A statistical theory for weak homogeneous turbulence

Citation for published version (APA):

Malfliet, W. P. M. (1971). *A statistical theory for weak homogeneous turbulence*. [Phd Thesis 1 (Research TU/e / Graduation TU/e), Electrical Engineering]. Technische Hogeschool Eindhoven. <https://doi.org/10.6100/IR143909>

DOI:

[10.6100/IR143909](https://doi.org/10.6100/IR143909)

Document status and date:

Published: 01/01/1971

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

A STATISTICAL THEORY FOR WEAK HOMOGENEOUS TURBULENCE

P R O E F S C H R I F T

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE
WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL TE EINDHOVEN
OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. IR. A. A. TH. M. VAN TRIER,
VOOR EEN COMMISSIE UIT DE SENAAT IN HET OPENBAAR TE VERDEDIGEN OP
DINSDAG 25 MEI 1971 DES NAMIDDAGS TE 4 UUR

DOOR

WILLY PETRUS MARIA MALFLIET

GEBOREN 17 JULI 1943 TE SINT-NIKLAAS (BELGIË)

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR
PROF. DR. H. BREMMER

"...the principal result of fifty years
of turbulent research is the recognition
of the profound difficulties of the
subject"

S.A. Orszag, J.Fluid Mech. 41 (1970) 363

Aan mijn ouders

SUMMARY	1
I INTRODUCTION	3
PART I - GENERAL ASPECTS	
II DERIVATION OF A MATHEMATICAL MODEL EQUATION	7
A. The case of an arbitrary number of dimensions	7
B. The one-dimensional model equation	13
III WAVE-CORRELATION FUNCTIONS FOR THE ONE-DIMENSIONAL MODEL	17
A. Second-order wave-correlation function	17
B. Third-order wave-correlation function	21
C. Fourth-order wave-correlation function	23
IV DERIVATION OF THE HIERARCHY FOR THE ONE-DIMENSIONAL MODEL	27
V THE BOGOLIUBOV EXPANSION METHOD AND THE DERIVATION OF A GENERAL KINETIC EQUATION	34
A. Determination of the first nonlinear correction term	34
B. Derivation of the lowest-order kinetic equation	40
C. Special cases of the kinetic equation	45
VI DISCUSSION OF THE ONE-DIMENSIONAL KINETIC EQUATION	49
A. Energy conservation	49
B. Nonlinear interaction between undamped stable waves	55
C. A special model equation for undamped stable waves	60
D. Nonlinear interaction between purely damped or between purely growing waves	65
E. Higher-order corrections	66
F. Other methods in weak turbulence theories	67

	page
VII EXTENSION OF THE THEORY IN THE CASE OF DISPERSION RELATIONS WITH SEVERAL BRANCHES	70
A. A second one-dimensional model, referring to a dispersion relation with two branches	70
B. Wave-correlation functions	72
C. The kinetic equation for the second one-dimen- sional model	75
VIII A THIRD ONE-DIMENSIONAL MODEL WITH PURELY DAMPED AND PURELY GROWING WAVES	84
A. The model equation and corresponding symmetry properties	84
B. Wave-correlation functions	86
C. The kinetic equations for the third one-dimen- sional model	88
PART II - APPLICATIONS	
IX HYDRODYNAMICAL TURBULENCE	94
A. Introduction	94
B. Basic equations	96
C. The Burgers' model equation; theory	100
D. The Burgers' model equation; numerical results	106
E. Three-dimensional incompressible hydrodynamical turbulence	110
F. Discussion of the theory of incompressible homogeneous turbulence	114
G. Nonlinear interaction between sound waves in a turbulent fluid	117
H. Sound waves with isotropic energy spectra	126
I. Sound waves influenced by viscosity	128
X A MODEL FOR MAGNETOHYDRODYNAMICAL TURBULENCE	131
A. Introduction	131
B. Derivation of a model system for magnetohydro- dynamical turbulence	132

	page
C. Derivation of the kinetic equations	134
D. Magnetohydrodynamical turbulence in the case of a magnetic diffusivity equal to the kine- matic viscosity	141
XI CONCLUSIONS	146
ACKNOWLEDGEMENTS	149
APPENDICES	150
REFERENCES	159

S U M M A R Y

The Bogoliubov expansion method, known from the kinetic theory in statistical mechanics, is introduced in the field of weak homogeneous turbulence. We first develop the mathematics of this method in its application to a one-dimensional model equation. It enables to close the hierarchy of the associated wave-correlation functions as well as to derive kinetic equations which statistically describe the nonlinear interaction of the different modes. The latter modes may have complex frequencies (connected with the linearized problem), corresponding to either stable or unstable modes, although the nonlinearity has to be small. For this model we discuss some general properties, such as energy conservation as well as special cases, e.g. resonant three-wave interaction. Next, two further model equations are investigated. The first one deals with the nonlinear interaction between damped waves travelling in opposite directions, the second one with the interaction between purely stable and purely unstable modes. The corresponding kinetic equations are obtained for either case.

Applications to the theory of hydrodynamical (HD) turbulence are considered. First of all, the theory has been worked out for the Burgers' model equation. The dynamical

equation for the energy spectrum of the latter shows the cascade of energy towards the higher wavenumbers; this is confirmed by numerical results. Such a cascade process also appears to exist for three-dimensional incompressible turbulence. Passing to compressible media, we study the nonlinear interaction between sound waves in a dissipative and in a nondissipative fluid. In the latter case a cascade of energy only proves to occur for isotropic energy spectra, whereas this condition is not required for the cascade in a dissipative fluid. Finally, some properties of a one-dimensional model of magnetohydrodynamical (MHD) turbulence are investigated, including those concerning the total energy.

C H A P T E R I

I N T R O D U C T I O N

Nonlinear phenomena in various media are getting more and more attention because just now the knowledge and the means to get a better insight in this field of research become available. We have in mind the growing facilities with computers and many experimental techniques (lasers for instance) to investigate the associated problems.

It is well known that most physical processes are essentially nonlinear so that, dealing with linear phenomena, these constitute more or less approximations with a limited validity. However, these linear or linearized problems are easily treated mathematically and can be solved in most cases, the boundary and (or) the initial conditions being given. The success of this simplified treatment is due to the applicability of the superposition principle, only valid for these linear problems; as a consequence, one can combine elementary solutions and satisfy as yet the same equations. Unfortunately such a principle does not exist for nonlinear problems and only in very limited cases one can solve them and find an explicit solution.

An important part of nonlinear physics concerns the

domain of turbulence, which we interpret here in a much broader sense than the known hydrodynamical turbulence. We call a system turbulent if it shows a large number of fluctuations (perturbations) which dynamically change in a random manner but not independent of each other. The interactions imply an energy transfer between the various fluctuations, which constitutes an essential property of turbulence and which is a direct consequence of the nonlinearity of the mathematical equations. Let us imagine two basic examples for the onset of turbulence. First, a large fluctuation amplitude is suddenly excited, nonlinear terms then immediately becoming important. Secondly, an instability develops in the system and after a certain time nonlinear effects play a role. In both cases one will be also interested in the relaxation towards a possible nonlinear equilibrium.

In many domains of physics one has to deal with this kind of problems. The investigation of turbulence is in particular of primordial interest in plasma physics. The main technical problem here is to suppress the instabilities in order to confine the plasma and to achieve thermonuclear fusion. It is commonly believed¹⁾ that a fusion reactor is feasible with a turbulent plasma in a stable nonlinear equilibrium. A related important problem concerns the influence of turbulence on the mean physical quantities, resulting in the so-called anomalous properties of the plasma.

Heisenberg²⁾ has introduced some general concepts to treat nonlinear problems theoretically: linearization, symmetry considerations, and statistical methods. Apart from them, one can mention the use of a computer or a mixed method, such as the synergetic approach of Kruskal and Zabusky³⁾.

In order to keep the discussion clear and simple, we mostly consider one-dimensional systems; first, on a rather formal basis (Part I), to illustrate the mathematical aspects of the introduced statistical theory and secondly, in part II, with reference to some applications in the field of hydrodynamical (HD) and magnetohydrodynamical (MHD) turbulence.

In chapter II we start with a general dynamical (vector) equation applicable to a very large class of turbulent systems. Next, the relevant mathematical equations are replaced by their Fourier transforms with respect to the space variables. In the Fourier space the energy sources are much better recognized than in the original one, and all further calculations are carried through much easier in the former space. We next linearize the problem then obtained which enables an analysis of the whole nonlinear problem by means of a model equation which essentially preserves the basic difficulties. Assuming further, in the linearized problem, only a single complex frequency associated with the single mode of only one physical quantity, the nonlinear problem can be investigated, after introducing a statistical description in accordance with Heisenberg's concepts.

Requiring the whole system to be homogeneous, we are able to introduce correlation functions, connected with the averaged products of the Fourier transforms of the relevant quantities (chapter III). In view of the assumed homogeneity these quantities satisfy various symmetry properties; moreover, when dealing with two- or three-dimensional problems, isotropy may also be required.

Two methods for calculating turbulent processes are available: either by straightforwardly solving the problem with a perturbation method⁴⁻⁶) and by deriving the statistical properties afterwards, or by starting with a statistical averaging and next using a perturbation method⁷). We have chosen for the second approach because, to us, it seems to be the most systematic one. Accordingly, a hierarchy of coupled equations referring to correlation functions of different orders (chapter IV) can be set up in the Fourier space.

The main purpose of this thesis is to find dynamical equations describing the energy spectrum of homogeneous turbulence. Therefore, one has to close the hierarchy, which implies a restriction to weak turbulence. Davidson⁷) succeeded in such a closure by the application of the multiple time-scales method⁸) which, however, can only be used if the fluctuations represented

by modes (waves) are stable and undamped, or at most very slightly unstable or damped.

In chapter V we shall introduce a perturbation method based on the Bogoliubov expansion method widely known in the kinetic theory of gases⁹⁾. With this technique we can admit all possible types of fluctuations, without violating the (energy) conservation laws (chapter VI).

The developed theory is further extended to two related model equations. The first one (chapter VII) refers to systems for which the relevant physical quantity satisfies a quadratic dispersion relation. Next, in chapter VIII we generalize the analysis of the original model equation for the case of a nonlinear interaction between pure, nonoscillatory, damped, and growing waves, these waves then being the only solutions of the dispersion relation for the linearized problem.

Finally, in part II, some important applications are investigated. Chapter IX deals with the problem of hydrodynamical turbulence without any mean motion. First of all, incompressible HD-turbulence is investigated. Most attention is drawn to the one-dimensional Burgers' equation¹⁰⁾, together with a discussion devoted mainly to Millionshtchikov's hypothesis¹¹⁾, which may be applied as a consequence of the assumed weak turbulence. This analysis is also extended to the three-dimensional case. Assuming a very small fluctuating density, the nonlinear interaction of sound waves (compressible fluid turbulence) could also be investigated. Finally, in chapter X, we discuss an extension of the Burgers' equation, introduced by Thomas¹²⁾, which concerns the combination of a random velocity field and a random magnetic field so as to describe MHD-turbulence by a simple model.

PART I - GENERAL ASPECTS

C H A P T E R II

DERIVATION OF A MATHEMATICAL MODEL EQUATION

A. The case of an arbitrary number of dimensions

In this section we intend to derive a general model equation in order to investigate some properties of turbulence. Therefore, we shall not consider specific problems or situations but only general mathematical and physical features connected with problems of turbulence. The established procedure can then easily be applied to specified problems.

We start from the equation

$$i \frac{\partial \underline{\psi}(\underline{r}, t)}{\partial t} = L(\underline{\nabla}) \underline{\psi}(\underline{r}, t) + \underline{B}(\underline{\nabla}, \underline{\psi}(\underline{r}, t); \underline{\nabla}, \underline{\psi}(\underline{r}, t)), \quad (\text{II.1})$$

where the column vector $\underline{\psi}(\underline{r}, t)$ describes the physical quantities of the system (velocity, pressure, mass density, etc.). The matrix L represents the effect of a linear differential operator and $\underline{B}$ that of a bilinear one, which both only contain

space derivatives (no time derivatives). The most general representations of their components (to the number of N) read

$$\{L(\underline{\nabla})\psi\}_\ell = \sum_{i;m;r} (\nabla_i)^m (L_i^m)_{\ell,r} \psi_r, \quad (II.1a)$$

$$B_\ell(\underline{\nabla}, \underline{\psi}; \underline{\nabla}', \underline{\psi}') = \sum_{i,j;r,s;m,n} A_{i,j;r,s}^{m,n} \ell \cdot (\nabla_i)^m \psi_r \cdot (\nabla'_j)^n \psi'_s, \quad (II.1b)$$

($m, n = 0, 1, 2, \dots$; $i, j = 1, 2, 3$; $\ell, r, s = 1, 2, \dots, N$).

Many problems can be described by a dynamical equation such as (II.1), for instance hydrodynamical media and plasmas. Therefore, we shall restrict ourselves to this kind of equations. For other authors (refs. 4, 13), too, this equation has been a relevant starting point.

We next suppose that the whole system, as formulated by eq. (II.1) is in a turbulent state and that the equation (II.1) also holds in this case. A large number of interacting fluctuations is then present. These fluctuations, as will be seen later, change irregularly in space ($\underline{r}$) and time (t) according to very complicated coupled equations such as to make it impossible to follow the detailed evolution of the system as a function of position and time. Also physically it makes sense to consider only averaged quantities. Now the question arises which kind of average we have to take: ensemble average, space or time average. From the experimental point of view only the two last ones are relevant. Theoretically, however, it is much easier to deal with ensemble averages. For convenience we assume all these averages to be equal (ergodic hypothesis). Thus we only need to consider the ensemble averages which we shall designate by the symbol $\langle \dots \rangle$.

In a turbulent state, the above-mentioned physical quantities are generally written as

$$\underline{\psi}(\underline{r}, t) = \underline{\phi}(\underline{r}, t) + \underline{\sigma}(\underline{r}, t), \quad (II.2)$$

where $\underline{\phi}$ describes the averaged and $\underline{\sigma}$ the fluctuating part.
Hence,

$$\underline{\phi}(\underline{r}, t) = \langle \underline{\psi}(\underline{r}, t) \rangle \quad (\text{II.3})$$

and

$$\langle \underline{\sigma}(\underline{r}, t) \rangle = 0. \quad (\text{II.4})$$

For convenience, we postulate that in the system, described by eq. (II.1), all quantities have been normalized such as to have vanishing averages. Hence, we have

$$\underline{\psi}(\underline{r}, t) = \underline{\sigma}(\underline{r}, t), \quad (\text{II.5})$$

$$\langle \underline{\psi}(\underline{r}, t) \rangle = 0. \quad (\text{II.6})$$

If (II.6) does not hold originally one has to split up the fundamental equation (II.1) with the aid of (II.2) into an averaged and a fluctuating part and the basic equation (II.1) will refer to this latter contribution (see appendix A). Thus dealing only with fluctuating quantities, it is useful to introduce the Fourier transform $\underline{\psi}(\underline{k}, t)$ according to

$$\underline{\psi}(\underline{r}, t) = \int_{-\infty}^{+\infty} \underline{\psi}(\underline{k}, t) e^{i \underline{k} \underline{r}} d\underline{k}, \quad (\text{II.7})$$

where

$$\underline{\psi}(\underline{k}, t) = \underline{\psi}^*(-\underline{k}, t) \quad (\text{reality condition}), \quad (\text{II.8})$$

and

$$\underline{\psi}(\underline{k}=0, t) = 0. \quad (\text{II.9})$$

The last equation expresses the fact that we deal only with fluctuating quantities having a vanishing spatial average. The existence of the transform (II.7) involves that, according to the definition of turbulence, we admit that all possible fluctuations can be represented by modes with different

wavenumbers $\underline{k} = |\underline{k}|$ (the so-called $\underline{k}$ -modes).

In the Fourier (or $\underline{k}$) space eq. (II.1) becomes:

$$i \frac{\partial \underline{\psi}(\underline{k}, t)}{\partial t} = H_0(\underline{k}) \underline{\psi}(\underline{k}, t) + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\underline{k}' d\underline{k}'' \delta(\underline{k} - \underline{k}' - \underline{k}'') \underline{H}_1[\underline{k}', \underline{k}''; \underline{\psi}(\underline{k}', t), \underline{\psi}(\underline{k}'', t)], \quad (\text{II.10})$$

if the following transformations into the Fourier space are used which, at the same time, define the operators H_0 and $\underline{H}_1$:

$$i \frac{\partial \underline{\psi}(\underline{r}, t)}{\partial t} \rightarrow \int_{-\infty}^{+\infty} i \frac{\partial \underline{\psi}(\underline{k}, t)}{\partial t} e^{i \underline{k} \underline{r}} d\underline{k}, \quad (\text{II.11})$$

$$L \underline{\psi}(\underline{r}, t) \rightarrow \int_{-\infty}^{+\infty} H_0(\underline{k}) \underline{\psi}(\underline{k}, t) e^{i \underline{k} \underline{r}} d\underline{k}, \quad (\text{II.12})$$

$$\underline{B}(\underline{V}, \underline{\psi}(\underline{r}, t); \underline{V}, \underline{\psi}(\underline{r}, t)) \rightarrow \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(\underline{k}' + \underline{k}'') \underline{r}} \times \\ \times \underline{H}_1[\underline{k}', \underline{k}''; \underline{\psi}(\underline{k}', t), \underline{\psi}(\underline{k}'', t)] d\underline{k}' d\underline{k}'' . \quad (\text{II.13})$$

The components of the new tensorial operator H_0 and the vectorial operator $\underline{H}_1$ in $\underline{k}$ -space are connected with the original operators L and $\underline{B}$ by the relations:

$$H_0(\underline{k}) \underline{\psi}(\underline{k}, t) = L(i \underline{k}) \underline{\psi}(\underline{k}, t), \quad (\text{II.12a})$$

$$\underline{H}_1[\underline{k}', \underline{k}''; \underline{\psi}(\underline{k}', t), \underline{\psi}(\underline{k}'', t)] = \underline{B}(i \underline{k}', \underline{\psi}(\underline{k}', t); i \underline{k}'', \underline{\psi}(\underline{k}'', t)). \quad (\text{II.13a})$$

In order to study the nonlinear equation (II.10) we first linearize the problem by neglecting the interaction between the different fluctuating amplitudes. Hence, we approximate (II.10) by taking $\underline{H}_1 = 0$ and the problem then reduces to solving the equation

$$i \frac{\partial \underline{h}(\underline{k}, t)}{\partial t} = H_0(\underline{k}) \underline{h}(\underline{k}, t), \quad (\text{II.14})$$

where the amplitude $\underline{h}(\underline{k}, t)$ of each $\underline{k}$ -mode changes independently of the other ones.

This homogeneous matrix equation is well known in ordinary mechanics and can easily be solved by the substitution $\underline{h}(\underline{k}, t) = \underline{h}(\underline{k}) e^{-i\omega(\underline{k})t}$ (normal-mode analysis). This results in the eigenvalue equation

$$\omega(\underline{k}) \underline{h}(\underline{k}) = H_0(\underline{k}) \underline{h}(\underline{k}), \quad (\text{II.15})$$

which only has nontrivial solutions if

$$\text{Det.}(\omega \delta_{ij} - [H_0]_{ij}) = 0, \quad (\text{II.16})$$

with $i, j = 1, 2, \dots, n$. $H_0(\underline{k})$ represents an $n \times n$ matrix, if n equals the number of components of the vectors $\underline{\psi}(\underline{r}, t)$ and $\underline{\psi}(\underline{k}, t)$. We suppose this secular equation to have n different roots (eigenvalues), namely ω_α ($\alpha = 1, 2, \dots, n$). Hence, the most general solution of eq. (II.14) can be represented by

$$\underline{h}(\underline{k}, t) = \sum_{\alpha=1}^n A_\alpha(\underline{k}) \underline{h}_\alpha(\underline{k}) e^{-i\omega_\alpha(\underline{k})t}, \quad (\text{II.17})$$

because of the applicability of the superposition principle for this linearized theory. Further, we assume that the set of eigenvectors $\{\underline{h}_\alpha(\underline{k})\}$ may constitute an orthonormal one. The construction of such a set strongly depends on the specific problem which one wants to treat, and is beyond the scope of this work. The orthonormality is expressed by the inner product

$$\underline{h}^\beta(\underline{k}) \cdot \underline{h}_\alpha(\underline{k}) = \delta_{\alpha\beta}, \quad (\text{II.18})$$

where the row vector $\underline{h}^\beta(\underline{k})$ marks the adjoint of the column

vector $\underline{h}_\beta(\underline{k})$, so that $(\underline{h}_\alpha(\underline{k}))_j = (\underline{h}^{\alpha*}(\underline{k}))_j$.

With the knowledge of this linearized problem, we are able to analyze the complete nonlinear equation (II.10). In fact, we take eq. (II.17) as an ansatz for its solution, replacing the amplitude $A_\alpha(\underline{k})$ by $A_\alpha(\underline{k}, t)$. This means that the nonlinearity is expressed by the time dependence of the amplitudes $A_\alpha(\underline{k}, t)$ ($\alpha = 1, 2, \dots, n$), so as to have

$$\underline{\psi}(\underline{k}, t) = \sum_{\alpha=1}^n A_\alpha(\underline{k}, t) \underline{h}_\alpha(\underline{k}) e^{-i\omega_\alpha(\underline{k})t}. \quad (\text{II.19})$$

The advantage of this ansatz concerns the property that all linear information is contained in $\omega_\alpha(\underline{k})$ ($\alpha = 1, 2, \dots, n$). Substituting eq. (II.19) into eq. (II.10), we easily obtain the following equation, in view of the property that the operators H_0 and H_1 , like L and B , are linear and bilinear with respect to each of their arguments $\underline{\psi}$:

$$\begin{aligned} & i \sum_{\alpha=1}^n \frac{\partial A_\alpha(\underline{k}, t)}{\partial t} \underline{h}_\alpha(\underline{k}) e^{-i\omega_\alpha(\underline{k})t} = \\ & = \sum_{\lambda, \gamma=1}^n \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\underline{k}' d\underline{k}'' \delta(\underline{k} - \underline{k}' - \underline{k}'') \underline{H}_1[\underline{k}', \underline{k}''; \underline{h}_\lambda(\underline{k}'), \underline{h}_\gamma(\underline{k}'')] \times \\ & \times A_\lambda(\underline{k}', t) A_\gamma(\underline{k}'', t) e^{-i\{\omega_\lambda(\underline{k}') + \omega_\gamma(\underline{k}'')\}t}. \end{aligned} \quad (\text{II.20})$$

Multiplying this last equation by the row vector $\underline{h}^\beta(\underline{k})$ we get, with the aid of the orthonormality condition (II.18), the final equation

$$\begin{aligned} & i \frac{\partial A_\beta(\underline{k}, t)}{\partial t} e^{-i\omega_\beta(\underline{k})t} = \\ & = \sum_{\lambda, \gamma=1}^n \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\underline{k}' d\underline{k}'' \delta(\underline{k} - \underline{k}' - \underline{k}'') \underline{h}^\beta(\underline{k}) \cdot \underline{H}_1[\underline{k}', \underline{k}''; \underline{h}_\lambda(\underline{k}'), \underline{h}_\gamma(\underline{k}'')] \times \\ & \times A_\lambda(\underline{k}', t) A_\gamma(\underline{k}'', t) e^{-i\{\omega_\lambda(\underline{k}') + \omega_\gamma(\underline{k}'')\}t}. \end{aligned} \quad (\text{II.21})$$

B. The one-dimensional model equation

Although the equation (II.21) is an exact one, derived from the dynamical equation (II.1), it remains complicated and difficult to deal with. Davidson¹³⁾ has considered this complete equation in the case of real eigenvalues $\omega_\alpha(\underline{k})$ ($\alpha = 1, 2, \dots, n$); even then the calculations are tedious. To avoid too great complications we shall reduce eq. (II.21) to a model equation which contains the essential properties of the preceding analysis, but which is more tractable than the original one. It turns out that the best one can do is to look at the one-dimensional analogue of eq. (II.21), with one single ω . In this case the original differential equation (II.1), with definitions (II.1a) and (II.1b), reduces to:

$$i \frac{\partial \psi(\underline{x}, t)}{\partial t} = \sum_m L^m \frac{\partial^m \psi(\underline{x}, t)}{\partial \underline{x}^m} + \sum_m \sum_n A^{mn} \frac{\partial^m \psi(\underline{x}, t)}{\partial \underline{x}^m} \frac{\partial^n \psi(\underline{x}, t)}{\partial \underline{x}^n} . \quad (\text{II.22})$$

Instead of eqs. (II.7), (II.8), and (II.9) we have now the one-dimensional formulae, taking into account that, in view of (II.18), the factor $\underline{h}(\underline{k})$ here reduces to the scalar $h(k) = 1$:

$$\psi(\underline{x}, t) = \int_{-\infty}^{+\infty} \psi(k, t) e^{ikx} dk = \int_{-\infty}^{+\infty} A(k, t) e^{ikx - i\omega(k)t} dk , \quad (\text{II.23})$$

with again

$$\psi(k, t) = \psi^*(-k, t) \quad (\text{II.24})$$

and

$$\psi(k=0, t) = A(k=0, t) = 0. \quad (\text{II.25})$$

As a consequence, all indices in (II.21) then disappear and

the summation sign can be omitted. In addition we put

$$\underline{h}(\underline{k}) \cdot \underline{H}_1 [\underline{k}', \underline{k}''; \underline{h}(\underline{k}'), \underline{h}(\underline{k}'')] = V(\underline{k}; \underline{k}', \underline{k}''). \quad (\text{II.26a})$$

As a matter of fact, the function V only depends on two variables, the dependence on $\underline{k}$ being redundant. On account of the expression $\underline{H}_1$ we then have

$$V(\underline{k}; \underline{k}', \underline{k}'') = \sum_{\underline{m}} \sum_{\underline{n}} A^{\underline{m}\underline{n}} (\underline{i}\underline{k}')^{\underline{m}} (\underline{i}\underline{k}'')^{\underline{n}}. \quad (\text{II.26b})$$

The parameter $\underline{k}$ recalls the interaction between three waves for which the relation $\underline{k} = \underline{k}' + \underline{k}''$ holds in general. Hence, we have

$$\begin{aligned} & i \frac{\partial A(\underline{k}, t)}{\partial t} e^{-i\omega(\underline{k})t} = \\ = & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\underline{k}' d\underline{k}'' \delta(\underline{k} - \underline{k}' - \underline{k}'') V(\underline{k}; \underline{k}', \underline{k}'') A(\underline{k}', t) A(\underline{k}'', t) \times \\ & \times e^{-i\{\omega(\underline{k}') + \omega(\underline{k}'')\}t}. \end{aligned} \quad (\text{II.27})$$

Eliminating the δ -function in eq. (II.27) we also obtain

$$\begin{aligned} & \frac{\partial A(\underline{k}, t)}{\partial t} = \\ = & -i \int_{-\infty}^{+\infty} d\underline{k}' V(\underline{k}; \underline{k}', \underline{k} - \underline{k}') A(\underline{k}', t) A(\underline{k} - \underline{k}', t) e^{-i\{\omega(\underline{k}') + \omega(\underline{k} - \underline{k}') - \omega(\underline{k})\}t}. \end{aligned} \quad (\text{II.28})$$

The amplitude of a particular $\underline{k}$ -mode thus proves to change in time due to a nonlinear interaction with all other modes $\underline{k}'$ and $(\underline{k} - \underline{k}')$, while the strength of their interaction is expressed by $V(\underline{k}; \underline{k}', \underline{k} - \underline{k}')$. Quite generally we remark that ω , which is the unique solution of a linear dispersion rela-

tion $\omega = \omega_0(k)$, represents a complex quantity

$$\omega(k) = \omega_R(k) - i\gamma(k). \quad (\text{II.29})$$

At this moment we assume $\gamma(k) > 0$ for all k , which means that we deal with damped modes (see (II.30)). Later on, we shall discuss under what conditions also unstable (growing) modes can be taken into account. We shall be able to treat also cases with $\gamma(k) = 0$ for all k , or also with $\omega_R(k) = 0$. The last case, for instance, holds for the Fourier transformed Burgers' equation used for hydrodynamical turbulence, which equation we shall discuss in chapter IX.

