutch) College lecture notes, Technical University Twente - Department of Mechanical Engineering (1994)


List of symbols

Notation conventions

The following notation conventions have been used:

- \( \mathbf{x} \): column vector
- \( \mathbf{x}^T \): transposed column vector \( \mathbf{x} \)
- \( \mathbf{G} \): matrix
- \( \mathbf{D} \mathbf{M} \): diagonal matrix with its diagonal terms
- \( \mathbf{G}^T \): transposed of matrix \( \mathbf{G} \)
- \( \mathbf{G}^{-1} \): inverse of matrix \( \mathbf{G} \)
- \( |x| \): absolute value of \( x \)
- \( ||\mathbf{x}||\): length of vector \( \mathbf{x} \)
- \( \mathbf{x}^1 \cdot \mathbf{x}^2 \): scalar product of \( \mathbf{x}^1 \) and \( \mathbf{x}^2 \)
- \( \mathbf{x}^1 \times \mathbf{x}^2 \): vector product of \( \mathbf{x}^1 \) and \( \mathbf{x}^2 \)
- \( D \mathcal{F}^x \): derivative of function \( \mathcal{F}^x \) to its argument
- \( D^2 \mathcal{F}^x \): second derivative of function \( \mathcal{F}^x \) to its argument
- \( D^2 \mathcal{F}^x \): second derivative of function \( \mathcal{F}^x \) to its argument
- \( D_{ei} \mathcal{F}^x \): derivative of function \( \mathcal{F}^x \) to \( e_i \)
- \( \sum_{i=1}^{I} ()_i \): summation over \( i \) from 1 up to \( I \)
- \( \prod_{i=1}^{I} ()_i \): multiplication over \( i \) from 1 up to \( I \)
- \( \dot{x} \): (first) time derivative of \( x \)
- \( \ddot{x} \): second time derivative of \( x \)
- \( \delta x \): variation of vector \( x \)
- \( \Delta x \): incremental change in \( x \)
- \( x^t \): value of \( x \) at time \( t \)
- \( \int_t () \, dt \): time integral
- \( \ln () \): natural logarithm
- \( \text{Sign} (x) \): sign of \( x \) ( +1 for \( x \geq 0 \) and -1 for \( x < 0 \))
- \( \mathcal{O} () \): order of error
List of Symbols

Greek symbols

$\alpha$              wrapped angle of wire on support
$\alpha_s$            angle between wire and the normal $\vec{n}_j^{\text{def}}$ of the defining plane
$\alpha_{\text{slip}}$ slipping angle
$\alpha_{i_{\text{lin}}}$ part of wrapped angle $\alpha_{i_{\text{lin}}}$ in line contact node $i$
$\alpha_{i_{\text{lin}}}^+$ part of wrapped angle $\alpha_{i_{\text{lin}}}$ in line contact node $i$
$\alpha_{i_{\text{lin}}}$ wrapped angle in line contact node $i$
$\alpha_{i_{\text{sur}}}$ part of wrapped angle in surface contact node $i$
$\alpha_{i_{\text{sur}}}^+$ part of wrapped angle in surface contact node $i$
$A_{i_{(+)}}^{kl}$ coefficients in relation between $\dot{x}_L^k$ and $\dot{x}_L^l$
$\alpha_{j_{\text{fly}}}$ wrapped angle of wire on flyer wheel $j$
$\beta, \beta_{i-}, \beta_{i+}$ angles in line contact node $i$
$\beta_{i_{\text{on}}}$ average angle of $\beta^-$ and $\beta^+$
$\gamma_i$ twist angle of the wire between node $i-1$ and $i$
$\delta_k$ kronecker delta
$\kappa_i$ angle in line contact node $i$
$\lambda_i$ friction factor in node $i$
$\lambda_j$ friction factor for flyer wheel $j$
$\lambda_{j_{\text{fly}}}^*$ adapted friction factor for flyer wheel $j$
$\xi_i$ angle of $\vec{g}_{i_{(+)}}^2$ and $\vec{r}_{i+1}$
$\rho$ specific mass
$\sigma_{\text{max}}$ maximum stress
$\sigma_y$ yield stress
$\phi_i$ wire angle in node $i$
$\psi_i$ angle defining local directions in node $i$
$\psi_{i+}$ special definition of $\psi_i$ in line contact node $i$
$\omega$ twist angle per length
$\varphi_i$ bending angle in node $i$
$\dot{\varphi}_i$ bending rate in node $i$
$\ddot{\varphi}_i$ bending rate in node $i$
$\theta$ angle used in friction description of line contact
Roman symbols

