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Brief Paper

Stabilisation of linear delay difference inclusions via time-varying control Lyapunov functions

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Abstract: The stabilisation of linear delay difference inclusions is often complicated by computational issues and the presence of constraints. In this study, to solve this problem, a receding horizon control scheme is proposed based on the Razumikhin approach and time-varying control Lyapunov functions. By allowing the control Lyapunov function to be time varying, the computational advantages of the Razumikhin approach can be exploited and at the same time the conservatism associated with this approach is avoided. Thus, a control scheme is obtained which takes constraints into account and requires solving on-line a low-dimensional semi-definite programming problem. The effectiveness of the proposed results is illustrated via an example that also shows the computational limitations of existing control strategies.

1 Introduction

Linear delay difference inclusions (DDIs) have the ability to model a wide variety of relevant linear processes in the discrete-time domain, such as uncertain systems with delay, systems with time-varying delay [1] and certain types of networked control systems [2–4]. Therefore the stabilisation of such systems, possibly subject to constraints, is a frequently studied problem. Within the context of Lyapunov theory, the most commonly used approach to solve this problem, called the Krasovskii approach, makes use of an augmentation of the state vector with all relevant delayed states. This yields a standard but higher-dimensional system without delay, and the classical stabilisation techniques apply to this system. For example, using this technique state-feedback controllers were obtained for uncertain systems with delay in [5–7], for uncertain systems with time-varying delay in [8, 9] and for uncertain singular systems with time-varying delay in [10]. Furthermore, control strategies that can handle constraints were developed for uncertain systems with delay in [11, 12] and for uncertain systems with time-varying delay in [13]. Unfortunately, all of the aforementioned approaches have an exponential dependence of the computational complexity on the size of the delay and are therefore not tractable for systems with large delays.

Alternatively, the Razumikhin approach for discrete-time systems does not involve the augmented system and, hence, has the potential to avoid the increase in complexity associated with the Krasovskii approach. Based on the Razumikhin approach a state feedback controller was obtained for systems with delay in [14] and for uncertain systems with delay in [1]. Furthermore, a control

strategy that takes constraints into account was developed for uncertain systems with delay in [15]. Interestingly, the control scheme proposed in [15] consists of an on-line optimisation-based component that is tractable for large delays because of an on-line Minkowski addition of sets. Unfortunately, the method also requires the off-line computation of a local static state-feedback controller and thus, as it is also the case for the control schemes proposed in [1, 14], remains computationally demanding. Furthermore, as the Razumikhin approach provides [1] sufficient but not necessary conditions for stability, any method based on this approach is inherently conservative.

Motivated by this conservatism and the computational shortcomings of existing synthesis methods, this paper investigates time-varying Lyapunov functions [16, 17] as a new tool for the stability analysis of linear DDIs. Thus, via a suitable modification of the Razumikhin approach, a set of necessary and sufficient Lyapunov-like conditions for exponential stability of linear DDIs is obtained. Then, this concept is used to design a control scheme that makes use of on-line optimisation. Similarly to [15], Minkowski set addition properties are used to obtain a computationally efficient algorithm. As the proposed technique no longer requires the off-line computation of a locally stabilising controller, but merely involves solving a low-dimensional semi-definite programming (SDP) problem, the overall synthesis method remains computationally tractable even for large delays. An example, for which existing control schemes are not tractable, illustrates the effectiveness of the proposed results.

Paper structure: Section 2 provides some preliminaries and the problem description. Then, time-varying Lyapunov

functions are considered in Section 3. Stabilisation of linear DDIs is studied in Sections 4 and 5 provides an illustrative example. Section 6 provides concluding remarks and the Appendix contains the proof of one of the main results.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$, define $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$, $\Pi_{\geq c} := \{k \in \Pi : k \geq c\}$ and similarly $\Pi_{\leq c}$. Furthermore, for every $c, d \in \mathbb{R}$ such that $c \geq d$ let $\Pi_{[c,d]} := \{k \in \Pi : c \leq k \leq d\}$ and similarly $\Pi_{(c,d)}$, $\Pi_{(c,d]}$ and $\Pi_{[c,d)}$. Let $\mathcal{S}_i \subset \mathbb{R}^n$, $i \in \mathbb{Z}_+$ denote arbitrary sets. Then, $\mathcal{S}_1^h := \mathcal{S}_1^{h-1} \times \mathcal{S}_1$ for any $h \in \mathbb{Z}_{\geq 1}$ and $\mathcal{S}_1^0 := \emptyset$. Furthermore, $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{x + y | x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$ denotes the Minkowski addition of $\mathcal{S}_1$ and $\mathcal{S}_2$ and we define $\bigoplus_{i=1}^N \mathcal{S}_i := \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_N$ for any $N \in \mathbb{Z}_{\geq 2}$. Let $\text{co}(\cdot)$ denote the convex hull.

