A generalized memoryless property

Kella, O.; Löpker, A.H.

Published: 01/01/2011

Citation for published version (APA):
A Generalized Memoryless Property

This report has been realized with support of NWO (grant 040.11.257)

O. Kella, A. Löpker
ISSN 1389-2355
A Generalized Memoryless Property

Offer Kella∗† Andreas Löpker‡

October 23, 2011

Abstract

We consider a generalized memoryless property which relates to Cantor’s second functional equation, study its properties and demonstrate various examples.

Keywords: Generalized memoryless property, Markov kernel, Cantor’s second functional equation.

AMS Subject Classification: Primary: 60K30 Secondary: 90B18, 60J25, 60J35, 60G44, 68M20.

1 Introduction and setup

Let \( \mathbb{R} \) denote the set of real numbers and \( \mathcal{B} \) be its Borel sets. Consider a Markov kernel \( P(x, A) \) where for each \( x \in \mathbb{R} \), \( P(x, \cdot) \) is a probability measure on \( (\mathbb{R}, \mathcal{B}) \). We say that the generalized memoryless property is satisfied if the following is satisfied for each nonnegative \( x,y \) and real \( z \):

\[
P(z, (x + y, \infty)) = P(z, (x, \infty))P(z + x, (y, \infty))
\]  

(1)

We note that if \( P(x, \cdot) = P(\cdot) \), that is, the Markov kernel is independent of the originating state, this is precisely the memoryless property of which the only solution is of the form \( P((x, \infty)) = \rho^x \) for some \( 0 \leq \rho \leq 1 \). That is, a random variable that has such a distribution is either almost surely (a.s.) zero, or a.s. infinite or has an exponential distribution.

To motivate the property given by (1) assume that in addition \( P(\cdot, A) \) is a Borel function for each \( A \in \mathcal{B} \) and let \( T_a \) (age) and \( T_r \) (remaining life) be a pair of random variables, where \( T_a \) has an arbitrary distribution and \( \mathbb{P}(T_r \in A|T_a) = P(T_a, A) \). That is, one interprets \( \mathbb{P}(T_r \in A|T_a = z) = P(z, A) \). Then (1) becomes

\[
\mathbb{P}(T_r > x + y|T_a = z) = \mathbb{P}(T_r > x|T_a = z)\mathbb{P}(T_r > y|T_a = z + x) .
\]  

(2)
In other words, in order for the a component having age $z$ to function at time $x + y$, it first has to function at time $x$. Then, independent of everything else, its age is modified to $z + x$ and, given that its age is now $z + x$, it has to function at time $y$. This model relates to Model II of [7]. In fact, given the condition that $P(0, (x, \infty)) > 0$ for each $x \geq 0$ this results in exactly the same model, even though we write it in more primitive terms. To see this, denote $1 - F(x) = P(0, (x, \infty))$ and observe that with $z = 0$, (2) becomes

$$1 - F(x + y) = (1 - F(x))P(T_r > y | T_z = x),$$

so that indeed

$$P(T_r > y | T_z = x) = \frac{1 - F(x + y)}{1 - F(x)},$$

which is equation (1) of [7] with the roles of $x$ and $y$ reversed. However, if we only assume that $P(z, (x, \infty)) > 0$ for all $0 < z \leq x$ then we will see that it is possible that $P(T_r > y | T_z = x) = s(x + y)/s(x)$ for some nonincreasing function $s$ for which $s(x) \to \infty$ as $x \downarrow 0$. We will also see that other possibilities may occur as well. We also mention that for the case where $P(z, (x, \infty)) > 0$ a more general model was considered in [9]. However, in this paper the authors suffice in pointing out some examples of this property but do not characterize the general form. In this generality, a characterization might not be possible.

To continue, for real $x$ and $y \geq x$ we denote $\mu(x, y) = P(x, (y - x, \infty))$ and observe that (1) becomes

$$\mu(z, y) = \mu(z, x)\mu(x, y)$$

for each $z \leq x \leq y$. If this was valid for all $x, y, z$ then (5) is called Cantor’s second functional equation. We note that in the latter case, if $\mu(z, y) \neq 0$ for some $z, y$ then $\mu(z, x) \neq 0$ and $\mu(x, y) \neq 0$ for all $x$. Since $\mu(x, y) = \mu(x, u)\mu(u, y)$ then $\mu(x, u) \neq 0$ for all $x, u$ and in particular if we denote $s(x) = \mu(z, x)$ then for any $x, y$ we have that $\mu(x, y) = s(y)/s(x)$. Thus, as is well known (e.g., see [1]), the only solutions of Cantor’s second functional equation are either $\mu(x, y) = 0$ for all $x, y$ or $\mu(x, y) = s(y)/s(x)$ for some function $s(\cdot)$ that never vanishes. In the second case it is clear that for all $x$, $\mu(x, x) = 1$ and that for any nonvanishing function $s(\cdot)$, $s(y)/s(x)$ obeys Cantor’s second functional equation.

