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A model reduction scheme with preserved optimal performance

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***Abstract*— This paper addresses a problem in control relevant model reduction. More specifically, a model reduction scheme is proposed that preserves the disturbance decoupling property of the to-be-reduced plant. It is shown that optimal feedback laws designed for the reduced system will actually be optimal for the non-reduced system. Moreover, a characterization is given for the minimal reduction order for which this property can be established. This can be extended to the design of observers for complex systems. The results are illustrated by a simulation example.**

I. Introduction

One of the most compelling applications of model reduction is to facilitate the synthesis of model based controllers for plants of high complexity. The most common strategies to achieve this are illustrated in Fig. 1, and can be referred to as an “optimize-then-reduce” and a “reduce-then-optimize” strategy. In the first approach a controller is synthesized for the full order system, where its complexity is subsequently reduced by approximating the controller by a simpler one. The second approach amounts to simplifying the to-be-controlled system by a model reduction technique followed by the subsequent synthesis of a controller on the basis of the simplified model. Often, in both approaches the model reduction is carried out in a manner that does not take the control objective into account. Consequently, there may be a considerable mismatch between control relevant properties of the full order model versus control relevant properties of the reduced order model. As noticed by many authors, this may be the case even when the reduced order model is a good approximation of the uncontrolled full order system.

Control relevant model reduction deals with the question of model approximation in which closed-loop performance criteria determine the quality of reduced order models. Model reduction strategies for control have been part of many earlier investigations [2], [3], [4], [7] and [10]. It is a general fact that model reduction of a to-be-controlled-plant generally degrades optimal achievable performance of the controlled system when the controller inferred from the reduced order model is implemented on the full order system. Usually, this performance degradation is justified and compensated by quantifying the robustness properties of the control system.

It is the purpose of this paper to investigate under what conditions optimal achievable performance can be left invariant in a model reduction scheme. More precisely, we address the general problem of disturbance decoupling for a linear time invariant system and propose a model reduction scheme in which optimal controllers of the reduced order system remain optimal for the full order system while, conversely, optimal controllers for the reduced order system also prove optimal for the full order system. In this manner, a model reduction scheme is developed that is specifically geared to leave disturbance decoupling properties of the (full order) plant invariant.

We provide a complete solution to this problem and characterize the minimal reduction degree for which disturbance decoupling of a full order plant can be maintained in the reduction procedure.

The paper is organized as follows. In Section II, the main problem will be formulated. Section III shows the main results of the approximation strategies for different controller and observer design problems. In Section IV, we will illustrate the results using a simulation example and conclusions are drawn in Section V. Notation and background information on geometric control theory is collected in the Appendix.

II. Problem Formulation

The problem that will be addressed in this paper amounts to developing a reduction strategy for a full order LTI system such that the disturbance decoupling properties of the system are preserved in the reduction procedure. Hence, in the context of this paper an optimal controller will be a controller that achieves a complete decoupling of a distinguished output variable from a disturbance that enters the system. With the proposed reduction strategy, the “reduce-then-optimize” approach of Fig. 1 will yield a controller (or observer) of low complexity, that after interconnection with the original full order system results in an optimal closed-loop behavior.

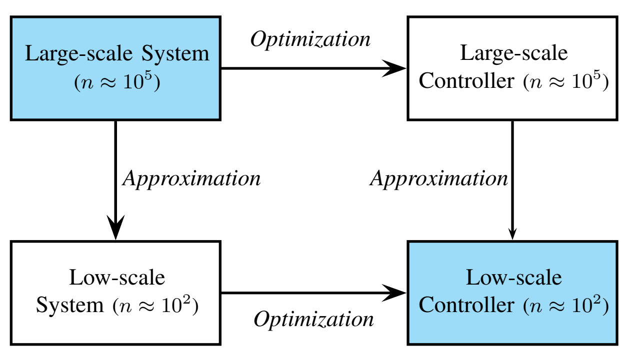

Fig. 1. Strategies for controller design for large-scale systems.

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The setting will be the following. Given is the system

$$\Sigma_P : \begin{cases} \dot{x} = Ax + Bu + Gd, \\ y = Cx, \\ z = Hx + Du, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n := \mathcal{X}$, $u(t) \in \mathbb{R}^u := \mathcal{U}$, $d(t) \in \mathbb{R}^d := \mathcal{D}$, $y(t) \in \mathbb{R}^y := \mathcal{Y}$ and $z(t) \in \mathbb{R}^z := \mathcal{Z}$ denote the state, control input, disturbance input, measured output and controlled output variable, respectively. We assume that there is not always a direct feed through from the control and disturbance inputs to the outputs in (1).