For a vector $x \in \mathbb{R}^n$ let $[x]_i$, $i \in \mathbb{Z}_{[1,n]}$ denote the i th component of x and let $\|x\| := \sqrt{\sum_{i=1}^n [x]_i^2}$. Let $\mathbf{x} := \{x(l) \in \mathbb{R}^n\}_{l \in \mathbb{Z}_+}$ denote an arbitrary sequence and define $\|\mathbf{x}\| := \sup\{\|x(l)\| : l \in \mathbb{Z}_+\}$. Furthermore, $\mathbf{x}_{[c_1, c_2]} := \{x(l)\}_{l \in \mathbb{Z}_{[c_1, c_2]}}$, with $c_1, c_2 \in \mathbb{Z}$, denotes a sequence that is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c_1, c_2]}$. Given a symmetric matrix $Z \in \mathbb{R}^{n \times n}$, let $Z \preceq 0$ and $Z \prec 0$ denote that Z is negative semi-definite and negative definite, respectively. Moreover, $*$ is used to denote the symmetric part of a matrix. Let $I_n \in \mathbb{R}^{n \times n}$ denote the n th-dimensional identity matrix.

2.1 Problem definition

Consider the DDI

$$x(k+1) \in f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}), \quad k \in \mathbb{Z}_+ \quad (1)$$

where $\mathbf{x}_{[k-h,k]} \in \mathbb{X}^{h+1}$ and $\mathbf{u}_{[k-h,k]} \in \mathbb{U}^{h+1}$ are sequences of (delayed) states and inputs, $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ define constraints on the states and inputs and $h \in \mathbb{Z}_+$ is the maximal delay. The DDI (1) is linear if the set-valued map $f : (\mathbb{R}^n)^{h+1} \times (\mathbb{R}^m)^{h+1} \rightrightarrows \mathbb{R}^n$ is of the form

$$f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) := \left\{ \sum_{i=-h}^0 A_i x(i) + B_i u(i) : (\{A_i, B_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{AB} \right\} \quad (2)$$

where the set $\mathcal{AB} \subset (\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m})^{h+1}$ is a non-empty and compact polytope. The main problem of interest in this paper is the stabilisation of the linear DDI (1) taking into account the constraints $\mathbb{X}$ and $\mathbb{U}$. To this end, a control law $\pi : \mathbb{X}^{h+1} \rightrightarrows \mathbb{U}$ will be used. The linear DDI (1) in closed loop with this control law yields

$$x(k+1) \in F_\pi(\mathbf{x}_{[k-h,k]}), \quad k \in \mathbb{Z}_+ \quad (3)$$

where

$$F_\pi(\mathbf{x}_{[-h,0]}) := \{f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) \mid u(0) \in \pi(\mathbf{x}_{[-h,0]})\}$$

In (3) and the remainder of the paper, for notational convenience, it is assumed that the initial input sequence $\mathbf{u}_{[-h,-1]} \in \mathbb{U}^h$ is dependent on the initial state sequence $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ only. As a consequence, it follows that at time $k \in \mathbb{Z}_+$, $\mathbf{u}_{[k-h,k-1]}$ is dependent on $\mathbf{x}_{[k-h,k]}$ only. Therefore the dependence of F_π on $\mathbf{u}_{[k-h,k-1]}$ can be omitted, as done in (3).

Let $\mathcal{S}(\mathbf{x}_{[-h,0]})$ denote the set of all trajectories of (3) starting from $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$. Furthermore, let $\Phi(\mathbf{x}_{[-h,0]}) := \{\phi(k, \mathbf{x}_{[-h,0]})\}_{k \in \mathbb{Z}_{\geq -h}} \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ denote a trajectory of (3), such that $\phi(k, \mathbf{x}_{[-h,0]}) = x(k)$ for all $k \in \mathbb{Z}_{[-h,0]}$ and $\phi(k+1, \mathbf{x}_{[-h,0]}) \in F_\pi(\Phi_{[k-h,k]}(\mathbf{x}_{[-h,0]}))$ for all $k \in \mathbb{Z}_+$.

Definition 1: (i) The DDI (3) is ‘exponentially stable in $\mathbb{X}$ ’ [or shortly, ES($\mathbb{X}$)] if for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]}), k) \in \mathbb{X}^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$ it holds that $\phi(k, \mathbf{x}_{[-h,0]}) \in \mathbb{X}$ and $\|\phi(k, \mathbf{x}_{[-h,0]})\| \leq c \|\mathbf{x}_{[-h,0]}\| \mu^k$ for some $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{(0,1)}$; (ii) the DDI (3) is ‘globally exponentially stable’ (GES) if it is ES($\mathbb{R}^n$).

In the remainder of the paper, a technique is proposed to find a control law π for the linear DDI (1) that renders the closed-loop system (3) ES($\mathbb{X}$).

