Boundary layer development in the flow field between a rotating and a stationary disk

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This paper discusses the development of boundary layers in the flow of a Newtonian fluid between two parallel, infinite disks. One of the disks is rotating at a constant angular velocity while the other remains stationary. An analytical series approximation and a numerical solution method are used to describe the velocity profiles of the flow. Both methods rely on the commonly used similarity transformation first proposed by Von Kármán [T. von Kármán, ZAMM 1, 233 (1921)]. For \( Re_h < 18 \), the power series analytically describe the complete velocity profile. With the numerical model a Batchelor type of flow was observed for \( Re_h > 300 \), with two boundary layers near the disks and a non-viscous core in the middle. A remarkable conclusion of the current work is the coincidence of the power series’ radius of convergence, a somewhat abstract mathematical notion, with the physically tangible concept of the boundary layer thickness. The coincidence shows a small deviation of only 2% to 4%. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.3698406]

I. INTRODUCTION

The flow of a fluid between two parallel, rotating disks has obtained a lot of attention since the pioneering study of Von Kármán in 1921.1 In his paper, Von Kármán described the flow of a Newtonian fluid flowing near an infinite disk rotating with angular velocity \( \Omega \). By adopting a similarity assumption for this axisymmetric configuration, the Navier-Stokes equations can be reduced from a system of three partial differential equations to a less complex system of three ordinary differential equations (ODEs). Although no full analytical solution of this system has been found yet, numerical calculations in subsequent years have shown that the azimuthal fluid velocity gradually decreases from the velocity of the disk to zero when moving further away from the disk. Furthermore, the rotating disk acts as a centrifugal fan by which fluid is sucked towards the disk in the axial direction and thrown radially outwards near the disk.2

The region in which these velocity changes occur will be termed here as the “Von Kármán boundary layer”. In literature nowadays, there is a tendency to refer to this layer as the “Ekman layer”. However, Ekman layers are boundary layers occurring at horizontal boundaries in rotating fluid systems: they establish the proper matching of the relative flow to the pertinent boundary conditions (in terms of velocity or shear stress). Their mathematical description is based on the assumption of small relative fluid velocities compared to the basic rotational speed of the system, thus comprising a balance between Coriolis, pressure, and viscous forces.3 The governing equations of Ekman layer flow are essentially linear, in contrast to the nonlinear equations describing the Von Kármán boundary layer flow.

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Two decades after the publication by Von Kármán, Bödewadt followed a similar analysis for the case of a fluid flowing above an infinite stationary disk, but rotating uniformly at constant angular velocity $\Omega$ infinitely far away from the disk.\(^4\) In this situation, the azimuthal velocity component gradually decreases to zero with a decreasing distance from the disk. Also, the centrifugal pressure created by the rotation of the fluid invokes a radial inflow of fluid near the disk and at the same time (due to continuity) an axial flow away from the disk. Here, the region of fluid in which the velocity changes take place is generally termed as the “Bödewadt boundary layer”.

In the early 1950s, Batchelor generalized Von Kármán’s solution method for the case of flow between two disks.\(^5\) This eventually led to a classical controversy in the field of fluid dynamics, when two completely different predictions for the flow of a fluid between two disks were put forward.

Batchelor\(^5\) suggested that when one disk was rotating at a constant angular velocity, while the other remained stationary, for moderate Reynolds numbers, three distinct layers would form between the disks. Near the rotating disk a Von Kármán boundary layer would form, whereas near the stationary disk a Bödewadt boundary layer would develop. According to Batchelor, these two boundary layers were separated by a core in which viscous effects were negligible. The fluid in this non-viscous core would have a constant angular velocity, a constant axial velocity, and zero radial velocity.

Stewartson\(^6\) opposed this view by predicting that indeed a Von Kármán boundary layer would develop near the rotating disk, but that the main body of the fluid would be at rest. In this situation, both the Bödewadt boundary layer and the non-viscous core would be absent.

The argument lasted for several decades until in 1977 a numerical study by Holodniok et al.\(^7\) showed that there is no unique solution of the system of ODEs. In fact, it was shown that depending on the Reynolds number up to five different solutions could be obtained.