With the aid of (II.23) and (II.29) the only relevant physical quantity can be written as

$$\psi(x, t) = \int_{-\infty}^{+\infty} A(k, t) e^{ikx - i\omega_R(k)t - \gamma(k)t} dk, \quad (\text{II.30})$$

where the following reality conditions must be satisfied:

$$A(k, t) = A^*(-k, t), \quad (\text{II.31a})$$

$$\omega_R(k) = -\omega_R(-k), \quad (\text{II.31b})$$

$$\gamma(k) = \gamma(-k). \quad (\text{II.31c})$$

The model equation (II.28) clearly applies only to systems for which these conditions hold and of which (II.31b) is in particular important. However, in many cases this condition is not fulfilled (for example Langmuir plasma waves). In chapter VIII we shall discuss how to treat problems for which $\omega_R(k) = \omega_R(-k)$.

The symmetry property of eq. (II.28) and the conditions (II.31) immediately leads to the relations

$$V(k; k', k-k') = V(k; k-k', k') = -V^*(-k; -k', k'-k). \quad (\text{II.32})$$

We finally remark that the model equation (II.28) is very often used in plasma physics to describe different turbulent phenomena (see, for instance: Galeev and Karpman⁴), Galeev and Sagdeev¹⁴) and Karpman¹⁵). Similar equations have been investigated by Benney and Saffman⁵), Coppi et al.⁶), and other authors (see section VI.C). They all treated stable or weakly unstable modes, satisfying the condition $|\gamma(k)| \ll |\omega_R(k)|$ throughout for all k .

The turbulent system, as described by the model equation (II.28), with the associated conditions and assumptions, is now suitable for a further analysis in order to get the desired physical information.

C H A P T E R   I I I

WAVE-CORRELATION FUNCTIONS FOR THE ONE-DIMENSIONAL MODEL

A. Second-order wave-correlation function

In chapter II we already noticed that, for the general system we are looking at, all ensemble averages vanish. For the present one-dimensional turbulent model system we then have in particular

$$\langle \psi(x,t) \rangle = \langle \int_{-\infty}^{+\infty} \psi(k,t) e^{ikx} dk \rangle = \langle \int_{-\infty}^{+\infty} A(k,t) e^{ikx-i\omega(k)t} dk \rangle = 0, \quad (\text{III.1})$$

from the inverse of which it follows that

$$\langle \psi(k,t) \rangle = \langle A(k,t) \rangle = 0 \text{ for all } k. \quad (\text{III.2})$$

Hence, the averages of ψ and A vanish in both x - and k -space which means that we are dealing with random modes (waves).

Our next item concerns the consideration of the ensemble average of the product of two ψ 's, simply called a

correlation between two physical quantities, at the same time but at different places:

$$R(x, \xi, t) = \langle \psi(x, t) \psi(x + \xi, t) \rangle. \quad (\text{III.3})$$

We call this function R a second-order correlation function and ξ the correlation distance. Other nomenclatures of the function R in the literature are: second-order mean value, mean value of double correlations, second-order moment, second-order cumulant, etc.

Starting from the general (one-dimensional) equation (II.22) it is possible to find a dynamical equation for the function R . One then has to multiply eq. (II.22) by $\psi(x + \xi, t)$ and to add the same expression obtained after interchanging $\psi(x, t)$ by $\psi(x + \xi, t)$. Taking next the ensemble average over these two expressions, we arrive at the desired result. However, this way of studying turbulence turns out to be rather cumbersome; later on we shall observe that the Fourier transform of R is a better tool to deal with theoretically. Nevertheless, the knowledge of the function (III.3) is very useful in order to compare it with empirical values, which are in particular obtained with the hot-wire anemometer technique.

The definition given in (III.3) can easily be extended for two or three dimensions, involving different components of the relevant two- or three-dimensional physical quantity. In general, statistical mixed correlations for different physical quantities are needed to study the various properties of the turbulent medium; further, in some problems one has to know such correlations even at different times. Quite generally functions of this type are tensors.

We now suppose that R is invariant with respect to common translations of the points x and $x + \xi$. This means the following: if, in a turbulent medium, we could measure R at a fixed time and for a sufficiently large number of pairs of points x and $x + \xi$ (ξ fixed), we would find a value for R , which is independent of the positions x of the points of observation. This situation is called homogeneous turbulence. For many practical cases, however, this is a rather strong

assumption, even in lowest approximation. We use this property of homogeneity merely as a mathematical idealization rather than as an agreement with a real physical situation. Thus we require that

$$R(x, \xi, t) \equiv R(\xi, t). \quad (\text{III.4})$$

It then proves to be useful to introduce also the Fourier transform for this second-order correlation function. Consequently we define

$$R(\xi, t) = \langle \psi(x, t) \psi(x + \xi, t) \rangle = \int_{-\infty}^{+\infty} E(k, t) e^{ik\xi} dk, \quad (\text{III.5})$$

where $E(k, t)$ may be called a second-order wave-correlation function, a second-order spectrum function, or a spectral density of the randomly fluctuating quantity $\psi(x, t)$. The usefulness of this latter function can be seen from the following example. Let $v(x, t)$ be the turbulent velocity of a one-dimensional fluid (see also section IX.C). Then the second-order correlation function reads

$$R(\xi, t) = \int_{-\infty}^{+\infty} E(k, t) e^{ik\xi} dk = \langle v(x, t) v(x + \xi, t) \rangle, \quad (\text{III.6})$$

so that in particular

$$\frac{1}{2} R(0, t) = \frac{1}{2} \int_{-\infty}^{+\infty} E(k, t) dk = \frac{1}{2} \langle v(x, t)^2 \rangle. \quad (\text{III.7})$$

One could observe that $\frac{1}{2} R(0, t)$ describes the turbulent kinetic energy per unit of mass density! In this case $E(k, t)$ is called the energy-spectrum function and expresses an energy density in k -space. It turns out that this function can be calculated much easier than the original $R(\xi, t)$ because $E(k, t)$ proves to be connected with the second-order wave correlation (see appendix B) according to:

$$\langle \psi(k_1, t) \psi(k_2, t) \rangle = \langle \psi(k_1, t) \psi^*(-k_2, t) \rangle = E(k_1, t) \delta(k_1 + k_2). \quad (\text{III.8})$$

This property, well known as the random phase approximation (refs. 4,6,7,16,etc.), is essentially a consequence of the assumed homogeneity of the turbulence. With the aid of eqs. (II.23), (II.24), and (II.31a,b,c) the quantity (III.8) can also be written as follows, taking into account its proportionality to $\delta(k_1 + k_2)$ so that k_2 may be replaced by $-k_1$ outside the delta-function:

$$\begin{aligned} \langle \psi(k_1, t) \psi(k_2, t) \rangle &= \\ &= \langle A(k_1, t) A(k_2, t) \rangle e^{-i\omega_R(k_1)t - \gamma(k_1)t - i\omega_R(k_2)t - \gamma(k_2)t} = \\ &= \langle A(k_1, t) A(k_2, t) \rangle e^{-2\gamma(k_1)t} = \langle A(k_1, t) A^*(-k_2, t) \rangle e^{-2\gamma(k_1)t} = \\ &= E(k_1, t) \delta(k_1 + k_2). \end{aligned} \quad (\text{III.9})$$

This relation suggests to define a new wave-correlation function G , directly connected with the second-order wave correlation E by

$$\langle A(k_1, t) A(k_2, t) \rangle = G(k_1, t) \delta(k_1 + k_2), \quad (\text{III.10})$$

which then involves

$$E(k, t) = G(k, t) e^{-2\gamma(k)t}. \quad (\text{III.11})$$

In the next section we shall see that, starting from a dynamical equation for $\langle A(k_1, t) A(k_2, t) \rangle$ and using the model equation (II.28), we are able to calculate $G(k, t)$ and hence $E(k, t)$ in view of (III.11). Knowing the spectrum of the turbulence expressed by $E(k, t)$ one can then find the correlation function $R(\xi, t)$, from which we get the necessary physical information.

Some useful properties of the second-order spectrum function $E(k, t)$ (and $G(k, t)$) are:

1) the even character according to

$$E(k, t) = E(-k, t), \quad (\text{III.12})$$

which follows from eq. (III.8), yielding

$$E(k_1, t) \delta(k_1 + k_2) = E(k_2, t) \delta(k_1 + k_2) = E(-k_1, t) \delta(k_1 + k_2),$$

2) $E(k, t)$ and $G(k, t)$ are real quantities. This is easily verified from eqs. (III.10, 11),

$$3) E(k, t) \text{ and } G(k, t) \geq 0 \text{ for all } k; \quad (\text{III.13})$$

this follows from the fact that the wave correlations in eq. (III.9) only differ from zero if $k_1 \sim -k_2$, so that $E(k, t)$ represents, if we drop the δ -function, the squared fluctuation amplitude. Therefore, in general, we call $E(k, t)$ the energy-spectrum function for waves (modes) of wavenumber k .

B. Third-order wave-correlation function

We now introduce the third-order correlation function defined by

$$\begin{aligned} U(\xi, \xi', t) &= \langle \psi(x, t) \psi(x + \xi, t) \psi(x + \xi', t) \rangle = \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(k_1, k_2, t) e^{ik_1 \xi + ik_2 \xi'} dk_1 dk_2, \quad (\text{III.14}) \end{aligned}$$

where $U(\xi, \xi', t)$, too, is supposed to be independent of x (homogeneous turbulence). Then, in view of the relation (II.23), involving $\psi(k, t) = A(k, t) e^{-i\omega(k)t}$, we find

$$\begin{aligned} & \langle \psi(k_1, t) \psi(k_2, t) \psi(k_3, t) \rangle = \\ & = \langle A(k_1, t) A(k_2, t) A(k_3, t) \rangle e^{-i\{\omega(k_1) + \omega(k_2) + \omega(k_3)\}t}. \end{aligned} \quad (\text{III.15})$$

Moreover, applying the inverse Fourier transforms

$$\psi(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \psi(x, t) e^{-ikx} \text{ for } k = k_1, k_2, k_3,$$

we also derive the following expression by averaging, with the aid of eq. (III.14):

$$\langle \psi(k_1, t) \psi(k_2, t) \psi(k_3, t) \rangle = S(k_1, k_2, t) \delta(k_1 + k_2 + k_3), \quad (\text{III.15a})$$

which can be compared with eq. (III.8) in the second-order case. We then may define a third-order wave-correlation function H according to

$$\langle A(k_1, t) A(k_2, t) A(k_3, t) \rangle = H(k_1, k_2, t) \delta(k_1 + k_2 + k_3), \quad (\text{III.16})$$

where

$$H(k_1, k_2, t) = S(k_1, k_2, t) e^{i\{\omega(k_1) + \omega(k_2) + \omega(k_3)\}t}. \quad (\text{III.16a})$$

Three useful properties are:

$$1. H^*(k_1, k_2, t) = H(-k_1, -k_2, t), \quad (\text{III.17})$$

which can easily be derived from eq. (III.16), applying eq. (II.31a).

2. The function $H(k_1, k_2, t)$ is invariant to all permutations of k_1, k_2 , and k_3 :

$$H(k_1, k_2, t) = H(k_3, k_1, t) = \dots \quad (\text{III.18})$$

This follows from the symmetrical dependence on k_1, k_2 , and k_3 of the left-hand side of eq. (III.16).

3. Under special circumstances, $H(k_1, k_2, t)$ vanishes identically, in particular in the case of Gaussian joint probability functions (see the property 4) at the end of the next section).

C. Fourth-order wave-correlation function

Concerning this wave-correlation function we use the fact that the four-point correlation function

$$\langle \psi(x, t) \psi(x+\xi, t) \psi(x+\xi', t) \psi(x+\xi'', t) \rangle$$

does, in general, not have a nondegenerating Fourier transform¹⁷⁾. In order to get such a transform we need the following function (the so-called 4th-order correlation function), containing a difference:

$$\begin{aligned} K(\xi, \xi', \xi'', t) &= \langle \psi(x, t) \psi(x+\xi, t) \psi(x+\xi', t) \psi(x+\xi'', t) \rangle - \\ &- \langle \psi(x, t) \psi(x+\xi, t) \rangle \langle \psi(x+\xi', t) \psi(x+\xi'', t) \rangle - \\ &- \langle \psi(x, t) \psi(x+\xi', t) \rangle \langle \psi(x+\xi, t) \psi(x+\xi'', t) \rangle - \\ &- \langle \psi(x, t) \psi(x+\xi'', t) \rangle \langle \psi(x+\xi, t) \psi(x+\xi', t) \rangle, \end{aligned} \quad (\text{III.19})$$

the Fourier transform L of which may be defined by

$$\begin{aligned} K(\xi, \xi', \xi'', t) &= \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{ik_1 \xi + ik_2 \xi' + ik_3 \xi''} L(k_1, k_2, k_3, t) dk_1 dk_2 dk_3. \end{aligned} \quad (\text{III.20})$$

By substituting the expressions

$$\psi(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \psi(x, t) e^{-ikx}$$

it is then easy to prove that

$$\begin{aligned} & \langle \psi(k_1, t) \psi(k_2, t) \psi(k_3, t) \psi(k_4, t) \rangle - \\ & - \langle \psi(k_1, t) \psi(k_2, t) \rangle \langle \psi(k_3, t) \psi(k_4, t) \rangle - \\ & - \langle \psi(k_1, t) \psi(k_3, t) \rangle \langle \psi(k_2, t) \psi(k_4, t) \rangle - \\ & - \langle \psi(k_1, t) \psi(k_4, t) \rangle \langle \psi(k_2, t) \psi(k_3, t) \rangle = \\ & = L(k_1, k_2, k_3, t) \delta(k_1 + k_2 + k_3 + k_4). \end{aligned} \quad (\text{III.21})$$

An application of (III.8) leads to the alternative form:

$$\begin{aligned} & \langle \psi(k_1, t) \psi(k_2, t) \psi(k_3, t) \psi(k_4, t) \rangle - \\ & - E(k_1, t) E(k_3, t) \delta(k_1 + k_2) \delta(k_3 + k_4) - \\ & - E(k_1, t) E(k_2, t) \delta(k_1 + k_3) \delta(k_2 + k_4) - \\ & - E(k_1, t) E(k_2, t) \delta(k_1 + k_4) \delta(k_2 + k_3) = \\ & = L(k_1, k_2, k_3, t) \delta(k_1 + k_2 + k_3 + k_4). \end{aligned}$$

Passing from $\psi(k, t)$ to $A(k, t) = \psi(k, t) e^{i\omega(k)t}$ and from $E(k, t)$ to $G(k, t) = E(k, t) e^{2\gamma(k)t}$, one finds, applying (II.29), (II.31b), and (II.31c), the following expression for the fourth-order wave-correlation function

$$\begin{aligned} & \langle A(k_1, t) A(k_2, t) A(k_3, t) A(k_4, t) \rangle = \\ & = F(k_1, k_2, k_3, t) \delta(k_1 + k_2 + k_3 + k_4) + \end{aligned}$$

$$\begin{aligned}
& + G(k_1)G(k_4)\delta(k_1+k_2)\delta(k_3+k_4)+G(k_1)G(k_2)\delta(k_1+k_3)\delta(k_2+k_4) + \\
& + G(k_1)G(k_3)\delta(k_1+k_4)\delta(k_2+k_3), \\
\end{aligned}
\tag{III.22}$$

in which

$$F(k_1, k_2, k_3, t) = L(k_1, k_2, k_3, t) e^{i\{\omega(k_1)+\omega(k_2)+\omega(k_3)+\omega(k_4)\}t}.
\tag{III.22a}$$

Properties of this fourth-order wave-correlation function F are

$$1) F^*(k_1, k_2, k_3, t) = F(-k_1, -k_2, -k_3, t)
\tag{III.23}$$

as may be verified from eq. (III.22), remembering (II.31a) and taking into account that the $G(k)$ are real even functions (see (III.12) and (II.31c)).

2) The wave-correlation functions $F(k_1, k_2, k_3, t)$ and $L(k_1, k_2, k_3, t)$ are again invariant to all permutations of k_i ($i = 1, 2, 3, 4$).

3) The fourth-order wave correlation (the left-hand side of eq. (III.22)) consists of the sum of a fourth-order wave-correlation function and of the three possible products of two second-order correlation functions.

4) The function $F(k_1, k_2, k_3, t)$ will vanish in special circumstances as can be explained as follows. Another way to explore our problem of ensemble averages of products of fluctuations in x -space is used in the method of statistics. There, an n -point joint probability distribution function is associated with each set of $\{\psi(x_i, t)\}$ with $i = 1, 2, 3, \dots$ and x_i all different. If these distribution functions were Gaussian (or normal) then all correlation functions, when defined as in (III.19) (that is, subtracting from the average of the n -point distribution the averages of all products in which the n points occur, correlated in smaller groups), vanish if

their order is higher than the second one. In the case of three points the function $U(\xi, \xi', t)$ of (III.14) then disappears and, consequently, $S(k_1, k_2, t)$ and $H(k_1, k_2, t)$. Also $K(\xi, \xi', \xi'', t)$ in eq. (III.19) would vanish, the fourth-order correlation then becoming related to the second-order one. In hydrodynamical turbulence this is called the cumulant discard hypothesis¹⁰⁾ or zero fourth-order cumulant hypothesis, first introduced by Millionshtchikov¹¹⁾. Translated into k -space, the Gaussian property involves that also $L(k_1, k_2, k_3, t)$ and $F(k_1, k_2, k_3, t)$ would vanish in that case.

All above considerations can be extended. Correlations of higher order will generally be represented as a sum of products of correlation functions. Further, one can introduce different physical quantities and more dimensions; the requirement of isotropic turbulence is needed throughout in order to get reasonable results.

CHAPTER IV

DERIVATION OF THE HIERARCHY FOR THE ONE-DIMENSIONAL MODEL

We return to our model equation (II.28), reading

$$\frac{\partial A(k,t)}{\partial t} = -i \int_{-\infty}^{+\infty} dk' V(k;k',k-k') A(k',t) A(k-k',t) e^{-i\{\omega(k')+\omega(k-k')-\omega(k)\}t} \quad (IV.1)$$

In view of eq. (II.25) we have

$$A(k=0,t) = 0, \quad (IV.2)$$

which implies

$$V(0;k',-k') = 0, \quad (IV.3)$$

since the assumption (II.9) amounts to $h(k=0,t) = 0$ for the linearized problem (see (II.26a)).

We already remarked that, in order to gain some physical information from eq. (IV.1), it seems to be necessary to calculate the second- and higher-order wave correlations dynamical-

ly, starting from that very equation. Therefore, we first derive

$$\begin{aligned} \frac{\partial}{\partial t} [A(k_1, t)A(k_2, t)] &= \left\{ \frac{\partial A(k_1, t)}{\partial t} A(k_2, t) \right\} + \{k_1 \neq k_2\} = \\ &= \left\{ -i \int_{-\infty}^{+\infty} dk' V(k_1; k', k_1 - k') A(k', t) A(k_1 - k', t) A(k_2, t) \times \right. \\ &\quad \left. \times e^{-i\{\omega(k') + \omega(k_1 - k') - \omega(k_1)\}t} \right\} + \{k_1 \neq k_2\}. \end{aligned} \quad (IV.4a)$$

In a similar way we find

$$\begin{aligned} \frac{\partial}{\partial t} [A(k_1, t)A(k_2, t)A(k_3, t)] &= \\ &= \left\{ -i \int_{-\infty}^{+\infty} dk_0 V(k_1; k_0, k_1 - k_0) A(k_0, t) A(k_1 - k_0, t) A(k_2, t) A(k_3, t) \times \right. \\ &\quad \left. \times e^{-i\{\omega(k_0) + \omega(k_1 - k_0) - \omega(k_1)\}t} \right\} + \{k_1 \neq k_2\} + \{k_1 \neq k_3\}. \end{aligned} \quad (IV.4b)$$

This procedure can be continued infinitely, but further equations are irrelevant for the present analysis.

Before introducing a perturbation method we immediately average all these equations over an ensemble in order to remove superfluous information. We repeat some formulae, obtained in the preceding chapter, which facilitate the averaging in question; for a short-hand notation we shall not mention the time dependence of the relevant quantities.

$$\langle A(k_1) \rangle = 0 \text{ for all } k_1 \text{ and } t, \quad (IV.5a)$$

$$\langle A(k_1)A(k_2) \rangle = G(k_1)\delta(k_1 + k_2), \quad (IV.5b)$$

$$\text{with } G(k_1) = G(-k_1) = G(k_2),$$

$$\langle A(k_1)A(k_2)A(k_3) \rangle = H(k_1, k_2)\delta(k_1 + k_2 + k_3), \quad (IV.5c)$$

with $H(k_1, k_2) = H(k_3, k_1) = \dots$ (invariance for permutations of k_i ($i = 1, 2, 3$)),

$$\begin{aligned} \langle A(k_1)A(k_2)A(k_3)A(k_4) \rangle &= F(k_1, k_2, k_3) \delta(k_1 + k_2 + k_3 + k_4) + \\ &+ G(k_1)G(k_4) \delta(k_1 + k_2) \delta(k_3 + k_4) + \\ &+ G(k_1)G(k_2) \delta(k_1 + k_3) \delta(k_2 + k_4) + \\ &+ G(k_1)G(k_3) \delta(k_1 + k_4) \delta(k_2 + k_3), \end{aligned} \quad (IV.5d)$$

($F(k_1, k_2, k_3)$ being invariant for permutations of k_i ($i = 1, 2, 3, 4$)).