2.2 Krasovskii approach

Suppose that the DDI (3) is polytopic and, therefore defined by a non-empty and compact polytope $\mathcal{A} \subset (\mathbb{R}^{n \times n})^{h+1}$. Consider the augmented state vector $\xi(k) := [x(k)^\top, \dots, x(k-h)^\top]^\top$, which yields the linear difference inclusion

$$\xi(k+1) \in \bar{F}(\xi(k)), \quad k \in \mathbb{Z}_+ \quad (4)$$

where $\bar{F}(\xi(k)) = \{\bar{A}\xi(k) : \bar{A} \in \bar{\mathcal{A}}\}$ with (see equation at the bottom of the page).

Let $\bar{\mathcal{S}}(\xi)$ denote the set of all trajectories of (4) starting from $\xi \in \mathbb{R}^{(h+1)n}$. Furthermore, let $\bar{\Phi}(\xi) := \{\bar{\phi}(k, \xi)\}_{k \in \mathbb{Z}_+} \in \bar{\mathcal{S}}(\xi)$ denote a trajectory of (4) such that $\bar{\phi}(0, \xi) = \xi$ and $\bar{\phi}(k+1, \xi) \in \bar{F}(\bar{\phi}(k, \xi))$ for all $k \in \mathbb{Z}_+$.

Then, it can be shown that [1] the DDI (3) is GES if and only if the augmented system (4) admits a standard Lyapunov function with exponential upper and lower bounds. When interpreted for the DDI (3), this Lyapunov function is called a Lyapunov–Krasovskii function (LKF) and satisfies the conditions corresponding to the Krasovskii approach. This shows that the Krasovskii approach provides non-conservative conditions for GES. Furthermore, the conditions corresponding to the Razumikhin approach can be shown to be sufficient but not necessary [1] for GES and the corresponding Lyapunov function is called a Lyapunov–Razumikhin function (LRF).

3 Time-varying Lyapunov functions

As the Lyapunov function corresponding to the Krasovskii approach is a function of the current state and all relevant delayed states, its complexity increases linearly with the size of the delay. On the other hand, any method based on the

$$\bar{\mathcal{A}} = \left\{ \begin{bmatrix} A_0 & \dots & A_{-h+1} & A_{-h} \\ I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I_n & 0 \end{bmatrix} \in \mathbb{R}^{(h+1)n \times (h+1)n} : (\{A_i\}_{i \in \mathbb{Z}_{[-h,0]}}) \in \mathcal{A} \right\}$$

Razumikhin approach is inherently conservative. Therefore in what follows, an additional degree of freedom is added to the Lyapunov function corresponding to the Razumikhin approach in order to reduce the conservatism associated with this approach while preserving its computational advantages. To this end, the concept of time-varying Lyapunov functions, see, e.g. [16, 17], will be considered. The following result exploits this concept for the stability analysis of the difference inclusion (4).

Theorem 1: Suppose that the difference inclusion (4) is linear. Then, (4) is GES ‘if and only if’ there exists a function $\tilde{V} : \mathbb{Z}_+ \times \mathbb{R}^{(h+1)n} \rightarrow \mathbb{R}_+$ such that the following conditions hold: (i) for all $(\xi, k) \in \mathbb{R}^{(h+1)n} \times \mathbb{Z}_+$

$$\bar{V}(k, \xi) = \xi^\top \bar{P}(k) \xi \quad (5a)$$

for some $(\bar{P}(k), \bar{c}_1, \bar{c}_2) \in \mathbb{R}^{(h+1)n \times (h+1)n} \times \mathbb{R}_{>0}^2$ such that $\bar{c}_1 I_{(h+1)n} \leq \bar{P}(k) \leq \bar{c}_2 I_{(h+1)n}$; (ii) for all $(\xi, \bar{\Phi}(\xi), k) \in \mathbb{R}^{(h+1)n} \times \bar{S}(\xi) \times \mathbb{Z}_+$

$$\bar{V}(k+1, \bar{\phi}(k+1, \xi)) \leq \rho \bar{V}(k, \bar{\phi}(k, \xi)) \quad (5b)$$

for some $\rho \in \mathbb{R}_{[0,1)}$.

The proof of Theorem 1 is an adaptation of its continuous-time counterpart in [18, Chapter 5.4.3] and is omitted here for brevity. A result similar to Theorem 1 was obtained in [16, Theorem 2]. However, the necessity of the quadratic structure was not established therein. When Theorem 1 is interpreted for the DDI (3), it provides a set of necessary and sufficient conditions for GES which correspond to the Krasovskii approach.

Next, using Theorem 1, a similar result is obtained for the DDI (3) based on the Razumikhin approach.

Theorem 2: Suppose that the DDI (3) is linear. Then, (3) is GES ‘if and only if’ there exists a function $V : \mathbb{Z}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that the following conditions hold: (i) for all $(x, k) \in \mathbb{R}^n \times \mathbb{Z}_+$

$$V(k, x) = x^\top P(k)x \quad (6a)$$

for some $(P(k), c_1, c_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}_{\geq 0}^2$ such that $c_1 I_n \preceq P(k) \preceq c_2 I_n$; (ii) for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]}), k) \in (\mathbb{R}^n)^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$

$$\begin{aligned} & V(k+1, \phi(k+1, \mathbf{x}_{[-h,0]})) \\ & \leq \rho \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(k+\theta, \phi(k+\theta, \mathbf{x}_{[-h,0]})) \end{aligned} \quad (6b)$$

for some $\rho \in \mathbb{R}_{[0,1)}$.