For the first class of control problems, it is assumed that the state variable is measured. That is, first suppose that $C = I$ and consider the following control problems:

Definition 2.1 (Disturbance Decoupling Problem): The disturbance decoupling problem (DDP) is said to be solvable for (1) if there exists $F : \mathcal{Y} \to \mathcal{U}$ such that the feedback law $u = Fx$ achieves a controlled system $\dot{x} = (A + BF)x + Gd$, $z = Hx$, where the output z does not depend on the disturbance d. ■

Hence, if DDP is solvable, the transfer function $T(s) = H(sI - A - BF)^{-1}G + D = 0$ for some feedback F. We will say that such a feedback achieves disturbance decoupling. Some variations on this problem include the possibility to assign the spectrum of the closed-loop state evolution matrix:

Definition 2.2 (DDP with Stability): The disturbance decoupling problem with stability (DDPS) is said to be solvable for (1) if DDP is solvable with a feedback $F : \mathcal{Y} \to \mathcal{U}$ that is stabilizing in the sense that $\lambda(A+BF) \subset \mathbb{C}_-$. ■

Definition 2.3 (DDP with Pole Placement): The disturbance decoupling problem with pole placement (DDPPP) is said to be solvable for (1) if DDP is solvable with a feedback $F : \mathcal{Y} \to \mathcal{U}$ such that the eigenvalues $\lambda(A+BF)$ can be located at arbitrary points in the complex plane. ■

The second class of problems involves the synthesis of observers that achieve disturbance decoupling. To formalize these problems, it is assumed that B and D are zero, implying that the influence of the control input u is neglected.

Definition 2.4 (DDEP): The disturbance decoupled estimation problem (DDEP) is said to be solvable if there exists an observer

$$\Sigma_O : \begin{cases} \dot{\tilde{x}} = A\tilde{x} + L(y - C\tilde{x}), \\ \tilde{z} = H\tilde{x}, \end{cases} \tag{2}$$

with an observer gain $L : \mathcal{Y} \to \mathcal{X}$ such that the estimation error $\epsilon(t) = z(t) - \tilde{z}(t)$ does not depend on the disturbance input d. ■

It is easily verified that the estimation error dynamics satisfies $\dot{e} = (A - LC)e + Gd$, $\epsilon = He$. If DDEP is solvable then the corresponding observer is said to achieve disturbance decoupling. The observer design problem can be extended so as to include additional properties of stability or pole placement on the spectrum $\lambda(A-LC)$ of the error dynamics. We refer to these problems with the acronyms DDEPS and DDEPPP, respectively.

Problem formulation:
Given Σ_P as in (1), find a reduced order system

$$\hat{\Sigma}_P : \begin{cases} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u + \hat{G}d, \\ y &= \hat{C}\hat{x}, \\ z &= \hat{H}\hat{x} + \hat{D}u + \hat{E}d, \end{cases}$$

such that:

DDP (or DDPS, DDPPP, DDEP) is solvable for Σ_P
if and only if
DDP (or DDPS, DDPPP, DDEP) is solvable for $\hat{\Sigma}_P$.

In addition, the obtained feedback (or observer) that achieves disturbance decoupling for $\hat{\Sigma}_P$ also achieves disturbance decoupling for Σ_P, and visa versa. ■

Obviously, a model order reduction scheme may or may not exhibit invariance of disturbance decoupling properties. Moreover, if a reduction scheme exhibits invariance of disturbance decoupling properties by reduction to order $r < n$, then this property may cease to exist by reduction orders $r' < r$.

For the introduced disturbance decoupling problems, we assume that the complete state vector is available as input for the to-be-designed controller (since C is assumed to be the identity matrix). Obviously, this is not possible in most practical cases. The problem where only partial state measurement is available for feedback is, in geometric control theory known as the disturbance decoupling problem with partial measurements (DDPM), is not dealt with in this paper (see e.g. [5], [6], [8]).

III. Main results

In this section, we first present the results for control relevant model reduction for the problems stated in Definition 2.1, 2.2 and 2.3. Afterwards, in Section III.B, reduction strategies are proposed that keep the property of disturbance decoupled estimation invariant.

A. Reduction for controller design

The interconnection structure that is used for the various types of disturbance decoupling problems is illustrated in the block scheme in Fig. 2. Here, Σ_P denotes the full order

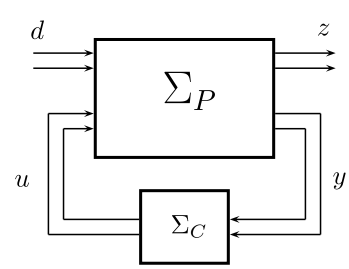

Fig. 2. Interconnection structure used in disturbance decoupling problems.

model of the plant and Σ_C is the to-be-designed controller. The disturbances acting on the system are given by d, u are the inputs that can be manipulated by the controller, y are the measurable outputs and z are the to-be-controlled outputs of the system.