We take the ensemble average of eq. (IV.1). With the aid of eqs. (IV.2, 3) for $k = 0$ and (IV.5a, b) for $k \neq 0$, both sides of the model equation then prove to vanish identically. This result is quite natural because this equation has been derived for a system containing only fluctuating quantities (see chapter II). Subsequently, the ensemble average of eq. (IV.4a) results in the following equation, applying the formulae (IV.5b, c):

$$\begin{aligned} \delta(k_1 + k_2) \frac{\partial G(k_1)}{\partial t} &= \left\{ -i \int_{-\infty}^{+\infty} dk' V(k_1; k', k_1 - k') H(k', k_2) \delta(k_1 + k_2) \times \right. \\ &\times e^{-i\{\omega(k') + \omega(k_1 - k') - \omega(k_1)\}t} \left. \right\} + \left\{ k_1 \neq k_2 \right\}. \end{aligned} \quad (IV.6)$$

Similarly, a dynamical equation for $H(k_1, k_2)$ is obtained from the ensemble average of eq. (IV.4b). With the aid of eqs. (IV.3) and (IV.5c, d) and the properties of invariance, one gets:

$$\begin{aligned} \delta(k_1 + k_2 + k_3) \frac{\partial H(k_1, k_2)}{\partial t} &= -i \delta(k_1 + k_2 + k_3) \times \\ &\times \left\{ V(k_1; -k_2, k_1 + k_2) G(k_2) G(k_3) e^{-i\{\omega(-k_2) + \omega(k_1 + k_2) - \omega(k_1)\}t} + \right. \end{aligned}$$

$$\begin{aligned}
& + V(k_1; -k_3, k_1+k_3) G(k_2) G(k_3) e^{-i\{\omega(-k_3)+\omega(k_1+k_3)-\omega(k_1)\}t} + \\
& + \int_{-\infty}^{+\infty} dk_0 V(k_1; k_0, k_1-k_0) F(k_0, k_2, k_3) e^{-i\{\omega(k_0)+\omega(k_1-k_0)-\omega(k_1)\}t} \Big\} - \\
& - i\delta(k_1+k_2+k_3) \left\{ k_1 \neq k_2 \right\} - i\delta(k_1+k_2+k_3) \left\{ k_2 \neq k_3 \right\} . \quad (IV.7)
\end{aligned}$$

If wanted, dynamical equations for higher-order wave-correlation functions can be derived in the same way.

We next integrate both sides of eq. (IV.6) over k_2 . Replacing then k_1 by k , we arrive at

$$\begin{aligned}
& \frac{\partial G(k)}{\partial t} = \\
& = \left\{ -i \int_{-\infty}^{+\infty} dk' V(k; k', k-k') H(k', -k) e^{-i\{\omega(k')+\omega(k-k')-\omega(k)\}t} \right\} + \\
& + \left\{ k \rightarrow -k \right\} . \quad (IV.8a)
\end{aligned}$$

To obtain an equation for the wave-correlation function $H(k', -k)$ which appears in eq. (IV.8a), we finally integrate the equation (IV.7) over k_3 , while replacing k_1 by k' and k_2 by $-k$:

$$\begin{aligned}
& \frac{\partial H(k', -k)}{\partial t} = \\
& = -2iV(k'; k, k'-k) e^{-i\{\omega(k)+\omega(k'-k)-\omega(k')\}t} G(k) G(k-k') - \\
& - 2iV(-k; -k', k'-k) e^{-i\{\omega(-k')+\omega(k'-k)-\omega(-k)\}t} G(k') G(k-k') - \\
& - 2iV(k-k'; k, -k') e^{-i\{\omega(k)+\omega(-k')-\omega(k-k')\}t} G(k) G(k') - \\
& - i \int_{-\infty}^{+\infty} dk_0 V(k'; k_0, k'-k_0) F(k_0, -k, k-k') e^{-i\{\omega(k_0)+\omega(k'-k_0)-\omega(k')\}t} -
\end{aligned}$$

$$\begin{aligned}
& -i \int_{-\infty}^{+\infty} dk_0 V(-k; k_0, -k-k_0) F(k_0, k', k-k') e^{-i\{\omega(k_0) + \omega(-k-k_0) - \omega(k)\}t} \\
& -i \int_{-\infty}^{+\infty} dk_0 V(k-k'; k_0, k-k'-k_0) F(k_0, -k, k') \times \\
& \quad \times e^{-i\{\omega(k_0) + \omega(k-k'-k_0) - \omega(k-k')\}t}
\end{aligned} \tag{IV.8b}$$

It can be easily checked that the required invariance of $H(k', -k)$ with respect to all permutations of the elements k' , $-k$ and $k-k'$ also holds for the sum of the first three and that of the last three terms in eq. (IV.8b).

We can continue the procedure, here explained for the lowest orders, when deriving a dynamical equation connecting the time derivative of some average in the k -space with an average of one order higher (that is, referring to products with an additional factor). Next we then obtain a dynamical equation for F and, consequently, for higher-order wave-correlation functions. The whole system we should get in this way is called a hierarchy. It consists of an infinite number of equations (IV.8). From the structure of the first equation of the hierarchy, that is (IV.8a), we observe that the second-order wave-correlation function G is connected with the third-order one H ; from eq. (IV.8b), the second one of the hierarchy, we notice that the function H in its turn is connected with fourth-order wave-correlation functions, etc. Quite generally: due to the nonlinearity of our model equation (IV.1), an n^{th} -order wave-correlation function becomes dynamically connected with an $(n+1)^{\text{th}}$ -order one (n equaling 2, 3, 4, ...).

All physical information is contained in this hierarchy. First of all we are interested in the second-order wave-correlation function $G(k, t)$ because it enables the investigation of energy properties (see chapter III). To know this function $G(k, t)$ exactly one has to solve an infinite number of nonlinearly coupled integro-differential equa-

tions, with an infinite number of initial conditions! Obviously, this is an impossible task to perform. We are faced here with the basic problem of studying turbulence with the aid of analytical statistical methods: how to find a both mathematically and physically suitable approximation method to derive an expression only for $G(k)$ (and, hence, $E(k)$) from the hierarchy (IV.8). If this method is found, we are able to understand some relevant physical quantities and properties of turbulence (energy transfer between the fluctuations for example). From the above given considerations it becomes clear that it is desirable to close the hierarchy. We then only have to look at a finite number of equations while knowing which effects we want to neglect. A justification for such a closure can be found as follows. In chapter III we already mentioned that all wave-correlation functions are defined such as to vanish, with the exception of the second-order one, if one has to deal with Gaussian (or normal) probability distributions. In practice, however, it appears that this assumption of "Gaussianity" can not be quite correct because an exact Gaussian distribution would involve a vanishing of the third-order (wave) correlation function H , occurring in the first equation of the hierarchy; any interaction would then be excluded and the problem would be linear! On the other hand, in the theory of statistics it is proved that under quite general conditions an arbitrary probability distribution tends to a normal or Gaussian one! This information suggests that we shall consider nearly normal (or quasi-normal) distribution functions.

In view of these remarks, leading to the desirability to close the system, and the possibility of dealing with a quasi-normal distribution, we shall require:

$$\text{2nd-order wave-correlation function } G \sim O(\varepsilon), \quad (\text{IV.9a})$$

$$\text{3rd-order wave-correlation function } H \sim O(\varepsilon^2), \quad (\text{IV.9b})$$

$$\text{4th-order wave-correlation function } F \sim O(\varepsilon^3), \quad (\text{IV.9c})$$

etc., where $\varepsilon < 1$.

Considering the hierarchy, it turns out that we have to take $G \sim O(\epsilon)$ and not $G \sim O(1)$ for instance because, otherwise, higher-order correlations, to be compared with powers of G , would also become of order unity. Hence, the fluctuations must be small, but large enough to allow non-linear interactions. This situation is defined as weak turbulence. Until now, the introduction of the assumptions (IV.9) seems the only practical hypothesis to treat the problem, when starting with a hierarchy formalism. The scheme (IV.9) enables to close the hierarchy of equations by neglecting terms of $O(\epsilon^{n+1})$; this results in a finite set of n equations for n unknown functions.

Attentive readers will immediately observe a striking similarity between the above analysis and the field of statistical kinetic theory, where one is confronted with the same problem. Indeed, our hierarchy (IV.8) has the same structure as the BBGKY-hierarchy (see refs. 9 and 19, for instance). Furthermore, the set of wave-correlation functions (IV.5) closely resembles the well-known cluster expansion (ref. 19, for instance). This is the reason why we intend to use the ideas of Bogoliubov⁹), who succeeded in deriving the Boltzmann and Landau kinetic equations from the BBGKY-hierarchy. With an assumption like (IV.9) some authors have obtained a kinetic equation for a plasma, viz. the Balescu-Lenard-Bogoliubov equation¹⁹). The similarity should have been complete if we had averaged quantities of $O(1)$.

C H A P T E R V

THE BOGOLIUBOV EXPANSION METHOD AND THE DERIVATION OF A GENERAL KINETIC EQUATION

A. Determination of the first nonlinear correction term

In this section we will discuss a new approximation method based on the mentioned Bogoliubov expansion technique. In earlier work²⁰⁾ we have first applied this method straightforwardly to the problem of hydrodynamic turbulence, dealing with Burgers' equation (see chapter IX), while later work refers to magnetohydrodynamic turbulence (see chapter X). In order to investigate all mathematical and physical consequences and to emphasize future possibilities we want to develop the theory along general lines. We postulate that the energy-spectrum function $E(k, t)$ should satisfy the following formal kinetic equation

$$\frac{\partial E(k, t)}{\partial t} = \sum_{n=0}^{\infty} \epsilon^n A_n(k; E) = A_0(k; E) + \epsilon A_1(k; E) + \epsilon^2 A_2(k; E) + \dots, \quad (V.1)$$

which replaces the hierarchy (IV.8) after eliminating all quantities but one. The time dependence of the right-hand side only appears through $E(k,t)$. Hence, the functions A_i ($i = 0, 1, 2, \dots$) are called functionals. Moreover, giving an initial condition $E(k, t=0)$ we would, in principle, be able to solve from eq. (V.1) the energy-spectrum function for each k at each time.

The following fundamental question now arises: how to construct a formal kinetic equation such as (V.1) from the hierarchy (IV.8). If we neglect all terms of $O(\epsilon^{n+1})$ in the hierarchy we must still choose $(n-1)$ appropriate initial conditions for the wave-correlation functions of j^{th} -order ($j = 3, 4, \dots, n+1$) in order to find an equation for $E(k)$ only! We can satisfy this latter requirement by assuming that the wave-correlation functions H, F , etc., initially (at $t=0$) only depend, in some way or other, on the energy-spectrum function $E(k, t=0)$ ($=G(k, t=0)$) at the same moment. This is expressed by representing these functions by

$$H(k_1, k_2; E(k_i, t=0)), F(k_1, k_2, k_3; E(k_i, t=0)), \dots \quad (V.2)$$

instead of $H(k_1, k_2; t=0)$, $F(k_1, k_2, k_3; t=0)$, ... (see appendix C for further explanation).

This synchronization hypothesis seems a reasonable choice, although one knows only a posteriori that it has been a good one. Later on we shall observe that, for instance, the conditions (IV.9) are satisfied (see eq. (V.14)). A weakly turbulent state also has to be assumed for eq. (V.1) (see (IV.9a)), i.e. $G(k) \sim E(k) \sim O(\epsilon)$. Therefore, we take the coefficients of the initial expansion (V.1) in the following way:

$$A_0 = 0 \text{ and } A_i \sim O(1) \text{ for all } i \text{ (} i = 1, 2, 3, \dots \text{)}. \quad (V.3)$$

For weakly turbulent systems we then have to start with

$$\frac{\partial E(k, t)}{\partial t} = \epsilon A_1(k; E) + \epsilon^2 A_2(k; E) + \dots \quad (V.4)$$

Because $\partial E/\partial t$ constitutes a power series with respect to ε we further introduce the following expansions, each first term of which is in agreement with (IV.9):

$$G = \varepsilon G^{(1)}, \quad (V.5a)$$

$$H = \varepsilon^2 H^{(2)} + \varepsilon^3 H^{(3)} + \dots \text{ with } H^{(i)} \sim O(1) \quad (i = 2, 3, \dots), \quad (V.5b)$$

$$F = \varepsilon^3 F^{(3)} + \varepsilon^4 F^{(4)} + \dots \text{ with } F^{(i)} \sim O(1) \quad (i = 3, 4, \dots). \quad (V.5c)$$

Obviously, all these "new" wave-correlation functions $H^{(i)}$, $F^{(i)}$, ... have to satisfy the same initial conditions chosen above and expressed by (V.2).

From the beginning of our analysis we supposed that a linear damping was present in the turbulent medium, given by

$$E^{(1)}(k, t) = G^{(1)}(k, t) e^{-2\gamma(k)t}, \quad (V.6)$$

in accordance with $E(k, t) = G(k, t) e^{-2\gamma(k)t}$.

In the hierarchy (IV.8) we only had to deal with the function $G(k, t)$, but in eq. (V.4) just with the other one $E(k, t)$. Therefore, we need a dynamical connection between both these functions, which trivially results from the relation:

$$\begin{aligned} \frac{\partial E^{(1)}(k, t)}{\partial t} &= \frac{\partial G^{(1)}(k, t)}{\partial t} e^{-2\gamma(k)t} - 2\gamma(k) G^{(1)}(k) e^{-2\gamma(k)t} = \\ &= A_1(k; E) + \varepsilon A_2(k; E) + \dots \end{aligned} \quad (V.7)$$

In order to facilitate the calculations we, moreover, introduce a formal expansion for $G^{(1)}(k, t)$:

$$\frac{\partial G^{(1)}(k, t)}{\partial t} = B_1(k) + \varepsilon B_2(k) + \dots, \quad (V.8)$$

with $B_i \sim O(1)$ for all i ($i = 1, 2, 3, \dots$).

As a matter of fact, this last expansion is only needed if the system consists of damped (possibly growing) modes. Substituting these expansions into eq. (V.7) we obtain the following relations by comparing each order of ϵ :

$$A_1(k; E) = -2\gamma(k)G^{(1)}(k, t)e^{-2\gamma(k)t} + B_1(k)e^{-2\gamma(k)t}, \quad (V.9a)$$

$$A_2(k; E) = B_2(k)e^{-2\gamma(k)t}, \quad (V.9b)$$

etc.

At a first glance some definitions in this section may appear to be somewhat artificial but a careful comparison with the hierarchy (IV.8) shows their justification.

Provided with all these additional definitions and assumptions for the relevant quantities of the hierarchy (IV.8) we can start our investigation for the time evolution of the system. Its first equation (IV.8a) is transformed, with the aid of eqs. (V.5b) and (V.8), into

$$\begin{aligned} \frac{\partial G^{(1)}(k)}{\partial t} &= B_1(k) + \epsilon B_2(k) + \dots = \\ &= \left\{ -i \int_{-\infty}^{+\infty} dk' V(k; k', k-k') [\epsilon H^{(2)}(k', -k) + \epsilon^2 H^{(3)}(k', -k) + \dots] \times \right. \\ &\quad \left. \times e^{-i\{\omega(k') + \omega(k-k') - \omega(k)\}t} \right\} + \{k \rightarrow -k\}. \end{aligned} \quad (V.10a)$$

The second equation of the hierarchy (IV.8b) gives

$$\begin{aligned} \epsilon^2 \frac{\partial H^{(2)}(k', -k)}{\partial t} + \epsilon^3 \frac{\partial H^{(3)}(k', -k)}{\partial t} + \dots = \\ = -2iV(k; k, k-k)\epsilon^2 G^{(1)}(k)G^{(1)}(k-k')e^{-i\{\omega(k) + \omega(k-k) - \omega(k')\}t} - \\ -2iV(-k; -k', k'-k)\epsilon^2 G^{(1)}(k')G^{(1)}(k-k')e^{-i\{\omega(-k') + \omega(k'-k) - \omega(-k)\}t} - \\ -2iV(k-k'; k, -k')\epsilon^2 G^{(1)}(k)G^{(1)}(k')e^{-i\{\omega(k) + \omega(-k') - \omega(k-k')\}t} + \end{aligned}$$

$$+ \{ \text{terms with } \varepsilon^3 F^{(3)} + \varepsilon^4 F^{(4)} + \dots \}, \quad (\text{V.10b})$$

and so on.

The next step is to compare the contributions of the various orders of ε for each equation (V.10). For $O(1)$ equation (V.10a) reduces to

$$B_1(k) = \left. \frac{\partial G^{(1)}}{\partial t} \right|_{O(1)} = 0. \quad (\text{V.11})$$

This means that, up to order 1, the wave-correlation function $G^{(1)}(k)$ does not vary in time. In physical terms, this function does not change yet due to the nonlinear interaction between all k -modes. Considering $O(\varepsilon)$ of eq. (V.10a), we get

$$\begin{aligned} & \left. \frac{\partial G^{(1)}}{\partial t} \right|_{O(\varepsilon)} = \varepsilon B_2(k) = \\ & = \left\{ -i \int_{-\infty}^{+\infty} dk' V(k; k', k-k') \varepsilon H^{(2)}(k', -k) \times \right. \\ & \times e^{-i\{\omega(k') + \omega(k-k') - \omega(k)\}t} \left. \right\} + \{k + -k\}. \end{aligned} \quad (\text{V.12})$$

In order to solve this last equation we need an expression for the time-dependent function $H^{(2)}(k', -k)$. The latter is found by determining the contribution of order ε^2 of eq. (V.10b):

$$\begin{aligned} & \left. \frac{\partial H^{(2)}(k', -k)}{\partial t} \right| = \\ & = -2iV(k'; k, k'-k) G^{(1)}(k) G^{(1)}(k-k') e^{-i\{\omega(k) + \omega(k'-k) - \omega(k')\}t} - \\ & - 2iV(-k; -k', k'-k) G^{(1)}(k') G^{(1)}(k-k') e^{-i\{\omega(-k') + \omega(k'-k) - \omega(-k)\}t} - \\ & - 2iV(k-k'; k, -k') G^{(1)}(k) G^{(1)}(k') e^{-i\{\omega(k) + \omega(-k') - \omega(k-k')\}t}. \end{aligned} \quad (\text{V.13})$$

This equation can be solved because the function $G^{(1)}(k)$ proved to be time-independent in the lowest-order approximation (see eq. (V.11)). Hence, one can immediately integrate (V.13), taking into account the assumed initial condition $H^{(2)}(k', -k, t=0) = H^{(2)}(k', -k; G(t=0))$.

This yields

$$\begin{aligned}
 H^{(2)}(k', -k, t) = & 2V(k'; k, k'-k)G^{(1)}(k)G^{(1)}(k-k') \times \\
 & \times \frac{e^{-i\{\omega(k)+\omega(k'-k)-\omega(k')\}t}}{\{\omega(k)+\omega(k'-k)-\omega(k')\}} + \\
 & + 2V(-k'; -k', k'-k)G^{(1)}(k')G^{(1)}(k-k') \times \\
 & \times \frac{e^{-i\{\omega(-k')+\omega(k'-k)-\omega(-k)\}t}}{\{\omega(-k')+\omega(k'-k)-\omega(-k)\}} + \\
 & + 2V(k-k'; k, -k')G^{(1)}(k)G^{(1)}(k') \times \\
 & \times \frac{e^{-i\{\omega(k)+\omega(-k')-\omega(k-k')\}t}}{\{\omega(k)+\omega(-k')-\omega(k-k')\}}. \quad (V.14)
 \end{aligned}$$

Substituting this result into (V.12) we obtain the following expression for $B_2(k)$:

$$\begin{aligned}
 B_2(k) = & \left\{ -2i \int_{-\infty}^{+\infty} dk' V(k; k', k-k') e^{-i\{\omega(k')+\omega(k-k')-\omega(k)\}t} \times \right. \\
 & \times \left[\left\{ V(k'; k, k'-k)G^{(1)}(k)G^{(1)}(k-k') \frac{e^{-i\{\omega(k)+\omega(k'-k)-\omega(k')\}t}}{\{\omega(k)+\omega(k'-k)-\omega(k')\}} \right\} + \right. \\
 & \left. \left. + \left\{ k' \neq -k \right\} + \left\{ k' \neq k-k' \right\} \right] \right\} + \left\{ k \rightarrow -k \right\}. \quad (V.15)
 \end{aligned}$$

B. Derivation of the lowest-order kinetic equation

At this stage we have found the first nonlinear correction term for $\partial G^{(1)}/\partial t$. We shall not continue our analysis calculating higher-order terms but, instead, derive more lucid expressions for the results obtained so far. It makes sense to define

$$W(k_3; k_1, k_2) = \omega(k_3) - \omega(k_1) - \omega(k_2), \quad (V.16)$$

so that eq. (V.15) can be represented by

$$\begin{aligned} B_2(k) = & \left\{ 2i \int_{-\infty}^{+\infty} dk' V(k; k', k-k') \left[V(k'; k, k'-k) G^{(1)}(k) G^{(1)}(k-k') \times \right. \right. \\ & \times \frac{e^{iW(k; k', k-k')t + iW(k'; k, k'-k)t}}{W(k'; k, k'-k)} + \\ & + V(-k; -k', k'-k) G^{(1)}(k') G^{(1)}(k-k') \times \\ & \times \frac{e^{iW(k; k', k-k')t + iW(-k; -k', k'-k)t}}{W(-k; -k', k'-k)} + \\ & + V(k-k'; k, -k') G^{(1)}(k) G^{(1)}(k') \times \\ & \times \left. \frac{e^{iW(k; k', k-k')t + iW(k-k'; k, -k')t}}{W(k-k'; k, -k')} \right] \Bigg\} + \\ & + \left\{ k \rightarrow -k \right\}. \end{aligned} \quad (V.17)$$

Before continuing the calculations we introduce some further useful abbreviations. We already defined the complex frequency $\omega(k)$ as

$$\omega(k) = \omega_R(k) - i\gamma(k) \quad (V.18)$$

with $\gamma(k) > 0$. This inequality prevents the occurrence of poles in the pertinent equations. In view of expression (V.16), we define the following combinations:

$$\Omega(k_3; k_1, k_2) = \omega_R(k_3) - \omega_R(k_1) - \omega_R(k_2), \quad (V.19)$$

$$\Gamma(k_3; k_1, k_2) = \gamma(k_3) - \gamma(k_1) - \gamma(k_2) \quad (V.20)$$

such that

$$W(k_3; k_1, k_2) = \Omega(k_3; k_1, k_2) - i\Gamma(k_3; k_1, k_2). \quad (V.21)$$

Obviously, the functions W, Ω , and Γ are invariant for interchanging of k_1 and k_2 . Also we shall make use of the invariant properties found earlier (see eqs. (II.31b,c)):

$$\omega_R(k) = -\omega_R(-k) \text{ and } \gamma(k) = \gamma(-k) \text{ for all } k. \quad (V.22)$$

With these definitions (V.18) - (V.22) and the properties of $V(k; k', k-k')$ (see eq. (II.32)), eq. (V.17) (in which the order of the terms is changed) reduces to

$$\begin{aligned} B_2(k) = & \left\{ 2 \int_{-\infty}^{+\infty} dk' \left[|V(k; k', k-k')|^2 G^{(1)}(k') G^{(1)}(k-k') \times \right. \right. \\ & \times \frac{e^{-2\gamma(k')t - 2\gamma(k-k')t + 2\gamma(k)t}}{\Gamma(-k; -k', k'-k) + i\Omega(-k; -k', k'-k)} - \\ & - V(k; k', k-k') V(k'; k, k'-k) G^{(1)}(k) G^{(1)}(k-k') \times \\ & \times \frac{e^{-2\gamma(k-k')t}}{\Gamma(k'; k, k'-k) + i\Omega(k'; k, k'-k)} - \\ & - V(k; k', k-k') V(k-k'; k, -k') G^{(1)}(k') G^{(1)}(k) \times \\ & \times \left. \left. \frac{e^{-2\gamma(k')t}}{\Gamma(k-k'; k, -k') + i\Omega(k-k'; k, -k')} \right] \right\} + \{k \rightarrow -k\}. \quad (V.23) \end{aligned}$$

Substituting next eq. (V.11) into eq. (V.9a) and eq. (V.23) into eq. (V.9b), remembering (V.6), we obtain:

$$A_1(k;E) = -2\gamma(k)E^{(1)}(k,t). \quad (V.24)$$

$$\begin{aligned} A_2(k;E) &= B_2(k)e^{-2\gamma(k)t} = \\ &= \left\{ 2 \int_{-\infty}^{+\infty} dk' \left[|V(k;k',k-k')|^2 E^{(1)}(k') E^{(1)}(k-k') \times \right. \right. \\ &\times \frac{1}{\Gamma(-k;-k',k'-k) + i\Omega(-k;-k',k'-k)} - \\ &- V(k;k',k-k')V(k';k,k'-k)E^{(1)}(k)E^{(1)}(k-k') \times \\ &\times \frac{1}{\Gamma(k';k,k'-k) + i\Omega(k';k,k'-k)} - \\ &- V(k;k',k-k')V(k-k';k,-k')E^{(1)}(k')E^{(1)}(k) \times \\ &\times \left. \left. \frac{1}{\Gamma(k-k';k,-k') + i\Omega(k-k';k,-k')} \right] \right\} + \{k \rightarrow -k\}, \quad (V.25) \end{aligned}$$

where, indeed, the functions $A_1(k;E)$ and $A_2(k;E)$ appear as functionals of $E(k,t)$ ($=\epsilon E^{(1)}(k,t)$), as we had required!