For the case of finite-sized disks, it was found by Brady and Durlofsky\(^8\) that the boundary conditions at the edge of the disk greatly determine the flow pattern actually emerging between the two disks: in an open disk configuration, the flow field would tend towards a Stewartson type of flow, while an enclosed disk would result in a Batchelor type of flow. Also, according to Brady and Durlofsky, the region of fluid in which the similarity solution applies, asymptotically shrinks towards the axis of rotation with an increasing Reynolds number. In fact, at a Reynolds number of 500, the obtained flow profiles for confined flow differ by 50% with the similarity solution at a radial position of 60% of the disk radius.

It should be noted that Brady and Durlofsky assumed that the fluid near the end wall was in inviscid motion. The inviscid fluid would thus conserve its angular momentum at the end wall while its radial outflow near the rotating disk was redirected in the axial direction. This indirectly implies that there is no effect on the fluid velocity in the interior of the cavity whether the shroud is co-rotating with the disk or not.

In a later study by Lopez,\(^9\) however, it was shown that the motion of the cylindrical shroud does have a profound impact on the fluid motion inside the cavity. When the full velocity field is calculated inside the cavity together with the no-slip conditions at the cylindrical end wall, it was found that a co-rotating shroud, on the one hand, resulted in Batchelor types of flow. This is in agreement with the results of Brady and Durlofsky\(^8\) where the fluid angular momentum is unchanged at the position of the shroud.

A stationary shroud, on the other hand, yields Stewartson types of flow. This difference is a direct result from the shape of the vortex lines; the vortex lines originate from the rotating disk and terminate at the singularity at the joint between the co-rotating shroud and the stator for Batchelor type of flow. For Stewartson types of flow, the vortex lines are terminated at the singularity at the joint between the rotating disc and the stationary shroud.\(^5\) Moreover, recent work by Lopez et al.\(^10\) has revealed that the self-similar solution is indeed able to describe the flow of a fluid near a confined rotating disk up to a radial position of 80% of the disc radius for Reynolds numbers up to the order $10^5$. Above this Reynolds number, circular axisymmetric waves propagate radially inwards through the fluid, breaking the self-similarity.\(^10\) It was also observed in the same study that instabilities in the form of spiral waves occur at Reynolds numbers greater than approximately $5 \times 10^5$. 
Today the study of rotating disk flow is still very relevant. Not only is the outlook of solving the full Navier-Stokes equations an appealing incentive, but a proper description of the flow pattern between two disks is also crucially important for the design of rotating machinery as, for example, the novel rotor-stator spinning disk reactor. The formation and behavior of different layers of fluid between the two disks have an enormous effect on mass and heat transfer in single phase and multiphase flows.

In this paper, the Navier-Stokes equations will be subjected to a Von Kármán transformation to yield a system of ODEs. This system of differential equations will then be solved with a numerical solution method and an analytical series approximation. From these solutions, the thicknesses of the Von Kármán and Bödewadt boundary layers are determined. As will be shown in the present work, a remarkable property of the power series approximation is the coincidence of its radius of convergence with the boundary layer thickness for both the Bödewadt and the Von Kármán boundary layers.

The paper is structured as follows: first, the flow geometry is sketched, the physical system and quantities are defined and the relevant mathematical equations are derived. This is followed by a discussion of the mathematical methods, including the Von Kármán similarity transformation, the numerical solution method, and the power series solution strategy. Next, the definitions of the Von Kármán and Bödewadt boundary layer thicknesses are discussed and four of these definitions are adopted. Then, the resulting velocity profiles from the numerical solution method are presented and they are used to calculate the boundary layer thicknesses. Finally, some concluding remarks are made.

II. PHYSICAL SYSTEM AND GOVERNING EQUATIONS

Figure 1 shows a schematic drawing of the flow geometry, including a rotating disk and a stationary disk with a disk spacing \( h \). The flow is conveniently described in terms of a cylindrical coordinate system \((\hat{r}, \theta, \hat{z})\), being the radial, azimuthal, and axial coordinates, respectively. The position \( \hat{r} = 0 \) is identified with the rotation axis and the plane \( \hat{z} = 0 \) is identified with the rotor. The fluid flow is then described by the velocity vector \( \hat{\mathbf{v}} = (\hat{v}_r, \hat{v}_\theta, \hat{v}_z) \). The fluid is taken to be Newtonian with a density \( \rho \) and a kinematic viscosity \( \nu \). The pressure is a function of position and is denoted by \( \hat{p} \). The physical quantities are nondimensionalized as follows:

\[
\begin{align*}
\hat{z} &\equiv \hat{z}h^{-1}, \quad \hat{r} \equiv \hat{r}h^{-1}, \quad \hat{p} \equiv \hat{p} \rho^{-1} \Omega^{-2} h^{-2}, \\
\hat{v} &\equiv (\hat{v}_r, \hat{v}_\theta, \hat{v}_z) = (\hat{\mathbf{v}}) / \Omega^{-1} h^{-1} = \hat{\mathbf{v}} / \Omega^{-1} h^{-1},
\end{align*}
\]

while the Reynolds number becomes

\[
Re_h \equiv \Omega h^2 \nu^{-1}.
\]

FIG. 1. Schematic drawing of the flow geometry, consisting of two parallel, infinite disks. The lower disk rotates with an angular velocity \( \Omega \), while the upper disk is stationary. The gap distance between the disks is \( h \).
For the case of rotational symmetry and steady flow, the three components of the Navier-Stokes equations in their dimensionless form can be written as

\begin{equation}
v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_0}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re_h} \left( \frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right),
\end{equation}

\begin{equation}
v_z \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} - \frac{v_z}{r} = \frac{1}{Re_h} \left( \frac{\partial^2 v_\theta}{\partial z^2} + \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \right),
\end{equation}

\begin{equation}
v_z \frac{1}{r} \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re_h} \left( \frac{\partial^2 v_z}{\partial z^2} + \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right),
\end{equation}

while the continuity equation is

\begin{equation}
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0.
\end{equation}

The boundary conditions at the rotating disk are

\begin{equation}v_z = 0 = (0, r, 0),
\end{equation}

while at the stationary disk, the flow has to satisfy

\begin{equation}v_z = 1 = (0, 0, 0).
\end{equation}

\section{III. METHODS OF SOLUTION}

The similarity transformation used by Von Kármán was based on the assumption that \( v_z = v_z(z) \) is a function of \( z \) only.\footnote{From this assumption, it can be derived that} \( v_r = r H^{(1)}(z), \)

\begin{equation}v_\theta = r G(z),\end{equation}

\begin{equation}v_z = -2H(z),\end{equation}

\begin{equation}p = -2 \left( H(z)^2 + Re_h^{-1} H^{(2)}(z) \right) + kr^2.\end{equation}

The coefficient \( k \) is unknown. The expression \( H^{(n)}(z) \) represents the \( n \)-th order derivative of \( H(z) \) with respect to \( z \).

With these transformations, the \( z \)-component of the Navier-Stokes equations (6) and the equation of continuity (7) are automatically satisfied. The \( r \) and \( \theta \)-components of the Navier-Stokes equations (4) and (5) can then be rewritten in the form of two ordinary differential equations

\begin{equation}H^{(4)} = -2Re_h \left( HH^{(3)} + GG^{(1)} \right),\end{equation}

\begin{equation}G^{(2)} = -2Re_h \left( HG^{(1)} - GH^{(1)} \right).
\end{equation}

These equations have to be solved subject to the boundary conditions (8) and (9) at the rotating and non-rotating disk, respectively. The boundary conditions are rewritten in terms of \( G \) and \( H \) so that they take the following form:

\begin{equation}H(0) = H^{(1)}(0) = 0, \quad G(0) = 1,\end{equation}
The system of Eqs. (14) and (15), subject to the boundary conditions (16) and (17) are solved in two different manners: a numerical solution method and an analytical solution method. The numerical solver, on the one hand, finds the numerical solution via the algorithm schematically depicted in Figure 2. The algorithm makes use of the boundary values (16) and the following initial values at the rotor:

\[ H^{(2)}(0) = \alpha, \]
\[ H^{(3)}(0) = \gamma, \]
\[ G^{(1)}(0) = \beta. \]

At a given Reynolds number, an estimate is made with a second-order polynomial model to calculate the initial values of \( \alpha, \beta, \) and \( \gamma \). These initial values are used to numerically integrate the differential equations, after which it is checked whether the boundary values at the stationary disk are satisfied.

