Effects of time delay in the synchronized motion of oscillators with Huygens' coupling
Pena Ramirez, J.; Alvarez Aguirre, A.; Fey, R.H.B.; Nijmeijer, H.

Published in:
3rd IFAC Conference on analysis and control of chaotic systems (3rd IFAC CHAOS Conference)

Published: 01/01/2012

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.
• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Effects of time delay in the synchronized motion of oscillators with Huygens’ coupling

Jonatan Peña Ramírez∗ Alejandro Alvarez Aguirre∗ Rob H. B. Fey∗ Henk Nijmeijer∗

∗ Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, the Netherlands (e-mail: j.pena@tue.nl, a.alvarez.aguirre@tue.nl, r.h.b.fey@tue.nl, h.nijmeijer@tue.nl)

Abstract: In this paper, the well known Huygens’ experiment on synchronization is examined, numerically and experimentally, from a control point of view. Namely, a suitable control input is applied to a pair of self-sustained oscillators such that the closed-loop system resembles a pair of oscillators with Huygens’ coupling. The case where the interaction between the oscillators and the control input is time delayed is analyzed. In particular, the onset of in-phase and anti-phase synchronization in the coupled/controlled oscillators is studied as a function of the coupling strength and the time delay. An eigenvalue-based numerical test is used as a first step to investigate the stability of the anti-phase synchronized motion and experimental validation is provided.

Keywords: Synchronization, self-sustained oscillators, time-delay, Huygens’ coupling.

1. INTRODUCTION

Synchronization is a phenomenon that can be observed in a wide class of (oscillating) systems, which are frequently encountered in biology, physics, engineering, and many other fields. Some familiar examples are the synchronized applause in a large audience, the synchronized motion of schools of fish, and the synchronized motion of pendulum clocks. These examples - amongst others - are nicely described in Strogatz (2003) and Pikovsky et al. (2001). In fact, the occurrence of synchronization in systems of very different nature suggests that the tendency towards synchronization is universal (Blekhman, 1988).

An interesting situation occurs when the interaction between the (oscillating) systems is not instantaneous but occurs after some (small) time delay. This is the case in for example biological oscillators (Kim et al., 2010), ecosystems (Haque et al., 2011), and remote coordination of robots (Alvarez-Aguirre, 2011).

Several studies have analyzed the influence of time delay in the synchronized motion of two oscillators with delayed coupling may experience a phase-flip bifurcation, i.e. an abrupt change from in-phase to anti-phase synchronization due to a variation in the amount of time delay. This phase-flip may find application in for instance coupled laser systems where transition from in-phase to anti-phase synchronization may lead to a high degree of constant output (Prasad et al., 2006).

In this work, the well known Huygens’ experiment on synchronization is examined from a control point of view. Different to the original Huygens’ setup, which consists of two pendulum clocks attached to a flexible wooden bar (Huygens’ coupling), in the present analysis, the pendulum clocks have been replaced by two self-sustained oscillators and the coupling structure, i.e. the wooden bar, has been replaced by a representative dynamical system. This dynamical system generates a suitable control input for the oscillators such that in closed loop the system resembles a pair of oscillators with Huygens’ coupling. Note that in this case, the oscillators do not have to be at the same location and moreover, the dynamical system generating the control input should be implemented separately, for instance a computer. Consequently, the possibility of communication time-delays (either in the oscillators or in the applied control input) comes into play. Then, this paper will focus in determining the influence of time delay in the limit behaviour of the (controlled) oscillators. In particular, the influence of variations in the size of the time delay in the onset of in-phase and anti-phase synchronization is studied. As a first step to investigate the stability of the anti-phase synchronized motion an eigenvalue-based numerical test is used. Experimental results are also provided.

The paper is organized as follows. First, in Section 2, the dynamical model of the coupled oscillators is presented. Next, Section 3 considers the case where only the input to the oscillators is time-delayed, whereas Section 4 addresses the case where both the input and output of the oscillators are time-delayed. Then, in Section 5, an experimental validation is presented. Finally, a discussion of obtained results and conclusions are formulated in Sections 6 and 7, respectively.
2. PRELIMINARIES

