Bouncing inflation in a nonlinear R2+R4 gravitational model

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terms (to the Einstein action) with $R^n > R^1$, the quadratic model, such modification of gravity results in $R^2$, continues up to now [9]. Obviously, the correction investigation of these models, which started in the 1980s model is Universe's evolution (see, e.g., [10,11], and numerous $R$-type nonlinear models with polynomial as well as $R^4$ ones. Active investigation of these models, which started in the 1980s [7,8], continues up to now [9]. Obviously, the correction terms (to the Einstein action) with $n > 1$ give the main contribution in the case of large $R$, e.g., in the early stages of the Universe’s evolution. As it was shown first in [1] for the quadratic model, such modification of gravity results in early inflation. On the other hand, function $f(R)$ may also contain negative degrees of $R$. For example, the simplest model is $R + R^{-1}$. In this case the correction term plays the main role for small $R$, e.g., at the late stage of the Universe’s evolution (see, e.g., [10,11], and numerous references therein). Such modification of gravity may result in the late-time acceleration of our Universe [12]. Nonlinear models with polynomial as well as $R^{-1}$-type correction terms have also been generalized to the multi-dimensional case (see, e.g., [10,11,13–18]).

It is well known that nonlinear models are equivalent to linear-curvature models with additional scalar field $\phi$ (dubbed the scalaron in [1]). This scalar field corresponds to additional degree of freedom of nonlinear models. The dynamics of this field (as well as a possibility of inflation of the Universe) is defined by potential $U(R(\phi), \phi)$ [see Eq. (2.6) below1], where $R = R(\phi)$ is a solution of Eq. (2.3): $\exp(A\phi) = df/dR$. Usually, models or particular cases of these models are considered where this equation has only one solution. In this case, potential $U$ is a one-valued function of $\phi$. However, in the most general case this equation has more than one solution and potential becomes a multivalued function with a number of branching points (see, e.g., [19]). Investigation of the dynamical behavior of scalar field and the Universe in such models (especially in the vicinity of the branching points) is not a trivial problem and may result in new important effects. Therefore, it is of interest to consider the models with multivalued potentials.

In the present paper, we study an example of such models. Here, $f(R)$ has both quadratic $R^2$ and quartic $R^4$ contributions. In this case Eq. (2.3) is a cubic equation with respect to $R$ and may have, in general, three real solutions/branches $R_i(\phi)$ ($i = 1, 2, 3$). We have investigated this model in our paper [16]. However, in this paper we considered a special case of one real solution in $D = 8$ spacetime. Now, we study the most interesting case of three real solutions. These solutions are fitted with each other in the branching points. There is also another type of matching points where one-valued solutions are fitted with the three-
valued solutions. In the vicinity of these points potential $U(\phi)$ is a monotonic function. Thus, these latter points are dubbed monotonic ones. The main aim of the paper consists in the investigation of dynamical behavior of the system in four-dimensional space-time in the vicinity of the branching and monotonic points. We show that dynamics is quite different for branching and monotonic points. The monotonic points are penetrable for the scalaron, while in the branching points the scalaron has bouncing behavior. There are branching points where the scalaron bounces an infinite number of times with decreasing amplitude and the Universe asymptotically approaches the de Sitter stage. Such accelerating behavior we call bouncing inflation. We should note that for this type of inflation, there is no need for original potential $U(\phi)$ to have a minimum or to check the slow-roll conditions. A necessary condition for such inflation is the existence of the branching points. This is a new type of inflation. We show that this inflation takes place both in the Einstein and Brans-Dicke frames. This is the main result of our paper. We think that the scalaron field and the Universe is investigated in Sec. V.

The paper is structured as follows. In Sec. II we study briefly the equivalence between an arbitrary nonlinear $f(\bar{R})$ theory and theory linear in another scalar curvature $R$ but which contains a scalaron field $\phi$. In Sec. III we consider a particular example of nonlinear model with curvature-quadratic and curvature-quartic correction terms and obtain solutions/branches $\bar{R}(\phi)$. The fitting procedure for these branches is proposed in Sec. IV. The dynamics of the scalaron and the Universe is investigated in Sec. V. Here, we parametrize the scalaron potential in such a way that it becomes a one-valued function. It provides the possibility to study the dynamical behavior of the system in the vicinity of the branching and monotonic points. We show that in the vicinity of the branching point the scalaron field bounces an infinite number of times with decreasing amplitude, and the Universe acquires the accelerating expansion approaching asymptotically to the de Sitter stage. Such accelerating expansion we call bouncing inflation. A brief discussion of the obtained results is presented in the concluding Sec. VI. In the Appendix, we show that bouncing inflation in the vicinity of the branching point takes place also in the Brans-Dicke frame.

**II. GENERAL SETUP**

It is well known that nonlinear theories $\mathcal{S} = \frac{1}{2\kappa_D^2} \int_M \sqrt{|g|} f(\bar{R})$, \(2.1\)

where $f(\bar{R})$ is an arbitrary smooth function of a scalar curvature $\bar{R} = R[\bar{g}]$ constructed from the $D$-dimensional metric $\bar{g}_{ab}$ for $a, b = 1, \ldots, D$, are equivalent to theories which are linear in scalar curvature $R$ but which contain an additional self-interacting scalar field. According to standard techniques [7,8], the corresponding $R$-linear theory has the action functional

$$ S = \frac{1}{2\kappa_D^2} \int_M \sqrt{|g|} \left[ R[g] - g^{ab} \phi_a \phi_b - 2U(\phi) \right], \tag{2.2} $$

where

$$ f'(\bar{R}) = \frac{df}{d\bar{R}} := e^{\phi A} > 0, \quad A := \sqrt{D - 2 \over \sqrt{D - 1}}, \tag{2.3} $$

and where the self-interaction potential $U(\phi)$ of the scalar field $\phi$ is given by

$$ U(\phi) = {1 \over 2} f(\bar{R})^{-(D-2)/2} \left[ \bar{R} f' - f \right], \tag{2.4} $$

$$ = {1 \over 2} e^{-R \phi} [\bar{R}(\phi)e^{A \phi} - f(\bar{R}(\phi))], \tag{2.5} $$