When we assume that (5) is satisfied only if $z \leq x \leq y$, the solution requires a bit more care.

## 2 Main observations

In order to consider the most general setup, let us assume until further notice that

$$\mu : \{(x, y) | x \leq y\} \to \mathbb{R}$$

(note: $\mathbb{R}$ rather than just $[0, 1]$) and that (5) is satisfied for $z \leq x \leq y$. Denote

$$b(z) = \begin{cases} 
  z & \text{if } \mu(z, x) \neq 0, x > z \\
  \sup\{x | \mu(z, x) \neq 0, x > z\} & \text{otherwise}
\end{cases}$$

(7)
and

\[ a(z) = \begin{cases} z & \text{if } \mu(x, z) = 0 \forall x < z \\ \inf\{x|\mu(x, z) \neq 0, \ x < z\} & \text{otherwise.} \end{cases} \]  

(8)

Now, when \( a(z) < b(z) \) we denote

\[ I(z) = \begin{cases} (a(z), b(z)) & \text{if } \mu(a(z), z) = 0 = \mu(z, b(z)) \\ (a(z), b(z]) & \text{if } \mu(a(z), z) = 0 \neq \mu(z, b(z)) \\ [a(z), b(z)) & \text{if } \mu(a(z), z) \neq 0 = \mu(z, b(z)) \\ [a(z), b(z)] & \text{if } \mu(a(z), z) \neq 0 \neq \mu(z, b(z)) \end{cases} \]  

(9)

and when \( a(z) = b(z) = z \) we let \( I(z) = \{z\} \), noting that from \( \mu(z, z) = \mu(z, z)\mu(z, z) \) necessarily \( \mu(z, z) \in \{0, 1\} \). Moreover, since

\[ \mu(u, v) = \mu(u, v)\mu(v, v) = \mu(u, u)\mu(u, v) \]  

(10)

for \( u \leq v \), it is easy to check that if \( I(z) \) is not a singleton, then necessarily \( \mu(z, z) = 1 \).

Consider now the following.

**Lemma 1.** Let \( \mu : \{(x, y)| \ x \leq y\} \to \mathbb{R} \) satisfy (5) for all \( z \leq x \leq y \). Then for \( x < y \), \( \mu(x, y) \neq 0 \) if and only if \( I(x) = I(y) \) and for every \( u, v \in I(x) \) with \( u \leq v \) we have that \( \mu(u, v) \neq 0 \).

**Proof.** For \( v \geq y \) we have that \( \mu(x, v) = \mu(x, y)\mu(y, v) \), so that \( \mu(x, v) \neq 0 \) if and only if \( \mu(y, v) \neq 0 \). For \( u \leq x \) we have by similar reasoning that \( \mu(u, x) \neq 0 \) if and only if \( \mu(u, y) \neq 0 \). For \( w \in (x, y) \) we have that \( \mu(x, y) = \mu(x, w)\mu(w, y) \) and thus \( \mu(x, w) \neq 0 \) and \( \mu(w, y) \neq 0 \). This implies that \( I(x) = I(y) \). Now, for every \( u \in I(x) \) we have that \( I(u) = I(x) \) and thus for any \( v \geq u \) with \( v \in I(x) \) it follows that \( v \in I(u) \) which implies that \( \mu(u, v) \neq 0 \). \(\square\)

**Theorem 1.** Under the conditions of Lemma 1 there exists a family \( \{I_{\theta}| \ \theta \in \Theta\} \) of necessarily at most countably many disjoint intervals (open, half open or closed) and possibly uncountably many singletons with \( \cup_{\theta \in \Theta} I_{\theta} = \mathbb{R} \) such that for each \( \theta \) for which \( I_{\theta} \) is not a singleton there exists a function \( s_{\theta} : I_{\theta} \to \mathbb{R} \) which is nonvanishing such that for every \( x, y \in I_{\theta} \) with \( x \leq y \) we have that \( \mu(x, y) = s_{\theta}(y)/s_{\theta}(x) \).