For the different types of disturbance decoupling problems addressed, we want to construct a (static) feedback controller Σ_C such that the influence of the disturbance d is not visible on the outputs z in the controlled system. We start with DDP, where no additional stability or pole placement requirements are imposed. This problem is solved as follows [1], [9]:

Lemma 3.1 (Disturbance decoupling problem):
Let $\mathcal{V}^* = \mathcal{V}^*(A, B, H, D)$ be associated with the system in (1). Then DDP is solvable for Σ_P if and only if $\operatorname{im} G \subset \mathcal{V}^*$. ■

For background information on the used notation and on controlled invariant subspaces, we refer to the Appendix. If DDP is solvable, then there exists a static feedback controller Σ_C defined by $u = Fy$, with $F \in \mathbb{R}^{y\times u}$ that achieved decoupling of d from z. The class of all such feedback matrices is denoted $\mathcal{F}(\Sigma_P)$.

We are interested in developing a reduction strategy to obtain a lower order approximate model $\hat{\Sigma}_P$ for the system in (1), such that the DDP property is preserved in the model reduction procedure. In addition, we require that the class of controllers $\mathcal{F}(\Sigma_P)$ that solve DDP for Σ_P is invariant in the reduction.

Let $\mathcal{V}^* = \mathcal{V}^*(A, B, H, D)$ be the controlled invariant subspace for Σ_P as defined in the Appendix. Consider the following reduced order model of order $\dim(\mathcal{V}^*) \leq n$

$$\hat{\Sigma}_P := \begin{cases} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u + \hat{G}d, \\ y &= \hat{C}\hat{x}, \\ z &= \hat{H}\hat{x} + \hat{D}u + \hat{E}d, \end{cases} \tag{3}$$

where we have the state space matrices:

$$\begin{aligned} &\hat{A} = \Pi_{\mathcal{V}^*} A|_{\mathcal{V}^*}, \quad &&\hat{B} = \Pi_{\mathcal{V}^*} B, \quad &&\hat{C} = C|_{\mathcal{V}^*}, \\ &\hat{G} = \Pi_{\mathcal{V}^*} G, &&\hat{H} = HA|_{\mathcal{V}^*}, &&\hat{D} = HB + D, \\ &\quad\text{and} &&\hat{E} = \|\Pi_{\mathcal{L}} G\| \ \mathbb{I}_{zd}. \end{aligned}$$

Here, $\Pi_{\mathcal{I}}$ and $|_{\mathcal{I}}$ are the canonical projections and restrictions on a subspace $\mathcal{I} \subset \mathcal{X}$ applied to the system matrices of the high-order model in (1), $\|\cdot\|$ denotes the matrix norm, which is the maximal singular value of the matrix, and $\mathcal{L}$ is any subspace of $\mathcal{X}$ such that $\mathcal{X} = \mathcal{V}^* \oplus \mathcal{L}$.
The matrix $\mathbb{I}_{zd} \in \mathbb{R}^{z\times d}$ equals $\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $[I \quad 0]$ or I when the dimension of z is larger, smaller or equal to the dimension of d, respectively.

From this state space representation, one can observe that the dimension of $\hat{x}$ is equal to $\dim(\mathcal{V}^*)$, since we have projected the original state vector onto $\mathcal{V}^*$. This has been illustrated in Fig. 3. The dimension of the reduced system in (3) is therefore $\dim \mathcal{V}^* \leq n$, and results in the following theorem:

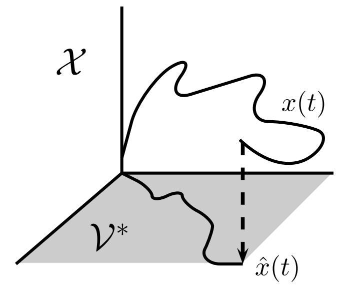

Fig. 3. Projection used in reduction for DDP from $\mathcal{X}$ onto $\mathcal{V}^*$.