The analytical solution method, on the other hand, searches for a set of two analytic functions \( H(z) \) and \( G(z) \) that are an exact and analytical solution of (14) and (15), subject to the boundary conditions (16) and (17). Any analytic function can be represented by an infinite power series in \( (z - z_0) \) in the region \( z_0 - R_c < z < z_0 + R_c \), where \( R_c \) is the radius of convergence of the power series around the initial point \( z_0 \). When it is assumed that the system of Eqs. (14) and (15) can be solved by an analytic function, these functions are of the form

\[ H(z) = \sum_{i=0}^{\infty} a_i z^i, \]
\[ G(z) = \sum_{i=0}^{\infty} b_i z^i, \]

with \( a_i \) and \( b_i \) coefficients to be defined. It should be noted that this form of \( H(z) \) and \( G(z) \) is determined by the definition of analytic functions and they do not correspond to a fitted polynomial.

These infinite series will be able to describe the velocity profiles analytically in some range \(-R_c < z < R_c\). The radius of convergence \( R_c \) can be calculated from

\[ R_c = \left( \lim_{i \to \infty} \sqrt[1]{|a_i|} \right)^{-1}. \]
After substituting the power series of Eqs. (21) and (22) into Eqs. (14) and (15), the following recurrence relations are obtained:

\[ a_{i+4} = -2 R e_h \sum_{j=0}^{i} \left( a_{i-j} a_{j+3} (j+1)(j+2)(j+3) + b_{i-j} b_{j+1} (j+1) \right) \frac{1}{(i+1)(i+2)(i+3)(i+4)}, \]  \hspace{1cm} (24)

\[ b_{i+2} = -2 R e_h \sum_{j=0}^{i} \left( a_{i-j} b_{j+1} (j+1) - b_{i-j} a_{j+1} (j+1) \right) \frac{1}{(i+1)(i+2)}. \]  \hspace{1cm} (25)

Again, it should be noted that an analytic function can be represented by an infinite power series and that any two polynomials are the same if and only if all of their coefficients are the same. Therefore, the set of two polynomials in Eqs. (21) and (22), with the recurrence relations (24) and (25) for their coefficients, is the only representation of all analytic functions that are exact, analytical solutions of the Navier-Stokes equations for this specific system.

From the initial values at the rotating disk at \( z = 0 \), it follows:

\[ a_0 = 0, \quad a_1 = 0, \quad a_2 = 2\alpha, \quad a_3 = 6\gamma, \quad b_0 = 1, \quad b_1 = \beta. \]  \hspace{1cm} (26)

As defined earlier, the plane \( z = 0 \) coincides with the rotating disk and the plane \( z = 1 \) with the non-rotating disk. The choice of the position of the plane \( z = 0 \) has an impact on the values of \( a_i \) and \( b_i \) and via Eq. (23) also on the radius of convergence. The radius of convergence, found when the plane \( z = 0 \) coincides with the rotor, is termed the radius of convergence of the power series near the rotating disk. Similarly, the choice can also be made to let the plane \( z = 0 \) coincide with the stationary disk and \( z = 1 \) with the rotating disk. The radius of convergence that is found when the plane \( z = 0 \) coincides with the stator is then termed the radius of convergence of the power series near the stationary disk. In accordance with the definition made earlier, the plane \( z = 0 \) will be identified with the rotating disk in all the following results, unless explicitly stated otherwise. In the cases when the plane \( z = 0 \) is identified with the non-rotating disk it follows that

\[ a_0 = 0, \quad a_1 = 0, \quad a_2 = 2\alpha, \quad a_3 = 6\gamma, \quad b_0 = 0, \quad b_1 = \beta. \]  \hspace{1cm} (27)

Please note that the only difference is the value of \( b_0 \), which equals zero at the stator side and unity at the rotor side.