$$ B := \frac{D}{\sqrt{(D - 2)(D - 1)}}. \tag{2.6} $$

The metrics $g_{ab}$, $\bar{g}_{ab}$ and the scalar curvatures $R$, $\bar{R}$ of the two theories (2.1) and (2.2) are conformally connected by the relations\(^2\)

$$ g_{ab} = \Omega^2 \bar{g}_{ab} = \left[ f'(\bar{R}) \right]^{2/(D-2)} \bar{g}_{ab} \tag{2.7} $$

and

$$ R = (f')^{2/(D-2)} \left\{ \bar{R} + \frac{D - 1}{D - 2} (f')^{-2} \bar{g}^{ab} \partial_a f' \partial_b f' + \frac{2}{D - 2} (f')^{-1} \bar{f}' \right\}, \tag{2.8} $$

via the scalar field $\phi = \ln[f'(\bar{R})]/A$. This scalar field $\phi$, known as the scalaron [1], carries an additional degree of freedom of original nonlinear model.

According to our definition (2.3), we consider the positive branch $f'(\bar{R}) > 0$. Although the negative $f' < 0$ branch can be considered as well (see, e.g., Refs. [8,10,15]). However, negative values of $f'(\bar{R})$ result in negative effective gravitational “constant” $G_{\text{eff}} = \kappa_D^2/f'$. Thus $f'$ should be positive for the graviton to carry positive kinetic energy (see, e.g., [6]).

From action (2.2), we obtain the equation of motion of the scalaron field $\phi$:

$$ \Box \phi - \frac{\partial U}{\partial \phi} = 0. \tag{2.9} $$

If scalaron potential $U(\phi)$ has a minimum in a point $\phi_0$
then we can define the mass squared of the scalaron [10]:
\[ m^2_{\phi} = \left. \frac{d^2U}{d\phi^2} \right|_{\phi_0} = \frac{1}{2(D-1)f''} \left( \frac{D-2}{f''} - \frac{2\ddot{R}}{f''} \right) \left|_{\phi_0} \right. > 0, \]  
(2.11)

A similar expression for the scalaron mass squared is also given, e.g., in [6]. The only difference consists of an additional conformal prefactor \( 1/(f'')^{2/(D-2)} \) originated from the conformal metric transformation\(^3\) (2.7). Up to this prefactor, the positiveness condition (2.11) of the mass squared coincides with the stability condition of de Sitter space in \( f(R) \) gravity with respect to inhomogeneous and homogeneous perturbations [6,21]. Additionally, to avoid the Dolgov-Kawasaki instability [22] (instability with respect to local perturbations), it is also required that \( f''(R) \geq 0 \) (see also [23]).

For further research, it is useful to introduce a new variable
\[ X := e^{A\phi} - 1. \]  
(2.12)

Then, potential (2.6) and its first derivative read, correspondingly,
\[ U(X) = \frac{1}{2}(X + 1)^{-B/A}[\ddot{R}(X + 1) - f] \]  
(2.13)

and
\[ \frac{dU}{dX} = -\frac{B}{2A}(X + 1)^{-(B/A)-1}[\ddot{R}(X + 1) - f] + \frac{1}{2}(X + 1)^{-B/A}\ddot{R}. \]  
(2.14)

To conclude this section, we would like to recall that in the multidimensional case, to avoid the effective four-dimensional fundamental constant variation, it is necessary to provide the mechanism of the internal spaces stabilization. In these models, the scale factors of the internal spaces play the role of additional scalar fields (geometrical moduli/gravexcitons [24]). To achieve their stabilization, an effective potential should have minima with respect to all scalar fields (gravexcitons and scalaron). Our previous analysis (see, e.g., [25]) shows that for a model of the form (2.2), the stabilization is possible only for the case of negative minimum of the potential \( U(\phi) \). However, it is not difficult to realize that it is impossible to freeze out the internal spaces in such an anti-de Sitter universe. Indeed, in these models scalar fields decrease their amplitude of oscillations around a minimum during the stage of expansion of the Universe [due to a friction term in dynamical equation of the form of (5.2) below] until the Universe reaches its maximum. Then, the Universe turns to the stage of contraction and the amplitudes of scalar fields start to increase again. Thus, geometrical moduli are not stabilized in such models. Therefore, in our present paper we do not investigate the problem of the extra dimension stabilization, but we focus our attention on the dynamics of the scalaron field and the Universe in four-dimensional case.

### III. THE \( R^2 + R^4 \)-MODEL

In this section we analyze a model with curvature-quadratic and curvature-quartic correction terms of the type
\[ f(\ddot{R}) = \dot{R} + \alpha \dddot{R}^2 + \gamma R^4 - 2\Lambda_D. \]  
(3.1)

We start our investigation for an arbitrary number of dimensions \( D \), but in the most particular examples we shall put \( D = 4 \) (unless stated otherwise). First of all, we define the relation between the scalar curvature \( \ddot{R} \) and the scalaron field \( \phi \). According to Eq. (2.3), we have
\[ f' = e^{A\phi} = 1 + 2\alpha\dddot{R} + 4\gamma R^3. \]  
(3.2)

The definition (2.3) \( f' = \exp(A\phi) \) clearly indicates that we choose the positive branch \( f' > 0 \). For our model (3.1), the surfaces \( f' = 0 \) as functions \( \ddot{R} = \ddot{R}(\alpha, \gamma) \) are given in Fig. 1. As it easily follows from Eq. (3.2), points where all three values \( \ddot{R}, \gamma, \alpha \) are positive correspond to the region \( f' > 0 \). Thus, this picture shows that we have one simply connected region \( f' > 0 \) and two disconnected regions \( f' < 0 \).