\( \ddot{a}_i \) acceleration vector of line contact node \( i \)
\( A \) cross-sectional area
\( A_{i,c} \) friction acceleration in line contact node \( i \)
\( A_{i,tra}^{cc,t} \) acceleration in the transversal direction
\( A_{i,tra}^{cc,prev} \) acceleration in the transversal direction in the previous iteration
\( A_{i,ion}^{cc,t} \) acceleration in the longitudinal direction
\( A_{i,ion}^{cc,prev} \) acceleration in the longit. direction in the previous iteration
\( c_j \) constant belonging to defining plane \( j \)
\( C_p \) constant used for calculation of wire tension
\( C_{i}^{d} \) first order acceleration coefficient of node \( i \) in eqs. of motion
\( C_{i}^{d2} \) second order acceleration coefficient of node \( i \) in eqs. of motion
\( C_{i}^{f} \) force coefficient of node \( i \) in equations of motion
\( C_{i}^{in} \) inertia plus damping coefficient of node \( i \) in equations of motion
\( C_{i}^{mk} \) mass coefficient of node \( i \) in equations of motion
\( C_{i}^{mk/2} \) \( \dot{C}_{i}^{mk}/2 \)
\( C_{i}^{trans} \) transversal stiffness in node \( i \)
\( c_i^b \) bending stiffness in node \( i \)
\( C_{kl}^{curv} \) curvature of the surface
\( C_{lin}^{curv} \) curvature of the contact line
\( C_{i}^{v} \) velocity coefficient of node \( i \) in equations of motion
\( C_{w} \) stiffness of the elastic spring in the flyer model
\( d \) diameter of the wire
\( D_{i}^{d} \) dependency of current velocity on acceleration
\( D_{i}^{f} \) dependency of current displacement on acceleration
\( D_{i,cur}^{sur,old} \) current/old distance to contact line
\( D_{i,cur}^{sur,old} \) current/old distance to end of contact line
\( D_{i,cur}^{int} \) interpolated distance to end of contact line
\( D_{i,n,cur}^{sur,old} \) current/old distance to contact line in normal plane
\( D_{i,n,cur}^{sur,old} \) current/old distance to end of contact line in normal plane
\( D_{i,n,cur}^{int} \) interpolated distance to end of contact line in normal plane
\( \varepsilon_i \)  
elongation of the wire between node \( i - 1 \) and \( i \)

\( \varepsilon^p \)  
plastic elongation in the plastic element

\( E \)  
modulus of elasticity

\( EI \)  
bending stiffness

\( f_{\text{lin}}^{\text{fric}} \)  
coefficient of friction for line contact

\( f_{\text{long}}^{\text{fric}} \)  
coefficient of friction in longitudinal direction

\( f_{\text{trans}}^{\text{fric}} \)  
coefficient of friction in transversal direction

\( f_{\text{sur}}^{\text{fric}} \)  
coefficient of friction for surface contact

\( f_{\text{w}}^{\text{fric}} \)  
coefficient of friction between flyer wheels and wire

\( f_{\text{fric}}^{\text{fric},ij} \)  
friction force component in node \( i \)

\( f_{\text{fric}}^{\phi i,j} \)  
external force component in node \( i \)