**IV. DEFINITIONS OF THE BOUNDARY LAYER THICKNESS**

There is no universal definition of the thickness of a boundary layer, because there is no sharp boundary between the boundary layer and a central region in a fluid. It is common practice, however, to define the thickness of a boundary layer as the position where velocity gradients are negligible. For both the Von Kármán and the Bödewadt boundary layer, two sets of definitions are adopted here. The first set of definitions is somewhat abstract and mathematical and does not have a clear physical meaning. According to the first set of definitions, the boundary layer thickness of both the Bödewadt layer \( \delta_B \) and the Von Kármán layer \( \delta_K \) are equal to the radii of convergence of the power series

\[ \delta_B^I \equiv R_c(\text{stator}), \]  \hspace{1cm} (28)

\[ \delta_K^I \equiv R_c(\text{rotor}). \]  \hspace{1cm} (29)

A second set of definitions for the two boundary layer thicknesses, with a clearer physical meaning can be introduced as illustrated in Figure 3. According to these definitions of the boundary layer thickness, the Bödewadt layer thickness equals that distance from the non-rotating disk at which the axial derivative of the azimuthal velocity function \( G(z) \) is zero,

\[ \left( \frac{\partial G(z)}{\partial z} \right)_{z=1-\delta_B^I} = 0. \]  \hspace{1cm} (30)
The definition of the Von Kármán layer thickness is similar to the definition commonly used in boundary layer theory: the thickness of this boundary layer is given by the position at which the azimuthal velocity function \( G(z) \) has developed for 99% from its value at the rotor, i.e., 1, to its constant value in the inviscid core \( G_{\text{core}} \),

\[
G(\delta_{K}^{II}) = 1 - 0.99(1 - G_{\text{core}}).
\]  

(31)

The value of \( G(z) \) in the non-viscous core equals \( G_{\text{core}} = 0.313 \), as reported by Dijkstra and van Heijst. The two boundary layer thicknesses \( \delta_{K}^{II} \) and \( \delta_{B}^{II} \) can thus be calculated from the numerical solution.

As pointed out by Von Kármán, the only relevant length scale in the description of the flow of an infinite fluid near an infinite rotating disk is \( \sqrt{\nu/\Omega} \). All other length scales should be proportional to this term, including the Von Kármán boundary layer thickness. Similarly, the only relevant length scale in the description of the flow of an infinite rotating fluid near an infinite stationary disk equals \( \sqrt{\nu/\Omega} \), as shown by Bödewadt. Again, all other length scales should be proportional to this quantity, including the Bödewadt boundary layer thickness.

When it is furthermore assumed that the flow between the two disks is of Batchelor type and that both boundary layers are not influencing each other, the azimuthal core velocity is \( \hat{u}_{\text{core}} = 0.313\hat{r} \). It then follows that the thicknesses of the Bödewadt layer and the Von Kármán layer must both be proportional to \( \sqrt{\nu/\Omega} \). Recalling the definition of the Reynolds number \( (Re_h = \Omega h^2 \nu^{-1}) \), it follows that

\[
\frac{\delta_{K}}{h} = \frac{\delta_{K}}{\hat{r}} \propto \frac{1}{\sqrt{Re_h}}.
\]

(32)

\[
\frac{\delta_{B}}{h} = \frac{\delta_{B}}{\hat{r}} \propto \frac{1}{\sqrt{Re_h}}.
\]

(33)

Also, when the Reynolds number approaches infinity, the boundary layers should disappear,

\[
\lim_{Re_h \to \infty} \delta_{K} = 0,
\]

(34)
FIG. 4. The vertical structure of the radial velocity component \( v_r = rH^{(1)}(z) \), calculated with the numerical solution method, shows both a Bödewadt and a Von Kármán boundary layer for all values of Reynolds. A non-viscous core starts to develop at \( Re_h \approx 300 \).

\[ \lim_{Re_h \to \infty} \delta_B = 0. \]  

(35)

It is thus clear that a plot of the boundary layer thickness versus \( Re_h^{-\frac{1}{2}} \) should result in a straight line with zero intercept for flows that are of Batchelor type.

V. RESULTS

A. Numerical calculation of velocity profiles

Numerical calculations were found to converge over the range 0 < \( Re_h < 975 \). The resulting velocity profiles are given in Figures 4–6 for the radial, azimuthal, and axial velocity, respectively. In these figures, velocity profiles are shown for \( Re_h \)-values of 10, 100, 300, and 975.

It can be seen from these figures that the flow develops from a flow where an inviscid core is absent to a Batchelor flow structure in which the Von Kármán boundary layer and the Bödewadt boundary layer are separated by a non-viscous core. The transition to Batchelor flow is completed at \( Re_h \approx 300 \).