Theorem 2 is proved in the Appendix and establishes that the Razumikhin approach is not conservative when the function is allowed to be time varying. Thus, necessary and sufficient Lyapunov-like conditions for the linear DDI (3) have been obtained based on the Razumikhin approach. Under some additional assumptions Theorems 1 and 2 also allow for a local variant, which is not presented here for brevity. A function that satisfies the hypothesis of Theorems 1 and 2 is called a quadratic time-varying LKF (qtvLKF) and quadratic time-varying LRF (qtvLRF), respectively. Fig. 1 provides a schematic overview of the conclusions of this section and summarises some of the results in [1].

Fig. 1 Existence of LRFs and LKFs when related to the stability of the DDI (3), which is assumed to be linear

$A \rightarrow B$ means that A implies B and $A \nrightarrow B$ means that A does not necessarily imply B

4 Time-varying control Lyapunov functions

In what follows, the conditions proposed in Theorem 2 are exploited to obtain a computationally tractable control scheme for linear DDIs. To this end, consider the following definition.

Definition 2: Suppose that there exists a control law $\pi : \mathbb{X}^{h+1} \rightrightarrows \mathbb{U}$ such that $F_\pi(\mathbf{x}_{[-h,0]}) \subseteq \mathbb{X}$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$. Furthermore, suppose that the function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfies (6a) and (6b) for the closed-loop system (3). Then, V is called a *quadratic time-varying control Lyapunov–Razumikhin function* (or shortly, qtvLRF) for the linear DDI (1).

Based on Definition 2 we propose the following optimisation problem.

Problem 1: Let $P(k) = c_2 I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$ and let $V(k, x) = x^\top P(k)x$. At time $k \in \mathbb{Z}_+$, suppose [This assumption merely implies that the controller is able to measure the current state and to store relevant past states, control actions and the corresponding matrices $P(k)$.] that $\{P(k+j)\}_{j \in \mathbb{Z}_{[-h,0]}}$, $\mathbf{x}_{[k-h,k]}$, $\mathbf{u}_{[k-h,k-1]}$ are known. Find a $(u(k), P(k+1)) \in \mathbb{R}^m \times \mathbb{R}^{n \times n}$ that satisfy

$$f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}) \subseteq \mathbb{X}, \quad u(k) \in \mathbb{U} \quad (7a)$$

$$c_1 I_n \preceq P(k+1) \preceq c_2 I_n \quad (7b)$$

$$V(k+1, x^+) \leq \rho \max_{\theta \in \mathbb{Z}_{[-h, 0]}} V(k+\theta, x(k+\theta)) \quad (7c)$$

for all $x^+ \in f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]})$.

Let $\pi(\mathbf{x}_{[k-h,k]}) := \{u(k) \in \mathbb{R}^m : \exists P(k+1) \in \mathbb{R}^{n \times n}, \text{ s.t. (7) holds}\}$.

Proposition 1: Consider the closed-loop system (3). Suppose that for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]})) \in \mathbb{X}^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]})$ Problem 1 is recursively feasible. Then, the closed-loop system (3) is ES($\mathbb{X}$).

Proof: Let $\hat{\rho} := \rho^{\frac{1}{h+1}} \in \mathbb{R}_{[0,1]}$. The fact that Problem 1 is, by assumption, recursively feasible for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]})) \in \mathbb{X}^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]})$ yields that $\phi(k, \mathbf{x}_{[-h,0]}) \in \mathbb{X}$ for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]}), k) \in \mathbb{X}^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Moreover, (7c) can be applied recursively, that is

$$V(k, \phi(k, \mathbf{x}_{[-h, 0]})) \leq \hat{\rho}^k \max_{\theta \in \mathbb{Z}_{[-h, 0]}} V(\theta, x(\theta))$$

Using that $P(k) = c_2 I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$, it follows from (7b) that

$$\|\phi(k, \mathbf{x}_{[-h,0]})\|^2 \leq \hat{\rho}^k \frac{c_2}{c_1} \|\mathbf{x}_{[-h,0]}\|^2$$

for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]}), k) \in \mathbb{X}^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]}) \times \mathbb{Z}_+$. Therefore the DDI (3) is ES($\mathbb{X}$) with $\mu := \sqrt{\hat{\rho}} \in \mathbb{R}_{[0,1]}$ and $c := \sqrt{c_2/c_1} \in \mathbb{R}_{\geq 1}$. $\square$

Proposition 1 is of the type ‘feasibility implies stability’. Hence, solving Problem 1 on-line for some initial conditions (assuming that it remains feasible) one does not obtain a qtvLRF but merely a function that satisfies (7) for the corresponding closed-loop trajectory, see [17] for more details. Therefore consider the following definition.