Equation (3.2) can be rewritten equivalently in the form
\[ \ddot{R}^3 + \frac{\alpha}{2\gamma} \dddot{R} - \frac{1}{4\gamma} X = 0, \]  
(3.3)

\[ \text{FIG. 1 (color online). The surfaces } f' = 0 \text{ as functions } \ddot{R} = \ddot{R}(\alpha, \gamma) \text{ for the model (3.1).} \]
Taking into account Eq. (3.5), the function \( X \) reads

\[
X(\theta) = 8\sqrt{q - q^3}\cosh(\theta). \tag{3.10}
\]

The three real solutions \( \tilde{R}_{1,2,3}(X) \) for \( Q < 0 \) are given as

\[
\tilde{R}_1 = s_1 + s_2, \\
\tilde{R}_2 = \frac{1}{2}(-1 + i\sqrt{3})s_1 + \frac{1}{2}(-1 - i\sqrt{3})s_2 = e^{i((2\pi)/3)}s_1 + e^{-i((2\pi)/3)}s_2, \tag{3.11}
\]

\[
\tilde{R}_3 = \frac{1}{2}(-1 - i\sqrt{3})s_1 + \frac{1}{2}(-1 + i\sqrt{3})s_2 = e^{-i((2\pi)/3)}s_1 + e^{i((2\pi)/3)}s_2,
\]

where we can fix the Riemann sheet of \( Q^{1/2} \) by setting in the definitions of \( s_{1,2} \):

\[
s_{1,2} := \left[ r \pm i|Q|^{1/2} \right]^{1/3}. \tag{3.12}
\]

A simple Mathematica calculation gives for Vieta’s relations from (3.11)

\[
\tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 = 0, \\
\tilde{R}_1\tilde{R}_2 + \tilde{R}_1\tilde{R}_3 + \tilde{R}_2\tilde{R}_3 = -3s_1s_2 = 3q, \\
\tilde{R}_1\tilde{R}_2\tilde{R}_3 = s_1^3 + s_2^3 = 2r. \tag{3.13}
\]

In order to work with explicitly real-valued \( \tilde{R} \), we rewrite \( s_{1,2} \) from (3.12) as follows:

\[
s_{1,2} = |b|^{1/3}e^{\pm i\theta/3}, \\
|b|^2 = r^2 + |Q| = r^2 - Q = -q^3, \tag{3.14}
\]

and get via (3.11)

\[
\tilde{R}_1 = s_1 + s_2 = 2|b|^{1/3}\cos(\theta/3), \\
\tilde{R}_2 = e^{i((2\pi)/3)}s_1 + e^{-i((2\pi)/3)}s_2 = 2|b|^{1/3}\cos(\theta/3 + 2\pi/3), \tag{3.15}
\]

\[
\tilde{R}_3 = e^{-i((2\pi)/3)}s_1 + e^{i((2\pi)/3)}s_2 = 2|b|^{1/3}\cos(\theta/3 - 2\pi/3),
\]


\[
\tilde{R}_k = 2|b|^{1/3}\cos(\theta + 2\pi k/3) = 2\sqrt{-q}\cos(\theta + 2\pi k/3),
\]

\[
k = -1, 0, 1. \tag{3.16}
\]

In order to understand the qualitative behavior of these three real-valued solutions as part of the global solution picture, we first note that, according to (3.3), we may interpret \( X \) as single-valued function

\[
X(\tilde{R}) = 4\gamma\tilde{R}^3 + 2\alpha\tilde{R}. \tag{3.17}
\]

and look what is happening when we change \((\alpha, \gamma)\). Obviously, the inverse function \( X(\tilde{R}) \) has three real-valued branches when \( X(\tilde{R}) \) is not a monotonic function, but instead has a minimum and a maximum, i.e. when

\[
\partial_{\tilde{R}}X := X' = 12\gamma\tilde{R}^2 + 2\alpha = 0 \Rightarrow \tilde{R}^2 = -\frac{\alpha}{6\gamma}. \tag{3.18}
\]

has two real solutions \( \tilde{R} = \pm\sqrt{-\alpha/(6\gamma)} \) and corresponding extrema\footnote{It is worth of noting that \( f''(\tilde{R}) = X'(\tilde{R}) = 0 \).}

\[
X(\tilde{R}) = \frac{4}{3}\alpha\tilde{R}. \tag{3.19}
\]

It should hold sign(\( \alpha \)) \neq sign(\( \gamma \)) in this case so that we find

\[
\gamma > 0, \quad \alpha < 0: \ X_{\text{max}} = X(\tilde{R}_+), \quad X_{\text{min}} = X(\tilde{R}_-), \\
\gamma < 0, \quad \alpha > 0: \ X_{\text{max}} = X(\tilde{R}_+), \quad X_{\text{min}} = X(\tilde{R}_-). \tag{3.20}
\]
The transition from the three-real-solution regime to the one-real-solution regime occurs when maximum and minimum coalesce at the inflection point

\[ \bar{R}_+ = \bar{R}_- = 0 \Rightarrow \alpha = 0, \quad \gamma \neq 0. \] (3.21)

(We note that here we consider the nondegenerate case \( \gamma \neq 0 \). Models with \( \gamma = 0 \) are degenerate ones and are characterized by quadratic scalar curvature terms only.) Because of \(-1 \leq X \leq +\infty\), we may consider the limit \( X \to +\infty \) where in leading approximation

\[ 4\gamma \bar{R}^3 = X \to +\infty \]

so that

\[ \bar{R}(X \to \infty) \to \text{sign}(\gamma) \times \infty. \] (3.23)

Leaving the restriction \( X \geq -1 \) for a moment aside, we have found that for \( \alpha \gamma < 0 \) there exist real solution branches \( \bar{R}_{1,2,3} \):

\[
\begin{align*}
\gamma > 0: \enspace & -\infty \leq \bar{R}_1 \leq \bar{R}_-, \quad -\infty \leq X \leq X_{\text{max}}, \\
\bar{R}_- & \leq \bar{R}_2 \leq \bar{R}_+, \quad X_{\text{max}} \geq X \geq X_{\text{min}}, \\
\bar{R}_+ & \leq \bar{R}_3 \leq +\infty, \quad X_{\text{min}} \leq X \leq +\infty,
\end{align*}
\]

\[
\gamma < 0: \enspace -\infty \leq \bar{R}_1 \leq \bar{R}_-, \quad +\infty \geq X \geq X_{\text{min}}, \\
\bar{R}_- & \leq \bar{R}_2 \leq \bar{R}_+, \quad X_{\text{min}} \leq X \leq X_{\text{max}}, \\
\bar{R}_+ & \leq \bar{R}_3 \leq +\infty, \quad X_{\text{max}} \geq X \geq -\infty.
\] (3.24)

It remains for each of these branches to check which of the solutions \( \bar{R}_k \) from (3.16) can be fit into this scheme. Finally, one will have to set the additional restriction \( X \geq -1 \) on the whole picture.