\( f_{\text{fric}}^{\phi i^{\text{w}},j} \)  
component of external force without friction in node \( i \)

\( f_{\text{fric}}^{\phi i,j} \)  
function of the generalised coordinates

\( f_{\text{fric}}^{\phi i} \)  
function of the generalised coordinates

\( f_{\text{fric}}^{\phi x,m} \)  
subfunction of the generalised coordinates

\( f_{\text{fric}}^{\phi x,0} \)  
subfunction of the generalised coordinates

\( f_{\text{fric}}^{\phi x,0,1} \)  
subfunction of the generalised coordinates

\( f_{\text{fric}}^{\phi x,c} \)  
subfunction of the generalised coordinates

\( f_{\text{fric}}^{\phi x,c,1}, f_{\text{fric}}^{\phi x,c,2} \)  
surface function of the generalised coordinates

\( f_{\text{fric}}^{\phi i,j} \)  
resulting force for coordinate \( j \) of node \( i \)

\( f_{\text{fric}}^{\phi l} \)  
external forces

\( f_{\text{fric}}^{\phi l} \)  
force vector in node \( j \) due to bending stiffness in node \( i \)

\( f_{\text{fric}}^{\phi l} \)  
perp. force vector in node \( j \) due to bending stiffness in node \( i \)

\( f_{\text{conv}}^{\phi l}(i-1) \)  
convective force acting on node \( i - 1 \)

\( f_{\text{conv}}^{\phi l}(i) \)  
convective force acting on node \( i \)

\( f_{\text{conv}}^{\phi l}(i+) \)  
convective force acting on node \( i + 1 \)

\( f_{\text{conv}}^{\phi l}(i+1) \)  
convective force acting on node \( i + 1 \)

\( f_{\text{fric}}^{\phi l} \)  
velocity dependent friction forces

\( f_{\text{fric}}^{\phi l} \)  
second order inertia plus damping forces
\( f_{L,0}^{in} \)  
subforce of \( f_{L}^{in} \)

\( f_{L,c}^{in} \)  
subforce of \( f_{L}^{in} \)

\( f_{L,1}^{in} \)  
subforce of \( f_{L}^{in} \)

\( f_{L,2}^{in} \)  
subforce of \( f_{L}^{in} \)

\( f_{d} \)  
damping forces

\( f_{vyd} \)  
velocity dependent external forces

\( f_{vi} \)  
velocity independent external forces

\( f_{vi,0} \)  
subforce of \( f_{vi}^{L} \)

\( f_{vi,c} \)  
subforce of \( f_{vi}^{L} \)

\( f_{vi,1} \)  
subforce of \( f_{vi}^{L} \)

\( f_{vi,2} \)  
subforce of \( f_{vi}^{L} \)

\( F_{ric}^{cir} \)  
friction force in circumferential direction

\( F_{ric}^{azi} \)  
friction force in axial direction

\( F_{ric}^{lon} \)  
friction force in the longitudinal direction

\( F_{lon}^{ric,act} \)  
actual friction force in the longitudinal direction

\( F_{tra}^{ric,act} \)  
actual friction force in the perpendicular direction

\( F_{surlin}^{ric} \)  
friction force for combined surface-line contact

\( f_{j}^{w} \)  
friction force between wire and flyer wheel \( j \)

\( F_{res}^{w} \)  
resulting force in flyer model

\( \vec{g}_{i} \)  
local base vector in node \( i \)

\( G \)  
shear modulus

\( GI \)  
torsional stiffness

\( G_{i} \)  
transformation matrix in node \( i \)