Consider two identical oscillatory (mechanical) systems of the form:

\[ \ddot{x}_i(t) + 2\xi \omega \dot{x}_i(t) + \omega^2 x_i(t) + f(x_i(t), \dot{x}_i(t)) = u(t), \quad i = 1, 2, \]  

where \( \omega \in \mathbb{R}^+ \) is the angular eigenfrequency of the unforced oscillators and \( \xi \in \mathbb{R}^+ \) is the dimensionless damping coefficient. The nonlinear term \( f(x_i(t), \dot{x}_i(t)) \) is designed such that each oscillator is converted into a self-sustained oscillator. The output of each oscillator is assumed to be given by

\[ y_i(t) = 2\xi \omega \dot{x}_i(t) + \omega^2 x_i(t) + f(x_i(t), \dot{x}_i(t)), \quad i = 1, 2. \]  

The control input \( u(t) \), which is the same for each oscillator, is generated by the dynamical system

\[ \dot{\eta}(t) = A\eta(t) + B(y_1(t) + y_2(t)) \]  

where \( \eta(t) = [\eta_1(t) \eta_2(t)]^T \) is the state vector,

\[ A = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/\gamma \end{bmatrix}, \]  

where \( \alpha_1 > 0 \) and \( \alpha_2 > 0 \) are positive parameters, and \( 1/\gamma \) is the coupling strength. The control input \( u(t) \) is now constructed as follows

\[ u(t) = [\omega^2 \ 2\xi \omega] \eta(t) = \omega^2 \eta_1(t) + 2\xi \omega \eta_2(t). \]  

The closed-loop system (1)-(5) describes a pair of nonlinear oscillators with Huygens’ coupling. This terminology is explained as follows. In Huygens’ experiment of pendulum clocks, each pendulum is affected by the vibrations of the coupling wooden bar, i.e. the pendula receive the same input. The vibrations in the coupling bar, which can be modeled as a suspended rigid bar of one degree of freedom (Pogromsky et al., 2003), are influenced by the forces exerted by the pendula. Therefore, Huygens’ model of pendulum clocks can be casted into the form (1)-(5). Nevertheless, it should be noted that in this case, the pendula have been replaced by two arbitrary self-sustained oscillators (1) and the coupling bar is represented by the dynamical system (3). Likewise, the coupling strength \( 1/\gamma \) can be associated to Huygens’ case, where the coupling strength is defined as a ratio of masses (mass of clocks/mass of wooden bar).

The need of having \( f(x_i(t), \dot{x}_i(t)) \) in (1) can be linked to Huygens’ pendulum clocks where the energy loss, due to friction, is compensated by an escapement mechanism. In the present case, the energy loss in system (1) is compensated by the nonlinear term

\[ f(x_i(t), \dot{x}_i(t)) = \alpha \dot{x}_i(t) \text{sign}(x_i(t) - x_{ref}), \quad i = 1, 2, \]  

where \( \alpha \) is a positive parameter satisfying \( \alpha > 2\omega \), the constant \( x_{ref} \in \mathbb{R}^+ \) represents a threshold displacement value, and \( \text{sign}(\cdot) \) denotes the sign function of a real number. However, other state dependent nonlinear functions can be considered as well.\(^1\)

In a previous publication (Peña-Ramírez et al., 2011), the delay-free case corresponding to system (1)-(6) has been considered and two synchronizing regimes are observed in the system, namely in-phase and anti-phase synchronization. It has been found that the steady-state synchronized motion depends on the coupling strength. For small values of \( \gamma \) in-phase synchronization has been observed, whereas for large values of \( \gamma \) anti-phase synchronization is likely to occur. The present work extends this analysis for the case where the interaction between the oscillators (1) and the coupling system (3) is time delayed. Two cases are considered: delayed unidirectional coupling, where only the control input (5) to the oscillators is delayed, and delayed bidirectional coupling where both the control input (5) and the output (2) of each oscillator are delayed, as schematically depicted in Figure 1.

3. DELAYED UNIDIRECTIONAL COUPLING

In this section, it is assumed that the oscillators (1) receive a delayed version of the control input (5), hence

\[ \ddot{x}_i(t) + 2\xi \omega \dot{x}_i(t) + \omega^2 x_i(t) + f(x_i(t), \dot{x}_i(t)) = u(t), \]  

for \( i = 1, 2 \) and

\[ u(t) = \omega^2 \eta_1(t - \tau) + 2\xi \omega \eta_2(t - \tau). \]  

In the sequel, this type of coupling is referred to as delayed unidirectional coupling. The resulting closed-loop (7),(8),(3) is given by

\[ \ddot{x}_i(t) - \omega^2 (x_i(t) - \eta_i(t - \tau)) - 2\xi \omega (\ddot{x}_i(t) - \eta_i(t - \tau)) - \alpha \dot{x}_i(t) \text{sign}(x_i(t) - x_{ref}), \quad i = 1, 2, \]  

\[ \eta_1(t) = -\alpha_1 \eta_1(t) - \alpha_2 \eta_2(t) + \gamma (y_1(t) + y_2(t)), \]  

with \( y_1(t) \) and \( y_2(t) \) as defined in (2).