For lower Reynolds numbers (i.e., \( Re_h < 300 \)), the flow gradually develops a non-viscous core. It is observed in the azimuthal velocity profile at \( Re_h \approx 100 \) and it appears when the Reynolds number is increased in the radial velocity profile at \( Re_h \approx 300 \).

The values of the coefficients \( \alpha, \beta, \) and \( \gamma \) in Eqs. (18)–(20), respectively, were also determined with the numerical solver. These values will be used in the series approximation, as will be discussed in Sec. V B. The values of \( \alpha, \beta, \) and \( \gamma \) as a function of the Reynolds number \( Re_h \) are depicted in Figure 7.

B. Power series solutions

A power series can only describe solutions of the system of ODEs within its radius of convergence. It can be seen in Figures 8 and 9 that the radii of convergence \( R_c \) are functions of the Reynolds
FIG. 5. The vertical structure of the azimuthal velocity component \( v_\theta = \alpha G(z) \), calculated with the numerical solution method. A non-viscous core is beginning to develop at \( Re_h \approx 100 \).

number, and also that the radius of convergence at the stator side is \( R_c < 1 \) when \( Re_h > 8 \), while at the rotor side this holds for \( Re_h > 18 \). For \( R_c < 1 \), the power series solution is no longer able to describe the velocity profile over the entire range of \( 0 \leq z \leq 1 \).

In Figures 10–12, the velocity profiles are depicted for both the numerical calculations and the power series approximations for \( Re_h = 5, 500, \) and 975. It can be seen that the power series solution gives similar results for \( Re_h = 5 \) over the entire range of \( z \) between the rotor and the stator. At \( Re_h \)

FIG. 6. The vertical structure of the axial velocity component \( v_z = -2H(z) \), calculated with the numerical solution method. It shows a development to Batchelor flow in the same way as the radial velocity, since \( v_r \propto v_z \).
FIG. 7. The coefficients (a) $\alpha$, (b) $\beta$, and (c) $\gamma$ as defined in Eqs. (18)–(20), respectively, are a function of the Reynolds number $Re_h$ and are required in the calculation of the power series approximation.

$Re_h = 500$, the velocity profiles are equally well described, but only within the ranges of convergence of both power series (i.e., those at the rotor and the stator). The velocity profiles can thus not be described by the power series in the range $0.119 < z < 0.789$. The same situation arises for $Re_h = 975$ where the power series are unable to describe the velocity profiles in the range $0.085 < z < 0.849$.

FIG. 8. The radius of convergence $R_c$ at the stator side decreases with increasing Reynolds number. $R_c = 1$ for $Re_h \approx 18$. 
C. Development of the boundary layers

The thickness $\delta_K^I$ of the Von Kármán boundary layer, based on the radius of convergence near the rotating disk ($z = 0$), is presented graphically as a function of $Re_h^{-1/2}$ in Figure 13. Only values of $Re_h > 300$ are considered, because above this Reynolds number, Batchelor flow was observed. A best linear fit on the 95% confidence interval yields $\delta_K^I = (2.6622 \pm 1.1 \times 10^{-4})/\sqrt{Re_h}$. Similarly, for the Bödewadt boundary layer based on the radius of convergence near the stationary disk
FIG. 11. Velocity profiles: (a) radial velocity, (b) tangential velocity, and (c) axial velocity. At $Re_h = 500$, the numerical solution and the power series approximation coincide only within the radii of convergence. The order of the approximation is 328.

$(z = 1)$, the results are displayed in Figure 14. A linear fit on the 95% confidence interval gives

$\delta_B = (4.7157 \pm 1.6 \times 10^{-3})/\sqrt{Re_h}$.

The thickness $\delta_K$ of the Von Kármán boundary layer, based on 99% development of the azimuthal velocity $v_\theta$ with respect to the inviscid core (see Fig. 3), is shown in Figure 15. Again for the same reason, only values of $Re_h > 300$ are considered. A linear fit on the 95% confidence interval

FIG. 12. Velocity profiles: (a) radial velocity, (b) tangential velocity, and (c) axial velocity. At $Re_h = 975$, the numerical solution and the power series approximation coincide only within the radii of convergence, but at the stator $(z = 1)$ it starts diverging when it approaches this radius. The order of the approximation is 284.
yields $\delta_{K}^{I} = (2.7203 \pm 1.7 \times 10^{-3})/\sqrt{Re_{h}}$. A similar conclusion can be drawn for the Bödewadt boundary layer thickness based on $\partial G(z)/\partial z = 0$ (see Fig. 3), as shown in Figure 16. Here, one finds $\delta_{B}^{II} = (4.9025 \pm 1.6 \times 10^{-3})/\sqrt{Re_{h}}$.