\( \vec{h}_{i}^{1} \)  
intermediate vector

\( \vec{h}_{i+}^{1} \)  
intermediate vector

\( H_{i}^{a} \)  
rotational velocity

\( H_{i}^{b} \)  
rotational velocity

\( i \)  
nodal point number

\( i-,i+ \)  
nodal point numbers in line contact node \( i \)
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<th>Description</th>
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<td>( I )</td>
<td>moment of inertia</td>
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<tr>
<td>( I^w_j )</td>
<td>moment of inertia of flyer wheel ( j )</td>
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<tr>
<td>( \bar{k}_i )</td>
<td>unit vector perpendicular to wire in node ( i )</td>
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<tr>
<td>( k_{Li}^{d,j} )</td>
<td>damping coefficients in node ( i )</td>
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<td>( k_{Li}^{f,j} )</td>
<td>friction damping coefficients in node ( i )</td>
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<tr>
<td>( k_{Li}^{d,j} )</td>
<td>damping coefficients in node ( i+ )</td>
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<tr>
<td>( k_i^{f} )</td>
<td>friction damping coefficient</td>
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<tr>
<td>( k_i^{b} )</td>
<td>bending damping in node ( i )</td>
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<tr>
<td>( k_{air} )</td>
<td>damping coefficient of air dereeler</td>
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<tr>
<td>( k^w )</td>
<td>material damping coefficient of wire in flyer</td>
</tr>
<tr>
<td>( k^{res} )</td>
<td>resulting damping coefficient in flyer</td>
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<tr>
<td>( K^d_L )</td>
<td>damping matrix</td>
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<tr>
<td>( K^d_{L0,0} )</td>
<td>submatrix of ( K^d_L )</td>
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<tr>
<td>( K^d_{Lc,c} )</td>
<td>submatrix of ( K^d_L )</td>
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<td>( K^d_{L1,1} )</td>
<td>submatrix of ( K^d_L )</td>
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<td>( K^d_{L2,2} )</td>
<td>submatrix of ( K^d_L )</td>
</tr>
<tr>
<td>( K^d_{Li} )</td>
<td>nodal damping matrix</td>
</tr>
<tr>
<td>( K^{fr}_i )</td>
<td>friction damping</td>
</tr>
<tr>
<td>( K^f_L )</td>
<td>friction damping matrix</td>
</tr>
<tr>
<td>( K^f_{Lc,c} )</td>
<td>submatrix of ( K^f_L )</td>
</tr>
<tr>
<td>( K^f_{L1,1} )</td>
<td>submatrix of ( K^f_L )</td>
</tr>
<tr>
<td>( K^{fr}_{Li} )</td>
<td>nodal friction damping matrix</td>
</tr>
<tr>
<td>( l_i )</td>
<td>wire coordinate of node ( i )</td>
</tr>
<tr>
<td>( l^A_i )</td>
<td>nodal distance between node ( i - 1 ) and ( i )</td>
</tr>
<tr>
<td>( L_j^w )</td>
<td>length of wire in flyer between wheel ( j ) and ( j + 1 )</td>
</tr>
<tr>
<td>( L^s )</td>
<td>distance between outer supports in three-point support</td>
</tr>
<tr>
<td>( L_{tot}^w )</td>
<td>total length of the wire in the flyer</td>
</tr>
<tr>
<td>( m_i )</td>
<td>mass of node ( i )</td>
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<tr>
<td>( m_{i-} )</td>
<td>mass of node ( i - ) in line contact node ( i )</td>
</tr>
<tr>
<td>( m_{i+} )</td>
<td>mass of node ( i+ ) in line contact node ( i )</td>
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<tr>
<td>( m_i^b )</td>
<td>bending mass in node ( i )</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>( m_{k_i} )</td>
<td>sum of mass and damping coefficients in node ( i ) or ( i^- )</td>
</tr>
<tr>
<td>( m_{k_{i+}} )</td>
<td>sum of mass and damping coefficients in node ( i+ )</td>
</tr>
<tr>
<td>( m_j^w )</td>
<td>mass of wire in flyer between wheel ( j ) and ( j+1 )</td>
</tr>
<tr>
<td>( m_j^{w+} )</td>
<td>adapted mass of wire in flyer between wheel ( j ) and ( j+1 )</td>
</tr>
<tr>
<td>( M^b )</td>
<td>bending moment</td>
</tr>
<tr>
<td>( M^t )</td>
<td>twisting moment</td>
</tr>
<tr>
<td>( M_i^j )</td>
<td>resulting mass matrix for coordinate ( j ) of node ( i )</td>
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<tr>
<td>( M_L )</td>
<td>mass matrix</td>
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<tr>
<td>( M_L^{0,0} )</td>
<td>submatrix of ( M_L )</td>
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<tr>
<td>( M_L^{c,c} )</td>
<td>submatrix of ( M_L )</td>
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<tr>
<td>( M_L^{1,1} )</td>
<td>submatrix of ( M_L )</td>
</tr>
<tr>
<td>( M_L^{2,2} )</td>
<td>submatrix of ( M_L )</td>
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<tr>
<td>( M_{Li} )</td>
<td>nodal mass matrix</td>
</tr>
<tr>
<td>( M_{res}^w )</td>
<td>resulting mass in the flyer model</td>
</tr>
<tr>
<td>( n )</td>
<td>number of nodes, flyer node number</td>
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<tr>
<td>( \vec{n}<em>{j</em>{\text{def}}} )</td>
<td>normal vector belonging to defining plane ( j )</td>
</tr>
<tr>
<td>( \vec{n}_i )</td>
<td>outward normal vector in line contact node ( i )</td>
</tr>
<tr>
<td>( P_{\text{norm}} )</td>
<td>normal force per wire length in line contact</td>
</tr>
<tr>
<td>( r_j^w )</td>
<td>radius of flyer wheel ( j )</td>
</tr>
<tr>
<td>( \vec{r}_i )</td>
<td>unit direction vector from node ( i-1 ) to ( i )</td>
</tr>
<tr>
<td>( \vec{r}_i^x )</td>
<td>projection vector of ( \vec{r}_i )</td>
</tr>
<tr>
<td>( \vec{r}<em>{i</em>{\text{lon}}} )</td>
<td>unit direction vector in the longitudinal direction</td>
</tr>
<tr>
<td>( \vec{r}<em>{i</em>{\text{tra}}} )</td>
<td>unit direction vector in the transversal direction</td>
</tr>
<tr>
<td>( R )</td>
<td>radius of curvature</td>
</tr>
<tr>
<td>( R_{cyl} )</td>
<td>radius of cylinder in line contact</td>
</tr>
<tr>
<td>( R_i^3 )</td>
<td>receiving coefficient of node ( i )</td>
</tr>
<tr>
<td>( s_{j_{\text{def}}} )</td>
<td>defining plane ( j )</td>
</tr>
<tr>
<td>( S_i^j )</td>
<td>sending coefficient of node ( i )</td>
</tr>
<tr>
<td>( S_{i_{3j}} )</td>
<td>sending and receiving coefficient of node ( i )</td>
</tr>
<tr>
<td>( \Delta s_{lip}^{\alpha} )</td>
<td>slip in circumferential direction</td>
</tr>
<tr>
<td>( \Delta s_{lip}^{\alpha} )</td>
<td>slip in axial direction</td>
</tr>
</tbody>
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LIST OF SYMBOLS