Because $E(k,t) \sim O(\epsilon)$, the ϵ -power series of the formal kinetic equation (V.4) appears due to products of energy-spectrum functions $E(k)$! Further calculations would have shown that $\epsilon^n A_n(k,E) \sim E^n = \epsilon^n \{E^{(1)}\}^n$.

With symmetry properties such as

$$\Gamma(k_3; k_1, k_2) = \Gamma(+k_3; +k_1, +k_2) = \Gamma(+k_3; -k_2, -k_1)$$

and

$$\Omega(k_3; k_1, k_2) = -\Omega(-k_3; -k_1, -k_2) = +\Omega(-k_1; -k_3, k_2) \dots \quad (V.26)$$

based on eq. (V.22), we can transform eq. (V.25) into a more symmetric form. After substitution of eqs. (V.24) and (V.25) into our formal kinetic equation (V.4), we obtain up to $O(\epsilon^2)$ (i.e. A_i neglected for $i \geq 3$):

$$\begin{aligned}
\frac{\partial E(k, t)}{\partial t} = & -2\gamma(k)E(k, t) + \\
& + 2 \int_{-\infty}^{+\infty} dk' \left\{ |V(k; k', k-k')|^2 E(k') E(k-k') \times \right. \\
& \times \frac{1}{\Gamma(k; k', k-k') - i\Omega(k; k', k-k')} - \\
& - V(k; k', k-k') V(k-k'; k, -k') E(k) E(k') \times \\
& \times \frac{1}{\Gamma(k-k'; k, -k') - i\Omega(k; k', k-k')} - \\
& - V(k; k', k-k') V(k'; k, k'-k) E(k) E(k-k') \times \\
& \times \left. \frac{1}{\Gamma(k'; k, k'-k) - i\Omega(k; k', k-k')} \right\} + 2 \int_{-\infty}^{+\infty} dk' \left\{ k \rightarrow -k \right\}. \quad (V.27)
\end{aligned}$$

To calculate the last term we there replace k' by $-k'$, again applying the symmetry properties (V.22), (V.26), and also (II.32). Then one finally arrives at the explicit kinetic equation:

$$\begin{aligned}
\frac{\partial E(k)}{\partial t} = & -2\gamma(k)E(k) + 4 \int_{-\infty}^{+\infty} dk' \left\{ |V(k; k', k-k')|^2 E(k') E(k-k') \times \right. \\
& \times \frac{\Gamma(k; k', k-k')}{\Gamma^2(k; k', k-k') + \Omega^2(k; k', k-k')} - \\
& - \operatorname{Re} [V(k; k', k-k') V(k-k'; k, -k')] E(k) E(k') \times \\
& \times \frac{\Gamma(k-k'; k, k')}{\Gamma^2(k-k'; k, k') + \Omega^2(k; k', k-k')} - \\
& - \operatorname{Re} [V(k-k'; k, k') V(k'; k, k'-k)] E(k) E(k-k') \times \\
& \times \frac{\Gamma(k'; k, k'-k)}{\Gamma^2(k'; k, k'-k) + \Omega^2(k; k', k-k')} \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& - \operatorname{Re} \left[V(k; k', k-k') V(k'; k, k'-k) \right] E(k) E(k-k') \times \\
& \times \frac{\Gamma(k'; k, k-k')}{\Gamma^2(k'; k, k-k') + \Omega^2(k; k', k-k')} + \\
& + \operatorname{Im} \left[V(k; k'; k-k') V(k-k'; k, -k') \right] E(k) E(k') \times \\
& \times \frac{\Omega(k; k', k-k')}{\Gamma^2(k-k'; k, k') + \Omega^2(k; k', k-k')} + \\
& + \operatorname{Im} \left[V(k; k', k-k') V(k'; k, k'-k) \right] E(k) E(k-k') \times \\
& \times \frac{\Omega(k; k', k-k')}{\Gamma^2(k'; k, k-k') + \Omega^2(k; k', k-k')} \Bigg\} . \tag{V.28}
\end{aligned}$$

Replacing still k' by $k-k'$ for terms containing $E(k')E(k-k')$, the latter can be combined with those containing $E(k)E(k')$. We thus finally obtain, up to second order, the following equation:

$$\begin{aligned}
\frac{\partial E(k)}{\partial t} = & -2\gamma(k)E(k) + 4 \int_{-\infty}^{+\infty} dk' \left\{ |V(k; k', k-k')|^2 E(k') E(k-k') \times \right. \\
& \times \frac{\Gamma(k; k', k-k')}{\Gamma^2(k; k', k-k') + \Omega^2(k; k', k-k')} - \\
& - 2\operatorname{Re} \left[V(k; k', k-k') V(k-k'; k, -k') \right] E(k) E(k') \times \\
& \times \frac{\Gamma(k-k'; k, k')}{\Gamma^2(k-k'; k, k') + \Omega^2(k; k', k-k')} + \\
& + 2\operatorname{Im} \left[V(k; k', k-k') V(k-k'; k, -k') \right] E(k) E(k') \times \\
& \times \left. \frac{\Omega(k; k', k-k')}{\Gamma^2(k-k'; k, k') + \Omega^2(k; k', k-k')} \right\} . \tag{V.29}
\end{aligned}$$

This is the most general form of the kinetic equation (to $O(\epsilon^2)$) found without any assumptions about $V(k; k', k-k')$.

C. Special cases of the kinetic equation

In many cases V is purely real or imaginary and, consequently, the last term in eq. (V.29) then vanishes. A very particular interesting case is obtained when V satisfies the relations

$$V(k-k'; k, -k') = V^*(k; k', k-k') = V(k'; k, k'-k) \quad (V.30a)$$

and thus

$$|V(k-k'; k, -k')|^2 = |V(k; k', k-k')|^2 = |V(k'; k, k'-k)|^2. \quad (V.30b)$$

The kinetic equation then becomes²²⁾:

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & -2\gamma(k)E(k) + 4 \int_{-\infty}^{+\infty} dk' |V(k; k', k-k')|^2 \left\{ E(k')E(k-k') \times \right. \\ & \times \frac{\Gamma(k; k', k-k')}{\Gamma^2(k; k', k-k') + \Omega^2(k; k', k-k')} - \\ & \left. - 2E(k)E(k') \frac{\Gamma(k-k'; k, k')}{\Gamma^2(k-k'; k, k') + \Omega^2(k; k', k-k')} \right\}, \quad (V.31) \end{aligned}$$

in which a common interaction strength $|V(k; k', k-k')|^2$, constituting a positive quantity, occurs throughout.

The first term on the right-hand side of eqs. (V.29) and (V.31) represents the linear dissipation term. The term with the integral in those equations describes the energy transfer due to the nonlinear interaction between different k -modes. The contribution of this transfer term to $E(k, t)$ depends on the frequency $\omega(k)$ and the properties of the turbulent system. In the last equation (V.31), however, it is only necessary to consider the signs of Γ in order to know whether energy is gained or lost for a particular k -mode. In any case it is to be noted that the first contribution to the integral term on the right-hand side of (V.29) and (V.31)

represents a contribution from two different modes (k' and $(k-k')$) to a given k -mode, while the remaining terms express a nonlinear dissipation (or growing) of the k -mode due to its coupling with a k' (or $(k-k')$) mode. By changing $\gamma(k)$ into $-\gamma(k)$ one obtains kinetic equations for purely unstable modes. Cases in which both kind of modes, damped as well as unstable, are present can be treated in the same way (see chapter VIII, for instance). The only limitation one has to satisfy is the condition that $E(k)$ remains of $O(\epsilon)$.

Before continuing with a profound discussion in the next chapter we shall give a general example in order to illustrate some remarkable features of these kinetic equations. Suppose we have a turbulent medium, in this context described by our general model equation (IV.1), where $\gamma(k) = \nu$ (a constant) and the relations (V.30) hold. Two cases will be considered:

$$1. \nu > 0.$$

In the linear problem this would mean that all k -modes are damped. We then have

$$\Gamma = -\nu \text{ for all different } k\text{-values.} \quad (V.32)$$

It is readily shown that eq. (V.31) here reduces to:

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & -2\nu E(k) + \\ & + 4 \int_{-\infty}^{+\infty} dk' |V|^2 \frac{\nu}{\nu^2 + \Omega^2(k; k', k-k')} [-E(k')E(k-k') + 2E(k)E(k')]. \end{aligned} \quad (V.33)$$

Hence, a particular k -mode loses energy to the k' and $(k-k')$ modes (for all k'), while the last term describes a nonlinear growth of the k -mode in question. This is a typical example of turbulence in dissipative media, where initially a large amplitude wave is excited as a consequence of which nonlinear effects appear immediately. After a

certain transition time, necessary to spread out the initial wave energy over many k -values and to satisfy the weak turbulence definition (IV.9), the equation (V.33) based on weak turbulence can be used to describe the further evolution of the system. Ultimately, all energy will be dissipated.

$$2. \nu < 0.$$

Here, the linearized system is unstable. Introducing $\mu = -\nu > 0$ in eq. (V.33) we then get

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & 2\mu E(k) + \\ & + 4 \int_{-\infty}^{+\infty} dk' |V|^2 \frac{\mu}{\mu^2 + \Omega^2(k; k', k-k')} [E(k')E(k-k') - 2E(k)E(k')], \end{aligned} \quad (V.34)$$

where μ is positive. Here, the physical picture is as follows: an unstable mode starts to grow and next nonlinear terms become important. Once again, eq. (V.34) describes the real situation of the system after a transition time, necessary to excite many different k -modes and to obtain a state of weak turbulence. Turbulence effects are thus introduced by a linear instability; this linear growth can only be decreased by the nonlinear dissipation term, i.e. the contribution described by the last contribution to the integral. If, nevertheless, $E(k) \sim O(\epsilon)$ cannot be preserved one must consider higher-order corrections (section VI.E).

In practice different turbulent physical quantities will contribute to the behaviour of a special one of them. By studying the turbulence in k -space, we are able to observe precisely from which k -modes the energy is coming and how it is distributed over the various k -modes of different quantities.

By integrating the obtained kinetic equations over k -space, one gets the time behaviour of $\int_{-\infty}^{+\infty} E(k) dk = \langle \psi^2 \rangle$, which variance constitutes the average of a squared turbulent quantity in x -space. In our example, we arrive at the

following equations, substituting $v = \pm \mu$ in eq. (V.33) (the upper sign then refers to the linearly stable, the lower sign to the linearly unstable case, μ being positive throughout):

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k) dk &= \frac{\partial}{\partial t} \langle \psi^2 \rangle = \mp 2\mu \langle \psi^2 \rangle + \\ &\mp 4 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' |V(k; k', k-k')|^2 \frac{\mu}{\mu^2 + \Omega^2(k; k', k-k')} \times \\ &\times [E(k')E(k-k') - 2E(k)E(k')] \end{aligned} \quad (V.35)$$

or, by changing the integration variables in the first contribution (k by $k+k'$ and next k' by $-k'$), at

$$\begin{aligned} \frac{\partial}{\partial t} \langle \psi^2 \rangle &= \\ &= \mp 2\mu \langle \psi^2 \rangle \pm 4 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' |V(k; k', k-k')|^2 E(k)E(k') \frac{\mu}{\mu^2 + \Omega^2(k; k', k-k')} ; \end{aligned} \quad (V.36)$$

the symmetries mentioned here follow from (V.26) and (V.30b). The linearly unstable system (lower signs) thus proves to be stabilized by the nonlinear effects and the damped system becomes damped much more slowly, this being due to turbulence! Taking into account the nonlinear phenomena due to turbulence, many important effects enter into the problem. In the next chapter we will elaborate on more particular cases and determine the consequences of the used expansion method.

C H A P T E R   V I

DISCUSSION OF THE ONE-DIMENSIONAL KINETIC EQUATION

Next we shall show some essential features of the general kinetic equation and the used expansion method and also investigate some specific cases which one can find in the literature.

A. Energy conservation

Usually, one is interested in the final energy balance, including turbulent effects. Therefore, we suppose that $\langle \psi^2(x,t) \rangle = R(0,t)$ (see (III.33)) describes the total turbulent energy of the model system per unit of volume. When the system is conservative (apart from the introduced linear dissipation) we should have

$$\begin{aligned} & \frac{\partial}{\partial t} \langle \psi(x,t)^2 \rangle = \\ & = \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k) dk = -2 \int_{-\infty}^{+\infty} \gamma(k) E(k) dk = \text{internal dissipation. (VI.1)} \end{aligned}$$

If this relation would hold for eq. (V.28) or (V.29), then the complete nonlinear transfer term only describes a permanent redistribution of energy between the different k-modes and no net loss or gain of energy on account of this interaction would occur.

Two ways to look for a conservative system such as (VI.1) are possible. First, with the aid of the first equation of the hierarchy (IV.8a), which is an exact equation, we are able to examine under what conditions the system is conservative. We shall speak of an "a priori" energy conservation when the condition in question is originally satisfied in the system. Secondly, one can investigate the kinetic equations derived in chapter V and look whether they lead to a conservative system or not.

We will derive the "a priori" condition and show next that it also constitutes a necessary condition for energy conservation when starting from the kinetic equations. With the aid of

$$E(k, t) = G(k, t) e^{-2\gamma(k)t}, \quad (VI.2)$$

the equation (IV.8a) is transformed into

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & -2\gamma(k)E(k) - \\ & -i \int_{-\infty}^{+\infty} dk' \left\{ V(k; k', k-k') H(k', -k) e^{-i\{\omega(k') + \omega(k-k') + \omega(-k)\}t} \right\} - \\ & - i \int_{-\infty}^{+\infty} dk' \left\{ k \rightarrow -k \right\}, \end{aligned} \quad (VI.3)$$

taking into account the relation $+i\omega(k) = -i\omega(-k)$, in view of eqs. (V.18) and (V.22).

Consequently, we also find

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k) dk = -2 \int_{-\infty}^{+\infty} \gamma(k) E(k) dk +$$

$$+ \left\{ -2i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk' dk V(k; k', k-k') H(k', -k) \times \right. \\ \left. \times e^{-i\{\omega(k') + \omega(k-k') + \omega(-k)\}t} \right\} \quad (VI.4)$$

after integrating over k , while changing the integration variables in the second integral.

The last term of eq. (VI.4) can also be written as follows:

$$\left\{ -2i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' dk'' V(k; k', k'') H(k', -k) \times \right. \\ \left. \times e^{-i\{\omega(k') + \omega(k'') + \omega(-k)\}t} \right\} \delta(k' + k'' - k). \quad (VI.5)$$

We observe that $H(k', -k)$ and the exponent are invariant for all permutations of two of the three quantities k', k'' , and $-k$. Hence, the mentioned term vanishes if

$$V(k; k', k'') + V(-k'; -k, k'') + V(-k''; k', -k) = 0 \text{ for } k' + k'' = k, \quad (VI.6)$$

that is, if

$$V(k; k', k-k') + V(-k'; -k, k-k') + V(k'-k; k', -k) = 0. \quad (VI.7)$$

The expressions (VI.6) or (VI.7) yield a sufficient condition for the energy conservation. Obviously, one can transform these expressions, represented in k -space, to the x -space, applying eqs. (II.22) and (II.26b). However, this would be quite an elaborate task and, once again, it proves to be much easier to work in the k or Fourier space when investigating matters such as the condition for an "a priori" energy conservation.

Next, the following question arises: does (VI.6) or (VI.7) also constitute a necessary condition for energy

conservation? In other words, the "a priori" condition (VI.6) or (VI.7) being given, can we then find back an "a posteriori" conservation of energy, making use of our kinetic equation (V.29)? In order to get the "a posteriori" conservation, we shall integrate the rigorous eq. (V.29) over k-space and then obtain

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k) dk = -2 \int_{-\infty}^{+\infty} \gamma(k) E(k) dk + \\
& + 4 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' dk'' \delta(k+k''-k) \left\{ |V(k; k', k'')|^2 E(k') E(k'') \times \right. \\
& \times \frac{\Gamma(k; k', k'')}{\Gamma^2(k; k', k'') + \Omega^2(k; k', k'')} - \\
& - 2 \operatorname{Re} [V(k; k', k'') V(k''; k, -k')] E(k) E(k') \times \\
& \times \frac{\Gamma(k''; k, k')}{\Gamma^2(k''; k, k') + \Omega^2(k; k', k'')} + \\
& \left. + 2 \operatorname{Im} [\dots] E(k) E(k') \frac{\Omega(k; k', k'')}{\Gamma^2(k''; k, k') + \Omega^2(k; k', k'')} \right\}. \quad (\text{VI.8})
\end{aligned}$$

In abbreviated form this may be written

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k) dk = -2 \int_{-\infty}^{+\infty} \gamma(k) E(k) dk + \int_{-\infty}^{+\infty} T_1(k) dk + \int_{-\infty}^{+\infty} T_2(k) dk, \quad (\text{VI.9})$$

where, apart from the k-integration, $T_1(k)$ describes the first two nonlinear contributions of eq. (VI.8) and $T_2(k)$ the last one. Next, assuming the "a priori" condition (VI.6) and eq. (II.32), we obtain:

$$\begin{aligned}
& |V(k; k', k'')|^2 = -V(k; k', k'') V(-k; -k', -k'') = \\
& = \operatorname{Re} [\bar{V}(k; k', k'') V(k'; k, -k'')] + \operatorname{Re} [\bar{V}(k; k', k'') V(k''; k, -k')] .
\end{aligned} \quad (\text{VI.10})$$

Substituting this expression into eq. (VI.8) we find, by rearranging the sequence of some terms and changing the integration variables k' and k'' in one of the contributions (this having no influence in view of the factor $\delta(k'+k''-k)$):

$$\begin{aligned}
& \int_{-\infty}^{+\infty} T_1(k) dk = 4 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' dk'' \delta(k'+k''-k) \times \\
& \times \left\{ \operatorname{Re} [\bar{V}(k; k', k'') V(k'; k, -k'')] E(k') E(k'') \times \right. \\
& \times \frac{\Gamma(k; k', k'')}{\Gamma^2(k; k', k'') + \Omega^2(k; k', k'')} - \\
& - \operatorname{Re} [\bar{V}(k; k', k'') V(k'; k, -k'')] E(k) E(k'') \times \\
& \times \frac{\Gamma(k'; k, k'')}{\Gamma^2(k'; k, k'') + \Omega^2(k; k', k'')} + \\
& + \operatorname{Re} [\bar{V}(k; k', k'') V(k''; k, -k')] E(k') E(k'') \times \\
& \times \frac{\Gamma(k; k', k'')}{\Gamma^2(k; k', k'') + \Omega^2(k; k', k'')} - \\
& - \operatorname{Re} [\bar{V}(k; k', k'') V(k''; k, -k')] E(k) E(k') \times \\
& \times \left. \frac{\Gamma(k''; k, k')}{\Gamma^2(k''; k, k') + \Omega^2(k; k', k'')} \right\}. \quad (\text{VI.11})
\end{aligned}$$

Replacing, in the second term, k by k' , k' by k , and k'' by $-k''$, the first two terms of this expression prove to cancel each other because

$$\Gamma(k; k', -k'') = \Gamma(k; k', k'') \quad (\text{VI.12a})$$

and

$$\Omega(k; k', k'') = -\Omega(k'; k, -k''). \quad (\text{VI.12b})$$

A similar reasoning shows how the two last terms of eq.

(VI.11) also cancel each other, so as to have $\int_{-\infty}^{+\infty} T_1(k) dk = 0$.

We next pass to the last term of eq. (VI.8), in which we account for the factor 2 by adding two equivalent expressions:

$$\begin{aligned} \int_{-\infty}^{+\infty} T_2(k) dk &= 4 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' dk'' \delta(k' + k'' - k) \times \\ &\times \operatorname{Im} [\bar{V}(k; k', k'') V(k''; k, -k')] E(k) E(k') \times \\ &\times \frac{\Omega(k; k', k'')}{\Gamma^2(k''; k, k') + \Omega^2(k; k', k'')} + \\ &+ \operatorname{Im} [\bar{V}(k; k', k'') V(k'; k, -k'')] E(k) E(k'') \times \\ &\times \frac{\Omega(k; k', k'')}{\Gamma^2(k'; k, k'') + \Omega^2(k; k', k'')} . \end{aligned} \quad (\text{VI.13})$$

We shall replace k' by $-k'$ and interchange k and k'' in the first term of this expression and, similarly, replace k'' by $-k''$ and interchange k and k' in the second term. In view of the symmetry properties of Γ and Ω the following common interaction factor then appears in the integrand:

$$\operatorname{Im} \{ V(k; k', k'') [\bar{V}(k''; -k', k) + V(k'; k, -k'')] \} . \quad (\text{VI.14})$$

In view of that "a priori" condition (VI.6), this common factor can also be represented by $\operatorname{Im} \{ -V(k; k', k'') V(-k; -k', -k'') \}$ or, remembering (II.32), by $\operatorname{Im} \{ V(k; k', k'') V^*(k; k', k'') \}$, which is a vanishing quantity, so that the integrand over T_2 as well as that over T_1 both become zero.

Summarizing, we have proved that the "a priori" energy conservation, as expressed by (VI.6) or (VI.7), is a necessary condition for an "a posteriori" conservation. Mathematically, the latter is expressed by

$$\int_{-\infty}^{+\infty} T_1(k) dk = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} T_2(k) dk = 0. \quad (\text{VI.15})$$

Similar conservation laws may also hold for other physical turbulent quantities. Such conditions justify the used approximation method and it could be an indication that for nonconservative systems also no physical information contained in the rigorous equations is lost. Finally, we notice that no such energy-conservation expression (VI.7) exists for the particular kinetic equation (V.31) because the relations (V.30a) would then imply that $V(-k'; -k, k' - k) = 0$, on account of (II.32)!

B. Nonlinear interaction between undamped stable waves

In this section we want to consider the very interesting case in which no linear growth or damping is present in the turbulent system. This case is already well known in the literature (refs. 4-7, 14, 15, and 23), in which various other perturbation methods have been used (see also section VI.F). Hence, $\gamma(k) = 0$ for all k and, consequently,

$$\omega(k) = \omega_R(k),$$

$$\Gamma(k_3; k_1, k_2) = 0 \text{ for all possible } k_i\text{-values } (i=1,2,3). \quad (\text{VI.16})$$

In order to arrive at a kinetic equation for stable and undamped waves one should, in principle, simply take the limit $\Gamma \rightarrow 0$ in the different kinetic equations (V.28), (V.29), and (V.31). However, the trouble is that we do not know how to take this limit because $\lim \Gamma \rightarrow +$ or -0 yields a different result! Therefore, we have to reconsider carefully our analysis of chapter V.