Definition 3: Let $\Phi(\mathbf{x}_{[-h,0]})$ denote a trajectory of the closed-loop system (3) for which Problem 1 is recursively feasible and let $V(k, \phi(k, \mathbf{x}_{[-h,0]}))$ denote the corresponding function. Then, $\{V(k, \phi(k, \mathbf{x}_{[-h,0]}))\}_{k \in \mathbb{Z}_{\geq -h}}$ is called a ‘quadratic trajectory-dependent control Lyapunov–Razumikhin function’ (or shortly, qtdcLRF) for the linear DDI (1).

It was established in Theorem 2 that any linear DDI that is GES admits a qtvLRF, which indicates the non-conservativeness of Problem 1. However, choosing the right variables out of the feasible set such that Problem 1 becomes recursively feasible is not straightforward. To facilitate this choice, consider adding the cost function

$$J(P(k+1), u(k)) := \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(k+\theta, x(k+\theta)) - V(k+1, x^+) \quad (8)$$

to Problem 1. Then, minimisation of (8) under the conditions (7) implies a maximisation of $V(k+1, x^+)$ and also guarantees that J is lower bounded by zero. In other words, minimising J achieves the least decrease in the value of $V(k+1, x^+)$. The example presented in Section 5 confirms that adding (8) to Problem 1 improves recursive feasibility.

In what follows, we focus on uncertain systems with delay. In this case each matrix A_i and B_i , $i \in \mathbb{Z}_{[-h,0]}$ is subject to some uncertainty. Therefore the matrix polytope $\mathcal{AB}$ is of the form

$$\mathcal{AB} := \mathcal{A}_{-h} \times \mathcal{B}_{-h} \times \cdots \times \mathcal{A}_0 \times \mathcal{B}_0$$

where $\mathcal{A}_i := \text{co}(\{\hat{A}_{i,l_i}\}_{l_i \in \mathbb{Z}_{[1,L_i]}})$ and $\mathcal{B}_i := \text{co}(\{\hat{B}_{i,l'_i}\}_{l'_i \in \mathbb{Z}_{[1,L'_i]}})$, for some $(L_i, L'_i) \in \mathbb{Z}_{\geq 1}^2$ and all $i \in \mathbb{Z}_{[-h,0]}$. In this case, the number of generators of the set $\mathcal{AB}$ increases exponentially with h , that is $L = \prod_{i=-h}^0 L_i L'_i$, and hence existing control schemes, such as [1, 5–14] are not computationally tractable for large delays. In [15], to make the on-line computation of the control update independent of h , the set-valued map (2) was replaced by

$$\hat{f}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) := \{B_0 u(0) + v : B_0 \in \mathcal{B}_0, v \in \mathcal{V}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,-1]})\} \quad (9)$$

where

$$\mathcal{V}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,-1]}) := \mathcal{A}_0 x(0) \oplus \left(\bigoplus_{i=-h}^{-1} \mathcal{A}_i x(i) \oplus \mathcal{B}_i u(i) \right)$$

Lemma 1: Let f and $\hat{f}$ be defined as in (2) and (9), respectively. Then

$$\hat{f}(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) \equiv f(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]})$$

for all $(\mathbf{x}_{[-h,0]}, \mathbf{u}_{[-h,0]}) \in (\mathbb{R}^n)^{h+1} \times (\mathbb{R}^m)^{h+1}$.

Proof: The claim follows straightforwardly from the commutativity and associativity of the Minkowski addition [19]. $\square$

The same approach applies to Problem 1. For our purpose it suffices to observe that the number of vertices spanning the set $\mathcal{V}$ is, in general, much smaller than all possible combinations of the vertices spanning $\mathcal{A}_i$ and $\mathcal{B}_i$. Hence, it is advantageous to consider the set-valued map $\hat{f}$. The interested reader is referred to [20, 21] for a detailed discussion on the reduction in complexity of Problem 1 owing to the use of $\hat{f}$ instead of f .

Remark 1: As Problem 1 does not, in contrast to the control scheme proposed in [15], require the construction of a local static state feedback controller, a control scheme is obtained that is not limited by the conservatism of the Razumikhin approach and that is computationally tractable independently of the size of the delay.