Theorem 3.1 (Reduction for DDP):
The following statements are equivalent:

1) DDP solvable for Σ_P in (1)
2) DDP solvable for $\hat{\Sigma}_P$ in (3)

Moreover, if $u = \hat{F}\hat{x}$ solves DDP for $\hat{\Sigma}_P$, then $u = Fx$, with $F|_{\mathcal{V}^*} = \hat{F}$ solves DDP for Σ_P. Conversely, if $u = Fx$ solves DDP for Σ_P, then $u = \hat{F}\hat{x}$, with $\hat{F} = F|_{\mathcal{V}^*}$, solves DDP for $\hat{\Sigma}_P$. ■

To prove this theorem, consider the following lemma:

Lemma 3.2:
The system $\dot{x} = Ax + Bd$, $z = Cx + Dd$ has transfer function $T(s) = C(sI - A)^{-1}B + D = 0$ if and only if there exists an A invariant subspace $\mathcal{L} \subset \mathcal{X}$ such that $\operatorname{im} B \subset \mathcal{L} \subset \ker C$ and $D = 0$. ■

Proof: ($1 \Rightarrow 2$) DDP solvable for Σ_P implies $\exists F$ such that $(A + BF)\mathcal{V}^* \subset \mathcal{V}^* \subset \ker(H + DF)$ and $\operatorname{im} G \subset \mathcal{V}^*$. Set $\hat{F} = F|_{\mathcal{V}^*}$. Then, $\hat{H} + \hat{D}\hat{F} = HA|_{\mathcal{V}^*} + (HB + D)\hat{F} = H(A|_{\mathcal{V}^*} + BF|_{\mathcal{V}^*}) + DF|_{\mathcal{V}^*} = H(A+BF)|_{\mathcal{V}^*} + DF|_{\mathcal{V}^*} \subset H\mathcal{V}^* + DF\mathcal{V}^* = (H + DF)\mathcal{V}^* = 0$, where we have used $\operatorname{im} H(A+BF)\mathcal{V}^* \subset H\mathcal{V}^* = 0$. Then, the feedback $u = \hat{F}\hat{x}$ establishes that $z = (\hat{H} + \hat{D}\hat{F})\hat{x} = 0$. Since $\operatorname{im} \hat{G} \subset \operatorname{im} G$, DDP is solvable for $\hat{\Sigma}_P$.
($2 \Rightarrow 1$) Suppose DDP is solvable for $\hat{\Sigma}_p$. Let $\hat{\mathcal{V}}^*$ be the largest controlled invariant subspace in $\hat{\Sigma}_p$ such that $(\hat{A} + \hat{B}\hat{F})\hat{\mathcal{V}}^* \subset \hat{\mathcal{V}}^* \subset \ker(\hat{H} + \hat{D}\hat{F})$ and $\operatorname{im} \hat{G} \subset \hat{\mathcal{V}}^*$ for some $\hat{F}$. From Lemma 3.2 we know that $\hat{E} = 0$. Redefine $\hat{F} := F|_{\mathcal{V}^*}$ with $F \in \mathcal{F}(\Sigma_P)$, so we observe that $\hat{H} + \hat{D}\hat{F} = H(A + BF)|_{\mathcal{V}^*} + DF|_{\mathcal{V}^*} \subset H(A + BF)\mathcal{V}^* + DF\mathcal{V}^* \subset (H + DF)\mathcal{V}^* = 0$, where the last equality follows from the definition of $\mathcal{V}^*$ and the fact that $F \in \mathcal{F}(\Sigma_P)$. Conclude that $\hat{F} = F|_{\mathcal{V}^*}$ solves DPP for $\hat{\Sigma}_P$. To prove that DDP is solvable for Σ_P observe that, by the previous construction, $(A + BF)\mathcal{V}^* \subset \mathcal{V}^* \subset \ker(H + DF)$ and since $\hat{E} = 0$ we have $\sigma_{\max}(\Pi_{\mathcal{L}} G) = 0$ and so $\Pi_{\mathcal{L}} G = 0$ implying that $\operatorname{im} G \subset \mathcal{V}^*$. By Lemma 3.1 we then have that DDP is solvable in Σ_P.

□

The following remarks pertain to Theorem 3.1:

- The proposed reduction strategy results in an approximation of order $\dim(\mathcal{V}^*)$. This model order is less or equal to the order of the full model, and preserves the desired closed-loop optimal performance.

- As depicted in Fig. 3, the projection onto $\mathcal{V}^*$ is used. In general, the dimension of $\mathcal{V}^*$ is not the lowest reduction order for which the DDP property remains invariant. The lowest achievable order is characterized as follows:

Theorem 3.2: The minimal achievable order of reduction possible, such that DDP solvability is preserved, is given by the smallest $\mathcal{V}$ subspace fulfilling the conditions:

$$r_{\min} = \min_{\mathcal{V}} \dim\{\mathcal{V} \mid \exists F \text{ such that } \operatorname{im} G \subset \mathcal{V} \text{ and } (A+BF)\mathcal{V} \subset \mathcal{V} \subset \ker(H+DF)\}. \blacksquare$$

- From the previous remark, we can conclude that there is a guaranteed performance degradation for all reduced order models of order $r < r_{\min}$.