To find the right answer we will give a physical argu-

ment. If $\gamma(k) = 0$ for all k we have

$$G(k) \equiv E(k) \quad (\text{VI.17})$$

and, consequently,

$$B_i(k) \equiv A_i(k), \quad (\text{VI.18})$$

and also

$$W(k_3; k_1, k_2) = \Omega(k_3; k_1, k_2). \quad (\text{VI.19})$$

Performing the same steps as in chapter V, we shall arrive at eq. (V.13), which then can be represented as follows, taking into account that $\omega(k) = \omega_R(k)$ now constitutes an odd function of ω :

$$\begin{aligned} \frac{\partial H^{(2)}(k', -k)}{\partial t} = & -2i \left\{ V(k'; k, k'-k) E^{(1)}(k) E^{(1)}(k-k') + \right. \\ & + V(-k; -k', k'-k) E^{(1)}(k') E^{(1)}(k-k') + \\ & \left. + V(k-k'; k, k') E^{(1)}(k) E^{(1)}(k') \right\} e^{-i\Omega(k; k', k-k')t}. \end{aligned} \quad (\text{VI.20})$$

We first observe that the proposed initial condition (V.2) seems useless in a certain sense because after integrating the equation (VI.20), using (V.2), we at once get poles $1/\Omega$ and the same difficulty as we stated above appears! However, we can allow other initial conditions as long as we obtain the same final answer, i.e. $e^2 A_2$ a functional of the energy-spectrum function. Taking $H^{(2)}(k', -k, t=0)=0$ we obtain the following solution of eq. (VI.20):

$$H^{(2)}(k', -k, t) = 2i \left\{ V(k'; k, k'-k) E^{(1)}(k) E^{(1)}(k-k') + \dots + \dots \right\} \times$$

$$\times \frac{e^{-i\Omega(k;k',k-k')t}}{i\Omega(k;k',k-k')} - 1. \quad (\text{VI.21})$$

With this special initial condition we have avoided the poles and a synchronization for $H^{(2)}$ is now achieved for $t \rightarrow \infty$ instead of $t = 0$ (see (VI.23)) because, up to this order, $E^{(1)}(k)$ is independent of time. Substitution of $H^{(2)}$ into the second-order contribution $\epsilon B_2 \equiv \epsilon A_2$ in (V.12) yields:

$$A_2(k, t) = \left\{ 2 \int_{-\infty}^{+\infty} dk' V(k; k', k-k') \left[\frac{1 - e^{-i\Omega(k;k',k-k')t}}{i\Omega(k;k',k-k')} \right] \times \right. \\ \left. \times [V(k'; k, k-k') E^{(1)}(k) E^{(1)}(k-k') + \dots + \dots] \right\} + \{k \rightarrow -k\}. \quad (\text{VI.22})$$

The required functional dependence of $\epsilon^2 A_2$ is obtained only for $t \rightarrow \infty$ in which case we can apply the property

$$\lim_{t \rightarrow \infty} \left(\frac{1 - e^{-i\Omega t}}{i\Omega} \right) = P \frac{1}{i\Omega} - \pi \delta(\Omega). \quad (\text{VI.23})$$

Hence,

$$\lim_{t \rightarrow \infty} \epsilon^2 A_2 = \left\{ 2 \int_{-\infty}^{+\infty} dk' V(k; k', k-k') \left[P \frac{1}{i\Omega} - \pi \delta(\Omega) \right] [\dots] \right\} + \{k \rightarrow -k\}, \quad (\text{VI.24})$$

where

$$\Omega(k; k', k-k') = \omega_R(k) - \omega_R(k') - \omega_R(k-k').$$

We could get the desired kinetic equation by repeating again all further calculations which previously resulted in eq. (V.29), starting with eq. (VI.24). It turns out that we obtain the same result if we take the limit $\Gamma \rightarrow +0$ in our general kinetic equation (V.28) or (V.29) which involves

$$\lim_{\Gamma \rightarrow +0} \frac{\Gamma}{\Gamma^2 + \Omega^2} = \pi \delta(\Omega) \text{ and } \lim_{\Gamma \rightarrow +0} \frac{\Omega}{\Omega^2 + \Gamma^2} = P \frac{1}{\Omega}. \quad (\text{VI.25})$$

The result for stable and undamped waves thus becomes

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & \int_{-\infty}^{+\infty} dk' \left\{ 4\pi \left[|V(k; k', k-k')|^2 E(k') E(k-k') - \right. \right. \\ & - 2\text{Re}(V(k; k', k-k') V(k-k'; k, -k')) E(k') E(k) \left. \right] \times \\ & \times \delta(\omega_R(k) - \omega_R(k') - \omega_R(k-k')) + 8\text{Im}(V(k; k', k-k') V(k-k'; k, -k')) \times \\ & \times E(k) E(k') P \frac{1}{\omega_R(k) - \omega_R(k') - \omega_R(k-k')} \left. \right\}. \end{aligned} \quad (\text{VI.26})$$

A similar equation was derived by Benney and Saffman⁵, Davidson⁷, and Karpman¹⁵). The last interaction term in eq. (VI.26) is of a nonresonant type. However, when the relations (V.30) are satisfied, we arrive at the well-known kinetic equation for resonant three-mode coupling:

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & 4\pi \int_{-\infty}^{+\infty} dk' |V|^2 \delta(\omega_R(k) - \omega_R(k') - \omega_R(k-k')) \times \\ & \times \{E(k') E(k-k') - 2E(k) E(k')\} = \\ = & 4\pi \int_{-\infty}^{+\infty} dk' |V|^2 \delta(\omega_R(k) - \omega_R(k') - \omega_R(k-k')) \times \\ & \times \{E(k') E(k-k') - E(k) E(k') - E(k) E(k-k')\}. \end{aligned} \quad (\text{VI.27})$$

Compared with the general kinetic equation (V.28) or (V.29), where a damping or growth of the modes had been presumed ($\gamma(k) \neq 0$), the nonlinear interaction described by eq. (VI.27)

may be weaker, because it only occurs for frequencies k' , satisfying (k being given) the relation

$$\omega_R(k) = \omega_R(k') + \omega_R(k-k'). \quad (\text{VI.28})$$

However, some authors distinguish between interaction of random waves, described by eq. (VI.27), and true turbulence, which they define as a situation where all modes interact in a nonlinear way (refs. 24,25). From our point of view of weak turbulence, we did not make this distinction although it can be defined mathematically. In real turbulent problems one wishes to deal with kinetic equations like (V.28) or (V.29) according to which each k -mode is coupled, without any restriction, with all other modes. Then, it may be more difficult to satisfy the necessary condition $E(k) \sim O(\epsilon)$. The interaction with random waves requires the extra condition (VI.28) which, in several cases, can even not be fulfilled; higher-order corrections are then to be taken into account (see section VI.E). In the latter case, the condition for $E(k)$ is much less stringent and hence less difficult to obey.

Along the same lines as discussed in section VI.A one can prove for eq. (VI.26) that the system is conserved "a posteriori" when an "a priori" condition such as (VI.6) or (VI.7) exists. The k -integration over the right-hand side of eq. (VI.26) then yields two individually vanishing parts, viz. the contribution with the δ -function and that with the P -symbol. The same kinetic equation (VI.26) or (VI.27) can also be used for a fair approximation if a small linear growth or damping is present in the system, i.e. when

$$|\gamma(k)| \ll |\omega_R(k)| \text{ for all } k. \quad (\text{VI.29})$$

Without any difficulty, one can add the usual linear dissipation or growth term to the right-hand side of the equations.

We shall now investigate a model equation closely related to the one studied already in this work. Instead of the equation (II.28) or (IV.1), reading as follows for undamped waves,

$$\frac{\partial A(k, t)}{\partial t} = -i \int_{-\infty}^{+\infty} dk' V(k; k', k-k') A(k', t) A(k-k', t) \times \\ \times e^{-i\{\omega_R(k') + \omega_R(k-k') - \omega_R(k)\}t} \quad (VI.30)$$

we shall start from the related equation

$$\frac{1}{\omega_R(k)} \frac{\partial A_O(k)}{\partial t} = -i \int_{-\infty}^{+\infty} dk' V_O(k; k', k-k') A_O(k', t) A_O(k-k', t) \times \\ \times e^{-i\{\omega_R(k') + \omega_R(k-k') - \omega_R(k)\}t} \quad (VI.31)$$

If one includes $\omega_R(k)$ in the interaction strength V_O , we are able to treat this last equation with the aid of the preceding investigations. We then have to substitute

$$V(k; k', k-k') = \omega_R(k) V_O(k; k', k-k'), \quad (VI.32)$$

where, in view of (II.31b), the condition (II.32) for V changes into

$$V_O(k; k', k-k') = V_O(k; k-k', k') = V_O^*(-k; -k', k'-k). \quad (VI.33)$$

The relevant kinetic equation (VI.26) in this case reads

$$\frac{\partial E_O(k)}{\partial t} = \int_{-\infty}^{+\infty} dk' \left\{ 4\pi \left[\omega_R^2(k) |V_O(k; k', k-k')|^2 E_O(k') E_O(k-k') - \right. \right.$$

$$\begin{aligned}
& -2\text{Re}\langle V_0(k; k', k-k') V_0(k-k'; k, -k') \rangle \omega_R(k) \omega_R(k-k') E_0(k) E_0(k') \Big] \times \\
& \times \delta(\omega_R(k) - \omega_R(k') - \omega_R(k-k')) + \\
& + 8\text{Im}(\dots) \omega_R(k) \omega_R(k-k') E_0(k) E_0(k') P \frac{1}{\{\omega_R(k) - \omega_R(k') - \omega_R(k-k')\}},
\end{aligned}
\tag{VI.34}$$

in which, in view of (III.9),

$$\langle A_0(k) A_0(k_1) \rangle = E_0(k) \delta(k+k_1).
\tag{VI.35}$$

Now it turns out to be useful to introduce the "action density" (refs. 5, 6, 14, and 26):

$$N(k) = \frac{E_0(k)}{|\omega_R(k)|} \text{ or } E_0(k) = S(k) \omega_R(k) N(k),
\tag{VI.36}$$

where

$$S(k) = \text{sign}(\omega_R(k)).
\tag{VI.37}$$

The following new interaction strength M may also be defined, without losing symmetry properties like (VI.33):

$$V_0(k; k', k-k') = \frac{M(k; k', k-k')}{\sqrt{|\omega_R(k)| |\omega_R(k')| |\omega_R(k-k')|}}.
\tag{VI.38}$$

Equation (VI.34) then reduces to:

$$\begin{aligned}
\frac{\partial N(k)}{\partial t} = & \int_{-\infty}^{+\infty} dk' \left\{ 4\pi \left[|M(k; k', k-k')|^2 N(k') N(k-k') - \right. \right. \\
& - 2\text{Re}(M(k; k', k-k') M(k-k'; k, -k')) S(k) S(k-k') N(k) N(k') \Big] \times \\
& \times \delta(\omega_R(k) - \omega_R(k') - \omega_R(k-k')) +
\end{aligned}$$

$$+8\text{Im}(\dots)S(k)S(k-k')N(k)N(k')P \frac{1}{\{\omega_R(k)-\omega_R(k')-\omega_R(k-k')\}} \Bigg\}, \quad (\text{VI.39})$$

Requiring again the simplifying condition (V.30a), the new interaction strength M has to satisfy

$$M^*(k;k',k-k') = M(k-k';k,-k') = M(k';k,k'-k), \quad (\text{VI.40a})$$

which is equivalent to

$$V_0^*(k;k',k-k') = V_0(k-k';k,-k') = V_0(k';k,k'-k). \quad (\text{VI.40b})$$

From eq. (VI.40a) we further deduce

$$|M(k;k',k-k')|^2 = |M(k-k';k,-k')|^2 = |M(k';k,k'-k)|^2. \quad (\text{VI.40c})$$

In view of these relations we can transform (VI.39) into the following kinetic equation, which closely resembles the one used in plasma physics describing weak Vlasov turbulence (refs. 6, 14, and 27):

$$\begin{aligned} \frac{\partial N(k)}{\partial t} &= 4\pi \int_{-\infty}^{+\infty} dk' |M(k;k',k-k')|^2 \{N(k')N(k-k') - \\ &- 2S(k)S(k-k')N(k)N(k')\} \delta(\omega_R(k)-\omega_R(k')-\omega_R(k-k')) = \\ &= 4\pi \int_{-\infty}^{+\infty} dk' |M(k;k',k-k')|^2 \{N(k')N(k-k') - \\ &- S(k)S(k-k')N(k)N(k') - S(k)S(k')N(k)N(k-k')\} \times \\ &\times \delta(\omega_R(k) - \omega_R(k') - \omega_R(k-k')). \end{aligned} \quad (\text{VI.41})$$

Model equations similar to (VI.30) can be treated

in the same way. Investigating Vlasov turbulence, for instance, one merely deals with a dielectric function rather than with a simple dispersion relation like we defined before (chapter II). The parameter $\frac{1}{\omega_R(k)}$ in eq. (VI.31) is then replaced by $\frac{\partial \epsilon(\omega)}{\partial \omega} \Big|_{\omega=\omega_R(k)}$ and $N(k)$ is, accordingly, defined as

$$N(k) = \left| \frac{\partial \epsilon(k, \omega)}{\partial \omega_R} \right| E(k). \quad (\text{VI.42})$$

We notice that the same kinetic equations (VI.39) or (VI.41) could be found by putting¹⁴⁾

$$\frac{A_0(k)}{|\omega_R(k)|^2} = C(k),$$

for which function the property (III.9) immediately leads to

$$\langle C(k)C(k_1) \rangle = N(k)\delta(k+k_1). \quad (\text{VI.43})$$

An important point turns out to be the "a priori" energy conservation condition (VI.7) for the kinetic equation (VI.41), which in this case characterizes purely resonant mode coupling. It reads:

$$\begin{aligned} \omega_R(k)V_0(k; k', k-k') + \omega_R(-k')V_0(-k'; -k, k-k') + \\ + \omega_R(k'-k)V_0(k'-k; -k, k') = 0. \end{aligned} \quad (\text{VI.44})$$

With the aid of (VI.33) and (VI.40b), this condition reduces to:

$$\omega_R(k) - \omega_R(k') - \omega_R(k-k') = 0. \quad (\text{VI.45})$$

We thus find a confirmation of the decay condition (VI.28) which resulted from the δ -function appearing in (VI.41)!

The conservation of energy according to (VI.1) (with $\gamma=0$), can now be expressed by

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} N(k) |\omega_R(k)| dk = 0, \quad (\text{VI.46a})$$

or by

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} S(k) \omega_R(k) N(k) dk = 0. \quad (\text{VI.46b})$$

However, the conservation of momentum, to be defined as

$$P = \int_{-\infty}^{+\infty} S(k) k N(k) dk, \quad (\text{VI.47})$$

also proves to be constant, because

$$k - k' - (k-k') = 0. \quad (\text{VI.48})$$

As a matter of fact, this conservation holds for any quantity $I(k)$ satisfying

$$I(k) - I(k') - I(k'') = 0, \quad (\text{VI.49})$$

as can be shown by substituting (VI.41) in the integral $\int_{-\infty}^{+\infty} S(k) I(k) \frac{\partial N(k)}{\partial t} dk$. The three contributions of (VI.41) then lead to three double integrals. By proper change of variables the sum of these double integrals can be reduced to one double integral, the integrand of which proves to be proportional to $I(k) - I(k') - I(k'')$, taking into account the symmetry properties (VI.40c). Hence, this integral vanishes in view of eq. (VI.49). For further properties (equilibria, H-functions, ...) we refer to the literature (refs. 6, 26, and 28).

D. Nonlinear interaction between purely damped or
between purely growing waves

A less important but nevertheless valuable case to consider arises if all modes satisfy $\omega_R(k) = 0$. This statement implies that the quantity $\Omega(k_3; k_1, k_2)$ then vanishes for all combinations k_i ($i = 1, 2, 3$), so that (V.21) reduces to

$$W(k_3; k_1, k_2) = -i\Gamma(k_3; k_1, k_2). \quad (\text{VI.49})$$

We encounter the same difficulties as mentioned in section VI.B by taking the limit $\Omega \rightarrow 0$ for the kinetic equations, found in chapter V because, similar to (VI.25),

$$\lim_{\Omega \rightarrow \pm 0} \frac{\Gamma}{\Gamma^2 + \Omega^2} = P \frac{1}{\Gamma} \text{ and } \lim_{\Omega \rightarrow \pm 0} \frac{\Omega}{\Gamma^2 + \Omega^2} = \pm \pi \delta(\Gamma). \quad (\text{VI.50})$$

Reconsidering the analysis of chapter V we cannot find any argument as to which sign should be preferred in the last integral. A reason may be that, until now, we have not found a physical example where this difficulty arises. Moreover, the term with $\pm \delta(\Gamma)$ is not important compared with the other $P \frac{1}{\Gamma}$ contributions. Therefore, we shall drop the δ -term altogether. Hence, the following kinetic equation (rigorous up to second order) is obtained from (V.29):

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & -2\gamma(k)E(k) + \\ & + 4 \int_{-\infty}^{+\infty} dk' \left\{ |V(k; k', k-k')|^2 E(k')E(k-k') P \frac{1}{\Gamma(k; k', k-k')} - \right. \\ & \left. - 2\text{Re} [V(k; k', k-k')V(k-k'; k, -k')] E(k)E(k') P \frac{1}{\Gamma(k-k'; k, k')} \right\}. \end{aligned} \quad (\text{VI.51})$$

Purely growing modes are also allowed here, provided that

the condition $E(k) \sim O(\epsilon)$ is preserved or, physically, that the growth should be limited by nonlinear effects. Moreover, this equation could again be simplified in the special case of the validity of eq. (V.30a).

We readily observe that all k-modes (except the few ones for which $\Gamma = 0$) interact nonlinearly, because there is also no supplementary condition like in the case of resonant three-mode coupling. In this case, therefore, a much stronger interaction may occur and a stronger condition has then to be imposed on the energy-spectrum function which has to remain small. An equation such as (VI.51) shall be used to describe hydrodynamical turbulence (chapter IX) and it will be extended to the theory of MHD-turbulence (chapter X).

E. Higher-order corrections

In the case for which the weak-turbulence definition, i.e. $E(k) \sim O(\epsilon)$ cannot be satisfied, one has to investigate the second nonlinear correction term $\epsilon^2 A_3(k)$ in the formal kinetic equation (V.4). However, there are different ways to do so and we are not sure that these complicated calculations are relevant physically. The same difficulties arise in the usual statistical kinetic theory, for instance when calculating collision correction terms for the Boltzmann equation. There is one exceptional case in which the higher-order term can easily be obtained. This happens if $A_2 = 0$. Such a first-order correction vanishes indeed if, simultaneously, the three-wave resonant interaction condition

$$\omega_R(k) = \omega_R(k') + \omega_R(k-k') \quad (\text{VI.52})$$

dealing with stable and undamped waves cannot be satisfied, while the interaction strength V becomes imaginary or real. With the aid of eqs. (IV.8b) and (V.5c), the contribution of $O(\epsilon^3)$ of eq. (V.10b), remembering that F starts with a

term of order ε^3 , proves to be:

$$\varepsilon^3 \frac{\partial H^{(3)}(k', -k)}{\partial t} = \left\{ -i \int_{-\infty}^{+\infty} dk_0 V(k'; k_0, k' - k_0) \varepsilon^3 F^{(3)}(k_0, -k, k - k') \times \right. \\ \left. \times e^{-i\{\omega_R(k_0) + \omega_R(k' - k_0) - \omega_R(k')\}t} \right\} + \left\{ k' \nabla - k \right\} + \left\{ k' \nabla k - k' \right\}. \quad (\text{VI.53})$$

As observed before, one could easily derive a dynamical equation for $\varepsilon^3 F^{(3)}$, the fourth-order correlation function. This equation contains fifth-order correlations which, in lowest order, split into a product of a factor of second order, viz. $\varepsilon G^{(1)}$ and a factor of third order, viz. $\varepsilon^2 H^{(2)}$. Following the same analysis as performed in chapter V one arrives at a nonlinear correction term $\varepsilon^3 A_3(k)$, which then contains products of three energy spectra $E(k)$ (in order to retain the required $O(\varepsilon^3)$) multiplied by a common δ -function. This δ -function requires a four-wave resonant condition, viz.

$$\omega_R(k) = \omega_R(k') + \omega_R(k'') + \omega_R(k''') \quad \text{if } k = k' + k'' + k'''. \quad (\text{VI.54})$$

Of course this result is also obtained by the other methods (ref. 23 and 29), which we shall discuss in the next section.

F. Other methods in weak-turbulence theories

Three different methods to derive a dynamical equation for the energy-spectrum function $E(k)$ are known from the literature. They all treat the particular case of stable and undamped waves. The first one, introduced by Hasselmann²³⁾ in the field of oceanography, and by Galeev and Karpman²⁴⁾ in that of plasma physics, consists of an obvious perturbation method for the fluctuating amplitude $A(k, t)$. This quantity is expanded there into a perturbation series by supposing a small interaction between different modes. After substituting the terms of this series in the relevant equations, all

these authors obtained solutions for every $A_i(k)$ and then could calculate the quantity $[A(k_1, t)A(k_2, t)]$. Next, they determined the ensemble average like we did above. Although they all neglected the triple-correlation function, assuming an exact Gaussian distribution, Galeev and Karpman nevertheless arrived, by taking the limit $t \rightarrow \infty$, at a kinetic equation equivalent to our (VI.27).

A slightly different method was also used by Coppi, Rosenbluth, and Sudan⁶⁾, considering right away

$$\frac{\partial}{\partial t} [A(k_1, t)A(k_2, t)].$$

The second method, introduced by Kovrizhnykh³⁰⁾ is based on the statistical principle of detailed balance, using semi-quantum-mechanical considerations. The advantage of this method concerns the simplicity of the derivation of the kinetic equations if the problem is symmetric with respect to the interaction strengths. This condition was also explored by Coppi et al.⁶⁾, studying nonlinear interactions of positive and negative energy modes in rarefied plasmas.

The third method, the most systematic one, is the multiple time-scales method introduced by Frieman³¹⁾ and Sandri⁶⁾ to derive kinetic equations in the domain of statistical mechanics. For problems in turbulence, this method was first used by Benney and Saffman⁵⁾ and refined by Davidson⁷⁾. These authors were faced with the same problem we posed: closing the hierarchy of equations. The freedom introduced by this method was reduced by eliminating the secular terms for $t \rightarrow \infty$. However, the method fails for the general case ($\gamma(k) \neq 0$): no secular behaviour then can appear because the system is damped or unstable. It turns out that the multiple time-scales method as applied in this theory reduces to an ordinary perturbation method like that of Galeev and Karpman.

Basically our method is closely connected with the last one and they are both used in the domain of statistical mechanics, leading to the same kinetic equations. However, the Bogoliubov expansion method does not require the appearance

of secular terms, which is prevented by the synchronization hypothesis.

Finally, we want to mention the application of another method of nonequilibrium statistical mechanics, introduced by the Brussels' school, explored by Balescu and Senatarski³²). Their method which aims to describe strong turbulence, falls beyond the scope of our work, although it seems related with the other methods of statistical mechanics.

C H A P T E R VII

EXTENSION OF THE THEORY IN THE CASE OF DISPERSION RELATIONS WITH SEVERAL BRANCHES

A. A second one-dimensional model, referring to a dispersion relation with two branches

In addition to the preceding analysis in which the introduced model implied odd real frequencies, we now wish to study the case of even real frequencies. This situation can be realized if one has to deal with a dispersion relation like (II.16), which has more than one root (or eigenvalue). For instance, some one-dimensional problems may lead to a quadratic dispersion relation of the form

$$\omega_R^2(k) = g(k^2) \text{ giving two branches, viz. } \omega_R(k) = \pm \sqrt{g(k^2)}, \quad (\text{VII.1})$$

for each of which we have

$$\omega_R(k) = \omega_R(-k). \quad (\text{VII.2})$$

The imaginary part of $\omega(k)$, too, is supposed to be an even

function of k :

$$\gamma(k) = \gamma(-k). \quad (\text{VII.3})$$

Similar to the analysis of chapter II, the most general solution of the nonlinear problem can be represented by

$$\psi(k, t) = \left[A^+(k, t) e^{+i\omega_R(k)t} + A^-(k, t) e^{-i\omega_R(k)t} \right] e^{-\gamma(k)t} = \quad (\text{VII.4})$$

$$= \sum_s A^s(k, t) e^{is\omega_R(k)t - \gamma(k)t} \quad (\text{VII.5})$$

Here, $\omega_R(k)$ and, similarly, $\gamma(k)$ is supposed to be a positive quantity for all k .

Applying the reality condition (II.24), viz. $\psi(k, t) = \psi^*(-k, t)$, to eqs. (VII.4) and (VII.5), we get

$$A^s(k, t) = A^{-s*}(-k, t) \text{ for } s = +, -. \quad (\text{VII.6})$$

It is easily recognized that for $k > 0$ the quantities $A^-(k, t)$ and $A^+(k, t)$ represent the amplitudes of waves travelling in the positive and the negative x -direction, respectively (for $k < 0$ the other way round).

