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General shell model for a rotating pretwisted blade

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Abstract

A novel dynamic model for a pretwisted rotating compressor blade mounted at an arbitrary stagger angle using general shell theory and including the rotational velocity is developed to study the eigenfrequencies and damping properties of the pretwisted rotating blade. The strain-displacement relation and constitutive model based on the general (thick) shell theory are applied to bring out the strain energy of the rotating blade. Using the Hamilton's principle, the variational form of the total energy is derived in order to obtain the corresponding weak form for the numerical simulation. The model is validated by comparing to literature results and Ansys results, showing good agreement. Parametric analyses are carried out to study the influence of the rotation velocity, the stagger angle and the radius of the disk on the eigenfrequencies of the pretwisted blade. Proportional damping is included into the proposed model to investigate the influence of rotational velocity on the damping characteristics of the pretwisted rotating blade system. It is shown that, due to inertial and Coriolis effects, damping decreases as the rotation velocity increases for the lower part of the velocity range considered and either decreases or increases depending on the mode order for higher velocities. Furthermore, frequency loci veering as a result of the rotation velocity is observed. The proposed model is an efficient and accurate tool for predicting the dynamic behavior of compressor blades of arbitrary thickness, stagger angle and pretwist, potentially during the early designing stage of turbomachinery.

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1. Introduction

The dynamic behavior of a rotating compressor blade has been studied by numerous researchers. In the early design stage, where detailed information about specific blades is not available yet, simplified blade models are often used. The typical routine is to simplify the real compressor blade, which has complex geometrical characteristics, into a simple geometry model only depending on several important parameters, such as the aspect ratio, pretwist angle, stagger angle, radius of the rotor disk, etc. The advantage of the simplified blade is to make it possible to use analytical or semi-analytical methods to solve the problem with good accuracy and fast calculation speed.

The dynamic behavior of compressor blades is either studied in interaction with the full rotor system including the blade arrays or for a single blade. Fast and efficient simplified models of disks with blade arrays can for example be found in [1, 2]. When looking at the specific modal behavior and the influence of damping treatments models of for a single blade are of interest. The present paper focusses on the dynamic response of a single blade. Due to their simplicity, beam models have extensively been applied in the literature [3, 4, 5, 6, 7, 8, 9] to study the dynamic response of a single rotor blade. These models can capture the flexible modes along the spanwise direction and torsional modes but not the in-plane modes and the flexible modes along the chordwise direction, because the beam model is a one-dimensional model.

In [10], thin shell theory is used to model a pretwisted rotating blade and study the influence of the rotation velocity, pretwisted angle, stagger angle etc. Leissa [11] provided a complete database of the vibration behavior of a non-rotating twisted blade. Rotating straight plates mounted at an arbitrary stagger angle are studied in [12, 13, 14, 15, 16], where several significant design parameters are varied and both modal and transient analyses are performed. Based on the beam theory, a pretwisted rotating blade with an attached mass is studied in [17].

Thin shell theory is used to investigate cylindrical [18] and conical [19] rotating panels with a twisted angle, where many parametric results are presented. In [20], a thin shell theory is applied to study the dynamic characteristics of a pretwisted rotating blade, where the equations of motion are only dependent on the transverse displacement and the specific mode shape functions are designed to satisfy the first few modes in order to reduce the computational cost due to the

usage of high order polynomial functions. The thick shell theory is employed in [21, 22] to study the non-rotating pretwisted plate. While in [23, 24], the vibration of the non-rotating twisted composite blade is investigated, where a detailed description about the two-dimensional shell theory is given.

In this paper, a novel dynamic model for a pretwisted rotating compressor blade mounted at an arbitrary stagger angle using general shell theory and including the rotational velocity is proposed. According to the thick shell assumption, three displacement and two rotation angle variables are used to describe the deformation of the blade in a rotating reference system, including the effect of the rotational stiffening and Coriolis acceleration. Through the Hamilton's principle, the variation of the total energy is obtained and the corresponding equivalent weak form is used for the numerical simulations. The proposed model is validated by comparison to reference data and parametric analyses with varying pretwist angle, stagger angle and aspect ratio are performed. Moreover, the influence of rotational velocity on the eigenfrequencies and modal loss factor is studied. Frequency loci veering is observed and a clear influence of the rotational velocity on the modal damping ratios is observed, including the effect of frequency loci veering. It is shown that, due to inertial and Coriolis effects, damping decreases as the rotation velocity increases for the lower part of the velocity range considered and either decreases or increases depending on the mode order for higher velocities. This model is an efficient numerical tool to evaluate the dynamic behavior and damping value of a rotating pretwisted blade, which would be quite useful in the early stage design stage of turbomachineries.

2. Dynamic modeling based on thick shell theory

As shown in Fig.1, a pretwisted blade is modeled as a cantilever twist shell, clamped to a rigid disk with a radius R , mounted with a stagger angle φ . The pretwist angle at the free end of the shell is denoted θ . The parameters of the twisted blade, of rectangular cross-section, are: the length of the blade along x -axis L ; the width b ; the thickness h ; the density ρ ; the Young's modulus E ; Poisson's ratio μ and the rotation velocity Ω .

Several coordinate systems are defined in this paper for dynamic modeling. The first is the XYZ - coordinate system, where the X -axis is along the spanwise direction of the blade, Y is the rotation axis and the Z -axis is perpendicular to the XY -plane following the right hand rule. The second is the $x'y'z'$ - coordinate system, where the origin lies in the root of the blade, the three axes are parallel to the XYZ -axes. With rotating the $x'y'z'$ coordinates around the x' -axis with an

angle φ , a new coordinate system $x y z$ is obtained with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. As a result, an arbitrary point on the blade can be simply expressed as

$$\mathbf{r}_0 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \quad (1)$$

Figure 1: The scheme of the rotating blade model.

To describe the twisted surface of the blade, an orthogonal coordinate system $x\beta n$ can be defined [10, 25], where the β -axis is perpendicular to the x -axis and equal to y -axis at $x = 0$, rotating at a constant twist rate $\tilde{\theta}_0 = \theta/L$ [rad/m], if a linear variation of the twist angle is assumed; the ζ -axis is the normal vector to the blade surface, i.e. the cross product of the x and β axes. Here, an arbitrary point on the middle surface can be written only using the $x\beta$ -coordinates

$$\mathbf{r}_0 = x\mathbf{i} + \beta \cos(\tilde{\theta}_0 x)\mathbf{j} + \beta \sin(\tilde{\theta}_0 x)\mathbf{k}. \quad (2)$$

Based on the theory of surfaces [26], the first fundamental forms are written as

$$\begin{cases} \hat{A} = |\mathbf{r}_{0,x}| = \sqrt{1 + \tilde{\theta}_0^2 \beta^2} \\ \hat{B} = |\mathbf{r}_{0,\beta}| = 1, \end{cases} \quad (3)$$

and the corresponding unit vectors along each orthogonal coordinate can be derived

$$\begin{cases} \mathbf{a}_1 = \frac{\mathbf{r}_{0,x}}{\hat{A}} = \frac{1}{\hat{A}}\mathbf{i} + \frac{-\beta\tilde{\theta}_0 \sin(\tilde{\theta}_0 x)}{\hat{A}}\mathbf{j} + \frac{\beta\tilde{\theta}_0 \cos(\tilde{\theta}_0 x)}{\hat{A}}\mathbf{k} \\ \mathbf{a}_2 = \frac{\mathbf{r}_{0,\beta}}{\hat{B}} = \cos(\tilde{\theta}_0 x)\mathbf{j} + \sin(\tilde{\theta}_0 x)\mathbf{k} \\ \mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2 = \frac{-\beta\tilde{\theta}_0}{\hat{A}}\mathbf{i} + \frac{-\sin(\tilde{\theta}_0 x)}{\hat{A}}\mathbf{j} + \frac{\cos(\tilde{\theta}_0 x)}{\hat{A}}\mathbf{k}. \end{cases} \quad (4)$$

This coordinate system can be used in thin shell theory to describe the deformation field on the blade, assuming small blade thickness [10]. However, this simplification will affect the accuracy to some extent, especially at large twist angles. Therefore, a general shell theory is applied to give a more accurate description with including the normal axis ζ , where the position vector of an arbitrary point on the pretwisted blade reads

$$\mathbf{r}' = \mathbf{r}_0 + \zeta \mathbf{a}_3, \quad (5)$$

where corresponding unit vectors are derived as

$$\begin{cases} \mathbf{g}_1 = \frac{\mathbf{r}'_x}{\hat{A}} = \frac{1}{\hat{A}}\mathbf{i} + \left[-\frac{\beta\tilde{\theta}_0 \sin(\tilde{\theta}_0 x)}{\hat{A}} - \frac{\zeta\tilde{\theta}_0 \cos(\tilde{\theta}_0 x)}{\hat{A}\hat{A}}\right]\mathbf{j} + \left[\frac{\beta\tilde{\theta}_0 \cos(\tilde{\theta}_0 x)}{\hat{A}} - \frac{\zeta\tilde{\theta}_0 \sin(\tilde{\theta}_0 x)}{\hat{A}\hat{A}}\right]\mathbf{k} \\ \mathbf{g}_2 = \frac{\mathbf{r}'_\beta}{\hat{B}} = \left[-\zeta \frac{\tilde{\theta}_0}{\hat{B}\hat{A}^3}\right]\mathbf{i} + \left[\frac{\cos(\tilde{\theta}_0 x)}{\hat{B}} + \frac{\zeta\beta\tilde{\theta}_0^2 \sin(\tilde{\theta}_0 x)}{\hat{B}\hat{A}^3}\right]\mathbf{j} + \left[\frac{\sin(\tilde{\theta}_0 x)}{\hat{B}} - \frac{\zeta\beta\tilde{\theta}_0^2 \cos(\tilde{\theta}_0 x)}{\hat{B}\hat{A}^3}\right]\mathbf{k} \\ \mathbf{g}_3 = \frac{\mathbf{r}'_\zeta}{|\mathbf{r}'_\zeta|} = \mathbf{a}_3 = \frac{-\beta\tilde{\theta}_0}{\hat{A}}\mathbf{i} + \frac{-\sin(\tilde{\theta}_0 x)}{\hat{A}}\mathbf{j} + \frac{\cos(\tilde{\theta}_0 x)}{\hat{A}}\mathbf{k} \end{cases} \quad (6)$$

and the corresponding first fundamental forms read $A = \sqrt{\hat{A}^2 + \zeta^2 \tilde{\theta}_0^2 / \hat{A}^2} = \hat{A}B$ and $B = \sqrt{1 + \zeta^2 \tilde{\theta}_0^2 / \hat{A}^4}$.