**Proof.** For an arbitrary \( z \in I_{\theta} \) define

\[ s_{\theta}(x) = \begin{cases} \mu(z, x) & \text{if } z \leq x \in I_{\theta} \\ 1/\mu(x, z) & \text{if } z \geq x \in I_{\theta} . \end{cases} \]  

(11)

Then, if \( z \leq x \leq y \) and \( x, y \in I_{\theta} \) then

\[ s_{\theta}(y) = \mu(z, y) = \mu(z, x)\mu(x, y) = s_{\theta}(x)\mu(x, y) . \]  

(12)

If \( x \leq y \leq z \) and \( x, y \in I_{\theta} \) then

\[ \frac{1}{s_{\theta}(x)} = \mu(x, z) = \mu(x, y)\mu(y, z) = \mu(x, y)\frac{1}{s_{\theta}(y)} . \]  

(13)
Finally, when \( x \leq z \leq y \) with \( x, y \in I_\theta \) then
\[
\mu(x, y) = \mu(x, z)\mu(z, y) = \frac{1}{s_\theta(x)}s_\theta(y).
\] (14)

In particular we observe that if \( \mu(x, y) \neq 0 \) for all \( x \leq y \) then \( I(x) = \mathbb{R} \) and for some nonvanishing function \( s : \mathbb{R} \to \mathbb{R} \) we have that \( \mu(x, y) = s(y)/s(x) \) for all \( x \leq y \). It seems as if this is the same solution of Cantor’s second functional equation until we recall that \( \mu(x, y) \) is undefined for \( x > y \).

Returning to the generalized memoryless property of (1) we recall that \( \mu(x, y) = P(x, (y - x, \infty)) \) and thus, for \( y \geq 0 \), \( P(x, y) = \mu(x, x + y) \). Hence we conclude the following.

**Theorem 2.** Assume that (1) is satisfied. Then there exists a family \( \{I_\theta \mid \theta \in \Theta\} \) of disjoint intervals and singletons with \( \bigcup_{\theta \in \Theta} I_\theta = \mathbb{R} \) such that for each \( \theta \) for which \( I_\theta \) is not a singleton there exists a nonincreasing right continuous strictly positive function \( s_\theta : I_\theta \to \mathbb{R} \) such that for every \( x \in I_\theta \) and \( y \geq 0 \) with \( x + y \in I_\theta \) we have that
\[
P(x, (y, \infty)) = \frac{s_\theta(x + y)}{s_\theta(x)}.
\]

**Remark 1.** We note that the same result holds, only with \( s_\theta \) being left continuous, for \( P(x, [y, \infty)) \) (left closed interval) if we replace (1) by the left closed version
\[
P(z, [x + y, \infty)) = P(z, [x, \infty))P(z + x, [y, \infty))
\] (15)
which, by taking \( y \downarrow u \), where \( u \geq x \), is also equivalent to
\[
P(z, (x + u, \infty)) = P(z, [x, \infty))P(z + x, (u, \infty))
\] (16)
whenever \( x, y \geq 0 \).

**Remark 2.** For the case \( \mu(x, y) = P(x, (y - x, \infty)) \) clearly \( \mu(x, y) \in [0, 1] \) for all \( x \leq y \). Therefore, from \( \mu(z, y) = \mu(z, x)\mu(x, y) \leq \mu(x, y) \) for \( z \leq x \leq y \) it follows that \( \mu(\cdot, y) \) is a nondecreasing function on \( (-\infty, y) \). Similarly \( \mu(x, \cdot) \) is nonincreasing on \( [x, \infty) \).

**Remark 3.** It is easy to check that these results remain unchanged if the domain of \( \mu \) is \( 0 \leq x \leq y \) rather than \( x \leq y \) or \( x \leq y \leq \infty \) or any other combination like this. Basically, whenever we have (5) for \( z \leq x \leq y \) these results apply.

**Remark 4.** When \( P(x, (y, \infty)) > 0 \) for every \( x \) and every \( y \geq 0 \), then there is some necessarily nonincreasing and right continuous function \( s : \mathbb{R} \to (0, \infty) \) for which \( \mu(x, (y, \infty)) = s(x + y)/s(x) \) for all \( x \) and \( y \geq 0 \). Moreover the function \( s \) is uniquely determined up to a constant multiple. We note that unlike in [7], it is possible that for some \( \theta \), \( I_\theta = (0, \infty) \) and \( s_\theta(x) \to \infty \) as \( x \downarrow 0 \). If \( [0, \infty) \subset I_\theta \) for some \( \theta \) then one has the model considered in [7].

4
Remark 5. It is also evident that the same structure holds when one reverses (1) to

\[ P(z, (-\infty, x + y)) = P(z, (-\infty, x))P(z + x, (-\infty, y)) \]

or to

\[ P(z, (-\infty, x + y)) = P(z, (-\infty, x))P(z + x, (-\infty, y)) \]

for all \( x, y \leq 0 \). In particular when \( P(z, (-\infty, 0)) = 0 \) and \(-z \leq x, y \leq 0 \). We will use this in the next section when modeling a certain growth collapse (additive increase multiplicative decrease) process with state dependent decrease ratios.