3.1 Limit behaviour for varying \( \gamma \) and \( \tau \).

Next, the dynamical behaviour of the coupled system (9) is studied as a function of the coupling strength and the time delay. The system’s response is simulated for \( \tau \in [0, 2.5] \) s, \( \gamma \in [0.04, 100] \) [—], and parameter values: \( \omega = 13.29 \) [rad/s], \( \xi = 0.3829 \) [—], \( \alpha = 10.1866 \) [1/s], \( \alpha_1 = 2204.4095 \gamma^{-1} \) [m/s], \( \alpha_2 = 35.9104 \gamma^{-1} \) [m], \( x_{ref} = 0.0025 \) [m], and initial conditions \( x_1(0) = 0.0027 \) [m] and \( x_2(0) = -0.0023 \) [m]. The remaining initial conditions are set to zero.

The simulation results are illustrated in Figure 2, where different colors have been used to show if the system has reached anti-phase synchronization (cyan), in-phase synchronization (orange), or unstable behaviour to infinity (brown). Clearly, the delay is a parameter that influences the limit behaviour of the coupled oscillators. For very small values of \( \tau \), the behaviour of the coupled oscillators is unaffected and the behaviour of the system is as described in Peña-Ramírez et al. (2011): small \( \gamma \) results in in-phase synchronization, larger \( \gamma \) results in anti-phase synchronization. However, when \( \tau \) increases, the synchronized behaviour of the system switches from in-phase to anti-phase and vice versa. It should also be noticed that there are also intermediate values of \( \tau \) for which

---

\(^1\) For example, \( f(x_i(t), \dot{x}_i(t)) = \lambda (\dot{x}_i^2(t) - 1) \dot{x}_i(t) \).
the coupled system will exhibit unstable oscillations to infinity, i.e., each oscillator becomes unstable. As depicted in Figure 2, for small values of $\gamma$, unstable behaviour does not seem to occur, whereas for large values of $\gamma$, the system will eventually become unstable.

3.2 Analysis of the anti-phase motion

An initial step in the stability analysis of the anti-phase synchronized motion in the coupled piece-wise linear oscillators (9) is conducted under the assumption that $x_i(t) < x_{ref}, \forall t \in [0, t_1]$ for some $t_1 > 0$ (Dilão, 2009). Then, the closed-loop system (9) becomes

$$\dot{x}_i(t) = -\omega^2(x_i(t) - \eta_1(t - \tau)) - 2\xi\omega(\dot{x}_i(t) - \dot{\eta}_1(t - \tau)) + \alpha x_i(t), \quad i = 1, 2,$$

(10a)

$$\ddot{\eta}_1(t) = -\alpha_1\eta_1(t) - \alpha_2\dot{\eta}_1(t) + \frac{1}{\gamma}(y_1(t) + y_2(t)),$$

(10b)

with $y_i(t) = (2\xi\omega - \alpha)x_i(t) + \omega^2x_i(t), i = 1, 2.$

Note that if the oscillators synchronize in anti-phase, then the coupling vanishes, i.e. $x_1(t) = -x_2(t)$ implies that $y_1(t) = -y_2(t)$ and consequently $\eta_1(t)$ and $\dot{\eta}_1(t)$ in (10b) tend to zero. The anti-phase synchronization errors are defined as follows

$$e_1(t) = x_1(t) + x_2(t), \quad \dot{e}_1(t) = \dot{x}_1(t) + \dot{x}_2(t),$$

(11a)

$$e_2(t) = \eta_1(t), \quad \dot{e}_2(t) = \dot{\eta}_1(t).$$

(11b)

The error dynamics can be written in the form

$$\dot{e}(t) = A_0e(t) + A_1e(t - \tau),$$

(12)

where the error state is given by $e = [e_1 \ e_2 \ \dot{e}_1 \ \dot{e}_2]^T$ and

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega^2 & 0 & a & 0 \\ \frac{1}{\gamma} & -\alpha_1 & b & -\alpha_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2\omega^2 & 0 & 4\xi\omega \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
The closed-loop (1,3,6,14) is given by

\[
\dot{x}_i(t) = -\omega^2(x_i(t) - \eta_i(t - \tau)) - 2\xi\omega(\dot{x}_i(t) - \dot{\eta}_i(t - \tau)) - \alpha_i\eta_i(t) - \alpha_2\dot{\eta}_i(t) + \frac{1}{\gamma}(y_i(t) + y_2(t)),
\]

with \(y_i(t), i = 1, 2\), as defined in (13).