The position of an arbitrary point on the blade after deformation reads

$$\mathbf{r} = \mathbf{r}' + \mathbf{u} = \mathbf{r}_0 + \zeta \mathbf{a}_3 + u \mathbf{a}_1 + v \mathbf{a}_2 + w \mathbf{a}_3, \quad (7)$$

where u , v and w are displacement components along unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, respectively.

Following the strain definition presented in [23] and considering the thick shell theory, the strain components in the local coordinate system ($\mathbf{g}_1$ $\mathbf{g}_2$ $\mathbf{g}_3$) can be calculated by

$$\begin{cases} \epsilon_{11} = \frac{u_x}{\hat{A}} \mathbf{g}_1 \\ \epsilon_{22} = \frac{u_\beta}{\hat{B}} \mathbf{g}_2 \\ \gamma_{12} = \frac{u_x}{\hat{A}} \mathbf{g}_2 \\ \gamma_{21} = \frac{u_\beta}{\hat{B}} \mathbf{g}_1 \\ \epsilon_{13} = \frac{u_x}{\hat{A}} \mathbf{g}_3 + \mathbf{u}_{,\zeta} \mathbf{g}_1 \\ \epsilon_{23} = \frac{u_x}{\hat{B}} \mathbf{g}_3 + \mathbf{u}_{,\zeta} \mathbf{g}_2. \end{cases} \quad (8)$$

After some derivations (see in Appendix A), the strain-displacement relations can be written as

$$\begin{cases} \epsilon_{11} = \frac{1}{A^2} [\hat{A}u_{,x} + \beta\tilde{\theta}_0^2 v + \zeta(\frac{\beta\tilde{\theta}_0^3}{\hat{A}^2} u - \frac{\tilde{\theta}_0}{\hat{A}} v_{,x} + \frac{\tilde{\theta}_0^2}{\hat{A}^2} w)] \\ \epsilon_{22} = \frac{1}{B^2} [v_{,\beta} + \zeta(\frac{\tilde{\theta}_0^2}{\hat{A}^4} w - \frac{\tilde{\theta}_0}{\hat{A}^2} u_{,\beta})] \\ \gamma_{12} = \frac{1}{AB} [v_{,x} - \frac{\beta\tilde{\theta}_0^2}{\hat{A}} u - \frac{\tilde{\theta}_0}{\hat{A}} w - \zeta(\frac{\tilde{\theta}_0}{\hat{A}^2} u_{,x} + \frac{\beta\tilde{\theta}_0^3}{\hat{A}^3} v)] \\ \gamma_{21} = \frac{1}{AB} [\hat{A}u_{,\beta} - \frac{\tilde{\theta}_0}{\hat{A}} w - \zeta(\frac{\tilde{\theta}_0}{\hat{A}} v_{,\beta})] \\ \epsilon_{13} = \frac{1}{A} [w_{,x} + \frac{\tilde{\theta}_0}{\hat{A}} v + \hat{A}u_{,\zeta} - \zeta(\frac{\tilde{\theta}_0}{\hat{A}} v_{,\zeta})] \\ \epsilon_{23} = \frac{1}{B} [w_{,\beta} + \frac{\tilde{\theta}_0}{\hat{A}^2} u + v_{,\zeta} - \zeta(\frac{\tilde{\theta}_0}{\hat{A}^2} u_{,\zeta})], \end{cases} \quad (9)$$

where $\epsilon_{12} = \gamma_{12} + \gamma_{21}$.

It should be noted that the unit vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ are not orthogonal while $\mathbf{g}_3$ is orthogonal to $\mathbf{g}_1$ and $\mathbf{g}_2$. Hence, the strain components derived in Eq. (9) have to be transferred to an orthogonal coordinate system. One possibility is to directly use the strain components in the local coordinate ($\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3$) as the approximate strain components in the coordinate ($\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$), i.e. $\epsilon_{ij}^{Appr} \approx \epsilon_{ij}$, when the thickness of the blade is small enough. This approximate method is called "Appr" mentioned in the numerical simulation section. Otherwise, the exact strain components need to be derived by using the tensor transformation law. As shown in [27, 28], the scalar multiplication can be calculated by

$$\delta_q^p = g^{pn} h_{qn}, \quad (10)$$

where p, q and $n = 1, 2, 3$; g^{pn} is the contravariant metric tensors in which each tensor is the corresponding element of the inverse of the matrix $[\mathbf{g}_p \cdot \mathbf{g}_n]_{3 \times 3}$ and $h_{qn} = \mathbf{a}_q \cdot \mathbf{g}_n$ is the covariant metric tensors (see in Appendix A). Thus, the new strain tensors can be derived by

$$\epsilon_{ij} = \delta_i^l \delta_j^m \epsilon_{lm}, \quad (11)$$

resulting in the strain components in the coordinate system ($\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$) as

$$\begin{cases} \epsilon_{11} = \frac{1}{(1-\zeta^2\tilde{\theta}_0^2/\hat{A}^4)} [\frac{u_{,x}}{\hat{A}} + \frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} v + \zeta(\frac{\tilde{\theta}_0}{\hat{A}^2} u_{,\beta} - \frac{\tilde{\theta}_0^2}{\hat{A}^4} w)] \\ \epsilon_{22} = \frac{1}{(1-\zeta^2\tilde{\theta}_0^2/\hat{A}^4)} [v_{,\beta} + \zeta(-\frac{\tilde{\theta}_0^2}{\hat{A}^4} w + \frac{\tilde{\theta}_0}{\hat{A}^3} v_{,x} - \frac{\beta\tilde{\theta}_0^3}{\hat{A}^4} u)] \\ \epsilon_{12} = \frac{1}{(1-\zeta^2\tilde{\theta}_0^2/\hat{A}^4)} [\frac{v_{,x}}{\hat{A}} - \frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} u + u_{,\beta} - \frac{2\tilde{\theta}_0}{\hat{A}^2} w + \zeta(\frac{\tilde{\theta}_0}{\hat{A}^3} u_{,x} + \frac{\beta\tilde{\theta}_0^3}{\hat{A}^4} v + \frac{\tilde{\theta}_0}{\hat{A}^2} v_{,\beta})] \\ \epsilon_{13} = \frac{1}{(1-\zeta^2\tilde{\theta}_0^2/\hat{A}^4)} [\frac{w_{,x}}{\hat{A}} + \frac{\tilde{\theta}_0}{\hat{A}^2} v + u_{,\zeta} + \zeta(\frac{\tilde{\theta}_0}{\hat{A}^2} w_{,\beta} + \frac{\tilde{\theta}_0^2}{\hat{A}^4} u)] \\ \epsilon_{23} = \frac{1}{(1-\zeta^2\tilde{\theta}_0^2/\hat{A}^4)} [w_{,\beta} + \frac{\tilde{\theta}_0}{\hat{A}^2} u + v_{,\zeta} + \zeta(\frac{\tilde{\theta}_0}{\hat{A}^3} w_{,x} + \frac{\tilde{\theta}_0^2}{\hat{A}^4} v)]. \end{cases} \quad (12)$$

For isotropic material the stress components can read

$$\begin{cases} \sigma_{11} = \frac{E}{1-\mu^2}(\varepsilon_{11} + \mu\varepsilon_{22}) \\ \sigma_{22} = \frac{E}{1-\mu^2}(\mu\varepsilon_{11} + \varepsilon_{22}) \\ \sigma_{12} = \frac{E}{2(1+\mu)}\varepsilon_{12} \\ \sigma_{13} = \frac{KE}{2(1+\mu)}\varepsilon_{13} \\ \sigma_{23} = \frac{KE}{2(1+\mu)}\varepsilon_{23}, \end{cases} \quad (13)$$

where $K = 5/6$ is the shear correction. The strain energy can be derived

$$\begin{aligned} U_s &= \int_V \frac{1}{2} \varepsilon_{ij} \sigma_{ij} \hat{A} \hat{B} dx d\beta d\zeta \\ &= \int_V \frac{1}{2} A B dx d\beta d\zeta \left\{ \frac{E}{1-\mu^2} [\varepsilon_{11}^2 + \mu\varepsilon_{11}\varepsilon_{22} + \varepsilon_{22}^2 + \mu\varepsilon_{11}\varepsilon_{22}] \right. \\ &\quad \left. + \frac{E}{2(1+\mu)} [\varepsilon_{12}^2 + K\varepsilon_{13}^2 + K\varepsilon_{23}^2] \right\}. \end{aligned} \quad (14)$$

To include the centrifugal and Coriolis effects, the resultant velocity due to the rotation is written in the $x y z$ coordinates as

$$\bar{\mathbf{V}} = \bar{V}_x \mathbf{i} + \bar{V}_y \mathbf{j} + \bar{V}_z \mathbf{k}, \quad (15)$$

where $\bar{V}_x$, $\bar{V}_y$ and $\bar{V}_z$ denote the velocity components along x , y and z -axes, respectively, reading

$$\begin{aligned} \bar{V}_x &= \Omega[\beta \sin(\varphi + \tilde{\theta}_0 x) + \frac{\zeta}{\hat{A}} \cos(\varphi + \tilde{\theta}_0 x) + \mathbf{u} \cdot \mathbf{j} \sin(\varphi + \tilde{\theta}_0 x) + \mathbf{u} \cdot \mathbf{k} \cos(\varphi + \tilde{\theta}_0 x)] \\ &= \Omega[\beta \sin \phi + \frac{\zeta}{\hat{A}} \cos \phi + \frac{\beta \tilde{\theta}_0}{\hat{A}} u \cos \phi + v \sin \phi + \frac{1}{\hat{A}} w \cos \phi], \end{aligned} \quad (16)$$

$$\bar{V}_y = -\bar{V} \sin(\varphi) \quad (17)$$

and

$$\bar{V}_z = -\bar{V} \cos(\varphi), \quad (18)$$

where $\phi = \varphi + \tilde{\theta}_0 x$ and $\bar{V} = \Omega(R + x - \zeta \frac{\beta \tilde{\theta}_0}{\hat{A}} + \frac{1}{\hat{A}} u - \frac{\beta \tilde{\theta}_0}{\hat{A}} w)$.