Consider now a possibly infinite interval \( I_\theta \) which is not a singleton and its corresponding positive valued function \( s_\theta \). Then for each \( z \in I_\theta \) and \( x \geq 0 \) such that \( z + x \in I_\theta \) we have that \( P(z, (x, \infty)) = s_\theta(z + x)/s_\theta(z) \). If \( x + z \not\in I_\theta \) then \( P(z, (x, \infty)) = 0 \) so that we may define \( s_\theta(y) = 0 \) for any \( y \not\in I_\theta \) which is on the right of \( I_\theta \) (if any) where necessarily \( s_\theta \) must be right continuous at \( z^*(\theta) = \sup\{z : z \in I_\theta \} \). Now, for \( z \in I_\theta \) and \( 0 < u < 1 \) denote

\[
t_\theta(u) = \inf \left\{ x \mid \frac{s_\theta(z + x)}{s_\theta(z)} \geq 1 - u, \ x \geq 0 \right\}
\]

\[
= -z + \inf \{ x \mid s_\theta(x) \leq s_\theta(z)u, \ x \geq z \}
\]

\[
= -z + \inf \{ x \mid s_\theta(x) \leq s_\theta(z)u \}
\]

Where the last equality follows since \( s_\theta(x) < s_\theta(z)u \) for every \( x \leq z \). It is standard that if \( U \sim \text{Uniform}(0, 1) \) then \( t_\theta(U) \) and \( t_\theta^*(U) \) have the distribution \( P(z, \cdot) \). Thus, if we denote \( t_\theta(v) = \inf \{ x \mid s_\theta(x) \leq v \} \) for \( v > \inf \{ z \mid z \in I_\theta \} \) then for every \( z \in I_\theta \) we have that \( t_\theta(s_\theta(z)U) - z \) has the distribution \( P(z, \cdot) \). Recalling \( T_a \) and \( T_r \) from Section 1, this implies the following.

Theorem 3. Assuming that \( T_a \) and \( U \sim \text{Uniform}(0, 1) \) are independent, then \((T_a, T_r) \mathbb{1}_{\{T_a \in I_\theta \}}\) and \((T_a, t_\theta(s_\theta(T_a)U) - T_a) \mathbb{1}_{\{T_a \in I_\theta \}}\) are identically distributed.

Clearly, when \( \Theta \) is countable then the immediate conclusion is that

\[
(T_a, T_r) \sim \left( T_a, \sum_\theta (t_\theta(s_\theta(T_a)U) - T_a) \mathbb{1}_{\{T_a \in I_\theta \}} \right).
\]

It is interesting to check when for a given \( \theta \) for which \( I_\theta \) is not a singleton the value of \( t_\theta(s_\theta(z)u) - z \) is independent of \( z \). That is, it is only a function of \( u \). The answer is not surprising.

Theorem 4. When \( I_\theta \) is not a singleton then \( t_\theta(s_\theta(z)u) - z \) is independent of \( z \in I_\theta \) if and only if for some \( 0 \leq \lambda_\theta < \infty \) and \( 0 < c_\theta < \infty \), \( s_\theta(z) = c_\theta e^{-\lambda_\theta z} \) for \( z \in I_\theta \).

Proof. Let \( f(u) = t_\theta(s_\theta(z)u) - z \) (independent of \( z \)) for every \( z \in I_\theta \) and \( 0 < u < 1 \). Note that since the right side is left continuous in \( u \), then so is \( f \) (as a function defined on
In particular $f$ is Borel. Denoting $X = f(U)$ we have that for every $z \in I_\theta$ and every $x,y \geq 0$, with $z + x + y \in I_\theta$,

$$P(X > x + y) = \frac{s_\theta(z + x + y)}{s_\theta(z)} = \frac{s_\theta(z + x)}{s_\theta(z)} \cdot \frac{s_\theta(z + x + y)}{s_\theta(z + x)} = P(X > x)P(X > y)$$

The equation $g(x + y) = g(x)g(y)$ for $x,y \geq 0$ under minor regularity conditions on $g$ implies that $g$ is either identically zero, identically one or exponential. Monotonicity, right or left continuity or even Lebesgue measurability are sufficient conditions. The standard proof can be easily modified to the case where $g$ is defined on and the equation is valid only when $x,y,x + y$ are in $[0,a)$ or $[0,a]$ for some $0 < a < \infty$, resulting in $g$ being identically zero, identically one or exponential on $[0,a)$ or $[0,a]$. When we know that $g$ is strictly positive and bounded above by one on $[0,a)$ or $[0,a]$, then zero is not an option and thus $g(x) = e^{-\lambda x}$ for some $0 \leq \lambda < \infty$. Thus, for every $z \in I_\theta$ and $x \geq 0$ such that $w = z + x \in \theta$ we have that for some $0 \leq \lambda_\theta < \infty$