**IV. THE FITTING PROCEDURE**

We start by considering a concrete example. For definiteness, let us assume \( \gamma > 0, \alpha < 0 \). The pairwise fitting of the various solution branches should be performed at points where \( Q = 0 \) and different branches of the three-solution sector are fitted with each other or to the branches of the one-solution sector. The points with \( Q = 0 \) correspond to the \( X \) values \( X_{\text{min}} \) and \( X_{\text{max}} \). Explicitly, we have from (3.19)

\[ X(\bar{R}_k) = \frac{4}{3} \alpha \bar{R}_k = \pm \frac{4}{3} \alpha \sqrt{-\frac{\alpha}{6\gamma}}, \] (4.1)

and for the concrete configuration \( \gamma > 0, \alpha < 0 \)

\[
\begin{align*}
X_{\text{max}} = X(\bar{R}_-) & = -\frac{4}{3} \alpha \sqrt{-\frac{\alpha}{6\gamma}} \geq 0, \\
X_{\text{min}} = X(\bar{R}_+) & = \frac{4}{3} \alpha \sqrt{-\frac{\alpha}{6\gamma}} \leq 0.
\end{align*}
\]

Next, we find from the defining Eq. (3.14) for the angle \( \vartheta \) that at \( Q = 0 \), it holds

\[ \cos(\vartheta) = \frac{r}{|b|} = \frac{r}{|r|} \] (4.3)

so that

\[
\begin{align*}
X_{\text{max}} & \geq 0 \Rightarrow r > 0 \Rightarrow \cos(\vartheta) = 1 \Rightarrow \vartheta = 2\pi m, \\
m & \in \mathbb{Z}, \\
X_{\text{min}} & \leq 0 \Rightarrow r < 0 \Rightarrow \cos(\vartheta) = -1 \Rightarrow \vartheta = \pi + 2\pi n, \\
n & \in \mathbb{Z}.
\end{align*}
\] (4.4)

Now, the fitting of the various solution branches can be performed as follows (see Fig. 2). We start with the branch \( \Gamma_1 := (R_+ \leq R \leq +\infty, X_{\text{min}} < X < +\infty) \). Moving in on this branch from \( X = +\infty \), we are working in the one-solution sector \( Q > 0 \) with

\[ \bar{R}(\Gamma_1; Q) = [r + Q^{1/2}]^{1/3} + [r - Q^{1/2}]^{1/3} \] (4.5)

until we hit \( Q = 0 \) at \( X = X_{\text{max}} \). At this point \( P_1 := (\Gamma_1, X = X_{\text{max}}) \in \Gamma_1 \), we have to perform the first fitting. Because of \( r > 0 \), we may choose

\[ \bar{R}(\Gamma_1; Q = 0) = 2b^{1/3} = 2|b|^{1/3} \] (4.6)

so that as simplest parameter choice in (3.16), we set

\[ P_1 = (\Gamma_1, X = X_{\text{max}}, Q = 0) \mapsto \vartheta = 0, \quad k = 0. \] (4.7)

Hence, the parametrization for \( \Gamma_1, Q < 0 \) will be given as

\[ \bar{R}(\Gamma_1, Q < 0) = 2|b|^{1/3} \cos(\vartheta/3). \] (4.8)

For later convenience, we have replaced here the \( \vartheta \) from Eqs. (3.15) and (3.16) by \( \vartheta \). The reason will become clear from the subsequent discussion. We note that on this \( \Gamma_1 \)

![FIG. 2. The schematic drawing of the real solution branches and the matching points \( P_{1,2,3} \). This figure shows that points \( P_{2,3} \) (correspondingly, \( \theta = \pi, 2\pi \)) and points \( P_{2,4} \) (correspondingly, \( \theta = 0, 3\pi \)) are of a different nature. So, \( P_{2,3} \) and \( P_{1,4} \) shall call branching points and monotonic points (in the sense that function \( \bar{R} \) is monotonic in the vicinity of these points), respectively.](image-url)
segment, we may set \( \theta = \theta \). Let us further move on \( \Gamma_1 \) until its end at \( X \) \(_{\text{min}} \), where again \( Q = 0 \). Because there was no other point with \( Q = 0 \) on this path, the smoothly changing \( \theta \) can at this local minimum \( P_2 = \Gamma_1 \cap \Gamma_3 = (X = X \) \(_{\text{min}}, R = \hat{R}_+ \) only take one of the values \( \theta = \pm \pi \). For definiteness, we choose it as \( \theta(P_2) = \pi \). Hence, it holds

\[
\hat{R}(P_2) = 2|b|^{1/3} \cos(\pi/3) = |b|^{1/3} = \sqrt{-q} \\
= \sqrt{-\alpha/(6\gamma)} = \hat{R}_+. \tag{4.9}
\]

as it should hold. For convenience, we may parametrize our movement on the cubic curve by simply further increasing \( \theta \). This gives for moving on \( \Gamma_2 = (\hat{R}_+ \cong R \cong \hat{R}_-, X \leq X \) \(_{\text{min}} \leq X \leq X \) \(_{\text{max}} \) \( R = \hat{R}_- \) a further increase of \( \theta \) by \( \pi \) up to \( \theta(P_3) = 2\pi \). Accordingly, we find the complete consistency

\[
\hat{R}(P_3) = 2|b|^{1/3} \cos(2\pi/3) = -|b|^{1/3} = -\sqrt{-q} \\
= -\sqrt{-\alpha/(6\gamma)} = \hat{R}_-. \tag{4.10}
\]