Consequently, the resultant velocity components in the local coordinate system $(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3)$ are written

$$\begin{cases} \bar{V}_1 = \dot{\mathbf{u}} \cdot \mathbf{a}_1 + \bar{\mathbf{V}} \cdot \mathbf{a}_1 \\ \bar{V}_2 = \dot{\mathbf{u}} \cdot \mathbf{a}_2 + \bar{\mathbf{V}} \cdot \mathbf{a}_2 \\ \bar{V}_3 = \dot{\mathbf{u}} \cdot \mathbf{a}_3 + \bar{\mathbf{V}} \cdot \mathbf{a}_3. \end{cases} \quad (19)$$

Inserting Eqs. (15)-(18) into Eq. (19), the velocity components can be expressed as

$$\begin{aligned} \bar{V}_1 &= \hat{A}\dot{u} + \frac{1}{\hat{A}}\bar{V}_x + \frac{\beta\tilde{\theta}_0 \sin(\tilde{\theta}_0 x)}{\hat{A}}\bar{V} \sin(\varphi) - \frac{\beta\tilde{\theta}_0 \cos(\tilde{\theta}_0 x)}{\hat{A}}\bar{V} \cos(\varphi) \\ &= \hat{A}\dot{u} + \frac{1}{\hat{A}}\bar{V}_x - \frac{\beta\tilde{\theta}_0}{\hat{A}}\cos \phi \bar{V} \end{aligned} \quad (20)$$

$$\begin{aligned} \bar{V}_2 &= \dot{v} - \cos(\tilde{\theta}_0 x) \sin(\varphi) \bar{V} - \sin(\tilde{\theta}_0 x) \cos(\varphi) \bar{V} \\ &= \dot{v} - \sin \phi \bar{V} \end{aligned} \quad (21)$$

$$\begin{aligned} \bar{V}_3 &= \dot{w} - \frac{\beta\tilde{\theta}_0}{\hat{A}}\bar{V}_x + \frac{\sin(\tilde{\theta}_0 x)}{\hat{A}} \sin(\varphi) \bar{V} - \frac{\cos(\tilde{\theta}_0 x)}{\hat{A}} \cos(\varphi) \bar{V} \\ &= \dot{w} - \frac{\beta\tilde{\theta}_0}{\hat{A}}\bar{V}_x - \frac{1}{\hat{A}} \cos \phi \bar{V}. \end{aligned} \quad (22)$$

As a result, the kinetic energy reads

$$T = \int_V \frac{1}{2} \rho \hat{A} \hat{B} \bar{V}_i^2 dx d\beta d\zeta. \quad (23)$$

Then, the centrifugal force components (surface force) along the x , y , z -axes are written

$$\begin{cases} F_x = \rho \Omega^2 (R + x - \zeta \frac{\beta\tilde{\theta}_0}{\hat{A}}) \\ F_y = \rho \Omega^2 (\beta \sin(\varphi + \theta_0 x) + \frac{\zeta}{\hat{A}} \cos \phi) \sin \varphi \\ F_z = \rho \Omega^2 (\beta \sin(\varphi + \theta_0 x) + \frac{\zeta}{\hat{A}} \cos \phi) \cos \varphi, \end{cases} \quad (24)$$

where $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$.

After projecting those force components into the $(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3)$ coordinates, the new force components in the orthogonal coordinate system read

$$\begin{cases} F_1 = \mathbf{F} \cdot \mathbf{a}_1 = \rho\Omega^2[(R+x - \zeta\frac{\beta\tilde{\theta}_0}{\hat{A}})\frac{1}{\hat{A}} + \frac{\beta^2\tilde{\theta}_0}{\hat{A}} \sin\phi \cos\phi + \frac{\zeta\beta\tilde{\theta}_0}{\hat{A}^2} \cos^2\phi] \\ F_2 = \mathbf{F} \cdot \mathbf{a}_2 = \rho\Omega^2[\beta \sin^2\phi + \frac{\zeta}{\hat{A}} \cos\phi \sin\phi] \\ F_3 = \mathbf{F} \cdot \mathbf{a}_3 = \rho\Omega^2[-(R+x - \zeta\frac{\beta\tilde{\theta}_0}{\hat{A}})\frac{\beta\tilde{\theta}_0}{\hat{A}} + \frac{\beta}{\hat{A}} \sin\phi \cos\phi + \frac{\zeta}{\hat{A}^2} \cos^2\phi] \end{cases} \quad (25)$$

In this paper, according to the thick shell model, the displacement components u , v and w are assumed as

$$\begin{cases} u = u_0 + \zeta\theta_\alpha \\ v = v_0 + \zeta\theta_\beta \\ w = w_0, \end{cases} \quad (26)$$

where u_0 , v_0 and w_0 denote three displacement components at the middle surface and θ_α and θ_β as two angles rotating with in-plane local coordinates. Thus, as referring to [29], the inward displacements in the $(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3)$ coordinates read $\Delta_1 = u_0 - \frac{1}{2}\hat{A}\theta_\alpha^2$, $\Delta_2 = v_0 - \frac{1}{2}\hat{B}\theta_\beta^2$ and $\Delta_3 = w_0$. The potential energy can be derived as

$$U_P = \int_V N_i \Delta_i \hat{A} \hat{B} dx d\beta d\zeta \quad (27)$$

where $i = 1, 2, 3$,

$$\begin{aligned} N_1 &= \int_x^L F_1 \hat{A} dx = \rho\Omega^2 \int_x^L [(R+x - \zeta\frac{\beta\tilde{\theta}_0}{\hat{A}})\frac{1}{\hat{A}} + \frac{\beta^2\tilde{\theta}_0}{\hat{A}} \sin\phi \cos\phi + \frac{\zeta\beta\tilde{\theta}_0}{\hat{A}^2} \cos^2\phi] \hat{A} dx \\ &= \rho\Omega^2 [((R - \zeta\frac{\beta\tilde{\theta}_0}{\hat{A}})L + L^2/2 - (R - \zeta\frac{\beta\tilde{\theta}_0}{\hat{A}})x - x^2/2) + \beta^2(\frac{\sin^2(\varphi + \tilde{\theta}_0 L)}{2} - \frac{\sin^2\phi}{2}) \\ &\quad + \frac{\zeta\beta}{\hat{A}}(\tilde{\theta}_0(L-x)/2 + \frac{\sin(2\varphi + 2\tilde{\theta}_0 L)}{4} - \frac{\sin(2\phi)}{4})] \end{aligned} \quad (28)$$

$$\begin{aligned} N_2 &= \int_\beta^{\pm b/2} F_2 \hat{B} d\beta = \rho\Omega^2 \int_\beta^{\pm b/2} (\beta \sin^2\phi + \frac{\zeta}{\hat{A}} \cos\phi \sin\phi) \hat{B} d\beta \\ &= \rho\Omega^2 [\sin^2\phi (b/2)^2/2 - \beta^2/2] + \int_\beta^{\pm b/2} \frac{\zeta}{\hat{A}} \cos\phi \sin\phi d\beta \end{aligned} \quad (29)$$

$$N_3 = F_3 \quad (30)$$

The virtual work done by the external transverse force is written as

$$\delta W = \int_V F_w \delta w dV, \quad (31)$$

where F_w is the external harmonic force normal to the blade surface.

Using Hamilton's principle in Eq. (32), the variational form of the total energy can be derived

$$\delta \int_{t_1}^{t_2} (T - U_S - U_P + W) dt = 0. \quad (32)$$

As stated in [28], for typical compressor blades, the twist rate and thickness are small compared to the width or length, the term $(1 - \zeta^2 \tilde{\theta}_0^2 / \hat{A}^4) \approx 1$. Thus, by inserting Eqs. (14), (23), (27) and (31) into Eq. (32), the simplified variational equation reads

$$\begin{aligned} & \delta \int_{t_1}^{t_2} \left\{ \int_V \frac{1}{2} \rho \bar{V}_i^2 \hat{A} \hat{B} dx d\beta d\zeta - \int_V \frac{1}{2} \varepsilon_{ij} \sigma_{ij} \hat{A} \hat{B} dx d\beta d\zeta - \int_V N_i \Delta_i \hat{A} \hat{B} dx d\beta d\zeta + \int_V F_w \delta w \hat{A} \hat{B} dx d\beta d\zeta \right\} dt \\ &= \int_{t_1}^{t_2} \int_V \left\{ \rho \bar{V}_i \delta \bar{V}_i - \frac{1}{2} \varepsilon_{ij} \delta \sigma_{ij} - \frac{1}{2} \sigma_{ij} \delta \varepsilon_{ij} - N_i \delta \Delta_i + F_w \delta w \right\} \hat{A} \hat{B} dx d\beta d\zeta dt = 0. \end{aligned} \quad (33)$$