To obtain a solution to Problem 1, consider the following SDP problem. Suppose that at each time $k \in \mathbb{Z}_+$ the set $\mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ is computed by the controller and let $\hat{v}_v(k) \in \mathbb{R}^n$, $l_v \in \mathbb{Z}_{[1,L_v(k)]}$ denote the corresponding vertices with $L_v(k) \in \mathbb{Z}_{\geq 1}$ the number of vertices at time $k \in \mathbb{Z}_+$ [note that at time $k \in \mathbb{Z}_+$ the variables $(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ are known and hence $\mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ can be computed via a Minkowski addition of sets]. Moreover, let $\gamma \in \mathbb{R}_{>0}$, $\Gamma \in \mathbb{R}_{\geq \gamma}$ and let $P(k) = \Gamma I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$. Consider the optimisation variables $(Z, u) \in \mathbb{R}^{n \times n} \times \mathbb{R}^m$ and the inequalities

$$\hat{B}_{0,l'_0} u + \hat{v}_{l_v}(k) \in \mathbb{X}, \quad u \in \mathbb{U} \quad (10a)$$

$$0 \leq Z - \Gamma^{-1} I_n, \quad 0 \leq \gamma^{-1} I_n - Z \quad (10b)$$

$$0 \leq \begin{bmatrix} \rho \max_{\theta \in \mathbb{Z}_{[-h,0]}} x(k+\theta)^\top P(k+\theta) x(k+\theta) & * \\ \hat{B}_{0,l'_0} u + \hat{v}_{l_v}(k) & Z \end{bmatrix} \quad (10c)$$

for all $(l'_0, l_v) \in \mathbb{Z}_{[0,L'_0]} \times \mathbb{Z}_{[0,L_v(k)]}$.

Lemma 2: At time $k \in \mathbb{Z}_+$ suppose that $\{P(k+j)\}_{j \in \mathbb{Z}_{[-h,0]}}$, $\{\hat{v}_{l_v}(k)\}_{l_v \in \mathbb{Z}_{[1,L_v(k)]}}$ and $(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ are known. If (Z, u) form a solution to (10), then $P(k+1) = Z^{-1}$ and $u(k) = u$ form a solution to (7) with $c_1 := \gamma$ and $c_2 := \Gamma$.

Proof: It follows from (10a) and Lemma 1 that $f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}) \subseteq \mathbb{X}$ and $u(k) \in \mathbb{U}$. Moreover, (10a) yields that $\gamma I_n \leq Z^{-1} = P(k+1) \leq \Gamma I_n$, which implies that (7b) holds. Applying the Schur complement to (10b) yields

$$(B_{0,l'_0} u + \hat{v}_{l_v}(k))^\top Z^{-1} (B_{0,l'_0} u + \hat{v}_{l_v}(k)) - \rho \max_{\theta \in \mathbb{Z}_{[-h,0]}} x(k+\theta)^\top P(k+\theta) x(k+\theta) \leq 0$$

which in view of Lemma 1 implies that (7c) holds with $P(k+1) = Z^{-1}$ and $u(k) = u$ and, hence, completes the proof. $\square$

The proposed approach provides an SDP-based receding horizon control scheme for linear DDIs. The resulting control law is stabilising if the corresponding optimisation problem is recursively feasible. As it was observed above, to facilitate the right choice of variables a cost function can be employed. For example, using Lemma 2 and some non-trivial facts about positive semi-definite matrices, see e.g. [22], it can be shown that a solution to Problem 1 that minimises the cost (8) is obtained by solving, via semi-definite programming, the following optimisation problem

$$\min_{(Z, u) \in \mathbb{R}^{n \times n} \times \mathbb{R}^m} \varepsilon(k) \quad (11)$$

subject to the linear matrix inequalities (10) and

$$\begin{bmatrix} \rho \max_{\theta \in \mathbb{Z}_{[-h, 0]}} x(k + \theta)^\top P(k + \theta) x(k + \theta) & * \\ \hat{B}_{0, l_0} u + \hat{v}_{l_v}(k) & Z \end{bmatrix} \preceq \varepsilon(k) I_{n+1}$$

for all $(l_0, l_v) \in \mathbb{Z}_{[0, L_0]} \times \mathbb{Z}_{[0, L_v(k)]}$. The optimisation algorithm (11) constitutes the main result of this paper and provides a stabilising controller for the linear DDI (1) taking into account the constraints $\mathbb{X}$ and $\mathbb{U}$.

5 Illustrative example

To illustrate the computational advantages of the control scheme proposed in this paper, consider the linear DDI (1) with $h = 3$. Define

$$\begin{aligned} \mathcal{A}_0 &:= \text{co} \left(\begin{bmatrix} 1.5 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.5 & -0.1 \\ 0.1 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.5 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}, \right. \\ &\quad \left. \begin{bmatrix} 1.5 & -0.1 \\ -0.1 & 0.3 \end{bmatrix} \right) \\ \mathcal{B}_0 &:= \text{co} \left(\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.25 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \right) \end{aligned}$$

$\mathcal{A}_i := 0.1 \mathcal{A}_0$ and $\mathcal{B}_i := 0.1 \mathcal{B}_0$, $i \in \mathbb{Z}_{[-3, -1]}$ such that $\mathcal{A}\mathcal{B} = \mathcal{A}_{-h} \times \dots \times \mathcal{B}_0$ and $L = 4^8 = 65\,536$. Furthermore, consider the constraints $\mathbb{X} = \mathbb{R}_{[-0.8, 0.8]} \times \mathbb{R}_{[-0.6, 0.6]}$ and $\mathbb{U} = \mathbb{R}_{[-0.6, 0.6]}$.