To make the results presented in (3) more accessible, we give the following example:

Example 3.1: Assume that we can apply an appropriate state transformation on the system in (1) such that $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, where $x \in \mathcal{X} = \mathcal{L} \oplus \mathcal{V}^*$, $x_2 \in \mathcal{V}^*$ and $x_1 \in \mathcal{L}$. We then have:

$$\Sigma_P : \begin{cases} \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] u + \left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right] d, \\ z \;=\; \left[\begin{smallmatrix} H_1 & H_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + Du, \\ y \;=\; \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]. \end{cases}$$

Then, the reduced order system leaving the DDP property invariant is given by:

$$\hat{\Sigma}_P : \begin{cases} \dot{\hat{x}} &= A_{22}\hat{x} + B_2 u + G_2 d, \\ z &= \left[\begin{smallmatrix} H_1 & H_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} A_{12} \\ A_{22} \end{smallmatrix}\right] \hat{x} + \left(\left[\begin{smallmatrix} H_1 & H_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] + D\right) u, \\ y &= \hat{x}, \end{cases}$$

and has an order $\dim(\mathcal{V}^*) \leq n$. $\blacksquare$

Now consider the problems formulated in Definition 2.2 and Definition 2.3. The solvability conditions for these problem are as follows [1], [9]:

Lemma 3.3 (DDPS):
Let $\mathcal{V}_g^* = \mathcal{V}_g^*(A, B, H, D)$ associated with the system Σ_P in (1) if and only if $\operatorname{im} G \subset \mathcal{V}_g^*$ and the pair (A, B) is stabilizable. $\blacksquare$

If DDPS is solvable, there exists a static feedback controller Σ_C such that $u = Fy$ with $F \in \mathcal{F}_g(\Sigma_P)$. Consider the reduced order system (3), but now with the state space matrices:

$$\begin{aligned} &\hat{A} = \Pi_{\mathcal{V}_g^*} A|_{\mathcal{V}_g^*}, \qquad \hat{B} = \Pi_{\mathcal{V}_g^*} B, \qquad \hat{C} = C|_{\mathcal{V}_g^*}, \\ &\hat{G} = \Pi_{\mathcal{V}_g^*} G, \qquad \hat{H} = HA|_{\mathcal{V}_g^*}, \qquad \hat{D} = HB + D, \\ &\qquad\qquad \text{and} \qquad \hat{E} = \|\Pi_{\mathcal{L}_g} G\| \; \mathbb{I}_{zd}, \end{aligned}$$

Here, $\mathcal{L}_g$ is any subspace of the $\mathcal{X}$ such that $\mathcal{X} = \mathcal{V}_g^* \oplus \mathcal{L}_g$ and $\mathbb{I}_{zd}$ is defined in a similar manner as before.
This reduced order system has complexity $\dim(\mathcal{V}_g^*)$ and yields the following result:

Theorem 3.3 (Reduction for DDPS):
Assume Σ_P is stabilizable. Then, equivalent are:

1) DDPS is solvable for Σ_P in (1)
2) DDPS is solvable for $\hat{\Sigma}_P$ in (3)

Moreover, if $u = \hat{F}\hat{x}$ solves DDPS for $\hat{\Sigma}_P$, then $u = Fx$, with $F|_{\mathcal{V}_g^*} = \hat{F}$ and $F|_{\mathcal{L}_g} = F_s|_{\mathcal{L}_g}$, with F_s *any* stabilizing feedback such that $\sigma(A + BF_s) \subset \mathbb{C}_-$, solves DDPS for Σ_P. Conversely, if $u = Fx$ solves DDPS for Σ_P, then $u = \hat{F}\hat{x}$, with $\hat{F} = F|_{\mathcal{V}_g^*}$, solves DDPS for $\hat{\Sigma}_P$. $\blacksquare$

The proof of this theorem is similar to the one in Theorem 3.1.

We also address the problem of disturbance decoupling with closed-loop pole placement, known as DDPPP. In this problem, we want to ensure that the closed-loop poles are located at arbitrary places in the complex plane. Known solvability conditions are given as follows:

Lemma 3.4 (DDPPP):
Let $\mathcal{R}^* = \mathcal{R}^*(A, B, H, D)$ be a subspace of $\mathcal{V}^*(A, B, H, D)$ associated with Σ_P. Then, DDPPP is solvable for Σ_P if and only if $\operatorname{im} G \subset \mathcal{R}^*$ and (A, B) is controllable. $\blacksquare$