The integration in the coordinate ζ and eliminating the static force terms, the

weak form used in the following numerical simulations is written as

$$\begin{aligned}
& \int_S \left\{ \left[-\hat{A} M u_{0,t} - 2\hat{A} M \Omega \cos \phi w_{0,t} - 2M \Omega \sin \phi v_{0,t} \right] u_0^q \right. \\
& + [M \Omega^2 \left(\frac{1}{\hat{A}} + \hat{A} \left(\frac{\beta \tilde{\theta}_0}{\hat{A}} \right)^2 \cos^2 \phi \right) u_0 + M \Omega^2 \beta \tilde{\theta}_0 \sin \phi \cos \phi v_0 - M \Omega^2 \frac{\beta \tilde{\theta}_0}{\hat{A}} \sin^2 \phi w_0] u_0^q \\
& + [-\hat{A} M v_{0,t} + 2M \Omega \sin \phi u_{0,t} - 2M \Omega \beta \theta_0 \sin \phi w_{0,t}] v_0^q \\
& + [\hat{A} M \Omega^2 \sin^2 \phi v_0 + M \Omega^2 \beta \tilde{\theta}_0 \sin \phi \cos \phi u_0 + M \Omega^2 \sin \phi \cos \phi w_0] v_0^q \\
& + [-\hat{A} M w_{0,t} + 2\hat{A} M \Omega \cos \phi u_{0,t} + 2M \Omega \beta \tilde{\theta}_0 \sin \phi v_{0,t}] w_0^q \\
& + [M \Omega^2 \left(\frac{1}{\hat{A}} \cos^2 \phi + \hat{A} \left(\frac{\beta \tilde{\theta}_0}{\hat{A}} \right)^2 \right) w_0 + M \Omega^2 \sin \phi \cos \phi v_0 - M \Omega^2 \frac{\beta \tilde{\theta}_0}{\hat{A}} \sin^2 \phi u_0] w_0^q \\
& + [-\hat{A} \rho I \theta_{\alpha,t} - 2\rho I \Omega \sin \phi \theta_{\beta,t}] \theta_\alpha^q \\
& + [\rho I \Omega^2 \left(\frac{1}{\hat{A}} + \hat{A} \left(\frac{\beta \tilde{\theta}_0}{\hat{A}} \right)^2 \cos^2 \phi \right) \theta_\alpha + \rho I \Omega^2 \beta \tilde{\theta}_0 \sin \phi \cos \phi \theta_\beta] \theta_\alpha^q \\
& + [-\hat{A} \rho I \theta_{\beta,t} + 2\rho I \Omega \sin \phi \theta_{\alpha,t}] \theta_\beta^q \\
& + [\hat{A} \rho I \Omega^2 \sin^2 \phi \theta_\beta + \rho I \Omega^2 \beta \tilde{\theta}_0 \sin \phi \cos \phi \theta_\alpha] \theta_\beta^q \Big\} \\
& + \hat{A} \hat{B} \Big\{ I_1 [Z_0 e_{11} e_{11}^q + Z_1 k_{11} k_{11}^q + Z_2 g_{11} g_{11}^q + Z_1 e_{11} g_{11}^q + Z_1 g_{11} e_{11}^q \\
& + Z_0 e_{22} e_{22}^q + Z_1 k_{22} k_{22}^q + Z_2 g_{22} g_{22}^q + Z_1 e_{22} g_{22}^q + Z_1 g_{22} e_{22}^q \\
& + \mu (Z_0 e_{11} e_{22}^q + Z_1 e_{11} g_{22}^q + Z_1 k_{11} k_{22}^q + Z_1 g_{11} e_{22}^q + Z_2 g_{11} g_{22}^q) \\
& + \mu (Z_0 e_{22} e_{11}^q + Z_1 e_{22} g_{11}^q + Z_1 k_{22} k_{11}^q + Z_1 g_{22} e_{11}^q + Z_2 g_{22} g_{11}^q)] \\
& + I_2 [Z_0 e_{12} e_{12}^q + Z_1 k_{12} k_{12}^q + Z_2 g_{12} g_{12}^q + Z_1 e_{12} g_{12}^q + Z_1 g_{12} e_{12}^q] \\
& + I_3 [Z_0 e_{13} e_{13}^q + Z_1 k_{13} k_{13}^q + Z_2 g_{13} g_{13}^q + Z_1 e_{13} g_{13}^q + Z_1 g_{13} e_{13}^q] \\
& + I_3 [Z_0 e_{23} e_{23}^q + Z_1 k_{23} k_{23}^q + Z_2 g_{23} g_{23}^q + Z_1 e_{23} g_{23}^q + Z_1 g_{23} e_{23}^q] \Big\} \\
& - \hat{A} \hat{B} \Big\{ M \Omega^2 [(RL + L^2/2 - Rx - x^2/2) + \beta^2 \left(\frac{\sin^2(\varphi + \tilde{\theta}_0 L)}{2} - \frac{\sin^2 \phi}{2} \right)] \theta_\alpha \theta_\alpha^q \\
& + M \Omega^2 \sin^2 \phi [(b/2)^2/2 - \beta^2/2] \theta_\beta \theta_\beta^q \Big\} + \hat{A} \hat{B} Z_0 F_w w_0^q \Big\} dS = 0, \tag{34}
\end{aligned}$$

where several constants are defined: $M = \rho h$, $I = h^3/12$, $D = Eh^3/12/(1 - \mu^2)$, $I_1 = E/(1 - \mu^2)$, $I_2 = E/2/(1 + \mu)$, $I_3 = KE/2/(1 + \mu)$, $Z_0 = h$, $Z_1 = h^3/12$ and $Z_2 = h^5/80$; u_0^q , v_0^q , w_0^q , θ_α^q and θ_β^q are test functions corresponding to u_0 , v_0 , w_0 , θ_α and θ_β , respectively; e_{ij} , k_{ij} , g_{ij} , e_{ij}^q , k_{ij}^q and g_{ij}^q can be found in Appendix A;

the derivation of the kinetic energy terms in the first curly bracket can be found in Appendix B.

If all terms associated with the ζ coordinate in Eq. (9) are removed, the model obtained is named 2D shell model in this paper and it can be seen as an enhanced thin shell model, where the rotation of the local reference system across the blade thickness is neglected (as in the thin shell theory) but the deformation variables are interpolated as given in Eq. (26). In order to assess the influence of this simplification, in the following section the results of this 2D shell model are compared to the general shell model presented above.

In this study, the weak form derived from the variational equations, Eq. (34), is used to perform the numerical simulation. The harmonic independent variables (displacement and rotation angle) are expressed as a sum of trial functions of x, y -coordinates

$$\left\{ \begin{array}{l} u_0 \approx \sum_{i=1}^g \hat{u}_0^i(t) \Upsilon^i(x, y) \\ v_0 \approx \sum_{i=1}^g \hat{v}_0^i(t) \Upsilon^i(x, y) \\ w_0 \approx \sum_{i=1}^g \hat{w}_0^i(t) \Upsilon^i(x, y) \\ \theta_\alpha \approx \sum_{i=1}^g \hat{\theta}_\alpha^i(t) \Upsilon^i(x, y) \\ \theta_\beta \approx \sum_{i=1}^g \hat{\theta}_\beta^i(t) \Upsilon^i(x, y), \end{array} \right. \quad (35)$$

where $\Upsilon^i(x, y)$ is a spatial function of x, y -coordinates; $\hat{u}_0^i(t)$, $\hat{v}_0^i(t)$, $\hat{w}_0^i(t)$, $\hat{\theta}_\alpha^i(t)$ and $\hat{\theta}_\beta^i(t)$ are temporal functions; in the Galerkin sense, the test functions are the same functions as the spatial functions, i.e. $u_0^q = v_0^q = w_0^q = \theta_\alpha^q = \theta_\beta^q = \Upsilon^i(x, y)$. The discretization procedure is implemented by inserting Eq. (35) into Eq. (34), resulting in a 2nd ordinary differential equation as

$$\begin{aligned}
& \begin{bmatrix} \mathbf{M}_u & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_v & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_w & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_\alpha & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_\beta \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_u \\ \ddot{\mathbf{x}}_v \\ \ddot{\mathbf{x}}_w \\ \ddot{\mathbf{x}}_{\theta\alpha} \\ \ddot{\mathbf{x}}_{\theta\beta} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{G}_{vuC} & \mathbf{G}_{wuC} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_{uvC} & \mathbf{0} & \mathbf{G}_{wvC} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}_{uwC} & \mathbf{G}_{vwC} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\beta\alpha C} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{\alpha\beta C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_u \\ \dot{\mathbf{x}}_v \\ \dot{\mathbf{x}}_w \\ \dot{\mathbf{x}}_{\theta\alpha} \\ \dot{\mathbf{x}}_{\theta\beta} \end{bmatrix} \\
& + \begin{bmatrix} \mathbf{K}_u & \mathbf{K}_{vuc} & \mathbf{K}_{wuc} & \mathbf{K}_{\alpha u} & \mathbf{K}_{\beta u} \\ \mathbf{K}_{uvC} & \mathbf{K}_v & \mathbf{K}_{wvC} & \mathbf{K}_{\alpha v} & \mathbf{K}_{\beta v} \\ \mathbf{K}_{uwC} & \mathbf{K}_{wvC} & \mathbf{K}_w & \mathbf{K}_{\alpha w} & \mathbf{K}_{\beta w} \\ \mathbf{K}_{u\alpha} & \mathbf{K}_{v\alpha} & \mathbf{K}_{w\alpha} & \mathbf{K}_\alpha & \mathbf{K}_{\beta\alpha C} \\ \mathbf{K}_{u\beta} & \mathbf{K}_{v\beta} & \mathbf{K}_{w\beta} & \mathbf{K}_{\alpha\beta C} & \mathbf{K}_\beta \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_v \\ \mathbf{x}_w \\ \mathbf{x}_{\theta\alpha} \\ \mathbf{x}_{\theta\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{D}_w \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad (36)
\end{aligned}$$

where $\mathbf{x}_u^T = [\hat{u}_0^1(t) \ \hat{u}_0^2(t) \cdots \ \hat{u}_0^g(t)]$; $\mathbf{x}_v^T = [\hat{v}_0^1(t) \ \hat{v}_0^2(t) \cdots \ \hat{v}_0^g(t)]$; $\mathbf{x}_w^T = [\hat{w}_0^1(t) \ \hat{w}_0^2(t) \cdots \ \hat{w}_0^g(t)]$; $\mathbf{x}_{\theta\alpha}^T = [\hat{\theta}_\alpha^1(t) \ \hat{\theta}_\alpha^2(t) \cdots \ \hat{\theta}_\alpha^g(t)]$; $\mathbf{x}_{\theta\beta}^T = [\hat{\theta}_\beta^1(t) \ \hat{\theta}_\beta^2(t) \cdots \ \hat{\theta}_\beta^g(t)]$; $\mathbf{G}_{uwC}/\mathbf{G}_{wuC}$, $\mathbf{G}_{vuC}/\mathbf{G}_{uvC}$, $\mathbf{G}_{wvC}/\mathbf{G}_{vwC}$ and $\mathbf{G}_{\alpha\beta C}/\mathbf{G}_{\beta\alpha C}$ are four pairs of coupling terms due to the Coriolis force; $\mathbf{K}_{vuc}/\mathbf{K}_{wvc}$, $\mathbf{K}_{vuc}/\mathbf{K}_{uvC}$, $\mathbf{K}_{wuc}/\mathbf{K}_{uwC}$ and $\mathbf{K}_{\alpha\beta C}/\mathbf{K}_{\beta\alpha C}$ are four pairs of coupling terms due to the centrifugal force; and $\mathbf{D}_w$ is the external force.