To stabilise the system under study, Problem 1 is used together with the cost (8). A solution to Problem 1 is obtained by solving at each time $k \in \mathbb{Z}_+$ the optimisation problem (11) with the initialisation $\rho = 0.95$, $\gamma = 0.5$,

$\Gamma = 5$ and $\{P(k) = \Gamma I_n\}_{k \in \mathbb{Z}_{[-3, 0]}}$. For a large variety of initial conditions, the control algorithm was able to stabilise the linear DDI (1). Fig. 2 shows the state trajectories and input values as a function of time for the initial conditions $\mathbf{x}_{[-3, 0]} = \{x(k) = [-0.5 \ 0.6]^\top\}_{k \in \mathbb{Z}_{[-3, 0]}}$ and $\mathbf{u}_{[-3, -1]} = \{u(k) = 0\}_{k \in \mathbb{Z}_{[-3, -1]}}$. Observe that the constraints are satisfied non-trivially at all times.

In Table 1 the dimension of the LMI that needs to be solved to stabilise the linear DDI (1) is provided for a selection of control solutions; the off-line synthesis methods presented in [5, 7], the on-line optimisation-based methods presented in [11, 15] and for the method presented in this paper. To obtain a fair comparison, the constraints were not taken into account for the numbers shown in Table 1. Furthermore, as the bottleneck for the control scheme of [15] is finding an off-line solution to an LMI of large dimensions (as opposed to the on-line component), these dimensions are shown in Table 1. Observe that the control schemes that are based on the Razumikhin approach, that is [15] and the method proposed in this paper, have a smaller complexity than their counterparts based on the Krasovskii approach.

The values in Table 1 clearly show the need for a technique that does not suffer from an exponential increase in complexity when the size of the delay increases, which justifies the approach presented in this paper. It should be emphasised that the comparison in Table 1 merely indicates the importance of the complexity issue for the stabilisation of linear DDIs and should not be used to draw any further conclusions regarding the various control schemes.

Table 1 Dimension of the LMI for various control approaches

Method	Type	Dimension of the LMI
[5]	off-line	$(589 \times 10^4 \times 589 \times 10^4)$
[7]	off-line	$(144 \times 10^4 \times 144 \times 10^4)$
[15]	off-line	$(655 \times 10^3 \times 655 \times 10^3)$
[11]	on-line	$(164 \times 10^4 \times 164 \times 10^4)$
Problem 1 without MA	on-line	$(197 \times 10^3 \times 197 \times 10^3)$
Problem 1 with MA	on-line	(100×100)

Fig. 2 Top-left: $[x(k)]_1$ as a function of time (—) and the constraints on $[x(k)]_1$ (---); Top-right: $[x(k)]_2$ as a function of time (—) and the constraints on $[x(k)]_2$ (---); Bottom: $u(k)$ as a function of time (—) and the constraints on $u(k)$ (---)

6 Conclusions

Based on the concept of time-varying Lyapunov functions, a modification of the Razumikhin approach was proposed that yields a set of necessary and sufficient conditions for exponential stability of linear DDIs. These conditions were exploited to obtain a control scheme for linear DDIs which, in contrast to other methods, remains computationally tractable for large delays.

7 References

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8 Appendix: Proof of Theorem 2

The fact that a function V satisfying (6) implies GES can be established similarly to Theorem 3.8 in [1]. Therefore only the converse is proved.

As the DDI (3) is GES if and only if [1] the augmented difference inclusion (4) is GES, it follows from Theorem 1 that the linear DDI (3) admits a function $\bar{V}$ that satisfies (5). The remainder of the proof relies on the construction of a function V that equals $\bar{V}$ except for some particular sequences of states. Therefore let $\mathbf{x}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, $\xi := [x^\top(0), \dots, x^\top(-h)]^\top$ and let $\bar{\Phi}(\xi)$ correspond to $\Phi(\mathbf{x}_{[-h,0]})$. For brevity $\phi(k)$ and $\bar{\phi}(k)$ are used to denote $\phi(k, \mathbf{x}_{[-h,0]})$ and $\bar{\phi}(k, \xi)$, respectively. Furthermore, observe that the equivalence of norms implies that $\|\xi\|^2 \leq c_3 \|\mathbf{x}_{[-h,0]}\|^2$ for some $c_3 \in \mathbb{R}_{>0}$.