By further increasing \( \theta \) up to \( \theta = 3\pi \), we reach the point \( P_4 = (X = X \) \(_{\text{min}}, Q = 0) \in \Gamma_3 \) with

\[
\hat{R}(P_4) = 2|b|^{1/3} \cos(3\pi/3) = -2|b|^{1/3} = -2|r|^{1/3}. \tag{4.11}
\]

Because of \( r < 0 \), we can smoothly fit it to the one-solution branch

\[
\hat{R}(\Gamma_4, Q) = [r + Q^{1/2}]^{1/3} + [r - Q^{1/2}]^{1/3} \tag{4.12}
\]

by setting trivially

\[
\hat{R}(P_4) = 2(-|r|)^{1/3} = -2|r|^{1/3}. \tag{4.13}
\]

Summarizing, we arrived at a very simple and transparent branch fitting picture, where all the movement in the three-solution sector can be parametrized by choosing the effective angle as \( \theta \in [0, 3\pi] \). Finally, we have to fit this picture in terms of smoothly varying \( \theta \in [0, 3\pi] \) with the three solutions \( \hat{R}_i \) from (3.16). For this purpose, we note that the single value \( \theta \in [0, \pi] \) in (3.16) is a projection of our smoothly varying \( \theta \in [0, 3\pi] \). Fixing an arbitrary \( \theta \), one easily finds the following correspondences:

\[
\theta(\Gamma_1, \theta) = \theta, \quad \theta(\Gamma_2, \theta) = 2\pi - \theta, \\
\theta(\Gamma_3, \theta) = 2\pi + \theta \tag{4.14}
\]

and hence

\[
\hat{R} \left[ \theta(\Gamma_1, \theta) \right] = 2|b|^{1/3} \cos \left( \frac{\theta}{3} \right) = \hat{R}_{(k=0)} = \hat{R}_3, \\
\hat{R} \left[ \theta(\Gamma_2, \theta) \right] = 2|b|^{1/3} \cos \left( \frac{2\pi - \theta}{3} \right) \\
= 2|b|^{1/3} \cos \left( \frac{\theta - 2\pi}{3} \right) = \hat{R}_{(k=-1)} = \hat{R}_2, \\
\hat{R} \left[ \theta(\Gamma_3, \theta) \right] = 2|b|^{1/3} \cos \left( \frac{2\pi + \theta}{3} \right) = \hat{R}_{(k=1)} = \hat{R}_1. \tag{4.15}
\]

Analogically, we can obtain rules for fitting procedure in the case \( \gamma < 0, \alpha > 0 \). So, all the fitting mechanism is clear now and can be used in further considerations.

V. DYNAMICS OF THE UNIVERSE AND SCALARON

To study the dynamics of the Universe in our model, we assume that the four-dimensional metric \( g \) in (2.7) is spatially flat Friedmann-Robertson-Walker one:

\[
g = -dt \otimes dt + a^2(t) d\vec{x} \otimes d\vec{x}. \tag{5.1}
\]

Thus, scalar curvatures \( R \) and \( \hat{R} \) and the scalaron \( \phi \) are functions of time. Therefore, Eq. (2.9) for homogeneous field \( \phi \) reads

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dU}{d\phi} = 0, \tag{5.2}
\]

where the Hubble parameter \( H = \dot{a}/a \) and the dots denote the differentiation with respect to time \( t \). Potential \( U \) is defined by Eq. (2.6). Because \( U \) depends on \( \hat{R} \), it which is a multivalued function of \( \phi \) (or, equivalently, of \( X \)). the potential \( U \) is also a multivalued function of \( X \) (see Fig. 3).

However, our previous analysis shows that we can avoid this problem making \( X \) and \( \hat{R} \) single-valued functions of a new field \( \theta \) (we recall that we consider the particular case of \( \alpha > 0 \) when \( \alpha < 0, \gamma > 0 \):

\[
X(\theta) = \begin{cases} \\
\sqrt{(1/\gamma)(2|\alpha|/3)^3} \cosh(\theta), & \theta < 0; \\
\sqrt{(1/\gamma)(2|\alpha|/3)^3} \cos(\theta), & 0 \leq \theta \leq 3\pi; \\
-\sqrt{(1/\gamma)(2|\alpha|/3)^3} \cosh(\theta - 3\pi), & \theta > 3\pi; 
\end{cases} \tag{5.3}
\]

and

\[
\hat{R}(\theta, \theta) = 2|b|^{1/3} \cos \left( \frac{\theta}{3} \right) = \hat{R}_{(k=0)} = \hat{R}_3, \\
\hat{R}(\theta, \theta) = 2|b|^{1/3} \cos \left( \frac{2\pi - \theta}{3} \right) \\
= 2|b|^{1/3} \cos \left( \frac{\theta - 2\pi}{3} \right) = \hat{R}_{(k=-1)} = \hat{R}_2, \\
\hat{R}(\theta, \theta) = 2|b|^{1/3} \cos \left( \frac{2\pi + \theta}{3} \right) = \hat{R}_{(k=1)} = \hat{R}_1. \tag{4.15}
\]

In spite of the divergency of \( d\hat{R}/dX \) in the branching points \( P_{\pm} \), the derivatives \( dU/dX \) are finite in these points in accordance with Eq. (2.14). Moreover, \( \hat{R} \) and \( X \) have the same values in branching points for different branches. Therefore, the branches arrive at the branching points with the same values of \( dU/dX \), and Fig. 3 clearly shows it.
The function $X = X(\theta)$ is schematically given in Fig. 4. It is necessary to keep in mind that we consider the case $f' > 0$ so $X > -1$. If we demand that $X_{\min} > -1$ (in opposite case our graphic $X(\theta)$ will be cut into two disconnected parts), then the parameters $\alpha$ and $\gamma$ should satisfy the inequality:

$$X_{\min} = -\sqrt{(1/\gamma)(2|\alpha|/3)^3} > -1 \Rightarrow |\alpha| \leq \frac{3\gamma}{2}^{1/3}. \quad (5.5)$$