In Figures 17 and 18, both definitions of the boundary layer thicknesses are compared for the Von Kármán and the Bödewadt boundary layers, respectively. It can be seen that for any of the definitions the boundary layers vanish when the Reynolds number approaches infinity. In addition, the boundary layer thickness is linear with $1/\sqrt{Re_{h}}$ or $\sqrt{\nu/\Omega}$, as would be expected of Von Kármán and Bödewadt
boundary layers. There is only a small deviation in the thickness of the Bödewadt boundary layer between both definitions: $\delta_{B}^{II} = 1.04\delta_{B}^{I}$. The deviation is even smaller when comparing the different definitions of the Von Kármán boundary layer: $\delta_{K}^{II} = 1.02\delta_{K}^{I}$. It is a striking feature of the power series that their radii of convergence so closely agree with the more physical definition of the thickness of the boundary layer. Especially since the radius of convergence is a somewhat abstract mathematical concept, whereas a boundary layer based on a negligible velocity gradient has a concrete physical meaning.
FIG. 17. The thickness $\delta_{II}^{K}$ of the Von Kármán boundary layer based on 99% development of the azimuthal velocity $v_\theta$ with respect to the inviscid core (see Fig. 3) and the thickness $\delta_{I}^{K}$ based on the radius of convergence $R_c$ near the rotor ($z = 0$) plotted versus $1/\sqrt{Re_h}$.

Although no full explanation to this coincidence can be given at this time, a preliminary analysis has been added in the Appendix. In this analysis, it is assumed that there are two mathematical singularities outside the fluid domain at distances equal to the boundary layer thicknesses away from both disks. From the analysis in the Appendix, it can clearly be seen that the hypothesis with the two singularities outside the fluid domain is able to describe the coincidence of $R_c$ and $\delta$ on a qualitative level but not on a quantitative level.

FIG. 18. The thickness $\delta_{II}^{B}$ of the Bödewadt boundary layer based on $\partial v_\theta/\partial z = 0$ (see Fig. 3) and the thickness $\delta_{I}^{B}$ based on the radius of convergence $R_c$ near the stator ($z = 1$) plotted versus $1/\sqrt{Re_h}$.
VI. CONCLUDING REMARKS

The flow of a Newtonian fluid in the region between an infinite rotating disk (angular velocity $\Omega$) and an infinite stationary disk with a disk spacing $h$ has been analyzed within the framework of the Von Kármán similarity assumption by numerical solution of the governing set of ODEs. With increasing Reynolds number $Re_h$, the flow develops from a situation where a Von Kármán boundary layer near the rotor and a Bödewadt boundary layer near the stator are touching, to a Batchelor type of flow in which the two boundary layers are separated by an inviscid core. The transition from one flow structure to the other occurs within the range $100 < Re_h < 300$. The non-viscous core first appears in the azimuthal velocity profile at $Re_h = 100$ and is present in the axial and radial velocity profile at $Re_h = 300$. For $Re_h > 300$, the flow is thus of a Batchelor type.

The flow profiles can also be calculated with an analytical power series approximation. The power series are able to completely describe the velocity profiles over the entire domain between the rotating disk and the stationary disk for $Re_h < 18$. In this situation, the radius of convergence $R_c$ at the stator side is larger than unity. When the Reynolds number is increased, the velocity profiles can only be described for $z < R_c$, i.e., within the radius of convergence of the power series. A remarkable feature of this radius of convergence is the fact that it coincides with the boundary layer thickness of both the Von Kármán and the Bödewadt boundary layers.

For the Von Kármán boundary layer near the rotating disk and the Bödewadt boundary layer near the stationary disk, four definitions are given for their thicknesses in the Batchelor regime:

- $\delta^I_K$ is the thickness of the Von Kármán layer based on the radius of convergence $R_c$ of the power series near the rotating disk;
- $\delta^I_B$ is the thickness of the Bödewadt layer based on the radius of convergence $R_c$ of the power series near the stationary disk;
- $\delta^{II}_K$ is the thickness of the Von Kármán layer based on the 99% development of the azimuthal velocity $v_\theta$ to the velocity in the inviscid core;
- $\delta^{II}_B$ is the thickness of the Bödewadt layer based on the condition $\partial G(z)/\partial z = 0$.