$$\frac{s_\theta(z + x)}{s_\theta(z)} = e^{-\lambda_\theta x} = \frac{e^{-\lambda_\theta(z+x)}}{e^{-\lambda_\theta z}} \quad (21)$$

which implies that for every $z,w \in I_\theta$

$$s_\theta(w)e^{\lambda_\theta w} = s_\theta(z)e^{\lambda_\theta z} \equiv c_\theta, \quad (22)$$

as required. \qed

## 3 Maximum at a random time of a continuous time Markov process with no positive jumps

Consider a continuous time right continuous Markov process $\{X(t)\}_{t \geq 0}$ with convex state space $\mathcal{X} \subset \mathbb{R}$, having no positive jumps and with generator $A$. As is customary, we denote $\mathbb{P}_x$ and $\mathbb{E}_x$ the distribution measure and the expected value when the process is initiated at $x \in \mathcal{X}$. Assuming its existence, let $f$ be strictly positive and nondecreasing function in the extended domain, which is bounded on $(-\infty,x] \cap \mathcal{X}$ for any $x \in \mathbb{R}$ and for which

$$M(t) = f(X(t)) \exp \left( -\int_0^t \frac{Af(X(s))}{f(X(s))} ds \right) \quad (23)$$

is a martingale with respect to the right continuous augmented filtration $\{\mathcal{F}_t \mid t \geq 0\}$ generated by $X$. A sufficient condition for the latter is that $f$ is bounded away from zero on $\mathcal{X}$ (e.g. [4], p.175). Furthermore we assume that $Af(x)$ is nonnegative for all $x \in \mathcal{X}$. Now, denote $\tau(y) = \inf\{t \mid X(t) > y\}$ (infinite if $X$ never exceeds $y$). Then $\tau(y)$ is right
continuous in \( y \) and it is easy to check that \( \sup_{0 \leq s \leq t} X(s) \leq y \) if and only if \( \tau(y) \geq t \). With \( a \land b = \min(a, b) \) it is well known that \( M(\tau(y) \land t) \) is also a martingale and moreover, our assumptions assure that it is also bounded. Finally, denoting \( \lambda(x) = Af(x)/f(x) \), then by the bounded convergence theorem we have that for \( y \geq x \) such that \( y \in \mathcal{X} \), if either \( \tau(y) < \infty \), almost surely or \( \int_0^\infty \lambda(X(s))ds = \infty \) on \( \{ \tau(y) = \infty \} \) then

\[
f(x) = \mathbb{E}_x M(0) = \mathbb{E}_x M(\tau(y)) = f(y)\mathbb{E}_x e^{-\int_0^{\tau(y)} \lambda(X(s))ds}.
\]

In particular this means that it is impossible to find a positive \( f \) in the extended domain of \( A \) such that \( Af \geq 0 \) and \( \int_0^{\tau(y)} \lambda(X(s))ds = \infty \) and moreover, our probability space so that there is), for \( x \).

Now, if we denote \( s(x) = 1/f(x) \), we have that

\[
\mathbb{E}_x e^{-\int_0^{\tau(y)} \lambda(X(s))ds} = \frac{s(y)}{s(x)}
\]

for every \( y \) for which \( \tau(y) < \infty \). If \( Z \) is a random variable (possibly infinite) such that \( \mathbb{P}_x (Z > t|\mathcal{F}_t) = e^{-\int_0^t \lambda(X(s))ds} \), then in fact

\[
\mathbb{E}_x e^{-\int_0^{\tau(y)} \lambda(X(s))ds} = \mathbb{P}_x (Z > \tau(y)) = \mathbb{P}_x \left( \max_{0 \leq t \leq Z} X(t) > y \right).
\]

Thus, we have that

\[
\mathbb{P}_x \left[ \max_{0 \leq t \leq Z} X(t) > y \right] = \frac{s(y)}{s(x)}
\]

for \( x \leq y \). If one assumes that \( U \) is an independent Uniform(0,1) random variable (if there isn’t then it is easy to artificially modify our probability space so that there is), then taking \( \mathcal{F}^U = \mathcal{F} \lor \sigma(U) \), we see that \( M \) is a martingale also with respect to this new filtration. Thus, if we let \( F(X, t) = 1 - e^{-\int_0^t \lambda(X(s))ds} \) and \( G(X, u) = \inf\{ t | F(X, t) \geq u \} \) then \( Z = G(X, U) \) has the correct conditional distribution.