The maximal value of $\theta$ (which is greater than $3\pi$ in the case $X_{\min} > -1$) is defined from the transcendental equation

$$\tilde{R}(\theta) = \begin{cases} 2\sqrt{|\alpha|/(6\gamma)} \cosh(\theta/3), & \theta < 0; \\ 2\sqrt{|\alpha|/(6\gamma)} \cosh(\theta/3), & 0 \leq \theta \leq 3\pi; \\ -2\sqrt{|\alpha|/(6\gamma)} \cosh((\theta/3) - \pi), & \theta > 3\pi. \end{cases} \quad (5.4)$$

Here, we introduce the one dimensional metric on the moduli space $G(\theta) = G^{\alpha\beta} = (d\phi/d\theta)^2$ with the corresponding Christoffel symbol $\Gamma(\theta) = \Gamma^{\alpha\beta}_{\gamma\delta} = (1/2)G^{(\alpha\beta)}G_{\gamma\delta} = (d^2\phi/d\theta^2)/(d\phi/d\theta)$. The form of the potential $U(\theta)$ [see Eqs. (2.4) and (2.6)] shows that this potential and its derivative $(dU/d\phi)$ for $X > -1$ is finite for $dU/d\phi$. Therefore, the extremum correspond either to the solutions of the Eq. (2.10) $dU/d\phi = 0$ for finite $dX/d\theta$ ($X > -1$) or to the solutions of the equation $dX/d\theta = 0$ ($X > -1$) for finite $dU/d\phi$. The form of the potential $U$ [see Eqs. (5.8)] shows that the potential $U(\theta)$ has extremal at the matching points $\theta = 0, \pi, 2\pi, 3\pi$. Additional extremum points are real solutions of the Eq. (2.10). For our model (3.1), this equation reads

$$\tilde{R}^4\gamma\left(\frac{D}{2} - 4\right) + \tilde{R}^2\alpha\left(\frac{D}{2} - 2\right) + \tilde{R}\left(\frac{D}{2} - 1\right) - DA_D = 0. \quad (5.11)$$

The form of this equation shows that there are two particular cases: $D = 8$ and $D = 4$. The $D = 8$ case was considered in [16]. Let us consider now the case $D = 4$:

$$\tilde{R}^4 - \frac{1}{2\gamma}\tilde{R} + \frac{2\Lambda_4}{\gamma} = 0. \quad (5.12)$$
It is worth of noting that parameter $\alpha$ disappeared from this equation. Thus $\alpha$ has no an influence on a number of additional extremum points. To solve this quartic equation, we should consider an auxiliary cubic equation

$$u^3 - \frac{8\Lambda_4}{\gamma} u - \frac{1}{4\gamma^2} = 0.$$  

(5.13)

The analysis of this equation can be performed in similar manner as we did it for the cubic Eq. (3.3). Let us introduce the notations:

$$\tilde{q} := -\frac{8}{3} \frac{\Lambda_4}{\gamma}, \quad \tilde{r} := \frac{1}{8} \frac{1}{\gamma},$$

$$\tilde{Q} := \tilde{r}^2 + \tilde{q}^3 = \frac{1}{\gamma^3} \left(1 + \frac{8\Lambda_4}{3}\right)^3.$$  

(5.14)

It make sense to consider two separate cases.

1. sign $\gamma = -\text{sign} \Lambda_4 \Rightarrow \tilde{Q} > 0$.

In this case we have only one real solution of Eq. (5.13):

$$u_1 = \left[\tilde{r} + \left(\frac{\tilde{r}^2}{\tilde{q}}\right)^{1/3} + \left[\tilde{r} - \left(\frac{\tilde{r}^2}{\tilde{q}}\right)^{1/3}\right]\right]/3 > 0.$$  

(5.15)

Then, solutions of the quartic (5.12) are the real roots of two quadratic equations

$$\tilde{R}^2 \mp \sqrt{u_1 \tilde{R} + \frac{1}{2} (u_1 \pm \epsilon \sqrt{u_1^3 + 3\tilde{q}})} = 0, \quad \epsilon = \text{sign} \gamma.$$  

(5.16)

Simple analysis shows that for any sign of $\gamma$ we obtain two real solutions:

$$\gamma < 0 \Rightarrow \tilde{R}_{1,2}^{(+) \gamma} = \pm \frac{1}{2} \sqrt{1/2} u_1^{1/2} \pm \sqrt{-1/2 u_1 + 1/2 (u_1^2 + 3\tilde{q})^{1/2}},$$

$$\gamma > 0 \Rightarrow \tilde{R}_{1,2}^{(-) \gamma} = \pm \frac{1}{2} u_1^{1/2} \pm \sqrt{-1/2 u_1 + 1/2 (u_1^2 + 3\tilde{q})^{1/2}}.$$  

(5.17)

2. sign $\gamma = \text{sign} \Lambda_4 \Rightarrow \tilde{Q} \leq 0$.

It is not difficult to show that in this case the real solutions of the form of (5.17) (where we should make the evident substitution $u_1^3 + 3\tilde{q} \rightarrow u_1^3 - 3|\tilde{q}|$) takes place if

$$\tilde{Q} > 0 \Rightarrow |\gamma|^{1/3} < \frac{3}{32 |\Lambda_4|}.$$  

(5.18)

Now, we want to investigate zeros of the potential $U$. For $f' \neq 0 \Rightarrow X \neq -1$, the condition of zeros of the potential (2.4) is

$$\tilde{R} f' - f = 0 \Rightarrow 3\gamma \tilde{R}^4 + \alpha \tilde{R}^2 + 2\Lambda_D = 0.$$  

(5.19)

Therefore, zeros are defined by equation

$$\tilde{R}^2 = -\frac{\alpha}{6\gamma} \pm \left[\left(\frac{\alpha}{6\gamma}\right)^2 - \frac{2\Lambda_D}{3\gamma}\right]^{1/2}.$$  

(5.20)

Obviously, the necessary conditions for zeros are

$$\gamma > 0 \Rightarrow \Lambda_D \leq \alpha^2/(24\gamma),$$

$$\gamma < 0 \Rightarrow \Lambda_D \geq -\alpha^2/(24|\gamma|).$$  

(5.21)

Additionally, we should check that the right-hand side of the Eq. (5.20) is positive.