\[ \Delta S_{\text{lon}}^{\text{slip}} \] slip in longitudinal direction
\[ \Delta S_{\text{lon}}^{\text{slip,prev}} \] slip in longitudinal direction in the previous iteration
\[ \Delta S_{\text{tras}}^{\text{slip}} \] slip in transversal direction
\[ \Delta S_{\text{tras}}^{\text{slip,prev}} \] slip in transversal direction in the previous iteration
\[ \Delta S_{\text{tot}}^{\text{slip}} \] total slip
\[ S_{\text{stiff}} \] stiffness of the wire used in sticking node
\[ t_i \] coordinate in the direction of \( \vec{t}_i \)
\[ \vec{t}_i \] unit direction vector of contact line in node \( i \)
\[ \Delta t \] time step
\[ \Delta t_c \] critical time step
\[ \Delta t_c' \] critical time step for the flyer model
\[ \Delta t_{\text{trans}} \] transversal critical time step
\[ \Delta t_{\text{long}} \] longitudinal critical time step
\[ \Delta t_{\text{prac}} \] practical value for critical time step
\[ \Delta t_b \] critical time step due to bending stiffness
\[ T_{\text{min}} \] smallest time period of the system
\[ T_i \] transmission coefficient of node \( i \)
\[ T_{(i-k)}(i) \] multiplication of transmission coefficients
\[ T_i \] tensile force between nodes \( i-1 \) and \( i \)
\[ T_{\theta} \] tensile force as a function of angle \( \theta \)
\[ T_{\text{stick}} \] tensile force in case of sticking node \( i \)
\[ T_{ji}^w \] tensile force in wire between flyer wheel \( j \) and mass \( m_{j-1}^w \)
\[ T^v \] maximum elastic tensile force
\[ \dot{v}_i \] velocity vector of line contact node \( i \)
\[ \dot{x}_{\text{wh}} \] velocity of flyer wheel \( j \)
\[ \bar{x}_i \] position vector of node \( i \)
\[ \bar{x}^a \] position vector of intersection point with defining plane
\[ x_i \] global coordinates of node \( i \)
\[ x_{\text{Li}}^j \] local coordinate of node \( i \)
\[ x_{Li} \] local coordinates of node \( i \)
\[ x_{L}^m \] local generalised coordinates
\[ x_{L}^0 \] local prescribed coordinates
\[ x_{L}^e \] local dependent coordinates
<table>
<thead>
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<th>Description</th>
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<td>$x_{L}^{m,1}$</td>
<td>first generalised coordinates</td>
</tr>
<tr>
<td>$x_{L}^{m,2}$</td>
<td>second generalised coordinates</td>
</tr>
<tr>
<td>$x^w$</td>
<td>wire displacement in the flyer</td>
</tr>
<tr>
<td>$x^b$</td>
<td>wire displacement between the buckling and the plastic element</td>
</tr>
<tr>
<td>$x^p$</td>
<td>wire displacement between the plastic element and the spring</td>
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LIST OF SYMBOLS
Samenvatting