### 3.1 Lévy processes

Sometimes, for various values of \( \alpha \), we may be lucky to find a function \( f \) satisfying the above conditions and for which \( \lambda(x) = \alpha \). In this case we immediately obtain the Laplace transform

\[
\mathbb{E}_x e^{-\alpha \tau(y)} = \frac{s(y)}{s(x)}.
\]

For a Lévy process with no positive jumps (in particular a Brownian motion) and

\[
\varphi(\alpha) = \log \mathbb{E}_0 e^{\alpha X(1)} = \alpha \alpha + \frac{\sigma^2}{2} \alpha^2 + \int_{(-\infty,0)} (e^{\alpha y} - 1 - \alpha y 1_{(-1,0)}(y)) \nu(dy)
\]

then the equation that needs to be solved is the following

\[
\alpha^2 f''(x) + \frac{\sigma^2}{2} f''(x) + \int_{(-\infty,0)} (f(x + y) - f(x) - y f'(x) 1_{(-1,0)}(y)) \nu(dy) = \alpha f(x).
\]
Fortunately, when $X$ is not nonincreasing (the negative of a subordinator or the zero function), then $\varphi$ has an inverse on $[\beta, \infty)$ when $\beta = \inf\{\alpha \mid \varphi(\alpha) > 0, \alpha > 0\}$. It is well known that $\beta = 0$ if $\varphi'(0) \geq 0$ and $\beta > 0$ otherwise. In this case, for every $x \leq y$, $\tau(y)$ is $\mathbb{P}_x$-a.s. finite and $f(x) = e^{\varphi^{-1}(\alpha)x}$ for $\alpha \geq \beta$ satisfies all the needed requirements and in particular solves (30). So, as is well known,

$$
\mathbb{E}_x e^{-\alpha \tau(y)} = \mathbb{E}_x e^{-\alpha \tau(y)} 1_{\{\tau(y) < \infty\}} = \frac{s(y)}{s(x)} = \frac{f(x)}{f(y)} = e^{-\varphi^{-1}(\alpha)(y-x)}, \quad (31)
$$
even if $\tau(y)$ is not $\mathbb{P}_x$-a.s. finite (that is, when $\varphi'(0) < 0$). In this particular case $Z \sim \exp(\alpha)$ (independent of $X$) and it follows as is also well known that

$$
\max_{0 \leq t \leq Z} X(t) - X(0) \sim \exp(\varphi^{-1}(\alpha)),
$$
so that this random variable obeys the standard (not-generalized) memoryless property.

### 3.2 Reflected Brownian motion

For the reflected Brownian motion on $\mathfrak{X} = [0, \infty)$ with general drift, the generator is the same as the one for Brownian motion, only that its domain is reduced to twice differentiable functions for which $f'(0) = 0$. In this case one needs to compute $\mu f''(x) + \frac{\sigma^2}{2} f''(x) = \alpha f(x)$ subject to $f'(0) = 0$ for $\alpha > 0$. It is easy to check that any positive constant multiple of the function

$$
f(x) = \frac{e^{a^+ x}}{a^+} + \frac{e^{a^- x}}{a^-} \quad (32)
$$

where $a^\pm = \sqrt{\mu^2 + 2\sigma^2 \alpha} \pm \mu$, would do the trick. In particular it is positive and increasing due to $0 < a^- < a^+$. In this case it is well known that $\tau(y) < \infty$ $\mathbb{P}_x$-a.s. for any $y > x$ regardless of the value of $\mu$, but it can also be inferred from this without resorting to anything else, by letting $\alpha \to 0$. Of course, this particular result is quite standard (e.g., problem 4 on p. 95 of [5]).

More generally of course, given a positive function $\lambda$ if it is possible to find some $f$ satisfying our assumptions for which $A f(x) = \lambda(x) f(x)$ for each $x \in \mathfrak{X}$, then if either $y > x$ is such that $\tau(y)$ is $\mathbb{P}_x$-a.s. finite or $\lambda$ is bounded away from zero, then (27) is satisfied.

### 3.3 A growth collapse process with generalized memoryless jumps

In this section we consider a piecewise deterministic Markov process $X_t$ with jumps that are governed by a jump measure with the generalized lack of memory property described above. See [2] and [8] for similar models.

Let $\{X(t)\}_{t \geq 0}$ be a Markov process on $\mathfrak{X} = [0, \infty)$ which is deterministically increasing with rate $r(x)$ between randomly occurring downward jumps. More specifically, we assume that inbetween jumps $dX_t = r(X_t) \, dt$, that $r(x)$ is positive and Lipschitz-continuous and that the time $t^*(x,y) = \int_x^y 1/r(u) \, du$ that is needed to reach the level $y$ from $x$ in the absence of any jumps is finite for all $x < y \in [0, \infty)$. Let $\kappa : \mathfrak{X} \to [0, \infty)$ denote the
state-dependent jump rate, i.e. if the process is in the state \( x \in \mathcal{X} \), then a jump occurs during the next \( \Delta t \) time units with probability \( \kappa(x) \Delta t + o(\Delta t) \) (and the probability to see more than one jump is \( o(\Delta t) \)). We assume \( \kappa \) to be bounded. Given that there is a jump at time \( t \), the process jumps from state \( x \in \mathcal{X} \) into some measurable \( A \subset [0,x) \) with probability \( \nu(x,A) \). We assume that for \( 0 \leq y \leq x \leq z \) the kernel \( \nu \) has the special property that