Let us consider now asymptotical behavior of the potential $U(\theta)$. Here, we want to investigate limits $\theta \rightarrow \theta_{\max}$ and $\theta \rightarrow -\infty$. In the former case we get

$$\theta \rightarrow \theta_{\max} \Rightarrow U(\theta) \rightarrow -\text{sign}(f(\theta_{\max})) \times \infty.$$  

(5.22)

In the latter case we obtain

$$\theta \rightarrow -\infty \Rightarrow U(\theta) \sim \exp\left(\frac{8 - D}{D - 2}\theta\right).$$

$$\begin{cases} +\infty, & D > 8; \\ \text{const} > 0, & D = 8; \\ +0, & 2 < D < 8; \end{cases}$$  

(5.23)

where we used Eqs. (5.3) and (5.4). This result coincides with conclusions of Appendix A in [16].

To illustrate the described above properties, we draw the potential $U(\theta)$ in Fig. 5 for the following parameters: $D = 4$, $\Lambda_4 = 0.1$, $\gamma = 1$, and $\alpha = -1$. These parameters contradict the inequalities (5.18) and (5.21). Therefore, $\theta = 0, \pi, 2\pi, 3\pi$ are the only extremum points of the potential $U(\theta)$, and zeros are absent. These parameters are also satisfy the condition (5.5). The absence of zeros means that all minima of the potential $U(\theta)$ are positive.

For our subsequent investigations, it is useful also to consider an effective force and mass squared of the field $\theta$. As it follows from Eq. (5.9), the effective force is

$$\begin{align*}
\text{FIG. 5. The form of the potential (2.6) as a function of } \theta \text{ in the case } D = 4, \Lambda_4 = 0.1, \gamma = 1, \text{ and } \alpha = -1. 
\text{For these values of the parameters, all extrema correspond to the matching points } \theta = 0, \pi, 2\pi, 3\pi. 
\text{In the branching points } \theta = \pi, 2\pi \text{ the potential has local maximum and local nonzero minimum, respectively, and the monotonic points } \theta = 0, 3\pi \text{ are the inflection ones. Potential tends asymptotically to } +\infty \text{ when } \theta \text{ goes to } \theta_{\max} \text{ and to zero when } \theta \rightarrow -\infty. 
\end{align*}$$
Varying Eq. (5.9) with respect to field $\theta$, we obtain dynamical equation for small fluctuation $\delta \theta$ where mass squared reads

$$m^2 \theta = G(\theta) \frac{d^2 U}{d\theta^2} + \frac{dG(\theta)}{d\theta} \frac{dU}{d\theta}. \quad (5.25)$$

In Fig. 6 we show the effective force and the mass squared as functions of $\theta$ for the potential drawn in Fig. 5. These figures indicate that field $\theta$ may have very nontrivial behavior. This nontriviality follows from two reasons. First, the field $\theta$ has noncanonical kinetic term which results in appearing of nonflat moduli space metric $G(\theta)$ and derivative of $G(\theta)$ in Eq. (5.9). Second, the function $G(\theta)$ has singular behavior at the matching points $\theta = 0, \pi, 2\pi, 3\pi$. Thus, our intuition does not work when we want to predict dynamical behavior of fields with equations of the form of (5.9) with potential drawn in Fig. 5, especially when field approaches the matching points. It is necessary to solve equations analytically or to investigate them numerically. Such analysis for our model is performed in the next subsection where we concentrate our attention to the most interesting case where all extrema correspond to the matching points $\theta = 0, \pi, 2\pi, 3\pi$.

### B. Dynamical behavior of the Universe and field $\theta$

Now, we intend to investigate dynamical behavior of scalar field $\theta$ and the scale factor $a$ in more detail. There are no analytic solutions for considered model. So, we use numerical calculations. To do it, we apply a Mathematica package proposed in [26] and adjusted to our models and notations in Appendix A of our paper [18]. According to these notations, all dimensional quantities in our graphics are given in the Planck units. Additionally, in the present paper we should remember that metric on the moduli space is not flat and defined in Eq. (5.9). For example, the canonical momenta and the kinetic energy read

$$P_\theta = \frac{a^3}{\kappa_4^2} G_{11} \dot{\theta} = \frac{a^3}{\kappa_4^2} \left( \frac{d\Phi}{d\theta} \right)^2 \dot{\theta},$$

$$E_{\text{kin}} = \frac{1}{2\kappa_4^2} G_{11} \dot{\theta}^2 = \frac{\kappa_4^2}{2a^6} G_{11}^2 \dot{\theta}^2 = \frac{1}{2\kappa_4^2} \left( \frac{d\Phi}{d\theta} \right)^2 \dot{\theta}^2,$$

where $8\pi G \equiv \kappa_4^2$ and $G$ is four-dimensional Newton constant. To understand the dynamics of the Universe, we shall also draw the Hubble parameter

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 3H^2 = \frac{1}{2} G_{11} \dot{\theta}^2 + U(\theta) \quad (5.27)$$

and the acceleration parameter

$$q = \frac{\ddot{a}}{H^2 a} = \frac{1}{6H^2} \left( -4 \times \frac{1}{2} G_{11} \dot{\theta}^2 + 2U(\theta) \right). \quad (5.28)$$

Figure 6 shows that the effective force changes its sign and the mass squared preserves the sign when $\theta$ crosses the branching points $\pi, 2\pi$ and vise versa, the effective force preserves the sign and the mass squared changes the sign when $\theta$ crosses the monotonic points $0, 3\pi$. Therefore, it make sense to consider these cases separately.

#### 1. Branching points $\theta = \pi, 2\pi$

First, we consider the dynamical behavior of the Universe and a scalaron in the vicinity of the branching point $\theta = 2\pi$ which is the local minimum of the potential in Fig. 5. The time evolution of a scalaron field $\theta$ and its kinetic energy $E_{\text{kin}}$ are drawn in Fig. 7. Here and in all pictures below, we use the same parameters as in Fig. 5. The time $t$ is measured in the Planck times and classical evolution starts at $t = 1$. For the initial value of $\theta$, we take $\theta_{\text{initial}} = 3.5$. We plot in Fig. 8 the evolution of the logarithms of the scale factor $a(t)$ (left panel) and the evolution of the Hubble parameter $H(t)$ (right panel) and in Fig. 9 the
evolution of the parameter of acceleration \( q(t) \) (left panel) and the equation of state parameter \( \omega(t) = \frac{1}{2q(t) + 1}/3 \) (right panel).