Het wikkelproces van deflectiespoelen is een belangrijk proces tijdens de productie van televisies. Tijdens dit proces wordt van een draad een deflectiespoel gewikkeld in een wikkelmal met behulp van een draaiend mechanisme. Omdat er behoefte bestaat om dit proces te simuleren, is een simulatiemodel (het SWING programma) ontwikkeld. Doel van het simuleren van het wikkelproces is zowel om de ontwikkeling van wikkelmallen te versnellen als om de ontwerpvaardigheden overdraagbaar te maken.

Het hoofdaandachtspunt tijdens de ontwikkeling van het simulatieprogramma was de snelle en nauwkeurige simulatie van de drie hoofdfactoren die van invloed zijn op het wikkelproces:

- De dynamica van de draad
- De beschrijving van het contact tussen de draad en de wikkelmal
- De beschrijving van de wrijving in het draad-mal contact

De theorie benodigd om de dynamica van de draad te beschrijven is deels gebaseerd op reeds bekende multibody dynamica en is verder uitgebreid om de contact- en wrijvingsrandvoorwaarden efficiënt te kunnen beschrijven. Dit is gerealiseerd door gebruik te maken van lokale absolute coordinaten hetgeen tot gevolg heeft dat er geen Lagrange multipliers benodigd zijn. Het wordt aangetoond dat een geschikte keuze van de richtingen van de lokale asenstelsels zelfs resulteert in een orde \( n \) methode. Op dit moment lijkt deze aanpak alleen mogelijk voor een mechanisme met dezelfde karakteristiek als de draad: het "enkelvoudige ketting" karakter.