3. Numerical simulation

The discretization procedure is implemented by COMSOL Multiphysics® 3.5a, where the variables are set to satisfy the geometric boundary conditions (i.e. clamped boundary conditions). For the eigenvalue calculations the force column in Eq. (36) (right hand side of the equality) is set to zero and the 2nd order ODE given is transformed to a 1st order ODE following the steps described in [14, 8]. The 1st order ODE can be written as

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{G} \end{bmatrix} \mathbf{q}(t), \quad (37)$$

where $\mathbf{q}^T = [\mathbf{x}^T \ \dot{\mathbf{x}}^T]$, with $\mathbf{x}^T = [\mathbf{x}_u^T \ \mathbf{x}_v^T \ \mathbf{x}_w^T \ \mathbf{x}_{\theta\alpha}^T \ \mathbf{x}_{\theta\beta}^T]$.

The eigenvalues obtained by solving the eigenvalue problem associated to this 1st order differential equation can be expressed as $\lambda_k = \mu_k + i\nu_k$ [30], where the real part is the exponential damping factor and the imaginary part is the damped eigenfrequency. Although damping is non-proportional, the damping values are very low which allows to write the eigenvalues in the following form: $\lambda_k = \omega_k \xi_k + i\omega_k \sqrt{1 - \xi_k^2}$ with ξ_k as the modal damping ratio and ω_k the undamped eigenfrequency.

In the eigenvalue analyses for the free vibration case the external forcing term is set to zero, but the Coriolis terms are included, leading to complex eigenvalues. Several non-dimensional parameters are defined to compare with the reference results. The non-dimensional rotation velocity and eigenvalues are used as $q = \Omega/\omega_1$ (herein ω_1 is the first eigenvalue of the blade model at the non-rotation situation) and $p^2 = \omega^2 12(1-\mu^2)\rho L^4/Eh^2$; the aspect ratio as $\bar{l} = L/b$; the thickness ratio as $\bar{t} = L/h$; the ratio of the radius to the length as $\bar{r} = R/L$.

3.1. Validation

The general shell model is validated by comparison to the results from previous literature. The first validation study is to calculate a non-rotation blade with the variation of the twist angle. The parameters of the blade are $\bar{l} = 1$, $\bar{t} = 100$, $\theta = 30^\circ/60^\circ/90^\circ$. A finite element model of the blade in ANSYS workbench is used for comparison. The details about the models used are summarized in Table 1. In Tables 2-4, the first 8 eigenfrequencies are compared with results in [22] and ANSYS workbench, respectively, both showing good agreements. The approximate model is also used to extract results "Appr", where the accuracy of approximate model is lower than the exact model but still gives a good agreement with reference data. In addition, results obtained from the 2D shell model are listed. For the twist angle $\theta = 30^\circ$, the 2D shell model shows acceptable accuracy as shown in Table 2; however, as the twist angle increases, some obvious discrepancies appear in the first eight eigenfrequencies, as shown in Tables 3 and 4. It can be seen that the 2D shell model predicts slightly higher eigenfrequencies than the reference model. This is expected, since the 2D shell model is stiffer than the general shell model because the ζ coordinate is not considered in the 2D shell model. For small twist angles, the eigenfrequencies predicted by the 2D shell model are very similar to the reference values and the prediction of the general shell model. However, as the twist angle increases the discrepancy between the eigenfrequencies predicted by the 2D model and the reference values becomes increasingly larger, while the general shell model still gives an accurate prediction of the eigenfrequencies. Therefore, the general shell theory should be used for moderate to large pretwist angles, which are often found in practice. One example is taken to show the efficiency of the general shell model: for the case of the pretwist angle $\theta = 60^\circ$, the computation time of the general shell model is 4.453686 s, while the total time with ANSYS is 37.565 s.

Table 1: Details of finite element models.

Model type	ANSYS	General shell/ approximate model/ 2D shell
Element type	Quad shell	2nd order triangular element
Element No.	7356	316
Number of Dofs	115593	7460

Table 2: Comparisons of the first 8 non-dimensional eigenfrequencies p , $\theta = 30^\circ$.

Mode No.	ANSYS	Reference [22]	General shell	Appr	2D shell
1	3.4120	3.4123	3.4124	3.4124	3.423
2	19.1535	19.149	19.1497	19.1488	19.3153
3	49.3554	49.361	49.3625	49.3511	49.3894
4	50.0457	50.051	50.0523	50.0522	50.0466
5	57.7773	57.748	57.7512	57.7500	57.9485
6	75.5745	75.764	75.7716	75.7636	75.8042
7	94.1668	94.143	94.1522	94.1677	94.2528
8	105.4680	105.51	105.5190	105.5124	105.525

Table 3: Comparisons of the first 8 non-dimensional eigenfrequencies p , $\theta = 60^\circ$.

Mode No.	ANSYS	Reference [22]	General shell	Appr	2D shell
1	3.2427	3.2439	3.2440	3.2443	3.3031
2	14.6236	14.623	14.6233	14.6232	15.0572
3	48.7050	48.685	48.6868	48.6821	49.2160
4	65.8017	65.851	65.8496	65.8351	65.9614
5	73.1679	73.234	73.2329	73.2251	73.1994
6	101.0775	100.96	100.9701	100.9547	101.6134
7	107.8003	107.84	107.8422	107.8003	107.8947
8	138.4014	138.77	138.7862	138.7551	138.7956

Table 4: Comparisons of the first 8 non-dimensional eigenfrequencies p , $\theta = 90^\circ$.

Mode No.	ANSYS	Reference [22]	General shell	Appr	2D shell
1	3.0312	3.0335	3.0329	3.0355	3.2701
2	10.9546	10.951	10.9557	10.9663	11.6225
3	39.9872	39.987	39.9823	39.9709	40.8919
4	70.8513	71.043	71.0375	71.0351	71.2311
5	79.5863	79.755	79.7505	79.7670	79.8076
6	86.3795	86.330	86.3370	86.3086	87.4514
7	124.2122	123.91	123.9135	123.8445	124.0703
8	146.7521	146.54	146.5570	146.5045	147.5416

The second step is to validate the eigenfrequencies of a pretwisted blade subjected to rotational velocity. The parameters of the blade are $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 0$, $\varphi = 0^\circ/45^\circ/90^\circ$ and $\theta = 60^\circ$. A convergence study of this blade mounted at $\varphi = 0^\circ$, rotating at $q = 1$, is plotted in Fig. 2, where the first 8 predicted eigenfrequencies approach the reference results which are calculated by a very fine mesh of 12480 quad elements and the relative difference is around 10^{-3} . Moreover, a comparison between the presented general shell model and reference data from [18] is made in Table 5. It shows good agreement in most of results, where the first few eigenfrequencies match very well except the case $\varphi = 90^\circ$, $q = 1$. The maximum difference is around 1.7%, because the reference data are obtained by the thin shell theory while the presented general shell model is based on the thick shell theory. In other words, with respect to the thin shell theory, the thick shell theory has more degrees of freedom to describe the structure, which leads to a relatively flexible structure, finally resulting in decreasing the eigenfrequencies. Observing Table 5, most eigenfrequencies predicted by the presented general shell model are lower than literature results.

Figure 2: Convergence study of a rotating pretwisted blade; $q = 1$, $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 1$, $\varphi = 30^\circ$ and $\theta = 60^\circ$.

Table 5: Comparisons of the non-dimensional eigenfrequencies p , $\theta = 60^\circ$, $\bar{l} = 2$, $\bar{t} = 40$ and $\bar{r} = 0$.

Mode No.	Reference [18]	General shell	Reference [18]	General shell	Reference [18]	General shell
	$\varphi = 0^\circ, q = 0.4$		$\varphi = 45^\circ, q = 0.4$		$\varphi = 90^\circ, q = 0.4$	
1	3.427	3.4195	3.494	3.4861	3.658	3.6487
2	15.21	15.1320	15.21	15.1199	15.20	15.1139
3	30.16	30.0198	30.08	29.9910	30.07	30.0213
4	50.07	49.5465	50.07	49.5374	50.06	49.5391
5	76.73	76.2791	76.66	76.2792	76.63	76.2806
6	79.88	79.1760	79.79	79.1652	79.72	79.1774
7	94.43	93.3033	94.40	93.2882	94.39	93.3006
8	119.8	117.9775	119.6	117.9707	119.5	117.9764
	$\varphi = 0^\circ, q = 1$		$\varphi = 45^\circ, q = 1$		$\varphi = 90^\circ, q = 1$	
1	3.487	3.4837	3.916	3.9011	4.816	4.7341
2	16.2	16.1113	16.17	16.0338	16.26	15.9965
3	30.17	30.1718	29.76	29.9927	30.26	30.1817
4	51.14	50.6449	51.16	50.5886	51.25	50.5993
5	77.00	76.6245	76.52	76.6252	76.51	76.6349
6	79.97	79.5618	79.44	79.4945	79.31	79.57707
7	94.60	93.5928	94.47	93.4945	95.01	93.5766
8	120.1	118.6516	119.3	118.6094	118.9	118.6456

3.2. Parametric analysis

The advantage of the proposed general shell model is that it is fast and easy to vary parameters, which is quite useful to investigate a number of possible blade configurations in the beginning design phase. In this section, a parametric analysis is performed to study the influence of the rotation velocity and stagger angle on the dynamic characteristics of the blade. In the following analyses, the basic parameters of the studied blade are: $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 1$, $\varphi = 30^\circ$ and $\theta = 60^\circ$.