Figure 7 demonstrates that scalar field \( \theta(t) \) bounces an infinite number of times with decreasing amplitude in the vicinity of the branching point \( \theta = 2\pi \). From Figs. 8 and 9, we see that the Universe asymptotically approaches the de Sitter stage: \( H \rightarrow \text{const}, \ q \rightarrow +1 \), and \( \omega \rightarrow -1 \). Such accelerating behavior we call bouncing inflation.

Concerning the dynamical behavior in the vicinity of the branching point \( \theta = \pi \), our analysis (similar performed above) shows that the scalaron field \( \theta \) cannot cross this local maximum regardless of the magnitude of initial velocity in the direction of \( \theta = \pi \). It bounces back from this point.

2. Monotonic points \( \theta = 0, 3\pi \)

Now, we want to investigate the dynamical behavior of the model in the vicinity of the monotonic points \( \theta = 0, 3\pi \) which are the points of inflection of the potential in Fig. 5. Figures 5 and 6 show that for both of these points the model has the similar dynamical behavior. Therefore, for definiteness, we consider the point \( \theta = 3\pi \). To investigate numerically the crossing of the monotonic point \( 3\pi \), it is necessary to take very small value of a step \( \Delta t \). It can be achieved if we choose very large value of the maximum number of steps. Thus, for the given value of the maximum number of steps, the closer to \( 3\pi \) the initial value \( \theta_{\text{initial}} \) is taken, the smaller step \( \Delta t \) we obtain. For our calculation, we choose \( \theta_{\text{initial}} = 9.513 \). Figure 10 demonstrates that scalar field \( \theta \) slowly crosses the monotonic point \( 3\pi \) with nearly zero kinetic energy.\(^6\) Then, just after the crossing, the kinetic energy has its maximum value and starts to decreases gradually when \( \theta \) moves to the direction \( 2\pi \).

Figures 11 and 12 demonstrate the behavior of the Universe before and after crossing \( 3\pi \). We do not show here the vicinity of the branching point \( 2\pi \) because when \( \theta \) approaches \( 2\pi \) the Universe has the bouncing inflation described above. Hence, there are 3 phases sequentially:

\(^6\)The derivative \( d\theta/dt \) goes to \( -\infty \) when \( \theta \rightarrow 3\pi \) (with different speed on different sides of \( 3\pi \)) but \( d\phi/d\theta = 0 \) at \( 3\pi \) and kinetic energy is finite [see Eq. (5.26)].
the short de Sitter-like stage during slow rolling in the vicinity of the inflection point before crossing, then decelerating expansion just after the crossing with gradual transition to the accelerating stage again when $C_1 = C_2$. Clearly, for another monotonic point $\theta = 0$, we get the similar crossing behavior (without the bouncing stage when $\theta \to -\infty$). Therefore, the monotonic points $\theta = 0$ and $\theta = 3\pi$ are penetrable for the scaloron field $\theta$.

FIG. 9. The parameter of acceleration $q(t)$ (left panel) and the equation of state parameter $\omega(t)$ (right panel) for the scale factor in Fig. 8.

FIG. 10. Dynamical behavior of scalar field $\theta(t)$ (left panel) and its time derivative $d\theta/dt$ (middle panel) and kinetic energy $E_{\text{kin}}(t)$ (right panel) for the case of crossing of the inflection point $\theta = 3\pi$.

FIG. 11. The time evolution of the logarithms of the scale factor $a(t)$ (left panel) and the Hubble parameter $H(t)$ (right panel) for the trajectory drawn in Fig. 10.
However, the branching points take also place in models during the early stages of the Universe evolution. We have found this type of inflation show that this inflation takes place both in the Einstein and Brans-Dicke frames. We also would like to thank Yi-Fu Cai for drawing our attention to their paper [28]. This work was supported in part by the “Cosmomicrophysics” programme of the Physics and Astronomy Division of the National Academy of Sciences of Ukraine.

APPENDIX: BOUNCING INFLATION IN THE BRANS-DICKE FRAME

According to Eq. (2.7), the four-dimensional Friedmann-Robertson-Walker metrics in the Einstein...
that where we choose the constant of integration in such a way
both frames we obtain, correspondingly,

\[ \frac{d\bar{t}}{dt} = dt / \sqrt{f'(t)}, \]  
\[ \bar{a}(\bar{t}) = a(t(\bar{t}))/\sqrt{f'(t(\bar{t}))}, \]  
which lead to the following equations:

\[ \bar{t} = \int_{t_0}^{t} \frac{dt}{\sqrt{X(t) + 1}} + 1, \]  

where we choose the constant of integration in such a way that \( \bar{t}(t = 1) = 1, \) and

\[ \bar{H}(\bar{t}) = \frac{d\bar{a}}{d\bar{t}} \bar{a} \]
\[ = \sqrt{X(t(\bar{t})) + 1} \left[ H(t(\bar{t})) - \frac{1}{2X(t(\bar{t})) + 1} \frac{dX}{dt}(t(\bar{t})) \right]. \]

From the latter equation, we get the relation between the Hubble parameters in both frames. We plot in Fig. 13 the logarithms of the scale factor \( \bar{a}(\bar{t}) \) and the Hubble parameter \( \bar{H}(\bar{t}) \) for the trajectory drawn in Fig. 7. These pictures clearly demonstrate that in the Brans-Dicke frame the Universe also has an asymptotical de Sitter stage when the scalaron field approaches the branching point \( \theta = 2\pi \).

It is not difficult to verify that because \( X(t \rightarrow +\infty) \rightarrow X_{\text{max}} \) and \( dX/dt(t \rightarrow +\infty) \rightarrow 0 \); we obtain the following relation for the asymptotic values of the Hubble parameters in both frames:

\[ \bar{H} = H\sqrt{X_{\text{max}} + 1}. \]