Here, the class of controllers Σ_C is given by $\mathcal{F}_{pp}(\Sigma_P)$. Consider the reduced order model (3) with the state space matrices:

$$\begin{aligned} &\hat{A} = \Pi_{\mathcal{R}^*} A|_{\mathcal{R}^*}, \qquad \hat{B} = \Pi_{\mathcal{R}^*} B, \qquad \hat{C} = C|_{\mathcal{R}^*}, \\ &\hat{G} = \Pi_{\mathcal{R}^*} G, \qquad \hat{H} = HA|_{\mathcal{R}^*}, \qquad \hat{D} = HB + D, \\ &\qquad\qquad \text{and} \qquad \hat{E} = \|\Pi_{\mathcal{L}_{pp}} G\| \; \mathbb{I}_{zd}, \end{aligned}$$

with $\mathcal{X} = \mathcal{R}^* \oplus \mathcal{L}_{pp}$. Without technical difficulties, we then obtain the following result:

Theorem 3.4 (Reduction for DDPPP):
Assume Σ_P is controllable. Then, equivalent are:

1) DDPPP is solvable for Σ_P in (1)
2) DDPPP is solvable for $\hat{\Sigma}_P$ in (3)

Moreover, if $u = \hat{F}\hat{x}$ solves DDPPP for $\hat{\Sigma}_P$ at pole location $\hat{\pi} \subset \mathbb{C}$, then $u = Fx$, with $F|_{\mathcal{R}^*} = \hat{F}$ solves DDPPP for Σ_P at pole locations $\pi \supset \hat{\pi}$. Conversely, if $u = Fx$ solves DDPPP for Σ_P at pole location $\pi \subset \mathbb{C}$, then $u = \hat{F}\hat{x}$, with $\hat{F} = F|_{\mathcal{R}^*}$, solves DDPPP for $\hat{\Sigma}_P$ at pole location $\hat{\pi} \subset \pi$. $\blacksquare$

The proof goes in a similar manner as for DDP and DDPS, and is therefore omitted in this paper. Obviously, the number of poles that can be placed in Σ_P is larger than in $\hat{\Sigma}_P$. For this full order system, the poles placed due to $F|_{\mathcal{R}^*} = \hat{F}$ are the same as the placed ones in $\hat{\Sigma}_P$ (namely $\hat{\pi}$). Note that the actual pole locations π and $\hat{\pi}$ can be chosen arbitrary due to the definition of DDPPP.

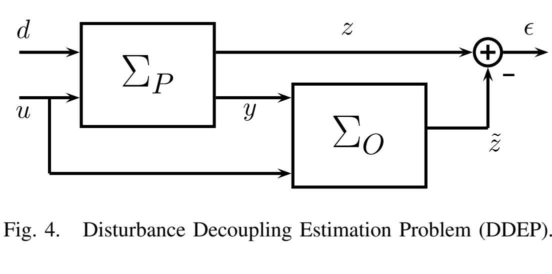

Fig. 4. Disturbance Decoupling Estimation Problem (DDEP).

B. Reduction for observer design

Not only the problem of model reduction for the design of controllers needs to be answered, but also the use of observers is crucial for large-scale systems. Here, we also do not want to lose information during the approximation step of the "reduce-then-optimize" strategy (see Fig. 1), that is relevant for the design of the observer.

The discussed observer design problem is depicted in Fig. 4, where Σ_P is the same large-scale system as in (1) and Σ_O is the low order to-be-designed observer as in (2). In contrast to the results in the previous subsection, this will not be a static system but will be dynamic. With this observer, we want to get an optimal estimate $\hat{z}$ for the original state z such that the influence of the input u as well as the disturbance d is not visible on the error ϵ. It is therefore called a Disturbance Decoupling Estimation Problem (DDEP). This problem is discussed extensively in geometric control theory (e.g. [8]), and is solvable if the following condition holds:

Lemma 3.5 (DDEP):
Given $\mathcal{S}^* = \mathcal{S}^*(A, G, C, 0)$ containing $\operatorname{im} G$ for the system Σ_P in (1). Then, DDEP is solvable for Σ_P if and only if $\mathcal{S}^* \cap \ker C \subset \ker H$. ■

For this observer design problem, we consider to use a reduced order model for the large-scale Σ_P as in (3) with the state space matrices:

$$\hat{A} = \Pi_{\mathcal{S}^*} A|_{\mathcal{S}^*}, \qquad \hat{C} = C|_{\mathcal{S}^*}, \qquad \hat{G} = \Pi_{\mathcal{S}^*} G,$$
$$\hat{H} = \Pi_{\mathcal{S}^*} H, \qquad \hat{E} = \|\Pi_{\mathcal{L}_O} G\| \ \mathbb{I}_{zd}$$

with $\mathcal{X} = \mathcal{S}^* \oplus \mathcal{L}_O$. In this case, we reduced the complexity of the original system from n towards $\dim(\mathcal{S}^*) \leq n$. In contrast to the case for DDP(S/PP), it is not possible to find a projection towards a lower dimensional conditioned invariant subspace, since the applied projection onto $\mathcal{S}^*$ is the smallest one. The chosen values for the matrices result in the following:

Theorem 3.5 (Reduction for DDEP):
The following statements are equivalent:

1) DDEP is solvable for Σ_P in (1)
2) DDEP is solvable for $\hat{\Sigma}_P$ in (3)

Moreover, if the observer gain $\hat{L}$ solves DDEP for $\hat{\Sigma}_P$, then L, with $\Pi_{\mathcal{S}^*} L = \hat{L}$ solves DDEP for Σ_P. Conversely, if L solves DDEP for Σ_P, then $\hat{L}$, with $\hat{L} = \Pi_{\mathcal{S}^*} L$, solves DDEP for $\hat{\Sigma}_P$. ■

The proof of this result can be obtained in a similar manner as done for the disturbance decoupling problems in the previous subsection. The results for DDEP can also be extended to the problems of DDEPS and DDEPPP [8], where the error spectrum of ϵ should be stable or should have poles within a certain subset of the complex plane, respectively.

IV. Simulation Example

To illustrate that the proposed reduction techniques indeed keep the desired closed-loop performance invariant, we apply the proposed strategy for DDP, discussed in Theorem 3.1, on a simulation example. The dynamical system Σ_P, that needs

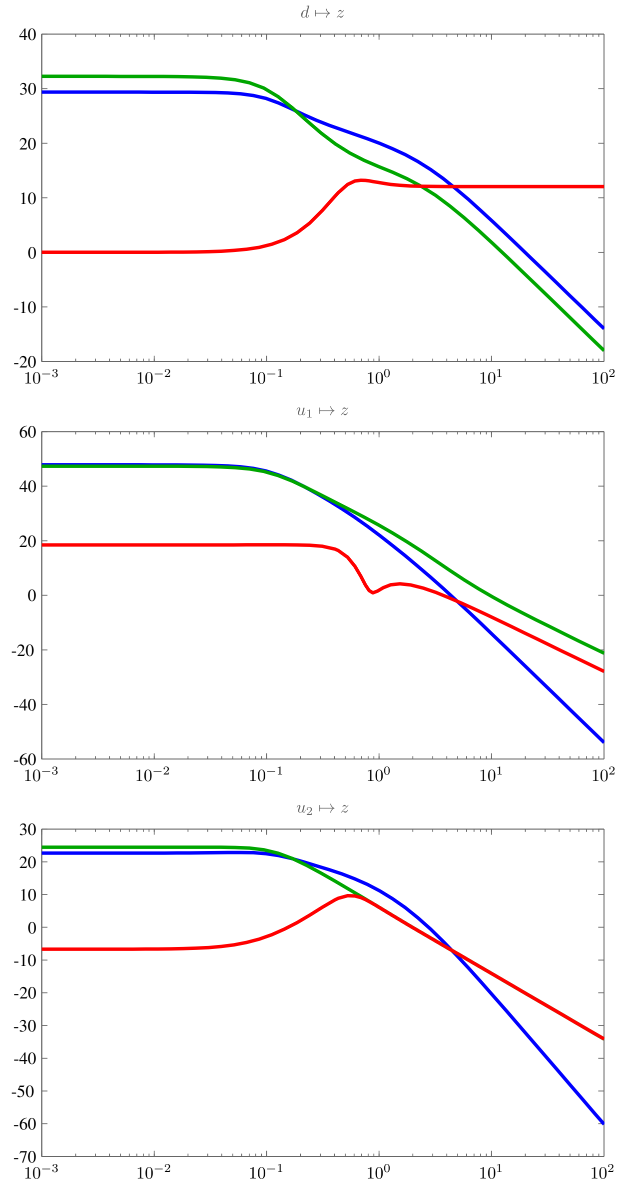

Fig. 5. Bode plots of open loop behavior for reduced order systems used to solve DDP. (Original system, Hankel approximation, Proposed approximation)

to be reduced as in (1), is chosen to have randomly generated state space matrices, one disturbance d acting on the system, two control inputs u and one measured output z. The "large-scale" system has a complexity of $\dim(\mathcal{X}) = 5$, which we want to reduce to an approximation of order 4. The open-loop bode plot for the large-scale system has been illustrated in blue in Fig. 5. To make comparisons between our proposed reduction scheme and classical reduction methods, we also applied optimal Hankel norm approximation resulting in the 4^{th} order approximant $\hat{\Sigma}_{P,\text{good}}$ in green. It is still possible to solve the DDP using this approximation, however due to the reduction method it is not possible to extend the found state feedback such that it can be connected to the original system. The reduced system $\hat{\Sigma}_{P,\text{proposed}}$ in (3) results in the open-loop bode plot in red, which does not contain similar dynamics as the original and, using Hankel approximation obtained, reduced order models. It does however leave the DDP solvability property invariant, so after reduction we are still able to find a Σ_C that can be interconnected with the original system and yields DDP for Σ_P.