Figure 3: Campbell diagram of a rotating pretwisted blade; $q = 0 \sim 5$, $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 1$, $\varphi = 30^\circ$ and $\theta = 60^\circ$.

Figure 4: Mode shapes of modes 8, 9 and 10 of the blade rotating at $q = 2.5, 3.5, 4.5$.

A typical Campbell diagram is plotted to study the change of the blade eigenfrequencies for varying rotation velocity $q = 0 \sim 5$ in Fig. 3. It is obvious that all eigenfrequencies increase when the rotation velocity increases as a result of the stiffening due to the centrifugal forces. Furthermore, frequency loci veering is observed: modes 2 and 3 exchange mode shapes around the rotation velocity $q = 4.5$; modes 9 and 10 exchange mode shapes at $q = 3$. Figure 4 shows mode shapes of modes 8, 9 and 10 at the rotation velocity from 2.5 to 4.5. It can be seen that mode 9 exchanges mode shapes with mode 10 when the rotation velocity changes from 2.5 to 3.5 and then, mode 9 exchanges mode shapes with mode 8 when the rotation velocity varies from 3.5 to 4.5.

The influence of the disk radius on the predicted eigenfrequencies is studied for $\bar{r}$ from 0 to 5. At the same rotation velocity, the larger disk radius generates a stronger centrifugal force (i.e. prestressed field) on the blade structure, which increases the stiffness of the blade, finally leading to higher eigenfrequencies. In Fig. 5, all eigenfrequencies increase as the disk radius increases.

Figure 5: The influence of the radius of the disk ($\bar{r} = 0, 0.5, 1, 1.5, 2, 5$) on the predicted eigenfrequencies; $q = 0.8, \bar{l} = 2, \bar{t} = 40, \varphi = 30^\circ$ and $\theta = 60^\circ$.

3.3. Damping analysis

Proportional damping is introduced into the current rotating blade model through a complex Young's modulus $E = E_0(1 + i\eta)$, where $i = \sqrt{-1}$ and η is the loss fac-

tor [31, 32]. Once the complex Young's modulus is applied, the corresponding complex eigenfrequencies can be obtained by modal analysis. Thus, the modal damping ratio can be calculated by $\xi_k = -\text{Real}\{\lambda_k\} / \sqrt{(\text{Real}\{\lambda_k\})^2 + (\text{Imag}\{\lambda_k\})^2}$, where λ_k is the k th eigenvalue; $\text{Real}\{\cdot\}$ and $\text{Imag}\{\cdot\}$ denote the real and imaginary parts of " $\cdot$ ". A damped pretwisted blade (pretwist angle $\theta = 60^\circ$) mounted at a 30° stagger angle is studied with the rotation velocity $q = 0.5 - 6.5$. Corresponding parameters are: $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 1$ and the loss factor is set 1%.

Both the modal damping ratios and exponential damping factors ($\eta_k \omega$) as a function of the rotation velocity are shown in Figs. 6 and 7. In Fig. 6, it can be observed that, in general, at the first eigenfrequencies (modes 3, 4, and 6-10), the modal damping ratios decrease with increasing rotation velocity due to the combined influence of the stiffening caused by the centrifugal forces and the Coriolis forces. In order to assess the influence of the Coriolis forces on the modal damping ratios, the exponential damping factors are plotted in Fig. 7. As seen Fig. 3, the frequency veering loci among modes 8-10 in the rotation velocity range from 2 to 3 leads to both mode shape exchanges and loss factor exchanges. In Fig. 7, modes 8 and 9 exchanges the modal damping ratio at $q = 2$ while modes 9 and 10 exchanges the modal damping ratio around $q = 3$. Keeping the mentioned loss factor exchange in mind, it can be seen that the exponential damping factor decreases with increasing rotational velocity in the first half of the velocity range considered. This is in agreement with the conclusions from previous work on multi-layered blades [33]. For higher rotational velocities the exponential damping factor increases for the first two modes as the rotational velocity increases and continues to decrease for the higher order modes. This result is also in agreement with the literature as well, since, as shown in [34], as the rotational velocity increases the loss factors can decrease, increase or be constant depending on the velocity range considered.

Figure 6: Modal damping ratios at the first 10 eigenfrequencies; $q = 0.5 \sim 6.5$, $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 1$, $\varphi = 30^\circ$ and $\theta = 60^\circ$.

Figure 7: Exponential damping factors at the first 10 eigenfrequencies; $q = 0.5 \sim 6.5$, $\bar{l} = 2$, $\bar{t} = 40$, $\bar{r} = 1$, $\varphi = 30^\circ$ and $\theta = 60^\circ$.

4. Conclusion

In this paper, a dynamic model of a pretwisted rotating blade based on the general shell theory is proposed to investigate the vibration behavior of the blade

at an arbitrary stagger angle and rotation velocity. Using the Hamilton's principle, the equivalent weak form of the governing equation is derived. This model allows to study the dynamic behavior of a rotating pretwisted blade of arbitrary thickness, stagger angle and pretwist. The validation is performed by comparison to literature and ANSYS results, showing good agreement. Moreover, the efficiency analysis demonstrates that the presented model has lower computational cost than the ANSYS solver. The influence of various parameters on the predicted eigenfrequencies is studied. The Campbell diagram reveals frequency loci veering phenomena which cause mode shape exchange when varying the rotation velocity. Finally, a damped rotating blade is studied by using a proportional damping model. A clear influence of the rotational velocity on the modal damping ratios is observed, including the effect of frequency loci veering. If the modal damping ratio exchanges due to frequency loci veering are kept in mind, it can be concluded that for the lower part of the rotational velocity range considered, the modal loss factors decrease as the rotational velocity increases. At higher rotational velocities the loss factors can either increase or decrease depending on the mode order. In short, the presented general shell model is an efficient and accurate tool to predict the eigenfrequencies and damping ratio of the pretwisted rotating blade in the early design stage of compressor blades.

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Nomenclature

A, B	First fundamental forms corresponding to unit vectors ($\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3$)
$\hat{A}, \hat{B}$	First fundamental forms corresponding to unit vectors ($\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$)
E	Blade length
$\mathbf{F}$	Centrifugal force
K	Shear correction
L	Blade length
R	Disk radius
T	Kinetic energy
U_p	Potential energy
U_s	Strain energy
$\bar{V}$	Velocity vector
b	Blade width
h	Blade thickness
$\bar{l}$	Aspect ratio L/b
p	Non-dimensional eigenfrequencies $\omega^2 12(1 - \mu^2)\rho L^4 / Eh^2$
q	Non-dimensional rotational velocity Ω/ω_1 , with ω_1 the first blade eigenfrequency
$\bar{t}$	Thickness ratio L/h
$\bar{r}$	Radius to length ratio R/L
$\mathbf{r}$	Position vector of an arbitrary point on the pretwisted blade after deformation
$\mathbf{r}_0$	Position vector of an arbitrary point on the middle surface of the blade
$\mathbf{r}'$	Position vector of an arbitrary point on the pretwisted blade
Ω	Rotation velocity
ϵ_{ij}	Strain components in the coordinate system ($\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3$)
ε_{ij}	Strain components in the coordinate system ($\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3$)

φ	Stagger angle
μ	Blade length
ρ	Density
σ_{ij}	Stress components in the coordinate system ($\mathbf{a}_1$ $\mathbf{a}_2$ $\mathbf{a}_3$)
θ	Pretwist angle at free end
$\tilde{\theta}_0$	Twist rate θ/L

Appendix A. Derivations of strain components

- The strain-displacement relation in the local coordinate ($\mathbf{g}_1 \ \mathbf{g}_2 \ \mathbf{g}_3$) can be written as $\mathbf{E} = \mathbf{DH}$, where

$$\mathbf{E} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \gamma_{12} & \gamma_{21} & \epsilon_{13} & \epsilon_{23} \end{bmatrix}^T,$$

$$\mathbf{H} = \begin{bmatrix} u & u_{,x} & u_{,\beta} & u_{,\zeta} & v & v_{,x} & v_{,\beta} & v_{,\zeta} & w & w_{,x} & w_{,\beta} & w_{,\zeta} \end{bmatrix}^T$$