In the case of two branches of the dispersion relation, we may introduce a new model equation, which constitutes a logical extension of eq. (II.28):

$$\begin{aligned} \frac{\partial A^s(k, t)}{\partial t} = & -i \sum_{s', s''} \int_{-\infty}^{+\infty} dk' v^{s, s', s''}(k; k', k-k') \times \\ & \times A^{s'}(k', t) A^{s''}(k-k', t) e^{i[s'\omega_R(k') + s''\omega_R(k-k') - s\omega_R(k)]t} \times \\ & \times e^{[\gamma(k) - \gamma(k-k') - \gamma(k')]t} \end{aligned} \quad (\text{VII.7})$$

for all combinations of values $+1$ or -1 of s, s', s'' .

This second-model equation describes all possible interactions

of the mentioned modes and is quite more general than the one introduced by Benney and Saffman⁵).

It will be clear from symmetry considerations that

$$v^{s,s',s''}(k;k',k-k') = v^{s,s'',s'}(k;k-k',k'). \quad (\text{VII.8a})$$

From eq. (VII.6) one also arrives, considering the complex conjugate of eq. (VII.7), at the relation

$$v^{s,s',s''}(k;k',k-k') = -v^{*-s,-s',-s''}(-k;-k',k'-k). \quad (\text{VII.8b})$$

B. Wave-correlation functions

Again we require that no averaged quantities should exist in the turbulent medium described by eq. (VII.7). Hence,

$$\langle \psi(x,t) \rangle = 0 \quad (\text{VII.9a})$$

so that

$$\langle \psi(k,t) \rangle = 0 \text{ and, therefore, } \langle A^s(k,t) \rangle = 0 \quad (\text{VII.9b})$$

for $s = +, -$ and all k .

Next, we consider the ensemble average over the product of two $\psi(x,t)$'s at different places, but at the same time. It was already shown in section III.A that

$$R(\xi,t) = \langle \psi(x,t) \psi(x+\xi,t) \rangle = \int_{-\infty}^{+\infty} E(k,t) e^{ik\xi} dk, \quad (\text{VII.10})$$

with

$$E(k_1,t) \delta(k_1+k_2) = \langle \psi(k_1,t) \psi(k_2,t) \rangle = \langle \psi(k_1,t) \psi^*(-k_2,t) \rangle, \quad (\text{VII.11})$$

if one deals with homogeneous turbulence, which means that the correlation function R is independent of x .

Substituting eq. (VII.5) into eq. (VII.11), we get the following expression for the energy-spectrum function:

$$E(k_1, t) \delta(k_1 + k_2) = \sum_{s_1, s_2} \langle A^{s_1}(k_1, t) A^{s_2}(k_2, t) \rangle \times \\ \times e^{i[s_1 \omega_R(k_1) + s_2 \omega_R(k_2)]t - \gamma(k_1)t - \gamma(k_2)t} \quad (\text{VII.12a})$$

Quite generally, we now postulate the following extension of eq. (III.10)

$$\langle A^{s_1}(k_1, t) A^{s_2}(k_2, t) \rangle = G^{s_1}(k_1) \delta(k_1 + k_2) \delta_{s_1, -s_2}. \quad (\text{VII.13})$$

The Kronecker delta excludes a coupling between waves traveling in opposite directions. This expression is also introduced in order to achieve an agreement with the linearized theory, where this kind of energy considerations is independent of time. Even in our nonlinear theory, this independence holds at lowest order!

In view of eq. (VII.13) we next obtain

$$E(k_1, t) \delta(k_1 + k_2) = \\ = \langle A^+(k_1, t) A^-(k_2, t) \rangle e^{i[\omega_R(k_1) - \omega_R(k_2)]t - \gamma(k_1)t - \gamma(k_2)t} + \\ + \langle A^-(k_1, t) A^+(k_2, t) \rangle e^{-i[\omega_R(k_1) - \omega_R(k_2)]t - \gamma(k_1)t - \gamma(k_2)t} = \\ = G^+(k_1, t) e^{-2\gamma(k_1)t} \delta(k_1 + k_2) + G^-(k_1, t) e^{-2\gamma(k_1)t} \delta(k_1 + k_2) = \\ = \{E^+(k_1, t) + E^-(k_1, t)\} \delta(k_1 + k_2), \quad (\text{VII.12b})$$

where E^+ and E^- have been defined in an obvious manner.

Further, the following properties result from eqs. (VII.6), (VII.12), and (VII.13):

$$a) E^S(k, t) = E^{-S}(-k, t) \text{ and } G^S(k, t) = G^{-S}(-k, t); \quad (\text{VII.14a})$$

this guarantees the even character of $E(k, t)$, as an extension of eq. (III.12);

$$b) E^{S*}(k, t) = E^S(k, t), \quad (\text{VII.14b})$$

which had to be expected in view of the reality of $E(k, t)$. We notice that it is not necessary to have the property of isotropy

$$E^S(k) = E^S(-k), \quad (\text{VII.15})$$

like in the first model (see eq. (III.12))!

Also higher-order wave-correlation functions are needed. The next relevant one is defined by the following extension of eq. (III.16):

$$\langle A^{s_1}(k_1) A^{s_2}(k_2) A^{s_3}(k_3) \rangle = H^{s_1, s_2, s_3}(k_1, k_2) \delta(k_1 + k_2 + k_3). \quad (\text{VII.16})$$

Obviously, this third-order wave-correlation function H^{s_1, s_2, s_3} is invariant for all permutations of the three quantities (k_1, s_1) , (k_2, s_2) and (k_3, s_3) .

For convenience we shall omit the t -dependence in the following equations. Then we arrive at the following definition as an extension of eq. (III.22)

$$\begin{aligned} \langle A^{s_1}(k_1) A^{s_2}(k_2) A^{s_3}(k_3) A^{s_4}(k_4) \rangle = \\ = F^{s_1, s_2, s_3, s_4}(k_1, k_2, k_3) \delta(k_1 + k_2 + k_3 + k_4) + \\ + G^{s_1}(k_1) G^{s_3}(k_3) \delta(k_1 + k_2) \delta(k_3 + k_4) \delta_{s_1, -s_2} \delta_{s_3, -s_4} + \\ + G^{s_1}(k_1) G^{s_2}(k_2) \delta(k_1 + k_3) \delta(k_2 + k_4) \delta_{s_1, -s_3} \delta_{s_2, -s_4} + \end{aligned}$$

$$+ G^{s_1}(k_1) G^{s_2}(k_2) \delta(k_1+k_4) \delta(k_2+k_3) \delta_{s_1, -s_4} \delta_{s_2, -s_3}, \quad (\text{VII.17})$$

where the function F is invariant for permutation of the relevant (k, s) values.

C. The kinetic equation for the second one-dimensional model

In this section we shall derive a closed equation for the energy-spectrum functions $E^+(k)$ and $E^-(k)$. First, we have to check whether the expression (VII.9) can really hold for all times, such that no nonvanishing averaged quantities are induced. Taking the ensemble average over eq. (VII.7) one obtains, with the aid of eq. (VII.13), a right-hand side proportional to $\delta(k)$, so that $\langle \partial A^s(k, t) / \partial t \rangle$ will vanish if the integrand vanishes for $k = 0$; this gives the condition

$$v^{s, s', s''}(0; k', -k') = 0 \text{ for all } k' \text{ and } s' = -s''. \quad (\text{VII.18})$$

To find an expression for $E^s(k)$ we first derive an equation for $G^s(k)$ which is immediately related to the ensemble average of the product of two fluctuation amplitudes. In a similar way, like in chapter IV, we obtain with the aid of eq. (VII.7)

$$\begin{aligned} \frac{\partial}{\partial t} [A^{s_1}(k_1) A^{s_2}(k_2)] = \\ = \left\{ -i \sum_{s', s''} \int_{-\infty}^{+\infty} dk' V^{s_1, s', s''}(k_1; k', k_1 - k') A^{s'}(k') A^{s''}(k_1 - k') A^{s_2}(k_2) \times \right. \\ \times e^{i[s' \omega_R(k') + s'' \omega_R(k_1 - k') - s_1 \omega_R(k_1)] t + [\gamma(k_1) - \gamma(k_1 - k') - \gamma(k')] t} \Big\} + \\ + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\} \end{aligned} \quad (\text{VII.19a})$$

and further

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[A^{s_1}(k_1) A^{s_2}(k_2) A^{s_3}(k_3) \right] = \\
 & = \left\{ -i \sum_{s', s''} \int_{-\infty}^{+\infty} dk' V^{s_1, s', s''}(k_1; k', k_1 - k') \times \right. \\
 & \quad \times A^{s'}(k') A^{s''}(k_1 - k') A^{s_2}(k_2) A^{s_3}(k_3) \times \\
 & \quad \times e^{i[s' \omega_R(k') + s'' \omega_R(k_1 - k') - s \omega_R(k_1)] t + [\gamma(k_1) - \gamma(k_1 - k') - \gamma(k')] t} \Big\} + \\
 & \quad + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\} + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_3 \\ k_1 & \longleftrightarrow & k_3 \end{matrix} \right\}. \quad (\text{VII.19b})
 \end{aligned}$$

Taking the ensemble average of eq. (VII.19a), using eqs. (VII.13) and (VII.16), one arrives at an equation which has a common $\delta(k_1 + k_2)$ function. Integrating the whole equation over the k_2 -space, and replacing (k_1, s_1) by (k, s) , one finally obtains:

$$\begin{aligned}
 & \frac{\partial}{\partial t} G^s(k) = \\
 & = \left\{ -i \sum_{s', s''} \int_{-\infty}^{+\infty} dk' V^{s, s', s''}(k; k', k - k') H^{s', s'', -s}(k', k - k') \times \right. \\
 & \quad \times e^{i[s' \omega_R(k') + s'' \omega_R(k - k') - s \omega_R(k)] t + [\gamma(k) - \gamma(k - k') - \gamma(k')] t} + \\
 & \quad + \left\{ \begin{matrix} s & \longleftrightarrow & -s \\ k & \longleftrightarrow & -k \end{matrix} \right\}, \quad (\text{VII.20a})
 \end{aligned}$$

Similarly, the ensemble average over eq. (VII.19b) gives, with the aid of eqs. (VII.16) and (VII.17), an equation in which all terms have now a common $\delta(k_1 + k_2 + k_3)$ function.

Integrating over the k_3 -space we then get, taking into account eq. (VII.18) and the property (VII.14a):

$$\begin{aligned}
 & \frac{\partial}{\partial t} H^{s_1, s_2, s_3}(k_1, k_2) = \\
 & = \left\{ -iV^{s_1, -s_2, -s_3}(k_1; -k_2, k_1+k_2) G^{s_2}(k_2) G^{s_3}(-k_1-k_2) \times \right. \\
 & \times e^{i[-s_2\omega_R(-k_2) - s_3\omega_R(k_1+k_2) - s_1\omega_R(k_1)]t + [\gamma(k_1) - \gamma(k_1+k_2) - \gamma(-k_2)]t} - \\
 & - iV^{s_1, -s_3, -s_2}(k_1; k_1+k_2, -k_2) G^{s_3}(-k_1-k_2) G^{s_2}(k_2) \times \\
 & \times e^{i[-s_3\omega_R(k_1+k_2) - s_2\omega_R(-k_2) - s_1\omega_R(k_1)]t + [\gamma(k_1) - \gamma(-k_2) - \gamma(k_1+k_2)]t} \Big\} + \\
 & + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\} + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_3 \\ k_1 & \longleftrightarrow & k_3 \end{matrix} \right\} + \text{terms with F.} \quad (\text{VII.20b}')
 \end{aligned}$$

In view of eq. (VII.9), this equation simplifies to:

$$\begin{aligned}
 & \frac{\partial}{\partial t} H^{s_1, s_2, s_3}(k_1, k_2) = \\
 & = \left\{ -2iV^{s_1, -s_2, -s_3}(k_1; -k_2, k_1+k_2) G^{s_2}(k_2) G^{s_3}(-k_1-k_2) \times \right. \\
 & \times e^{i[-s_2\omega_R(-k_2) - s_3\omega_R(k_1+k_2) - s_1\omega_R(k_1)]t + [\gamma(k_1) - \gamma(k_1+k_2) - \gamma(-k_2)]t} \Big\} + \\
 & + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\} + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_3 \\ k_1 & \longleftrightarrow & k_3 \end{matrix} \right\} + \text{terms with F.} \quad (\text{VII.20b})
 \end{aligned}$$

In both equations, the relation

$$k_1 + k_2 + k_3 = 0 \quad (\text{VII.21})$$

must be taken into account.

Of course, this hierarchy (VII.20) closely resembles the one obtained in chapter IV. To calculate the corresponding kinetic equation we first introduce the concept of weak turbulence:

$$G^S(k) = \varepsilon G_1^S(k), \quad (\text{VII.22a})$$

$$H^{s_1, s_2, s_3}(k_1, k_2) = \varepsilon^2 H_2^{s_1, s_2, s_3}(k_1, k_2) + \varepsilon^3 \dots, \quad (\text{VII.22b})$$

$$F^{s_1, s_2, s_3, s_4}(k_1, k_2, k_3) = \varepsilon^3 F_3^{s_1, s_2, s_3, s_4} + \dots \quad (\text{VII.22c})$$

All wave-correlation functions defined here with a subscript are of $O(1)$!

The formal kinetic equation will now be written as:

$$\frac{\partial E_1^S(k)}{\partial t} = A_1(k; E_1) + \varepsilon A_2(k; E_1) + \dots, \quad (\text{VII.23})^{\dagger}$$

where the A_i 's ($i = 1, 2, \dots$) are functionals of both energy-spectrum functions $E_1^+(k)$ and $E_1^-(k)$.

In view of the definitions (see eqs. (VII.12a,b))

$$E_1^S(k, t) = G_1^S(k, t) e^{-2\gamma(k)t} \quad (\text{VII.24})$$

it, again, seems useful to introduce (compare eq. (V.8))

$$\frac{\partial G_1^S(k)}{\partial t} = B_1(k) + \varepsilon B_2(k) + \dots, \quad (\text{VII.25})$$

which results in the following expressions

$$A_1(k; E) = -2\gamma(k) E_1^S(k) + B_1(k) e^{-2\gamma(k)t}, \quad (\text{VII.26a})$$

$$A_2(k; E) = B_2(k) e^{-2\gamma(k)t}. \quad (\text{VII.26b})$$

[†] Here, we do not assign an index to E ; it means that both $E^+(k)$ and $E^-(k)$ are to be included in the corresponding equations.

Substituting eqs. (VII.22a), (VII.22b), and (VII.25) into the first equation of the hierarchy (VII.20a), one finds, in $O(1)$:

$$B_1(k) = \left. \frac{\partial G_1^s(k)}{\partial \tau} \right|_{O(1)} = 0 \text{ for } s = +, -.$$

At this order, the function $G^s(k)$ does not change yet due to nonlinear interaction with all other modes.

The next order reads:

$$\begin{aligned} B_2(k) = & \\ = & \left\{ -i \int_{s', s''}^{+\infty} dk' V^{s, s', s''}(k; k', k-k') H_2^{s', s'', -s}(k', k-k') \times \right. \\ & \times e^{i[s' \omega_R(k') + s'' \omega_R(k-k') - s \omega_R(k)] \tau + [\gamma(k) - \gamma(k-k') - \gamma(k')] \tau} \Big\} + \\ & + \left\{ \begin{matrix} s \longleftrightarrow -s \\ k \longleftrightarrow -k \end{matrix} \right\} \quad (s = +, -). \end{aligned} \quad (\text{VII.28})$$

To get a dynamical equation for the particular triple-order wave-correlation function appearing in the last equation, we replace, in eq. (VII.20b), (k_1, s_1) , (k_2, s_2) , and (k_3, s_3) by (k', s') , $(k-k', s'')$, and $(-k, -s)$, respectively. Then we obtain in lowest order the following equation, using eq. (VII.22b)

$$\begin{aligned} & \frac{\partial H_2^{s', s'', -s}(k', k-k')}{\partial \tau} = \\ = & \left\{ -2i V^{s', -s'', s}(k'; k'-k, k) G_1^{s''}(k-k') G_1^{-s}(-k) \times \right. \\ & \times e^{i[-s'' \omega_R(k'-k) + s \omega_R(k) - s' \omega_R(k')] \tau + [\gamma(k') - \gamma(k) - \gamma(k'-k)] \tau} \Big\} + \\ & + \left\{ \begin{matrix} s' \longleftrightarrow s'' \\ k' \longleftrightarrow k-k' \end{matrix} \right\} + \left\{ \begin{matrix} s' \longleftrightarrow -s \\ k' \longleftrightarrow -k \end{matrix} \right\}. \end{aligned} \quad (\text{VII.29})$$

At lowest order (see eq. (VII.27)), the function G^s is independent of t and one thus immediately can integrate eq. (VII.29). Using the synchronization hypothesis

$$H_2^{s', s'', -s}(k', k-k'; t=0) = H_2^{s', s'', -s}(k', k-k'; G(t=0)) \quad (\text{VII.30})$$

and eqs. (VII.2), (VII.3), and (VII.14a), we arrive at an approximation up to order ε , viz.

$$\begin{aligned} H_2^{s', s'', -s}(k', k-k') = & \\ = & \left\{ -2i v^{s', -s'', s}(k'; k'-k, k) G_1^s(k) G_1^{s''}(k-k') \times \right. \\ \times & \frac{e^{i[s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]t + [\gamma(k') - \gamma(k-k') - \gamma(k)]t}}{[\gamma(k') - \gamma(k-k') - \gamma(k)] + i[s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]} \Big\} + \\ & + \left\{ \begin{matrix} s' \longleftrightarrow s'' \\ k' \longleftrightarrow k-k' \end{matrix} \right\} + \left\{ \begin{matrix} s' \longleftrightarrow -s \\ k' \longleftrightarrow -k \end{matrix} \right\}. \end{aligned} \quad (\text{VII.31})$$

After substitution of this last equation into eq. (VII.28) one arrives, with the aid of eqs. (VII.24), (VII.26b), and (VII.8b), at the following expression for eq. (VI.26b):

$$\begin{aligned} A_2(k; E) = & \left\{ -2 \sum_{s', s''} \int_{-\infty}^{+\infty} dk' \times \right. \\ \times & \left[\frac{v^{s, s', s''}(k; k', k-k') v^{s', -s'', s}(k'; k'-k, k) E^s(k) E^{s''}(k-k')}{[\gamma(k') - \gamma(k-k') - \gamma(k)] + i[s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]} + \right. \\ + & \frac{v^{s, s', s''}(k; k', k-k') v^{s'', -s', s}(k-k'; -k', k) E^s(k) E^{s'}(k')}{[\gamma(k-k') - \gamma(k') - \gamma(k)] + i[s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]} - \\ - & \left. \frac{|v^{s, s', s''}(k; k', k-k')|^2 E^{s'}(k') E^{s''}(k-k')}{[\gamma(k) - \gamma(k-k') - \gamma(k')] + i[s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]} \right] \Big\} + \\ & + \left\{ \begin{matrix} s \longleftrightarrow -s \\ k \longleftrightarrow -k \end{matrix} \right\}, \end{aligned} \quad (\text{VII.32})$$

where, as desired, the function A_2 is a functional of the energy-spectrum functions $E^s(s = +, -)$. When calculating the last term { } we replace, in accordance with eq. (VII.8b), (s, k) by $(-s, -k)$, (s', k') by $(-s', -k')$, and $(s'', k-k')$ by $(-s'', k'-k)$. Substituting then eqs. (VII.32) and (VII.26a) with (VII.27) into eq. (VII.23), we arrive at the wanted kinetic equations after multiplying numerator and denominator of each term by the complex conjugate of the denominator (the imaginary parts of all contributions then cancel as they should since E^s is real):

$$\begin{aligned} \frac{\partial E^s(k)}{\partial t} = & -2\gamma(k)E^s(k) + \\ + & 4 \int_{s', s''}^{+\infty} dk' \left\{ |V^{s, s', s''}(k; k', k-k')|^2 E^{s'}(k') E^{s''}(k-k') \times \right. \\ \times & \frac{[\gamma(k) - \gamma(k-k') - \gamma(k')]}{[\gamma(k) - \gamma(k-k') - \gamma(k')]^2 + [s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]^2} - \\ - & \operatorname{Re} [V^{s, s', s''}(k; k', k-k') V^{s', -s'', s}(k'; k'-k, k)] E^s(k) E^{s''}(k-k') \times \\ \times & \frac{[\gamma(k') - \gamma(k-k') - \gamma(k)]}{[\gamma(k') - \gamma(k-k') - \gamma(k)]^2 + [s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]^2} - \\ - & \operatorname{Im} [V^{s, s', s''}(k; k', k-k') V^{s', -s'', s}(k'; k'-k, k)] E^s(k) E^{s''}(k-k') \times \\ \times & \frac{[s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]}{[\gamma(k') - \gamma(k-k') - \gamma(k)]^2 + [s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]^2} - \\ - & \operatorname{Re} [V^{s, s', s''}(k; k', k-k') V^{s'', -s', s}(k-k'; -k', k)] E^s(k) E^{s'}(k') \times \\ \times & \frac{[\gamma(k-k') - \gamma(k') - \gamma(k)]}{[\gamma(k-k') - \gamma(k') - \gamma(k)]^2 + [s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')]^2} \end{aligned}$$

$$\begin{aligned}
& - \operatorname{Im} [V^{s, s', s''}(k; k', k-k') V^{s'', -s', s}(k-k'; -k', k)] E^s(k) E^{s'}(k') \times \\
& \times \frac{[\bar{s} \omega_R(k) - s' \omega_R(k') - s'' \omega_R(k-k')]}{[\gamma(k-k') - \gamma(k') - \gamma(k)]^2 + [\bar{s} \omega_R(k) - s' \omega_R(k') - s'' \omega_R(k-k')]^2} \Big\} (s=+, -).
\end{aligned}
\tag{VII.33}$$

These equations seem to be rather difficult and very complicated. However, in most cases the interaction strength V is a purely real function; all terms with $\operatorname{Im}[\dots]$ then vanish. Further, one can show the identity of the second and third of the remaining terms in the integral by passing in the former from the integration and summation variables (k', s', s'') to $(k-k', s'', s')$. Equation (VII.33) then reduces to:

$$\begin{aligned}
& \frac{\partial E^s(k)}{\partial t} = -2\gamma(k) E^s(k) + \\
& + 4 \sum_{s', s''} \int_{-\infty}^{+\infty} dk' \Big\{ [V^{s, s', s''}(k; k', k-k')]^2 E^{s'}(k') E^{s''}(k-k') \times \\
& \times \frac{[\gamma(k) - \gamma(k-k') - \gamma(k')]}{[\gamma(k) - \gamma(k-k') - \gamma(k')]^2 + [\bar{s} \omega_R(k) - s' \omega_R(k') - s'' \omega_R(k-k')]^2} - \\
& - 2V^{s, s', s''}(k; k', k-k') V^{s'', -s', s}(k-k'; -k', k) E^s(k) E^{s'}(k') \times \\
& \times \frac{[\gamma(k-k') - \gamma(k') - \gamma(k)]}{[\gamma(k-k') - \gamma(k') - \gamma(k)]^2 + [\bar{s} \omega_R(k) - s' \omega_R(k') - s'' \omega_R(k-k')]^2} \Big\}.
\end{aligned}
\tag{VII.34}$$

In view of eq. (VII.8) this equation has a great similarity with the previously derived kinetic equation (V.29), omitting there the last term (V has to be real). This similarity holds in spite of the fact that the energy-spectrum functions $E^s(k)$ are not even functions of k !

Without growth or damping (see section VI.B) one had

to take the limit $[\gamma(k_1) - \gamma(k_2) - \gamma(k_3)] \rightarrow +0$ in order to find the correct answer. Doing so, we obtain from eq. (VII.34):

$$\begin{aligned} \frac{\partial E^s(k)}{\partial t} = & 4\pi \sum_{s', s''} \int_{-\infty}^{+\infty} dk' \times \\ & \times \left\{ [V^{s, s', s''}(k; k', k-k')]^2 E^{s'}(k') E^{s''}(k-k') - \right. \\ & - 2 [V^{s, s', s''}(k; k', k-k') V^{s'', -s', s}(k-k'; -k', k)] E^s(k) E^{s'}(k') \left. \right\} \times \\ & \times \delta(s\omega_R(k) - s'\omega_R(k') - s''\omega_R(k-k')). \end{aligned} \quad (\text{VII.35})$$

The first term of this equation, describing the interaction of two different modes (k', s') and $(k-k', s'')$, constitutes again a positive quantity, even for arbitrary values of s' and s'' . The remaining term may represent a nonlinear growth or damping.

The kinetic equation (VII.35) allows several three-wave resonant interactions. We do not go into detail but refer to the application of this theory in chapter IX. Finally, we notice that, in view of eqs. (VII.2) and (VII.6), the case of vanishing $\omega_R(k)$ frequencies is irrelevant for this analysis.