4.1 Limit behaviour for varying \(\gamma\) and \(\tau\).

The response of system (15) is numerically investigated as a function of \(\gamma\) and the time delay \(\tau\). The same parameter values are used as in the unidirectional case discussed in Section 3. The simulation results are depicted in Figure 4. Among others, transitions from in-phase to anti-phase synchronization and vice versa again can be clearly seen. An important difference between the results obtained for the unidirectional delayed coupling and the present coupling is that the regions where the in-phase or anti-phase synchronized motion of the oscillators are stable, are smaller and therefore, the instability region is larger. Note that for small \(\gamma\) and moderate to large \(\tau\), the oscillators will synchronize in-phase.

4.2 Analysis of the anti-phase motion

An initial step in the stability analysis of the anti-phase synchronized motion is again carried out by using the same assumption as in Subsection 3.2.

Writing the anti-phase error dynamics as a set of linear first order differential equations yields

\[
\dot{\epsilon}(t) = A_0\epsilon(t) + A_1\epsilon(t - \tau),
\]

where

\[
A_0 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & \omega^2 & 0 & 1 \\
-\omega^2 & 0 & a & 0 \\
0 & -\alpha_1 & 0 & -\alpha_2
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2\omega^2 & 0 & 4\xi\omega \\
\frac{1}{\gamma} & \omega^2 & 0 & b
\end{bmatrix}
\]

where \(e = [\epsilon_1, \epsilon_2, \dot{\epsilon}_1, \dot{\epsilon}_2]^T\) is the state vector, with \(\epsilon_1\) and \(\epsilon_2\) as defined in (11), \(a = -(2\xi\omega - \alpha)\), and \(b = \frac{1}{\gamma}(2\xi\omega - \alpha)\).

Next, the same eigenvalue-based numerical test as already used for the delayed unidirectional coupling case is carried out. The results are shown in Figure 5. In this case, the stability regions identified by DDE-BIFTOOL almost exactly match the stability regions resulting from the simulations.

5. EXPERIMENTAL VALIDATION

The effect of time delay in the synchronized motion of oscillators with delayed Huygens’ coupling is experimentally studied. Figure 6 shows the essentials of the experimental setup, which consists of two (actuated) oscillators. Each oscillator has a maximum stroke of 5 [mm] and is equipped with a voice coil actuator and a linear variable differential transformer, which acts as a translational displacement sensor. Additional information regarding the platform is provided in Peña-Ramírez et al. (2011).
In the first experiment, the controllers of the oscillators are adjusted such that the oscillator’s dynamic behaviour is described by (9a), i.e. the case of delayed unidirectional coupling is considered. Obviously, the experimental set-up introduces some uncertainty in the dynamics intended by the model. The dynamical system (9b) is implemented in software. The parameter values in the experiment are assumed to be $\omega = 13.29 \text{ [rad s}^{-1}]$, $\xi = 0.3829 \text{ [\%]}$, $\alpha = 10.1866 \left[\frac{1}{s}\right]$, $\omega_1 = \frac{2\pi}{2.45 \text{ [mm]}}$, $\omega_2 = \frac{2\pi}{2.3 \text{ [mm]}}$, $x_{ref} = 2.5 \text{ [mm]}$, $\gamma = 50 \text{ [\%]}$ and initial conditions $x_1(0) = 2.4 \text{ [mm]}$ and $x_2(0) = 2.15 \text{ [mm]}$. The remaining initial conditions are set to zero.

In order to better appreciate the influence of the time delay in the synchronized motion of the oscillators, the experiment is started by considering $\tau = 0 \text{ [s]}$\footnote{The actual time delay in the set-up is negligible.}. After a short transient behaviour, the oscillators synchronize in anti-phase as depicted in Figure 7b. At $t = 35 \text{ [s]}$, a unidirectional time delay of $\tau = 0.155 \text{ [s]}$ is induced and as a consequence the oscillators synchronize in-phase, as illustrated in Figures 7a and 7c.

Note that besides the change from anti-phase to in-phase synchronization, the oscillation frequency $\Omega$ (measured from peak to peak) also changes as shown in Figure 7d. This is further explained in Section 6. Figure 7e shows the control input (8) applied to the oscillators. It can be seen that the control input (almost) vanishes when anti-phase synchronization is achieved, whereas for the in-phase case, the control input becomes periodic.