and

$$\mathbf{D} = \begin{bmatrix} \frac{\partial \mathbf{a}_1}{\partial \beta} \frac{\mathbf{g}_3}{B} + \frac{\partial \mathbf{a}_1}{\partial \zeta} \mathbf{g}_2 & \frac{\partial \mathbf{a}_1}{\partial x} \frac{\mathbf{g}_3}{A} + \frac{\partial \mathbf{a}_1}{\partial \zeta} \mathbf{g}_1 & \frac{\partial \mathbf{a}_1}{\partial \beta} \frac{\mathbf{g}_1}{B} & \frac{\partial \mathbf{a}_1}{\partial x} \frac{\mathbf{g}_2}{A} & \frac{\partial \mathbf{a}_1}{\partial \beta} \frac{\mathbf{g}_2}{B} & \frac{\partial \mathbf{a}_1}{\partial x} \frac{\mathbf{g}_1}{A} \\ 0 & \frac{\mathbf{a}_1 \mathbf{g}_3}{A} & 0 & \frac{\mathbf{a}_1 \mathbf{g}_2}{A} & 0 & \frac{\mathbf{a}_1 \mathbf{g}_1}{A} \\ \frac{\mathbf{a}_1 \mathbf{g}_3}{B} & 0 & \frac{\mathbf{a}_1 \mathbf{g}_1}{B} & 0 & \frac{\mathbf{a}_1 \mathbf{g}_2}{B} & 0 \\ \mathbf{a}_1 \mathbf{g}_2 & \mathbf{a}_1 \mathbf{g}_1 & 0 & 0 & 0 & 0 \\ \frac{\partial \mathbf{a}_2}{\partial \beta} \frac{\mathbf{g}_3}{B} + \frac{\partial \mathbf{a}_2}{\partial \zeta} \mathbf{g}_2 & \frac{\partial \mathbf{a}_2}{\partial x} \frac{\mathbf{g}_3}{A} + \frac{\partial \mathbf{a}_2}{\partial \zeta} \mathbf{g}_1 & \frac{\partial \mathbf{a}_2}{\partial \beta} \frac{\mathbf{g}_1}{B} & \frac{\partial \mathbf{a}_2}{\partial x} \frac{\mathbf{g}_2}{A} & \frac{\partial \mathbf{a}_2}{\partial \beta} \frac{\mathbf{g}_2}{B} & \frac{\partial \mathbf{a}_2}{\partial x} \frac{\mathbf{g}_1}{A} \\ 0 & \frac{\mathbf{a}_2 \mathbf{g}_3}{A} & 0 & \frac{\mathbf{a}_2 \mathbf{g}_2}{A} & 0 & \frac{\mathbf{a}_2 \mathbf{g}_1}{A} \\ \frac{\mathbf{a}_2 \mathbf{g}_3}{B} & 0 & \frac{\mathbf{a}_2 \mathbf{g}_1}{B} & 0 & \frac{\mathbf{a}_2 \mathbf{g}_2}{B} & 0 \\ \mathbf{a}_2 \mathbf{g}_2 & \mathbf{a}_2 \mathbf{g}_1 & 0 & 0 & 0 & 0 \\ \frac{\partial \mathbf{a}_3}{\partial \beta} \frac{\mathbf{g}_3}{B} + \frac{\partial \mathbf{a}_3}{\partial \zeta} \mathbf{g}_2 & \frac{\partial \mathbf{a}_3}{\partial x} \frac{\mathbf{g}_3}{A} + \frac{\partial \mathbf{a}_3}{\partial \zeta} \mathbf{g}_1 & \frac{\partial \mathbf{a}_3}{\partial \beta} \frac{\mathbf{g}_1}{B} & \frac{\partial \mathbf{a}_3}{\partial x} \frac{\mathbf{g}_2}{A} & \frac{\partial \mathbf{a}_3}{\partial \beta} \frac{\mathbf{g}_2}{B} & \frac{\partial \mathbf{a}_3}{\partial x} \frac{\mathbf{g}_1}{A} \\ 0 & \frac{\mathbf{a}_3 \mathbf{g}_3}{A} & 0 & \frac{\mathbf{a}_3 \mathbf{g}_2}{A} & 0 & \frac{\mathbf{a}_3 \mathbf{g}_1}{A} \\ \frac{\mathbf{a}_3 \mathbf{g}_3}{B} & 0 & \frac{\mathbf{a}_3 \mathbf{g}_1}{B} & 0 & \frac{\mathbf{a}_3 \mathbf{g}_2}{B} & 0 \\ \mathbf{a}_3 \mathbf{g}_2 & \mathbf{a}_3 \mathbf{g}_1 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

- The scalar multiplication δ_q^p is derived as

$$\left[\delta_q^p \right]_{3 \times 3} = \frac{1}{(1 - \zeta^2 \tilde{\theta}_0^2 / \hat{A}^4)} \begin{bmatrix} \frac{\hat{A}}{\hat{A}^3} & \frac{\zeta \tilde{\theta}_0 \hat{A}}{\hat{A}^3} & 0 \\ \frac{\zeta \tilde{\theta}_0 \hat{A}}{\hat{A}^3} & \frac{\hat{A}}{\hat{A}^3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- By using the thick shell theory, the simplified strain components can be written as $\epsilon_{ij} = e_{ij} + \zeta g_{ij} + \zeta^2 k_{ij}$ and corresponding test function as $\epsilon_{ij}^q = e_{ij}^q + \zeta g_{ij}^q + \zeta^2 k_{ij}^q$ ($i = 1, 2$ and $j = 1, 2, 3$), where the strain components in ζ^0 ,

ζ^1 and ζ^2 terms read

$$\begin{aligned}
e_{11} &= \frac{1}{\hat{A}} u_{0,x} + \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} v_0, & e_{11}^q &= \frac{1}{\hat{A}} u_{0,x}^q + \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} v_0^q, \\
e_{22} &= v_{0,\beta}, & e_{22}^q &= v_{0,\beta}^q, \\
e_{12} &= \frac{1}{\hat{A}} v_{0,x} - \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} u_0 + u_{0,\beta} - \frac{2\tilde{\theta}_0}{\hat{A}^2} w_0, \\
e_{12}^q &= \frac{1}{\hat{A}} v_{0,x}^q - \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} u_0^q + u_{0,\beta}^q - \frac{2\tilde{\theta}_0}{\hat{A}^2} w_0^q, \\
e_{13} &= \frac{1}{\hat{A}} w_{0,x} + \frac{\tilde{\theta}_0}{\hat{A}^2} v_0 + u_{0,\zeta}, & e_{13}^q &= \frac{1}{\hat{A}} w_{0,x}^q + \frac{\tilde{\theta}_0}{\hat{A}^2} v_0^q + u_{0,\zeta}^q, \\
e_{23} &= w_{0,\beta} + \frac{\tilde{\theta}_0}{\hat{A}^2} u_0 + v_{0,\zeta}, & e_{23}^q &= w_{0,\beta}^q + \frac{\tilde{\theta}_0}{\hat{A}^2} u_0^q + v_{0,\zeta}^q,
\end{aligned}$$

$$\begin{aligned}
k_{11} &= \frac{1}{\hat{A}} \theta_{\alpha,x} + \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} \theta_\beta + \frac{\tilde{\theta}_0}{\hat{A}^2} u_{0,\beta} - \frac{\tilde{\theta}_0^2}{\hat{A}^4} w_0, \\
k_{11}^q &= \frac{1}{\hat{A}} \theta_{\alpha,x}^q + \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} \theta_\beta^q + \frac{\tilde{\theta}_0}{\hat{A}^2} u_{0,\beta}^q - \frac{\tilde{\theta}_0^2}{\hat{A}^4} w_0^q, \\
k_{22} &= \theta_{\beta,\beta} - \frac{\tilde{\theta}_0^2}{\hat{A}^4} w_0 + \frac{\tilde{\theta}_0}{\hat{A}^3} v_{0,x} - \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} u_0, & k_{22}^q &= \theta_{\beta,\beta}^q - \frac{\tilde{\theta}_0^2}{\hat{A}^4} w_0^q + \frac{\tilde{\theta}_0}{\hat{A}^3} v_{0,x}^q - \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} u_0^q, \\
k_{12} &= \frac{1}{\hat{A}} \theta_{\beta,x} - \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} \theta_\alpha + \theta_{\alpha,\beta} + \frac{\tilde{\theta}_0}{\hat{A}^3} u_{0,x} + \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} v_0 + \frac{\tilde{\theta}_0}{\hat{A}^2} v_{0,\beta}, \\
k_{12}^q &= \frac{1}{\hat{A}} \theta_{\beta,x}^q - \frac{\beta \tilde{\theta}_0^2}{\hat{A}^2} \theta_\alpha^q + \theta_{\alpha,\beta}^q + \frac{\tilde{\theta}_0}{\hat{A}^3} u_{0,x}^q + \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} v_0^q + \frac{\tilde{\theta}_0}{\hat{A}^2} v_{0,\beta}^q, \\
k_{13} &= \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_\beta + \frac{\tilde{\theta}_0}{\hat{A}^2} w_{0,\beta} + \frac{\tilde{\theta}_0^2}{\hat{A}^4} u_0, & k_{13}^q &= \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_\beta^q + \frac{\tilde{\theta}_0}{\hat{A}^2} w_{0,\beta}^q + \frac{\tilde{\theta}_0^2}{\hat{A}^4} u_0^q, \\
k_{23} &= \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_\alpha + \frac{\tilde{\theta}_0}{\hat{A}^3} w_{0,x} + \frac{\tilde{\theta}_0^2}{\hat{A}^4} v_0, & k_{23}^q &= \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_\alpha^q + \frac{\tilde{\theta}_0}{\hat{A}^3} w_{0,x}^q + \frac{\tilde{\theta}_0^2}{\hat{A}^4} v_0^q,
\end{aligned}$$

$$\begin{aligned}
g_{11} &= \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_{\alpha\beta}, & g_{11}^q &= \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_{\alpha\beta}^q, \\
g_{22} &= \frac{\tilde{\theta}_0}{\hat{A}^3} \theta_{\beta,x} - \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} \theta_\alpha, & g_{22}^q &= \frac{\tilde{\theta}_0}{\hat{A}^3} \theta_{\beta,x}^q - \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} \theta_\alpha^q, \\
g_{12} &= \frac{\tilde{\theta}_0}{\hat{A}^3} \theta_{\alpha,x} + \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} \theta_\beta + \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_{\beta\beta}, \\
g_{12}^q &= \frac{\tilde{\theta}_0}{\hat{A}^3} \theta_{\alpha,x}^q + \frac{\beta \tilde{\theta}_0^3}{\hat{A}^4} \theta_\beta^q + \frac{\tilde{\theta}_0}{\hat{A}^2} \theta_{\beta\beta}^q, \\
g_{13} &= \frac{\tilde{\theta}_0^2}{\hat{A}^4} \theta_\alpha, & g_{13}^q &= \frac{\tilde{\theta}_0^2}{\hat{A}^4} \theta_\alpha^q, \\
g_{23} &= \frac{\tilde{\theta}_0^2}{\hat{A}^4} \theta_\beta, & g_{23}^q &= \frac{\tilde{\theta}_0^2}{\hat{A}^4} \theta_\beta^q.
\end{aligned}$$