C H A P T E R   V I I I

A THIRD ONE-DIMENSIONAL MODEL WITH PURELY DAMPED AND PURELY GROWING MODES

A. The model equation and corresponding symmetry properties

In the preceding section a kinetic equation has been derived for the variance of the fluctuating amplitudes of two waves, travelling in opposite directions. In contrast with chapter II, where we introduced the first-model equation, the real frequency was an even function of k (see eq. (VII.2)) and, moreover, these modes had a common dissipation factor $e^{-\gamma(k)t}$. Although the forthcoming problem is already studied for stable and undamped waves^{1,3}), it seems very interesting to study the case where two independent k -modes exist. They may consist of purely damped modes and purely unstable ones (with equal rates of damping and growth, respectively), all of which will interact with each other. This third model turns out to be a "simple" extension of the theory developed in chapter II.

Mathematically, we have to start from the following expressions:

$$\psi(k, t) = A_1(k, t)e^{\gamma(k)t} + A_2(k, t)e^{-\gamma(k)t} = \quad (\text{VIII.1})$$

$$= \sum_{s=1,2} A_s(k, t)e^{-\gamma_s(k)t}, \quad (\text{VIII.2})$$

with the definitions, $\gamma(k)$ being positive,

$$\gamma_1(k) = -\gamma(k), \quad \gamma_2(k) = +\gamma(k), \quad \text{while } \gamma_s(k) = \gamma_s(-k) \text{ for } s=1,2. \quad (\text{VIII.3})$$

For convenience, we choose the same absolute value for the growth and damping rate although other possibilities might also be treated. The reality condition (II.24) for $\psi(k, t)$ here results in

$$A_s(k) = A_s^*(-k) \quad (s = 1, 2). \quad (\text{VIII.4})$$

Again, for simplicity, we shall omit the t -dependence.

A third-model equation can be set up, describing the interaction of a particular $A_s(k)$ amplitude with all other modes, including the independent ones.

Similar to eq. (VII.7) we assume the model equation:

$$\begin{aligned} \frac{\partial A_s(k)}{\partial t} = & -i \sum_{s', s''} \int_{-\infty}^{+\infty} dk' V_{s, s', s''}(k; k', k-k') \times \\ & \times A_{s'}(k') A_{s''}(k-k') e^{[\gamma_s(k) - \gamma_{s'}(k') - \gamma_{s''}(k-k')]t} \end{aligned} \quad (\text{VIII.5})$$

With the aid of eq. (VIII.4), applied to eq. (VIII.5), one obtains the following property for the interaction strength:

$$V_{s, s', s''}(k; k', k-k') = -V_{s, s', s''}^*(-k; -k', k'-k). \quad (\text{VIII.6})$$

This relation completely resembles the first and the last term of the relation (II.32), valid for the interaction

strength of the first model, but not the one of chapter VII (see eq. (VII.8b)). From symmetry considerations we infer that

$$V_{s,s',s''}(k;k',k-k') = V_{s,s'',s'}(k;k-k',k'), \quad (\text{VIII.7})$$

which is now the same relation as eq. (VII.8a). Both relations (VIII.6) and (VIII.7) are also equivalent to the corresponding ones of chapter II (see eq. (II.32)).

B. Wave-correlation functions

Just like in chapters II and VII we do not allow the presence of physical quantities with a nonvanishing average in the system. Hence,

$$\langle A_s(k,t) \rangle = 0 \text{ for } s = 1,2 \text{ and all } k. \quad (\text{VIII.8})$$

Next, the following definitions are obviously suggested, taking into account the independence of the two k -modes. From eqs. (II.25), (VIII.1), (VIII.2), and (VIII.4) we obtain:

$$\begin{aligned} \langle \psi(k_1,t) \psi(k_2,t) \rangle &= \langle \psi(k_1,t) \psi^*(-k_2,t) \rangle = \\ &= E(k_1) \delta(k_1+k_2) = \sum_s G_s(k_1) e^{-2\gamma_s(k_1)t} \delta(k_1+k_2) = \\ &= \sum_s E_s(k_1) \delta(k_1+k_2), \end{aligned} \quad (\text{VIII.9})$$

where we have introduced the following assumption leading to the definitions of G_s :

$$\langle A_{s_1}(k_1) A_{s_2}(k_2) \rangle = \langle A_{s_1}(k_1) A_{s_2}^*(-k_2) \rangle = G_{s_1}(k_1) \delta(k_1+k_2) \delta_{s_1,s_2}. \quad (\text{VIII.10})$$

Hence, the total energy-spectrum function $E(k)$ consists of the sum of the energy-spectrum functions of the stable and unstable modes. Obviously, the last expression completely agrees with the corresponding one of chapter III (see eq. (III.10)). Similarly, the same properties for the energy-spectrum function, as derived in chapter III, also hold for the present spectrum functions $E_s(k)$ and $G_s(k)$. Therefore, the other wave-correlation functions are easily derived without any difficulty. Summarizing, we have

$$\langle A_{s_1}(k_1) A_{s_2}(k_2) \rangle = G_{s_1}(k_1) \delta(k_1 + k_2) \delta_{s_1, s_2} \quad (\text{VIII.11a})$$

with

$$G_s(k_1) = G_s(-k_1) = G_s(k_2), \quad (\text{VIII.11a}')$$

$$\langle A_{s_1}(k_1) A_{s_2}(k_2) A_{s_3}(k_3) \rangle = H_{s_1, s_2, s_3}(k_1, k_2) \delta(k_1 + k_2 + k_3), \quad (\text{VIII.11b})$$

and

$$\begin{aligned} \langle A_{s_1}(k_1) A_{s_2}(k_2) A_{s_3}(k_3) A_{s_4}(k_4) \rangle = \\ = F_{s_1, s_2, s_3, s_4}(k_1, k_2, k_3) \delta(k_1 + k_2 + k_3 + k_4) + \\ + G_{s_1}(k_1) G_{s_3}(k_3) \delta(k_1 + k_2) \delta(k_3 + k_4) \delta_{s_1, s_2} \delta_{s_3, s_4} + \\ + G_{s_1}(k_1) G_{s_2}(k_2) \delta(k_1 + k_3) \delta(k_2 + k_4) \delta_{s_1, s_3} \delta_{s_2, s_4} + \\ + G_{s_1}(k_1) G_{s_2}(k_2) \delta(k_1 + k_4) \delta(k_2 + k_3) \delta_{s_1, s_4} \delta_{s_2, s_3}. \quad (\text{VIII.11c}) \end{aligned}$$

The last two relations, analogous to eqs. (VII.16) and (VII.17), only differ from the latter in view of the modified Kronecker deltas.

With the aid of all these definitions and properties, we are able to apply the theory of chapter V or VII.

C. The kinetic equations for the third one-dimensional model

In view of eq. (VIII.8) one finds, with the aid of eq. (VIII.11a) and, by taking the ensemble average over the model equation (VIII.5), the following requirement for the interaction strength:

$$V_{s,s',s''}(0;k',-k') = 0 \text{ for } s' = s''. \quad (\text{VIII.12})$$

Similar to eqs. (IV.4a,b) and (VII.19a,b), an hierarchy can be established as follows:

$$\begin{aligned} \frac{\partial}{\partial t} [A_{s_1}(k_1)A_{s_2}(k_2)] &= \left\{ -i \sum_{s',s''} \int_{-\infty}^{+\infty} dk' \times \right. \\ &\times V_{s_1,s',s''}(k_1;k',k_1-k') A_{s'}(k') A_{s''}(k_1-k') A_{s_2}(k_2) \times \\ &\times e^{\left[\gamma_{s_1}(k_1) - \gamma_{s'}(k') - \gamma_{s''}(k_1-k') \right] t} \Big\} + \\ &+ \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\}, \end{aligned} \quad (\text{VIII.13a})$$

and

$$\begin{aligned} \frac{\partial}{\partial t} [A_{s_1}(k_1)A_{s_2}(k_2)A_{s_3}(k_3)] &= \left\{ -i \sum_{s',s''} \int_{-\infty}^{+\infty} dk' \times \right. \\ &\times V_{s_1,s',s''}(k_1;k',k_1-k') A_{s'}(k') A_{s''}(k_1-k') A_{s_2}(k_2) \times \\ &\times A_{s_3}(k_3) e^{\left[\gamma_{s_1}(k_1) - \gamma_{s'}(k') - \gamma_{s''}(k_1-k') \right] t} \Big\} + \\ &+ \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\} + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_3 \\ k_1 & \longleftrightarrow & k_3 \end{matrix} \right\}. \end{aligned} \quad (\text{VIII.13b})$$

Taking next the ensemble average over eq. (VIII.13a) for $s_2 = s_1$ we obtain, using eqs. (VIII.11a,b) and integrating over the k_2 -space:

$$\begin{aligned} \frac{\partial}{\partial t} G_s(k) = & \\ = \left\{ -i \sum_{s', s''} \int_{-\infty}^{+\infty} dk' V_{s, s', s''}(k; k', k-k') H_{s', s'', s}(k', k-k') \times \right. & \\ \times e^{\left[\gamma_s(k) - \gamma_{s'}(k') - \gamma_{s''}(k-k') \right] t} \Big\} + \{k \leftrightarrow -k\}, & \quad (\text{VIII.14a}) \end{aligned}$$

where k_1 , as usual, is replaced by k .

The last term between brackets guarantees the property of the second-order wave-correlation function $G_s(k)$ to be even.

Using eqs. (VIII.7), (VIII.11b,c), and the symmetry property (VIII.11a'), the average over the second equation (VIII.13b) of the hierarchy proves to result in

$$\begin{aligned} \frac{\partial}{\partial t} H_{s_1, s_2, s_3}(k_1, k_2) = & \\ = \left\{ -2i V_{s_1, s_2, s_3}(k_1; -k_2, -k_3) G_{s_2}(k_2) G_{s_3}(-k_3) \times \right. & \\ \times e^{\left[\gamma_{s_1}(k_1) - \gamma_{s_2}(-k_2) - \gamma_{s_3}(k_3) \right] t} \Big\} + & \\ + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_2 \\ k_1 & \longleftrightarrow & k_2 \end{matrix} \right\} + \left\{ \begin{matrix} s_1 & \longleftrightarrow & s_3 \\ k_1 & \longleftrightarrow & k_3 \end{matrix} \right\} + \text{term containing } F, & \quad (\text{VIII.14b}) \end{aligned}$$

with $k_1 + k_2 + k_3 = 0$.

Now, the relevant third-order wave-correlation function $H_{s', s'', s}(k', k-k')$ needed in eq. (VIII.14a) can be calculated from the last equation by replacing (k_1, s_1) , (k_2, s_2) , and (k_3, s_3) by (k', s') , $(k-k', s'')$, and $(-k, s)$, respectively; we neglect the F -term in order to get a closed system. Taking account of the even property of $\gamma_s(k)$ (see eq. (VIII.3)),

the result then reads

$$\begin{aligned}
 & \frac{\partial}{\partial t} H_{s', s'', s}(k', k-k') = \\
 & = \left\{ -2iV_{s', s'', s}(k'; k'-k, k) G_{s''}(k-k') G_s(k) \times \right. \\
 & \times e^{\left[\gamma_{s'}(k') - \gamma_{s''}(k-k') - \gamma_s(k) \right] t} \left\{ \begin{matrix} s' \longleftrightarrow s'' \\ k' \longleftrightarrow k-k' \end{matrix} \right\} + \left\{ \begin{matrix} s' \longleftrightarrow s \\ k' \longleftrightarrow -k \end{matrix} \right\} \right\}.
 \end{aligned}
 \tag{VIII.15}$$

Henceforth, the analysis becomes similar to that of chapter V. Remembering (see eq. (VIII.9))

$$E_s(k) = G_s(k) e^{-2\gamma_s(k)t} \tag{VIII.16}$$

we finally obtain, with the aid of the integral of (VIII.15) up to second order,

$$\begin{aligned}
 & \frac{\partial E_s(k)}{\partial t} = -2\gamma_s(k) E_s(k) - \\
 & -2 \left\{ \sum_{s', s''} \int_{-\infty}^{+\infty} dk' \left[V_{s, s', s''}(k; k', k-k') V_{s', s'', s}(k'; k'-k, k) \times \right. \right. \\
 & \times \frac{E_{s''}(k-k') E_s(k)}{\left[\gamma_{s'}(k') - \gamma_{s''}(k-k') - \gamma_s(k) \right]} + \\
 & + V_{s, s', s''}(k; k', k-k') V_{s'', s', s}(k-k'; -k', k) \times \\
 & \times \frac{E_{s'}(k') E_s(k)}{\left[\gamma_{s''}(k-k') - \gamma_{s'}(k') - \gamma_s(k) \right]} + \\
 & + V_{s, s', s''}(k; k', k-k') V_{s, s'', s}(-k; k'-k, -k') \times
 \end{aligned}$$

$$\times \frac{E_{s''}(k-k')E_s(k')}{[\gamma_s(k)-\gamma_{s'}(k')-\gamma_{s''}(k-k')]} \Big] \Big\} - 2 \{k \leftrightarrow -k\} \quad (s=1,2). \quad (\text{VIII.17})$$

This equation can as yet be reduced to the following one, using the relations (VIII.6) and (VIII.7), while changing the variables of the first term, according to $(k',s') \longleftrightarrow (k-k',s'')$:

$$\begin{aligned} \frac{\partial E_s(k)}{\partial t} = & -2\gamma_s(k)E_s(k) + 2 \left\{ \sum_{s',s''} \int_{-\infty}^{+\infty} dk' \times \right. \\ & \times \left[|V_{s,s',s''}(k;k',k-k')|^2 \frac{E_{s''}(k-k')E_s(k')}{[\gamma_s(k)-\gamma_{s'}(k')-\gamma_{s''}(k-k')]} - \right. \\ & - 2V_{s,s',s''}(k;k',k-k')V_{s'',s',s}(k-k';-k',k) \times \\ & \left. \left. \times \frac{E_{s'}(k')E_s(k)}{[\gamma_{s''}(k-k')-\gamma_{s'}(k')-\gamma_s(k)]} \right] \right\} + 2 \{k \leftrightarrow -k\}. \quad (\text{VIII.18}) \end{aligned}$$

It is easily shown that the term with $k \leftrightarrow -k$ guarantees the reality of the energy-spectrum functions $E_s(k)$; one has to change all signs of the different k - and k' -values and use the relation (VIII.6). If the interaction strength V would be real, the result becomes:

$$\begin{aligned} \frac{\partial E_s(k)}{\partial t} = & -2\gamma_s(k)E_s(k) + 4 \sum_{s',s''} \int_{-\infty}^{+\infty} dk' \times \\ & \times \left[(V_{s,s',s''}(k;k',k-k'))^2 \frac{E_{s'}(k')E_{s''}(k-k')}{[\gamma_s(k)-\gamma_{s'}(k')-\gamma_{s''}(k-k')]} - \right. \\ & - 2V_{s,s',s''}(k;k',k-k')V_{s'',s',s}(k-k';-k',k) \times \\ & \left. \times \frac{E_{s'}(k')E_s(k)}{[\gamma_{s''}(k-k')-\gamma_{s'}(k')-\gamma_s(k)]} \right]. \quad (\text{VIII.19}) \end{aligned}$$

Quite generally, as is to be expected from the structure of the third-model equation, three different interactions contribute to the energy-spectrum function of a particular (k, s) wave. First of all, it is seen from the nonlinear term of eq. (VIII.19) that a process of self-interaction occurs if $s' = s'' = s$ (s fixed). Secondly, a mixed interaction takes place for $s' = s$, $s'' = -s$ and $s'' = s$, $s' = -s$. Finally, one observes a strange interaction if $s' = -s$ and also if $s'' = -s$.

The sign of the first contribution to the integral is determined only by that of $[\gamma_s(k) - \gamma_s(k') - \gamma_s(k-k')]$. Unfortunately, it is not possible to split up the positive and negative contributions precisely, because the γ 's are k -dependent. However, one can remark for the growing mode ($s=1$) that it loses energy by interaction with the two damped modes ($s' = s'' = 2$). Vice versa, the damped mode ($s = 2$) gains energy by its interaction with the growing modes.

As to the second contribution to the integral of eq. (VIII.19) nothing can be said if a symmetry condition for V , like in our first model, is missing (see eq. (V.30a)). Let us assume for a moment that a symmetry condition exists for all relevant k - and s -values. One can then easily extend the example given at the end of chapter II, according to which the $\gamma_s(k)$ are supposed to be independent of k ; hence, in the present situation

$$\gamma_1(k) = -\gamma \text{ and } \gamma_2(k) = \gamma. \quad (\text{VIII.20})$$

For the unstable modes the kinetic equation (VIII.19) reduces to:

$$\begin{aligned} \frac{\partial E_1(k)}{\partial t} = & 2\gamma E_1(k) + 4 \int_{-\infty}^{+\infty} dk' \frac{|V|^2}{\gamma} \times \\ & \times \left\{ \left[E_1(k-k') E_1(k') - 2E_1(k-k') E_2(k') - \frac{E_2(k-k') E_2(k')}{3} \right] - \right. \\ & \left. - \frac{8}{3} E_1(k') E_1(k) \right\}. \end{aligned} \quad (\text{VIII.21a})$$

The other one reads:

$$\begin{aligned} \frac{\partial E_2(k)}{\partial t} = & -2\gamma E_2(k) + 4 \int_{-\infty}^{+\infty} dk' \frac{|V|^2}{\gamma} \times \\ & \times \left\{ \left[-E_2(k-k')E_2(k') + 2E_1(k-k')E_2(k') + \frac{E_1(k-k')E_1(k')}{3} \right] + \right. \\ & \left. + \frac{8}{3} E_2(k')E_2(k) \right\}. \end{aligned} \quad (\text{VIII.21b})$$

With regard to the self-interaction, these equations are similar to eqs. (V.33) and (V.34). According to eq. (VIII.21a) the unstable modes lose energy due to the mixed interaction and to the strange interaction with the damped modes (third and fourth term of the right-hand side of eq. (VIII.21a)). Further, the last term of (VIII.21a) represents a nonlinear damping rate. Vice versa, the damped modes gain energy by these just mentioned processes, as expressed by (VIII.21b).

PART II - APPLICATIONS

C H A P T E R IX

HYDRODYNAMICAL TURBULENCE

A. Introduction

It is well recognized that the problem of hydrodynamical turbulence remains one of the few areas of classical physics for which a satisfactory theory has not yet been found. In his lectures about homogeneous turbulence, P.G. Saffman³³) even declared: "...our real understanding of the mechanics of turbulence appears to be practically nil".

From the experimental point of view hydrodynamical turbulence is known to consist of irregular (random) motions of fluid elements. These elements, causing local fluctuations and called eddies, interact rather strongly with each other. A specific property of them concerns the fact that they cannot propagate like waves. It turns out that this description corresponds to incompressible fluid turbulence. The observed small eddies are associated with small-scale

motions and thus in the Fourier space, with respect to the space variables, with large wavenumber values. Vice versa, large eddies (or large-scale motions) correspond to small wavenumbers.

Our main aim is to study the structure of such a turbulent fluid with the aid of our statistical theory as developed for weak turbulence. We shall restrict ourselves to the evolution of a random velocity field, decaying due to viscosity effects, without external sources and without a mean motion. Dynamical equations for the energy-spectrum functions for the kinetic energy of a particular eddy will be derived, in order to reveal the energy transfer between the different eddies.

In section B we shall investigate the general equations of hydrodynamics for fluctuations, which may be either transverse or longitudinal with respect to the direction of their wavenumber vector. The former ones, corresponding to the above-mentioned incompressible eddies, are treated in the two next sections. The first one, section C, concerns a simple one-dimensional model, i.e. the well-known Burgers' equation¹⁰⁾ for fluid turbulence; it leads to an insight into some essential properties of real turbulence. The results are then illustrated by numerical calculations in section D. The theory of section C is next extended in section E for the general case of three-dimensional incompressible turbulence. In section F we discuss our results while comparing them with some obtained by other authors. The crucial point in all statistical theories based on a hierarchy of wave-correlation functions (see, for instance, refs. 18,34), that is the closure hypothesis first introduced by Millionshtchikov¹¹⁾, is considered in particular. This so-called quasi-normal (or nearly Gaussian) hypothesis was also introduced in our approach developed in part I, such in order to get a not too complicated theory for weak turbulence.

We next pass to the case of compressible turbulence which is associated with the nonlinear interaction of lon-

gitudinal (random) modes, which turn out to be sound waves. First, section G quite generally deals with a nondissipative fluid. The simplification introduced by assuming isotropic spectra is considered in section H. Finally, some considerations are devoted to the case including a viscous dissipation (section I).

B. Basic equations

We consider the following set of equations known from the ordinary theory of hydrodynamics:

1. The Navier-Stokes equation

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = - \frac{\underline{\nabla} p}{\rho} + \frac{\nu}{\rho} \nabla^2 \underline{v}. \quad (\text{IX.1})$$

2. The continuity equation

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0. \quad (\text{IX.2})$$

3. A nonspecified equation for the pressure in which this quantity is supposed to be only a function of the mass density ρ :

$$p = f(\rho). \quad (\text{IX.3})$$

The resulting five scalar equations form a consistent set which can be represented by the general form of the basic vector equation (II.1).

In this chapter we shall not treat equations more general than the present ones. It will appear that only special models, abstracted from eqs. (IX.1,2,3), can be treated reasonably.

Under certain circumstances, in the fluid disturbances will originate that are caused either by an external agent (a grid for example) or by an internal one (instabilities).

We shall call it a laminar fluid if these perturbations remain small and damp out later on, due to viscosity effects. But, if they grow, nonlinear terms become important and turbulence arises. A measure for this process is given by the Reynolds' number: it represents the ratio between the nonlinear term and the viscosity term in the Navier-Stokes equation. Formally written, we obtain:

$$Re = \frac{\text{nonlinear term}}{\text{viscosity term}} \approx \frac{u^2/L}{(\nu'/\rho)(u/L^2)} = \frac{uL}{\nu'/\rho} = \frac{uL}{\nu}, \quad (\text{IX.4})$$

where u and L represent characteristic values for the velocity and for the distance along which the fluid pattern changes drastically, respectively. Turbulence will appear, if this dimensionless number is large enough; the nonlinear terms then cannot be neglected anymore. For different experimental situations one can always indicate a critical Reynolds' number for which the laminar fluid changes into a turbulent one.

From our point of view we are not interested in the onset of turbulence, but merely in the decay period of fully developed weak turbulence. This implies that an infinite number of fluctuations is already present and that we are able to apply our general statistical theory of part I. As before, we postulate that the equations mentioned there hold for the turbulent fluid we are interested in.

First we assume

$$\langle \underline{v}(\underline{r}, t) \rangle = 0, \quad (\text{IX.5})$$

Which means that no averaged velocity is allowed. Further, we define the mass density by

$$\rho = \rho_0(1 + \eta), \quad (\text{IX.6})$$

with

$$\langle \rho \rangle = \rho_0 \text{ and } \langle \eta(\underline{r}, t) \rangle = 0. \quad (\text{IX.7})$$

The averaged mass density ρ_0 is supposed to be constant in time and space, while the dimensionless quantity $\eta(\underline{r}, t)$ has to satisfy the relation

$$|\eta| \ll 1. \quad (\text{IX.8})$$

Hence, the pressure term in eq. (X.1) can be written, taking into account eq. (IX.3), as

$$-\frac{\nabla p}{\rho} = -\frac{f'(\rho)\nabla\eta}{1+\eta} \approx -f'(\rho_0)\nabla\eta. \quad (\text{IX.9})$$

Moreover, the coefficient in the viscosity term may be represented by

$$\frac{\nu'}{\rho} \approx \frac{\nu'}{\rho_0} = \nu, \quad (\text{IX.10})$$

ν being known as the kinematic viscosity.

With these expressions in mind, we shall first investigate the linearized equivalents of eqs. (IX.1)-(IX.3), viz.

$$\frac{\partial \underline{v}_L}{\partial t} = -f'(\rho_0)\nabla\eta_L + \nu\nabla^2\underline{v}_L, \quad (\text{IX.11})$$

$$\frac{\partial \eta_L}{\partial t} + \nabla_L \cdot \underline{v}_L = 0, \quad (\text{IX.12})$$

in which the pressure term is eliminated; all relevant quantities are labelled here with the subscript L, which should recall the linearization.