The numerical results presented in Section 3 are in agreement with these results. Note that for $\gamma = 50 \text{ [\%]}$ and $\tau = 0 \text{ [s]}$, the numerical results presented in Figures 2 and 3 indicates that anti-phase synchronization is likely to occur, whereas for $\tau = 0.155 \text{ [s]}$, in-phase synchronization is expected to occur.

In a second experiment, the case of bidirectional delayed coupling is considered. Then, the oscillators are adjusted according to model (15a). The dynamical system (15b) is again implemented in software. The parameter values are the same as used in the first experiment, except for the nonzero initial conditions, which now are $x_1(0) = 2.45 \text{ [mm]}$ and $x_2(0) = -2.3 \text{ [mm]}$, and the coupling strength is computed for $\gamma = 10 \text{ [\%]}$.

Again, at the beginning of the experiment, the time delay is assumed to be zero. The time series corresponding to $x_1$ and $x_2$ is depicted in Figure 8a. From Figure 8b it can be observed that after the transient behaviour the oscillators synchronize in-phase. Then, at $t = 15 \text{ [s]}$ a time delay of $\tau = 0.043 \text{ [s]}$ is induced in the oscillators. Consequently, the synchronized motion switches from in-phase to anti-phase as depicted in Figure 8c. Note that now the transition is slower in comparison with the transition observed in the first experiment. This is also the case in computer simulations. Figure 8d shows the oscillation frequency $\Omega$, which increases to the value of the frequency of the individual self-driven oscillators. In theory, the interaction between the oscillators should vanish and consequently the control $u(t)$, see (14), should also vanish, but due to the small differences between the oscillators in the experiment, the control input remains oscillating with small amplitude as illustrated in Figure 8e.

6. DISCUSSION

- The influence of time delay in the appearance of transitions from in-phase to anti-phase synchronization and vice versa, occurring in delayed coupled oscillators, has been already studied for the case of diffusive couplings, see e.g. Prasad et al. (2006). However,
to the best of our knowledge, this is the first time that these transitions are studied in the context of Huygens’ coupling.

- The synchronization problem addressed here may seem artificial, but it may find interesting industrial applications like for example in the control of vibrations during the start-up phase of two generators/industrial motors, which are placed close to each other. Initially, the generators/motors can be forced to synchronize in anti-phase, this will reduce the amount of vibrations in the supporting structure and when the generators/motors are operating at nominal speed, they can be forced to synchronize in-phase in order to reduce the consumption of energy. According to the results presented here, this can be done by using the same control law and just varying a parameter, namely the delay. Moreover, further studies of the proposed delayed Huygens’ coupling may lead to understand similar phenomena occurring in other fields. For example in neurosystems, where it has been found that time delay induces transitions from in-phase to anti-phase synchronization in two coupled excitable neurons (Scholl et al., 2009).

- The analysis presented here focused mainly in the synchronizing limit behaviour of the oscillators. However, it is worthwhile mentioning that for our system, other dynamical limit behaviours exist, like amplitude death (Fathihcan, 2009), where the oscillations of both oscillators decay. This has not only been observed in computer simulations but also in experiments (not included here).

- The experimental results have revealed that when the oscillators synchronize in anti-phase, the oscillation frequency corresponds to the frequency of the uncoupled self-driven oscillators. This results from the fact that in anti-phase, the coupling vanishes (as in the original Huygens’ experiment) and, consequently, the oscillators run uncoupled (in the ideal case). When the oscillators synchronize in-phase, it has been found that the oscillation frequency can be tuned by properly choosing parameter $\alpha_1$ in (9b) or (15b).

7. CONCLUSIONS

In this paper, Huygens' synchronization has been addressed from a control point of view. In particular, the influence of time delay in the synchronized motion of a pair of self-sustained oscillators with delayed Huygens’ coupling has been numerically and experimentally investigated. It has been shown that the time delay is a bifurcation parameter leading to transitions from 0 to $\pi$ and vice versa in the phase difference of the coupled oscillators. An eigenvalue-based numerical test has been used as a first step to study the stability of the anti-phase synchronized motion. Moreover, the experimental and numerical results are in good agreement. The richness of dynamical behaviour of the analyzed coupled system and the similitude of the obtained results with results found in other research fields, should be enough reason to further investigate the effect of time delay in systems with Huygens’ coupling. A formal theoretical analysis of the stability of the limit behaviours presented here is still missing.

REFERENCES