The thicknesses of these boundary layers all correspond to the physical necessity that the boundary layer thickness should vanish when the Reynolds number approaches infinity and they are linear with $1/\sqrt{Re_h}$ and $\sqrt{v/\Omega}$. This leads to the following relations:

$$
\begin{align*}
\delta^I_K &= \frac{2.6622}{\sqrt{Re_h}}, & \delta^{II}_K &= \frac{2.7203}{\sqrt{Re_h}}, & \delta^I_B &= \frac{4.7157}{\sqrt{Re_h}}, & \delta^{II}_B &= \frac{4.9025}{\sqrt{Re_h}}.
\end{align*}
$$

Apparently, the mathematical concept of the radius of convergence of the series solution in this specific configuration closely follows the theoretical predictions on the boundary layer thickness based on the commonly used physical characteristics.

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APPENDIX: ANALYSIS ON THE COINCIDENCE OF THE BOUNDARY LAYER THICKNESSES AND THE RADII OF CONVERGENCE

A function $g(z)$ is analytic on an interval $z_0 - R_c < z < z_0 + R_c$ when it can be written as a power series of the form

$$
g(z) = \sum_{i=0}^{\infty} c_i (z - z_0)^i.
$$

(A1)
in which the radius of convergence \((R_c)\) is given by

\[
R_c = \left( \limsup_{i \to \infty} \sqrt{|c_i|} \right)^{-1}.
\]  

(A2)

One way, but maybe not the only way, in which a function can be said to be not analytic at a point \(z^*\) is when the function value itself and thus all of its derivatives approach infinity as \(z\) approaches \(z^*\). The point \(z^*\) is termed a singularity. When it is assumed that two singularities are responsible for the functions \(G(z)\) and \(H(z)\) to be non-analytic at the position of the end of the boundary layer, we are left with two options:

1. At least one singularity is located in the fluid domain, \(0 \leq z \leq 1\). This would mean that there would be a singularity at \(z = \delta^I_K\) and/or one at \(z = 1 - \delta^I_B\);
2. Both singularities are located outside of the fluid domain, \(z < 0\) and \(z > 1\). This would mean that there would be a singularity at \(z = -\delta^I_K\) and one at \(z = 1 + \delta^I_B\).

When it is furthermore assumed that neither the fluid velocity, nor its gradients are allowed to approach infinite values, only option 2 is left. When in this situation the functions \(f_1(z_0) = z_0 + R_c(z_0)\) and \(f_2(z_0) = z_0 - R_c(z_0)\) are plotted in the domain \(0 \leq z_0 \leq 1\), as is schematically depicted in Figure 19, it can be found from simple geometrical considerations that the following must hold with respect to the derivatives of \(f_1\) and \(f_2\) with respect to \(z_0\):

\[
f'_1(z_0) = \begin{cases} 
  2 & \text{if } 0 \leq z_0 < 0.5(1 + \delta^I_B - \delta^I_K) \\
  0 & \text{if } 0.5(1 + \delta^I_B - \delta^I_K) < z_0 \leq 1 
\end{cases}.
\]  

(A3)

\[
f'_2(z_0) = \begin{cases} 
  0 & \text{if } 0 \leq z_0 < 0.5(1 + \delta^I_B - \delta^I_K) \\
  2 & \text{if } 0.5(1 + \delta^I_B - \delta^I_K) < z_0 \leq 1 
\end{cases}.
\]  

(A4)

In Figure 20, an example is given of \(f_1(z_0)\) and \(f_2(z_0)\) for \(Re_h = 975\) as well as the graph that would result from the purely theoretical considerations given above. It can be clearly seen that the situation qualitatively resembles the theoretical situation but is quantitatively different.
FIG. 20. (Light grey line, blue) Theoretical value of $f_1(z_0)$, (light grey markers, blue) numerical values of $f_1(z_0)$, (dark grey line, red) theoretical value of $f_2(z_0)$, and (dark grey markers, red) numerical values of $f_2(z_0)$.