\[
\nu(z,y) = \nu(z,x)\nu(x,y)
\]

holds (compare with (5)). Here we write \( \nu(x,y) \) for \( \nu(x,[0,y]) \). It is then easy to see that a similar situation as in Section 1 is present (let \( P(z,A) = \nu(-z,-A) \) for \( z \leq 0 \) and \( A \subseteq (x,0] \)). It follows that there exists a family \( \{I_\theta| \theta \in \Theta\} \) of disjoint intervals and singletons with

\[
\cup_{\theta \in \Theta} I_\theta = \{0, \infty\}
\]

such that for each \( \theta \) for which \( I_\theta \) is not a singleton there exists a function \( s_\theta : I_\theta \to \mathbb{R} \) which is nonvanishing such that for every \( x,y \in I_\theta \) with \( x \leq y \) we have that

\[
\nu(x,y) = \frac{s_\theta(y)}{s_\theta(x)}.
\]

Note that \( s_\theta(y) : I_\theta \to [0,\infty) \) is nondecreasing and is not necessarily bounded. The infinitesimal generator of the Markov process \( X_t \) is given by

\[
\mathcal{A}f(x) = r(x)f'(x) + \kappa(x) \int_0^x (f(y) - f(x))\nu(x,dy).
\]

We assume that the domain \( \mathcal{D}_\mathcal{A} \) of \( \mathcal{A} \) consists of functions \( f \) that are absolutely continuous and for which the expectation of \( \sum_{0 < T_i \leq t} |f(X_{T_i}) - f(X_{T_{i-1}})| \) is finite for every \( t \geq 0 \), where \( T_i \) denotes the \( i \)th jump time (see [3]).

The following Lemma generalizes formula (28) in [8].

**Lemma 2.** Suppose that \( r(x), \kappa(x), \lambda(x) \) and \( s_\theta(x) \) are differentiable for \( x \in I_\theta \). Define the functions \( a(x) = r'(x) + r(x)\xi(x) - \lambda(x) - \kappa(x) \) and \( b(x) = \lambda'(x) + \lambda(x)\xi(x) \), where

\[
\xi(x) = \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)} \quad \text{if} \quad \kappa(x) \neq 0 \quad \text{and} \quad \xi(x) = 0 \quad \text{otherwise}.
\]

Any twice differentiable solution \( f \) with \( f'(x)s_\theta(x) \) being continuous of

\[
r(x)f''(x) + a(x)f'(x) - b(x)f(x) = 0,
\]

fulfils \( \mathcal{A}f(x) = \lambda(x)f(x) \).

**Proof.** The process \( X_t \), if started in the state \( x \in I_\theta \) will leave \( I_\theta \) only at the moment when it passes through the upper boundary \( z^*(\theta) \) and \( \nu(x,y) = 0 \) for \( y < z^*(\theta) \). If \( x \in I_\theta \) we may hence write

\[
\mathcal{A}f(x) = r(x)f'(x) + \frac{\kappa(x)}{s_\theta(x)} \int_{z^*(\theta)}^x \int_x^y f'(u) du s_\theta(dy), \quad x \in I_\theta.
\]

Applying Fubini’s theorem we can write this as

\[
\mathcal{A}f(x) = r(x)f'(x) - \frac{\kappa(x)}{s_\theta(x)} \int_{z^*(\theta)}^x f'(u)s_\theta(u) du, \quad x \in I_\theta.
\]
Then $Af(x) = \lambda(x)f(x)$ is equivalent to

$$
\kappa(x) \int_{z_*}^x f'(u)s_\theta(u) \, du = s_\theta(x) \left( r(x)f'(x) - \lambda(x)f(x) \right).
$$

(37)

Differentiation yields

$$
\frac{\kappa'(x)}{s_\theta(x)} \int_{z_*}^x f'(u)s_\theta(u) \, du = r(x)f''(x) + (r'(x) + r(x)\frac{s_\theta'(x)}{s_\theta(x)})
\]

$$
- \lambda(x) - \kappa(x))f'(x) - (\lambda'(x) + \lambda(x)\frac{s_\theta'(x)}{s_\theta(x)})f(x).
$$