V. Conclusions

The design of controllers for large-scale systems often results in strategies consisting of disjoint steps, namely, approximation, obtaining a reduced order approximation, and optimization, resulting in the controller, which does not always guarantee that the desired closed-loop performance can be obtained. In this paper, we therefore focus on a model reduction scheme that keeps optimal performance invariant after the reduction. More precisely, we have been focusing on disturbance decoupling problems, with or without requirements on pole placement of closed-loop stability. We have presented reduction strategies that keep DDP solvability invariant, such that the design of controllers can be done using low order approximations, which can be interconnected with the original large-scale system. In the proposed reduction strategy, the system complexity has been decreased from the original order n towards the dimension of the largest controlled invariant subspace of the system, which is strictly smaller than n. Also a condition for the minimal order that can be obtained such that DDP solvability can be preserved is presented in this paper. The problem of design of observers for large-scale systems is also addressed, where we have provided a reduction strategy such that DDEP solvability is kept invariant.

The results presented in this paper are proving that the principle of control relevant reduction strategies, using the addressed problems in geometric control theory, indeed works. With the first extensions to observer or estimation problems, we try to get a good understanding such that model reduction schemes for systems with partial measurements (DDPM) can be researched. This extends the possible reduction strategies towards models for real physical systems where we can not feedback the complete state space.

Appendix

This appendix consists of notation and required background on geometric control theory. Consider the following system:

$$\Sigma: \ \dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{4}$$

with $x \in \mathcal{X}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. Let $x(t; x_0, u)$ denote the state trajectory of (4) evaluated at time t corresponding to initial condition $x(0) = x_0$ and input u.
Controlled invariant subspaces are defined as:

Definition A.1: Call $\mathcal{V} \subset \mathcal{X}$ controlled invariant if $\forall x_0 \in \mathcal{V}$, $\exists u$ such that $x(t; x_0, u) \in \mathcal{V}$ for all $t \geq 0$. ■

This results in the following equivalent conditions:

1) $\mathcal{V}$ is a controlled invariant subspace
2) $A\mathcal{V} \subset \mathcal{V} + \operatorname{im} B$
3) $\exists F$ such that $(A + BF)\mathcal{V} \subset \mathcal{V}$

It is well known that $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ is controlled invariant whenever $\mathcal{V}_1$ and $\mathcal{V}_2$ are controlled invariant. That is, the property of controlled invariance is closed under addition. For the system (4) we define $\mathcal{V}^* = \mathcal{V}^*(A, B, C, D)$ as the largest subspace $\mathcal{V} \subset \mathcal{X}$ for which there exists $F : \mathcal{X} \to \mathcal{U}$ such that

$$(A + BF)\mathcal{V} \subset \mathcal{V} \subset \ker(C + DF).$$

This subspace is well defined [6], [9] in the sense that is only depends on (A, B, C, D).

Controlled invariant subspaces that are also stable are denoted by $\mathcal{V}_g$ and have the property that $(A + BF)\mathcal{V} \subset \mathcal{V}$ with $\lambda((A + BF)|_{\mathcal{V}}) \subset \mathbb{C}_-$.

Definition A.2: A subspace $\mathcal{R} \subset \mathcal{X}$ is said to be a *controllability subspace* of Σ if for all $x_0, x_1 \in \mathcal{R}$ there exists $T > 0$ and $u : \mathbb{T} \to \mathcal{U}$ such that the state trajectory $x(t)$ of (4) with input u satisfies $x(0) = x_0$, $x(T) = x_1$ and $x(t) \in \mathcal{R}$ for all $t \in \mathbb{T}$. ■

The largest controllability subspace of Σ is denoted by $\mathcal{R}^*(A, B, C, D)$. Conditioned invariant subspaces are dual to controlled invariant subspaces:

Definition A.3: A subspace $\mathcal{S} \subset \mathcal{X}$ is called conditioned invariant if $A(\mathcal{S} \cap \ker C) \subset \mathcal{S}$. For the system (4) we define $\mathcal{S}^* = \mathcal{S}^*(A, B, C, D)$ as the *smallest* subspace $\mathcal{S} \subset \mathcal{X}$ for which there exists $L : \mathcal{Y} \to \mathcal{X}$ such that

$$(A + LC)\mathcal{S} \subset \mathcal{S} \quad \text{and} \quad \operatorname{im}(B + LD) \subset \mathcal{S}. \qquad ■$$

Well known is that when $\mathcal{S}_1$ and $\mathcal{S}_2$ are conditioned invariant, also the intersection $\mathcal{S}_1 \cap \mathcal{S}_2$ is conditioned invariant.

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