AppendixB. Derivations of the kinetic energy

The kinetic energy can be derived as

$$\begin{aligned}
T &= \int_V \frac{1}{2} \rho \hat{A} \hat{B} \bar{V}_i^2 dx d\beta d\zeta \\
&= \int_V \left[\frac{1}{2} \rho \hat{A} \hat{B} (\bar{V}_1^2 + \bar{V}_2^2 + \bar{V}_3^2) \right] dx d\beta d\zeta \\
&= \int_V \left[\frac{1}{2} \rho \hat{A} \hat{B} \right] dx d\beta d\zeta \left\{ \dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \bar{V}_x^2 + \bar{V}_1^2 \right. \\
&\quad \left. + 2 \frac{1}{\hat{A}} \dot{u} \bar{V}_x - 2 \frac{\beta \tilde{\theta}_0}{\hat{A}} \cos \phi \dot{u} \bar{V}_1 - 2 \sin \phi \dot{v} \bar{V}_1 \right. \\
&\quad \left. - 2 \frac{\beta \tilde{\theta}_0}{\hat{A}} \dot{w} \bar{V}_x - 2 \frac{1}{\hat{A}} \cos \phi \dot{w} \bar{V}_1 \right\}.
\end{aligned}$$

AppendixC. Equations of motion by using the 2D shell model

By using the 2D shell model, equations of motion corresponding to each variable are given

For u_0 :

$$\begin{aligned}
& [-\hat{A}Mu_{0,tt} - 2\hat{A}M\Omega \cos \phi w_{0,t} - 2M\Omega \sin \phi v_{0,t}]u_0^q \\
& + [M\Omega^2(\frac{1}{\hat{A}} + \hat{A}\left(\frac{\beta\tilde{\theta}_0}{\hat{A}}\right)^2 \cos^2 \phi)u_0 + M\Omega^2\beta\tilde{\theta}_0 \sin \phi \cos \phi v_0 - M\Omega^2\frac{\beta\tilde{\theta}_0}{\hat{A}} \sin^2 \phi w_0]u_0^q \\
& + \left\{ I_1 h \left[\frac{1}{\hat{A}} u_{0,\alpha} + \frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} v_0 + \mu v_{0,\beta} \right] u_{0,\alpha}^q + I_2 h \left[v_{0,\alpha} - \frac{\beta\tilde{\theta}_0^2}{\hat{A}} u_0 - \frac{2\tilde{\theta}_0}{\hat{A}} w_0 + \hat{A} u_{0,\beta} \right] u_{0,\beta}^q \right. \\
& + I_2 h \left[-\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} v_{0,\alpha} + \hat{A} \left(\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \right)^2 u_0 + \frac{2\beta\tilde{\theta}_0^3}{\hat{A}^3} w_0 - \frac{\beta\tilde{\theta}_0^2}{\hat{A}} u_{0,\beta} \right] u_0^q \\
& \left. + I_3 h \left[\frac{\tilde{\theta}_0}{\hat{A}} w_{0,\beta} + \hat{A} \left(\frac{\tilde{\theta}_0}{\hat{A}^2} \right)^2 u_0 + \frac{\tilde{\theta}_0}{\hat{A}} \theta_\beta \right] u_0^q \right\} = 0
\end{aligned}$$

For v_0 :

$$\begin{aligned}
& [-\hat{A}Mv_{0,tt} + 2M\Omega \sin \phi u_{0,t} - 2M\Omega\beta\tilde{\theta}_0 \sin \phi w_{0,t}]v_0^q \\
& + [\hat{A}M\Omega^2 \sin^2 \phi v_0 + M\Omega^2\beta\tilde{\theta}_0 \sin \phi \cos \phi u_0 + M\Omega^2 \sin \phi \cos \phi w_0]v_0^q \\
& + \left\{ I_1 h \left[\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} u_{0,\alpha} + \hat{A} \left(\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \right)^2 v_0 + \mu \frac{\beta\tilde{\theta}_0^2}{\hat{A}} v_{0,\beta} \right] v_0^q + I_1 h [\hat{A}v_{0,\beta} + \mu u_{0,\alpha} + \mu \frac{\beta\tilde{\theta}_0^2}{\hat{A}} v_0] v_{0,\beta}^q \right. \\
& \left. + I_2 h \left[\frac{1}{\hat{A}} v_{0,\alpha} - \frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} u_0 - \frac{2\tilde{\theta}_0}{\hat{A}^2} w_0 + u_{0,\beta} \right] v_{0,\alpha}^q + I_3 h \left[\frac{\tilde{\theta}_0}{\hat{A}^2} w_{0,\alpha} + \hat{A} \left(\frac{\tilde{\theta}_0}{\hat{A}^2} \right)^2 v_0 + \frac{\tilde{\theta}_0}{\hat{A}} \theta_\alpha \right] v_0^q \right\} = 0
\end{aligned}$$

For w_0 :

$$\begin{aligned}
& [-\hat{A}Mw_{0,tt} + 2\hat{A}M\Omega \cos \phi u_{0,t} + 2M\Omega\beta\tilde{\theta}_0 \sin \phi v_{0,t}]w_0^q \\
& + [M\Omega^2(\frac{1}{\hat{A}} \cos^2 \phi + \hat{A}\left(\frac{\beta\tilde{\theta}_0}{\hat{A}}\right)^2)w_0 + M\Omega^2 \sin \phi \cos \phi v_0 - M\Omega^2\frac{\beta\tilde{\theta}_0}{\hat{A}} \sin^2 \phi u_0]w_0^q \\
& + \left\{ I_2 h \left[-\frac{2\tilde{\theta}_0}{\hat{A}^2} v_{0,\alpha} + \frac{2\beta\tilde{\theta}_0^3}{\hat{A}^3} u_0 + \frac{4\tilde{\theta}_0^2}{\hat{A}^3} w_0 - \frac{2\tilde{\theta}_0}{\hat{A}} u_{0,\beta} \right] w_0^q \right. \\
& \left. + I_3 h \left[\frac{1}{\hat{A}} w_{0,\alpha} + \frac{\tilde{\theta}_0}{\hat{A}^2} v_0 + \theta_\alpha \right] w_{0,\alpha}^q + I_3 h \left[\hat{A}w_{0,\beta} + \frac{\tilde{\theta}_0}{\hat{A}} u_0 + \hat{A}\theta_\beta \right] w_{0,\beta}^q \right\} = 0
\end{aligned}$$

For θ_α :

$$\begin{aligned}
& [-\hat{A}\rho I\theta_{\alpha,tt} - 2\rho I\Omega \sin \phi \theta_{\beta,t}] \theta_\alpha^q \\
& + [\rho I\Omega^2 \left(\frac{1}{\hat{A}} + \hat{A} \left(\frac{\beta\tilde{\theta}_0}{\hat{A}} \right)^2 \cos^2 \phi \right) \theta_\alpha + \rho I\Omega^2 \beta\tilde{\theta}_0 \sin \phi \cos \phi \theta_\beta] \theta_\alpha^q \\
& \left\{ D \left[\frac{1}{\hat{A}} \theta_{\alpha,\alpha} + \frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \theta_\beta + \mu \theta_{\beta,\beta} \right] \theta_{\alpha,\alpha}^q + I_2 I \left[-\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \theta_{\beta,\alpha} + \hat{A} \left(\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \right)^2 \theta_\alpha - \frac{\beta\tilde{\theta}_0^2}{\hat{A}} \theta_{\alpha,\beta} \right] \theta_\alpha^q \right. \\
& \left. + I_3 h \left[w_{0,\alpha} + \frac{\tilde{\theta}_0}{\hat{A}} v_0 + \hat{A} \theta_\alpha \right] \theta_\alpha^q + I_3 I \hat{A} \left(\frac{\tilde{\theta}_0}{\hat{A}^2} \right)^2 \theta_\alpha \theta_\alpha^q + I_2 I \left[\theta_{\beta,\alpha} - \frac{\beta\tilde{\theta}_0^2}{\hat{A}} \theta_\alpha + \hat{A} \theta_{\alpha,\beta} \right] \theta_{\alpha,\beta}^q \right\} = 0
\end{aligned}$$

For θ_β :

$$\begin{aligned}
& [-\hat{A}\rho I\theta_{\beta,tt} + 2\rho I\Omega \sin \phi \theta_{\alpha,t}] \theta_\beta^q \\
& + [\hat{A}\rho I\Omega^2 \sin^2 \phi \theta_\beta + \rho \frac{I}{12} \Omega^2 \beta\tilde{\theta}_0 \sin \phi \cos \phi \theta_\alpha] \theta_\beta^q \\
& + \left\{ D \left[\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \theta_{\alpha,\alpha} + \hat{A} \left(\frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \right)^2 \theta_\beta + \mu \frac{\beta\tilde{\theta}_0^2}{\hat{A}} \theta_{\beta,\beta} \right] \theta_\beta^q + D [\hat{A} \theta_{\beta,\beta} + \mu \theta_{\alpha,\alpha} + \mu \frac{\beta\tilde{\theta}_0^2}{\hat{A}} \theta_\beta] \theta_{\beta,\beta}^q \right. \\
& \left. + I_3 h [\hat{A} w_{0,\beta} + \frac{\tilde{\theta}_0}{\hat{A}} u_0 + \hat{A} \theta_\beta] \theta_\beta^q + I_3 I \hat{A} \left(\frac{\tilde{\theta}_0}{\hat{A}^2} \right)^2 \theta_\beta \theta_\beta^q + I_2 I \left[\frac{1}{\hat{A}} \theta_{\beta,\alpha} - \frac{\beta\tilde{\theta}_0^2}{\hat{A}^2} \theta_\alpha + \theta_{\alpha,\beta} \right] \theta_{\beta,\alpha}^q \right\} = 0.
\end{aligned}$$

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