If $\kappa(x) \neq 0$ then we divide (37) by $\kappa(x)$ and obtain (35) with $\xi(x) = \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)}$. If $\kappa(x) = 0$ then it follows from (37) that

$$
r(x)f''(x) + (r'(x) - \lambda(x))f'(x) - \lambda'(x)f(x) = 0,
$$

which is (35) with $\xi(x) = 0$. □

As is described earlier in the section via (27), the probability that the maximum process $\max_{0\leq t\leq Z} X(t)$ exceeds $y$, given $X(0) = x$, satisfies the generalized lack of memory property when $Z$ is defined right before (26). More precisely,

**Corollary 1.** Fix a $\theta \in \Theta$ and suppose that $f \in D_\Delta$ is bounded away from zero (or is such that $M(t)$ in (23) is a martingale) and solves equation (35) in $I_\theta$. Then

$$
P_x \left[ \max_{0\leq t\leq Z} X(t) > y \right] = \frac{f(x)}{f(y)},
$$

for all $x, y \in I_\theta$ with $x \leq y$, where $Z$ be a random variable, such that $\mathbb{P}_x(Z > t | \mathcal{F}_t) = e^{-\int_0^t \lambda(X(s)) \, ds}$.

In general (35) is not easy to solve and closed form solutions may be obtained only in certain cases. We provide two examples, where the coefficients $a(x)$ and $b(x)$ are such that a solution can be given.

**Example 1.** Equation (35) reduces to a differential equation with constant coefficients if

$$
\frac{r'(x)}{r(x)} + \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)} = C
$$

and

$$
\frac{\lambda(x)}{r(x)} \left( \frac{\lambda'(x)}{\lambda(x)} + \frac{s_\theta'(x)}{s_\theta(x)} - \frac{\kappa'(x)}{\kappa(x)} \right) = D.
$$

For example suppose that $\lambda(x) = c_1 e^{ax}, \kappa(x) = c_2 e^{ax}, r(x) = c_3 e^{ax}, s_\theta(x) = c_4 e^{bx},$ with $c_1, c_2, c_3, c_4, \beta \geq 0$ and $\alpha \in \mathbb{R}$. Then (35) reads

$$
f''(x) + \left( \beta - \frac{c_1 + c_2}{c_3} \right)f'(x) - \frac{c_1 \beta}{c_3}f(x) = 0,
$$

for all $x \in I_\theta$. □
which is solved by \( f(x) = Ae^{a^{-x}} + Be^{a^x} \), where

\[
a^\pm = \frac{1}{2} \left( \beta - \frac{c_1 + c_2}{c_3} \pm \sqrt{\left( \beta - \frac{c_1 + c_2}{c_3} \right)^2 + 4 \frac{c_1 \beta}{c_3}} \right).
\]

If we set \( f(z_*(\theta)) = 1 \) (w.l.o.g.), then \( f'(z_*(\theta)) = \lambda(z_*(\theta)) / r(z_*(\theta)) = c_1 / c_3 \). This leads to the final solution

\[
f(x) = \frac{a^+ - c_1}{c_3} e^{a^- (x-z_*(\theta))} + \frac{c_1}{c_3} - a^- e^{a^+ (x-z_*(\theta))}.
\]

**Example 2.** This example is a generalization of Example (A), Section 4.1 in [8]. Suppose that the jump measure \( \nu(x,y) = s_\theta(y)/s_\theta(x) \) is defined such that for some \( \alpha > 0 \) \( s_\theta(x)\lambda(x) = \alpha \kappa(x) \). Then \( \xi(x) = -\lambda'(x)/\lambda(x) \) and as a consequence the second coefficient \( b(x) \) is zero (while \( a(x) = r'(x) - r(x)\lambda'(x)/\lambda(x) - \lambda(x) - \kappa(x) \)). Hence (35) becomes

\[
r(x)f''(x) + a(x)f'(x) = 0, \quad (38)
\]

which is solved by

\[
f(x) = f(z_*(\theta)) + f'(z_*(\theta)) \int_{z_*(\theta)}^x \frac{\lambda(u)}{\lambda(z_*(\theta))} e^{\int_{z_*(\theta)}^u \frac{\lambda(w) + \alpha(w)}{r(w)} dw} du.
\]

Note that since \( A f(x) = \lambda(x)f(x) \) it follows that \( \lambda(z_*(\theta))f(z_*(\theta)) = r(z_*(\theta))f'(z_*(\theta)) \) and hence, choosing w.l.o.g. \( f(z_*(\theta)) = 1 \), we obtain the solution

\[
f(x) = 1 + \int_{z_*(\theta)}^x \frac{\lambda(u)}{r(u)} e^{\int_{z_*(\theta)}^u \frac{\lambda(w) + \alpha(w)}{r(w)} dw} du.
\]

**References**