Een algoritme ter beschrijving van het oppervlakcontact is ontwikkeld dat het mogelijk maakt om snel en nauwkeurig contact te detecteren tussen draad en wikkelmal inclusief smalle segmentvleugels. Voor dit doel wordt gebruik
gemaakt van vier knooppunt types. Twee van deze types zijn niet-materiële knooppunten, knooppunten zonder vaste positie ten opzichte van de draad. De oppervlak beschrijving kan worden gegenereerd binnen een CAD programma zoals, in ons geval, UNIGRAPHICS.

Een speciale wrijvingsbeschrijving is ontwikkeld. Deze is benodigd om de wrijving tussen de draad en de wikkelmal te simuleren. De speciale eigenschappen van dit contact, zoals bijvoorbeeld het grote verschil in draadstijfheid in de lengterichting en de richting loodrecht daarop en de grote omgeslagen hoek van de draad om sommige oppervlakken met een kleine kromtestraal, stellen speciale eisen aan deze wrijvingsbeschrijving.

Enige voorbeelden van de simulatieresultaten van het SWING programma worden gepresenteerd en de functionaliteit van het SWING programma wordt geverifieerd door resultaten te vergelijken met experimentele resultaten.
Stellingen

behorende bij het proefschrift

Simulation of
Deflection Coil Winding

Theory and verification of
SWING

van

Ruud Voncken

1. Indien in plaats van een diagonale massamatrix een consistente lokale massamatrix gekozen zou worden voor de wikkeldraad, zou dit vanwege de modellering van de draad met oneindig stijve schakels resulteren in een onrealistisch dynamisch buiggedrag (zie hoofdstuk 1 van dit proefschrift).

2. Het toestaan van slip in draadrichting voor een niet slippend knooppunt is bij de modellering van het wikkelproces een eenvoudig en toelaatbaar alternatief voor het toepassen van een Lagrange multiplier (zie hoofdstuk 6 van dit proefschrift).

3. Zowel een zeer lange als een zeer korte literatuurlijst wijzen op een grote inspanning van de auteur. In het eerste geval duidt dit op een grote leesinspanning, in het tweede geval op een grote onderzoeksinspanning.

4. Het zou het rechtsgevoel van de burger ten goede komen indien het aantal wetten gehalveerd zou worden en er met dubbele inspanning toegezien zou worden op naleving van de resterende wetten.

5. Om het cellentekort te verminderen dienen de celstraffen verlengd te worden.
6. Behalve dat simulaties door bedrijven steeds meer gebruikt zullen worden om de werkelijkheid te voorspellen (hoofdstuk 1 van dit proefschrift), zullen ze ook steeds meer gebruikt gaan worden om de werkelijkheid te vervangen (virtual reality). In de nabije toekomst zal dit het einde betekenen van het monopolie van het betaald voetbal op het leveren van commercieel interessante voetbalbeelden.

7. Politici streven als gevolg van de behoefte aan zelfprofilering teveel naar het ontwikkelen van nieuw beleid in plaats van het verbeteren van de uitvoering van bestaand beleid.

8. Promoveren naast een fulltime baan schaadt de lichamelijke, geestelijke en sociale gezondheid.

9. Impliciete problemen zijn moeilijker op te lossen dan expliciete. Het beste voorbeeld is het voorspellen van de toekomst. Ofschoon theoretisch mogelijk, is dit praktisch gezien onmogelijk, niet op de laatste plaats vanwege het feit dat de voorspelling zelf ook invloed heeft op de toekomst.

10. De best haalbare conclusie voor toegepast onderzoek is dat het bereikte resultaat voldoende goed overeenstemt met de praktijk. De best haalbare conclusie voor zuiver wetenschappelijk onderzoek daarentegen is dat er nog meer onderzoek benodigd is.