Next, using all of the above, for all $k \in \mathbb{Z}_{\geq 1}$, if $\|\bar{\phi}(k)\| \neq 0$ let

$$P(k) := \begin{cases} \frac{\bar{V}(k, \bar{\phi}(k))}{\|\phi(k)\|^2} I_n, & \phi(k) \in \mathbb{R}^n \setminus \mathcal{N}(k, \bar{\phi}(k)) \\ \bar{c}_2 c_3 I_n, & \phi(k) \in \mathcal{N}(k, \bar{\phi}(k)) \end{cases}$$

and, if $\|\bar{\phi}(k)\| = 0$, let $P(k) := \bar{c}_2 c_3 I_n$. Above $\mathcal{N}(k, \bar{\phi}(k)) := \{x \in \mathbb{R}^n : \|x\|^2 < (1/\bar{c}_2 c_3) \|\bar{\phi}(k)\|\}$. Moreover, let $P(k) := \bar{c}_2 c_3 I_n$ for all $k \in \mathbb{Z}_{[-h,0]}$.

Consider any $\xi \in \mathbb{R}^{(h+1)n}$ and any $\bar{\Phi}(\xi) \in \bar{\mathcal{S}}(\xi)$ [with corresponding $\mathbf{x}_{[-h,0]}$ and $\Phi(\mathbf{x}_{[-h,0]})$]. If $\|\bar{\phi}(k)\| \neq 0$ and $\phi(k) \in \mathbb{R}^n \setminus \mathcal{N}(k, \bar{\phi}(k))$, then (5a) and the definition of $\mathcal{N}$ yield

$$\bar{c}_1 I_n \leq \frac{\bar{c}_1 \|\bar{\phi}(k)\|^2}{\|\phi(k)\|^2} I_n \leq P(k) \leq \frac{\bar{c}_2 \|\bar{\phi}(k)\|^2}{\|\phi(k)\|^2} I_n \leq (\bar{c}_2)^2 c_3 I_n$$

Moreover, if $\phi(k) \in \mathcal{N}(\bar{\phi}(k))$, then $\bar{c}_2 c_3 I_n \leq P(k) \leq \bar{c}_2 c_3 I_n$. Therefore $P(k)$ is well defined for all $k \in \mathbb{Z}_{\geq -h}$ and satisfies (6a) with $c_1 := \min\{\bar{c}_1, \bar{c}_2 c_3\} \in \mathbb{R}_{>0}$ and $c_2 := \max\{(\bar{c}_2)^2 c_3, \bar{c}_2 c_3\} \in \mathbb{R}_{\geq c_1}$.

Next, consider the time-varying function $V(k, x) = x^\top P(k)x$, $k \in \mathbb{Z}_{\geq -h}$, defined by the matrix $P(k)$ as given above. In what follows, it will be shown, using three different cases, that the candidate function V satisfies (6b). Firstly, consider any $k \in \mathbb{Z}_{\geq h}$ and suppose that there exists a $\theta_M \in \mathbb{Z}_{[-h,0]}$ such that $\phi(k + \theta_M) \notin \mathcal{N}(k + \theta_M, \bar{\phi}(k + \theta_M))$. Then

$$\begin{aligned} V(k+1, \phi(k+1)) &\leq \bar{V}(k+1, \bar{\phi}(k+1)) \\ &\leq \bar{\rho} \bar{V}(k + \theta_M, \bar{\phi}(k + \theta_M)) \\ &= \bar{\rho} V(k + \theta_M, \phi(k + \theta_M)) \\ &\leq \bar{\rho} \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(k + \theta, \phi(k + \theta)) \end{aligned}$$

for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]})) \in (\mathbb{R}^n)^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]})$. Next, consider any $k \in \mathbb{Z}_{\geq h}$ and suppose that $\phi(k + \theta) \in \mathcal{N}(k + \theta, \bar{\phi}(k + \theta))$ for all $\theta \in \mathbb{Z}_{[-h,0]}$. Then

$$\begin{aligned} V(k+1, \phi(k+1)) &\leq \bar{V}(k+1, \bar{\phi}(k+1)) \\ &\leq \bar{\rho} \bar{V}(k, \bar{\phi}(k)) \leq \bar{\rho} \bar{c}_2 \|\bar{\phi}(k)\|^2 \\ &\leq \bar{\rho} \bar{c}_2 c_3 \max_{\theta \in \mathbb{Z}_{[-h,0]}} \|\phi(k + \theta)\|^2 \\ &= \bar{\rho} \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(k + \theta, \phi(k + \theta)) \end{aligned}$$

for all $(\mathbf{x}_{[-h,0]}, \Phi(\mathbf{x}_{[-h,0]})) \in (\mathbb{R}^n)^{h+1} \times \mathcal{S}(\mathbf{x}_{[-h,0]})$. Thirdly, let $k \in \mathbb{Z}_{[0,h-1]}$. Then for $k = 1$, (6b) follows from the second case considered above. Moreover, for each $k \in \mathbb{Z}_{[1,h-1]}$, (6b) follows from the second case if $\phi(k + \theta) \in \mathcal{N}(\tilde{\phi}(k + \theta))$

for all $\theta \in \mathbb{Z}_{[-k+1,0]}$ and (6b) follows from the first case otherwise. Thus, it has been established that the candidate function V satisfies (6b) with $\rho := \bar{\rho} \in \mathbb{R}_{[0,1]}$, which completes the proof. $\square$