Introducing next the relevant Fourier transforms

$$\underline{v}_L(\underline{r}, t) = \int_{-\infty}^{+\infty} \underline{v}_L(\underline{k}, t) e^{i\underline{k}\cdot\underline{r}} d\underline{k}, \quad (\text{IX.13a})$$

$$\eta_L(\underline{r}, t) = \int_{-\infty}^{+\infty} \eta_L(\underline{k}, t) e^{i\underline{k}\cdot\underline{r}} d\underline{k}, \quad (\text{IX.14a})$$

in which

$$\underline{v}_L^*(\underline{k}, t) = \underline{v}_L(-\underline{k}, t), \quad (\text{IX.13b})$$

$$\eta_L^*(\underline{k}, t) = \eta_L(-\underline{k}, t), \quad (\text{IX.14b})$$

we arrive at the following linearized equations in $\underline{k}$ -space:

$$\frac{\partial \underline{v}_L(\underline{k})}{\partial t} = -i f'(\rho_0) \underline{k} \eta_L(\underline{k}) - v k^2 \underline{v}_L(\underline{k}), \quad (\text{IX.15})$$

$$\frac{\partial \eta_L(\underline{k})}{\partial t} + i \underline{k} \underline{v}_L(\underline{k}) = 0. \quad (\text{IX.16})$$

It is now possible to classify the velocity fluctuations into transverse and longitudinal modes and to consider the corresponding special cases of the just mentioned equations.

1. For transverse velocity fluctuations, defined by $\underline{k} \cdot \underline{v}(\underline{k}) = 0$, we derive from the latter equations:

$$\frac{\partial}{\partial t} \underline{k} \wedge \underline{v}_L(\underline{k}) = -v k^2 \underline{k} \wedge \underline{v}_L(\underline{k}), \quad (\text{IX.17})$$

and

$$\frac{\partial}{\partial t} \eta_L(\underline{k}) = 0. \quad (\text{IX.18})$$

Taking $\eta_L(\underline{k}) \equiv 0$, we trivially satisfy eq. (IX.18) and we get essentially a solenoidal turbulent velocity field through eq. (IX.12). It is then necessary, however, to define a pressure, not depending on the mass density (see eq. (IX.9)). Properly speaking, the original set of equations (IX.1) - (IX.3) in this case reduces to

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} = -\nabla(p/\rho_0) + \nu \nabla^2 \underline{v}, \quad (\text{IX.19})$$

$$\nabla \cdot \underline{v} = 0. \quad (\text{IX.20})$$

This set describes the problem of incompressible turbulence. We shall study this kind of turbulence in sections IX.C,D,E,

and F.

2. For the case of longitudinal velocity fluctuations, defined by $\underline{k} // \underline{v}(\underline{k})$, we derive from (IX.15) and (IX.16):

$$\frac{\partial \underline{k} \underline{v}_L(\underline{k})}{\partial t} = -i f'(\rho_0) k^2 \eta_L(\underline{k}) - v k^2 (\underline{k} \underline{v}_L(\underline{k})), \quad (\text{IX.21})$$

$$\frac{\partial \eta_L(\underline{k})}{\partial t} + i \underline{k} \underline{v}_L(\underline{k}) = 0. \quad (\text{IX.22})$$

This case of compressible turbulence will be treated in sections IX.G, H, and I.

C. The Burgers' model equation; theory

The following fluid equations, viz. (IX.19) and (IX.20), were derived for transverse incompressible fluctuations:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \underline{\nabla}) \underline{v} = - \underline{\nabla} (p / \rho_0) + \nu \nabla^2 \underline{v} \quad (\text{IX.23})$$

and

$$\underline{\nabla} \cdot \underline{v} = 0. \quad (\text{IX.24})$$

Again, we postulate

$$\langle p / \rho_0 \rangle = \langle p \rangle = 0, \quad (\text{IX.25})$$

and

$$\langle \underline{v} \rangle = 0. \quad (\text{IX.26})$$

In view of these two last expressions, the averaged equation of motion (IX.23) results into

$$\langle (\underline{v} \underline{\nabla}) \underline{v} \rangle = 0. \quad (\text{IX.27})$$

It would be tempting to investigate homogeneous turbulence by starting from this simple set of equations. Very

soon, however, it turns out that the latter are very difficult to deal with mathematically (see section E), and it is hard to interpret their results physically. Therefore, it is usual (refs. 35,36,37,38) to test all mathematical methods and techniques for the one-dimensional equivalent of eqs. (IX.23) and (IX.24), namely the pressureless Navier-Stokes equation (incompressible case) or Burgers' equation¹⁰:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \nu \frac{\partial^2 v}{\partial x^2} = 0; \quad (\text{IX.28})$$

the one-dimensional velocity $v(x,t)$ is a fluctuating (or random) quantity here. Fortunately it is well known that this model equation also conserves the most physical properties of the three-dimensional case (refs. 33,39).

A linearization of the Burgers' equation, following the procedure of section III.A, results in a dispersion relation with one branch $\omega(k)$, viz.

$$\omega(k) = -i\nu k^2. \quad (\text{IX.29})$$

Hence, the velocity $v(x,t)$ reads in the Fourier space:

$$v(x,t) = \int_{-\infty}^{+\infty} v(k,t) e^{ikx - \nu k^2 t} dk \quad (\text{IX.30})$$

with again (compare (II.31a), (III.2), and (IV.2)),

$$v(k,t) = v^*(-k,t), \quad (\text{IX.31})$$

$$\langle v(k,t) \rangle = 0 \text{ and } v(k=0,t) = 0. \quad (\text{IX.32})$$

According to (IX.30) the Fourier transform of the complete Burgers' equation (IX.28) becomes:

$$\frac{\partial v(k,t)}{\partial t} = -\frac{ik}{2} \int_{-\infty}^{+\infty} dk' v(k',t) v(k-k',t) e^{-2\nu k'(k'-k)t}, \quad (\text{IX.33})$$

which can most conveniently be verified by writing $\frac{\partial}{\partial x}(v^2/2)$ for $v \frac{\partial v}{\partial x}$.

Attentive readers immediately remark the striking similarity between this Fourier transformed equation and our first model equation (II.28), if one defines the relations:

$$V(k; k', k-k') = \frac{k}{2}, \quad (\text{IX.34})$$

$$\omega(k) = \omega_R(k) - i\gamma(k) \text{ with } \omega_R(k) \equiv 0 \text{ and } \gamma(k) = vk^2, \quad (\text{IX.35})$$

$$A(k, t) = v(k, t). \quad (\text{IX.36})$$

Before continuing this analysis we shall deduce some further information from eq. (IX.33), which is still exact. Defining

$$g(k, k', t) = e^{-2vk'(k'-k)t}, \quad (\text{IX.37})$$

which quantity behaves like a weight function, and

$$\alpha = |2vk'(k'-k)| \quad (\text{IX.38})$$

it is possible to distinct three classes, characterized by the role of the parameter α in eq. (IX.33), viz.

First class: $0 < k' < k$; hence, $g(k, k', t) = e^{\alpha t}$.

Second class: $k' = 0$ and $k' = k$; hence, $g = 1$;
this class does not occur in view of eq. (IX.32).

Third class: all other values of k' ; hence,
 $g(k, k', t) = e^{-\alpha t}$.

We infer that the most important contributions to $\partial v(k, t)/\partial t$ arise from wavenumbers belonging to the first class, if t is "large enough". Moreover, if, at a time $t = t_0$, only modes with a wavenumber between $0 < k < k_c$ do exist, a mode with a wavenumber k_c is caused at a time $t = t_0 + \Delta t_0$, due to non-linear interaction. This is in agreement with the (three-

dimensional) concept that small eddies, corresponding to large values of k , would be produced by nonlinear interaction from larger eddies, corresponding to smaller values ($k' < k$). Besides, we notice that the maximum of $g(k, k', t)$ (for a fixed time) is attained for $k' = k/2$, which is a well-known fact in the nonlinear one-mode theory (the appearance of harmonics).

To arrive at a kinetic equation for the energy-spectrum function one can apply the whole technique of chapters IV and V. First of all we remark that, also in view of eqs. (IX.25) - (IX.27), no averaged quantities are generated because (see eq. (IV.3))

$$V(0; k', -k') = 0, \quad (\text{IX.39})$$

in view of eq. (IX.34).

We next introduce, analogous to eqs. (III.10) and (III.11), the definitions

$$G(k_1) \delta(k_1 + k_2) = \langle v(k_1, t) v(k_2, t) \rangle, \quad (\text{IX.40})$$

$$E(k, t) = G(k, t) e^{-2\nu k^2 t}, \quad (\text{IX.41})$$

with the properties

$$E(k) = E(-k) \geq 0 \text{ for all } k. \quad (\text{IX.42})$$

This enables us to write down immediately the special kinetic equation of section VI.D, describing the interaction between purely damped modes for our Burgers' model. In fact, with the aid of eqs. (IX.34) and (IX.35), we find the following representation of (VI.51)

$$\begin{aligned} \frac{\partial E(k, t)}{\partial t} = & -2\nu k^2 E(k, t) + \\ & + P \int_{-\infty}^{+\infty} dk' \left[k^2 \frac{E(k', t) E(k-k', t)}{2\nu k' (k-k')} + \frac{(k-k') E(k, t) E(k', t)}{\nu k'} \right], \end{aligned} \quad (\text{IX.43})$$

in which k is supposed to be a positive quantity.

Before discussing this equation we will put it in a more convenient form by applying the following properties:

a) The principal-value symbol P can be omitted because $E(k) \sim O(k^2)$ for $k \sim 0$. This follows from a Taylor series expansion around $k \sim 0$:

$$E(k, t) = E(k=0, t) + k \left. \frac{\partial E(k, t)}{\partial k} \right|_{k=0} + \frac{1}{2} k^2 \left. \frac{\partial^2 E(k, t)}{\partial k^2} \right|_{k=0} + O(k^3). \quad (IX.44)$$

Indeed, in view of the relations (IX.32) and (IX.42), the first and the second term of this expansion vanish. As a consequence, the integrand of eq. (IX.43) has no singularity.

b) The first contribution to the integral in (IX.43) consists of one positive and two negative parts ($k > 0$), its integrand being positive for $0 < k' < k$, but negative for $k < k' < \infty$ and for $-\infty < k' < 0$. Hence, we can split up this contribution into a combination of three positive integrals.

c) The function $E(k')$ being an even function of k' , a part of the last integral contribution of (IX.43), viz.

$$k \int_{-\infty}^{+\infty} \frac{E(k, t) E(k', t)}{\sqrt{k'}} \quad (IX.45)$$

will vanish.

Taking into account all these considerations we arrive at the following fundamental kinetic equation for the energy-spectrum function²⁰), passing yet from k' to $-k'$ in the integral extending over the negative k' -values:

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & -2\sqrt{k} k^2 E(k) + \int_0^k \frac{k^2}{2\sqrt{k'}(k-k')} E(k') E(k-k') dk' - \\ & - \int_k^\infty \frac{k^2}{2\sqrt{k'}(k'-k)} E(k') E(k'-k) dk' - \int_0^\infty \frac{k^2}{2\sqrt{k'}(k+k')} E(k') E(k+k') dk' - \end{aligned}$$

$$- \frac{E(k)}{\nu} \int_{-\infty}^{+\infty} E(k') dk'. \quad (IX.46)$$

For simplicity's sake, we omitted the t -dependence in $E(k, t)$. To interpret the associated physical processes, we will discuss each term separately.

The first term on the right-hand side describes the ordinary linear dissipation due to viscosity. The smaller the fluctuations (large k), the more energy is dissipated.

The second term, the "cascade term", represents the energy supply from all lower k' -modes ($0 < k' < k$) to one particular k -mode. It physically means that the large fluctuations transfer energy to the smaller ones. This is an explicit proof of the existence of the well-known cascade process in turbulence, intuitively introduced by L. Onsager⁴⁰) and later on determined by hypothetical transfer terms (see ref. 41).

The third and fourth term describe the loss of energy of the k -mode in question due to a coupling of two modes, the wavenumbers of which differ by k . The last, fifth contribution in eq. (IX.46) represents an additional damping, which is proportional to the total turbulent energy $\langle v(x, t)^2 \rangle$ (see eq. (III.7)).

Because $E(k)$ tends to zero for large k -values, the third, fourth, and fifth term of eq. (IX.46) become negligible in this k -region. Then, the "cascade term" is the dominating one in the nonlinear interaction process!

In hydrodynamical turbulence it is customary to represent an equation like (IX.46) in the following form

$$\frac{\partial E(k, t)}{\partial t} = -2\nu k^2 E(k, t) + T(k, t), \quad (IX.47)$$

$T(k, t)$ being called the inertial transfer term.

Now, the "a priori" energy condition (VI.7) is satisfied trivially since it requires, in view of the relation (IX.34),

$$\frac{k}{2} - \frac{k'}{2} + \frac{(k' - k)}{2} = 0. \quad (\text{IX.48})$$

Hence, in view of the analysis of section VI.A, we find

$$\int_{-\infty}^{+\infty} T(k, t) dk = 0 \quad (\text{IX.49})$$

and next, integrating (IX.47) with respect to k ,

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k, t) dk = \frac{\partial}{\partial t} \langle v^2(x, t) \rangle = -2\nu \int_{-\infty}^{+\infty} k^2 E(k) dk. \quad (\text{IX.50})$$

This equation expresses the conservation of energy, apart from viscous dissipation, because no external sources are assumed in this turbulent system. At each time, the transfer term $T(k, t)$ represents the redistribution of wave energy in k -space.

D. The Burgers' model equation; numerical results

To illustrate the above theory we performed some numerical calculations on the Electrologica X8-computer of the "Elektronisch Rekencentrum" of the University of Utrecht. For this purpose our dynamical equation was first made dimensionless, with the aid of the following substitutions³⁶⁾

$$t = \frac{\tau}{\nu k_0^2}, \quad (\text{IX.51a})$$

$$k = k_0 K, \quad (\text{IX.51b})$$

$$Re^2 = \frac{E_0}{\nu^2 k_0}, \quad (\text{IX.51c})$$

where k_0 and E_0 are units to enable the introduction of proper normalizations for the wavenumber and the energy-spectrum, respectively. Moreover, the parameter Re^2 can be

interpreted as the squared Reynolds' number for the new normalization. Further, we define

$$E(k) = E(k_0 \kappa) = E_1(\kappa), \quad (\text{IX.51d})$$

and, accordingly, for $E(k')$, etc.

In terms of these new quantities, eqs. (IX.46) and (IX.47) become as follows, if we yet pass from the variable κ' to $-\kappa'$ in the third-integral term of eq. (IX.46):

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(\frac{E_1(\kappa)}{E_0} \right) &= -2\kappa^2 \frac{E_1(\kappa)}{E_0} + \frac{\text{Re}^2}{2} \int_{-\infty}^{+\infty} \frac{\kappa^2}{\kappa'(\kappa - \kappa')} \frac{E_1(\kappa')}{E_0} \frac{E_1(\kappa - \kappa')}{E_0} d\kappa' - \\ &- \text{Re}^2 \frac{E_1(\kappa)}{E_0} \int_{-\infty}^{+\infty} \frac{E_1(\kappa')}{E_0} d\kappa' = -2\kappa^2 \frac{E_1(\kappa)}{E_0} + T_1(\kappa, \tau), \quad (\text{IX.52}) \end{aligned}$$

in which

$$T_1(\kappa, \tau) = \frac{\nu \text{Re}^2}{E_0^2 k_0} T \left(k_0 \kappa, \frac{\tau}{\nu k_0^2} \right). \quad (\text{IX.52a})$$

In order to solve this equation, we started with the initial condition

$$\frac{E_1(\kappa, \tau=0)}{E_0} = \kappa^2 e^{-\kappa^2}, \quad (\text{IX.53})$$

which satisfies the conditions $E_1(\kappa)/E_0 \sim O(\kappa^2)$ for $\kappa \rightarrow 0$ and $E_1(\kappa)/E_0 \rightarrow 0$ for $\kappa \rightarrow \infty$, as required. The results are reproduced on page 108 for $\text{Re} = 2$ and $\text{Re} = 4$.

In fig. 1 we sketched the linearized problem, leaving out the whole transfer term T_1 in eq. (IX.52). The elementary solution of this problem reads:

$$\left(\frac{E_1(\kappa, \tau)}{E_0} \right)_{\text{linear}} = \kappa^2 e^{-\kappa^2} e^{-2\kappa^2 \tau}. \quad (\text{IX.54})$$

Obviously, for arbitrary $\Delta \tau > 0$, the following relation is always satisfied:

Figs. 1,2,3. Evolution of the energy spectrum for an initial value $\frac{E_l(k, \tau=0)}{E_0} = k^2 e^{-k^2}$ and various Reynolds' numbers.

$$\left(\frac{E_1(\kappa, \tau + \Delta\tau)}{E_0} \right)_{\text{linear}} < \left(\frac{E_1(\kappa, \tau)}{E_0} \right) \text{ for } 0 < \kappa < \infty. \quad (\text{IX.55})$$

Hence, in this picture no modes can increase due to interaction with other ones. On the contrary, the other figures (2 and 3), which take into account the nonlinearity of the problem, clearly show the "cascade" of energy to higher modes. In fig. 2, the nonlinear effects are not so important, because the Reynolds' number is too small; the cascade process is rather limited there and the dissipation terms quickly dominate the other contributions. The larger the Reynolds' number, the better the cascade process is established. In this way, one can imagine the following experimental picture of this kind of turbulence: energy from external agents is mainly put into the smaller k -values. By means of the cascade process, energy is transferred from these lower k -modes to the higher ones and finally dissipated by viscous effects (see fig. 4). This is also in agreement with the already mentioned concept of three-dimensional homogeneous

Fig. 4. Schematic illustration of the energy transport for HD-turbulence.

turbulence. As an example we mention the generation of atmospheric turbulence by the solar radiation which shows how turbulence can be maintained during a long period, even after the external energy supply has stopped. Mathematically, such a process is described by the following extension of (IX.47):

$$\frac{\partial E(k)}{\partial t} = -2\nu k^2 E(k) + T(k,t) + H_{\text{ext}}(k,t). \quad (\text{IX.56})$$

Further, we observe from fig. 2 and 3 that the energy spectrum strongly changes during a short-time interval after $\tau = 0$ and then reaches more or less an "equilibrium range", especially in fig. 3. The Wiener-Hermite theory of Meecham and Siegel³⁹, too, predicts such fast initial changes which are followed by an "equilibrium range". These phenomena occur due to the nonvanishing value of $T(k, \tau=0)$! Indeed, this function is completely determined by the initial spectra at $\tau = 0$. This is also the reason why our results do not correspond with those from Kawahara³⁶), where $T(k, \tau=0) = 0$.

Finally, we mention that in the "equilibrium range" we have not found a $k^{-\alpha}$ spectrum, with well-defined α , but a decay proportional to a negative k -power which, in this range, increases continuously from about k^{-1} up to k^{-3} . However, our curves quite well fit some experimental results concerning weak or not strong hydrodynamical turbulence (of course three-dimensional) (refs. 42, 43, and 44). Also, we notice a striking similarity between our numerical figures and the corresponding one obtained by Kraichnan (see ref. 45, fig. 4) for his three-dimensional model of turbulence.

E. Three-dimensional incompressible hydrodynamical turbulence

The set of equations (IX.23) - (IX.24) constitutes the starting point here, with the assumptions (IX.25) - (IX.27). For convenience we present them again, introducing the nor-

malized pressure $P = p / \rho_0$:

1) the Navier-Stokes equation

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} = - \nabla P + \nu \nabla^2 \underline{u}, \quad (\text{IX.57})$$

2) the continuity equation

$$\nabla \cdot \underline{u} = 0. \quad (\text{IX.58})$$

The relevant Fourier transforms will be defined by:

$$\underline{u}(\underline{r}, t) = \int_{-\infty}^{+\infty} \underline{u}(\underline{k}, t) e^{i \underline{k} \cdot \underline{r} - \nu k^2 t} d\underline{k}, \quad (\text{IX.59a})$$

with

$$\underline{u}(\underline{k}, t) = \underline{u}^*(-\underline{k}, t), \quad (\text{IX.59b})$$

and

$$P(\underline{r}, t) = \int_{-\infty}^{+\infty} P(\underline{k}, t) e^{i \underline{k} \cdot \underline{r} - \nu k^2 t} d\underline{k}, \quad (\text{IX.60a})$$

with

$$P(\underline{k}, t) = P^*(-\underline{k}, t). \quad (\text{IX.60b})$$

The last equation is formally introduced in order to facilitate the elimination of the pressure term. Further, we again do not allow nonvanishing averaged quantities. Hence,

$$\underline{u}(\underline{k}=0, t) = 0, \quad (\text{IX.61a})$$

$$P(\underline{k}=0, t) = 0. \quad (\text{IX.61b})$$

Substituting eqs. (IX.59a) and (IX.60a) into the basic equations, one arrives at the Fourier transformed set:

$$\begin{aligned} & \frac{\partial \underline{u}(\underline{k}, t)}{\partial t} = \\ & = -i \int_{-\infty}^{+\infty} d\underline{k}' [(\underline{k} - \underline{k}') \underline{u}(\underline{k}', t)] \underline{u}(\underline{k} - \underline{k}', t) e^{-2\nu \underline{k}' \cdot (\underline{k}' - \underline{k}) t} - i \underline{k} P(\underline{k}, t), \end{aligned} \quad (\text{IX.62})$$

$$\underline{k} \cdot \underline{u}(\underline{k}, t) = 0 = \sum_{i=1}^3 k_i u_i(\underline{k}, t). \quad (\text{IX.63})$$

Imposing this last equation on the former one a relation between $P(\underline{k})$ and $\underline{u}(\underline{k})$ is obtained. We can thus eliminate the pressure term and the result reads:

$$\begin{aligned} \frac{\partial \underline{u}(\underline{k}, t)}{\partial t} = \\ = -i \int_{-\infty}^{+\infty} d\underline{k}' \left[\underline{k} \underline{u}(\underline{k}', t) \right] \left[\underline{u}(\underline{k} - \underline{k}', t) - \frac{\underline{k}}{k^2} (\underline{k} \underline{u}(\underline{k} - \underline{k}', t)) \right] e^{-2v\underline{k}'(\underline{k}' - \underline{k})t}, \end{aligned} \quad (\text{IX.64})$$

In order to find an energy spectrum function from this equation it is not sufficient to require homogeneous turbulence, but the turbulence should also be isotropic (no dependence of the averaged quantities on any direction). The wave-correlation function of second order⁴¹⁾ then obtains the form:

$$\langle u_i(\underline{k}_1) u_j(\underline{k}_2) \rangle = \frac{E(k_1)}{2} \left(\delta_{ij} - \frac{k_{1i} k_{1j}}{k_1^2} \right) \delta(\underline{k}_1 + \underline{k}_2) \quad (\text{IX.65})$$

with $|\underline{k}| = k$. In view of eq. (IX.63) this function has to be a divergenceless tensor according to

$$\sum_{i=1}^3 \langle k_{1i} u_i(\underline{k}_1) u_j(\underline{k}_2) \rangle = 0, \quad (\text{IX.66})$$

which relation is satisfied indeed.

Another property reads

$$\sum_{i=1}^3 \langle u_i(\underline{k}_1) u_i(\underline{k}_2) \rangle = E(k_1) \delta(\underline{k}_1 + \underline{k}_2) \quad (\text{IX.67})$$

which represents, as required the proper definition for the energy-spectrum function. In this model it is also possible to derive the other properties for $E(k)$, mentioned in chapter III.

Starting from the basic equation (IX.64) and the rele-

vant definitions and conditions, one can derive a dynamical equation for $E(k)$, repeating all earlier calculations (see chapters IV and V). But, as a matter of fact, most of this very tedious algebra has already been done in the literature (refs. 1, 46, and 47). With the aid of the corresponding calculations, the final kinetic equation for the energy-spectrum function becomes:

$$\frac{\partial E(k)}{\partial t} = -2vk^2 E(k) + \frac{1}{2} k^2 P \int_{-\infty}^{+\infty} \frac{a(\underline{k}, \underline{k}')}{2v\underline{k}'(\underline{k}-\underline{k}')} E(k') E(|\underline{k}-\underline{k}'|) d\underline{k}' -$$

$$- k^2 P \int_{-\infty}^{+\infty} \frac{b(\underline{k}, \underline{k}')}{2v\underline{k}\underline{k}'} E(k) E(k') d\underline{k}', \quad (IX.68)$$

where the following abbreviations have been introduced:

$$a(\underline{k}, \underline{k}') = \frac{1}{2} \left[1 - 2 \frac{(\underline{k}\underline{k}')^2 (\underline{k}\underline{k}'')^2}{k^4 k'^2 k''^2} + \frac{(\underline{k}\underline{k}') (\underline{k}\underline{k}'') (\underline{k}'\underline{k}'')}{k^2 k'^2 k''^2} \right], \quad (IX.69a)$$

$$b(\underline{k}, \underline{k}') = \left[\frac{(\underline{k}\underline{k}')^3}{k^4 k'^2} - \frac{(\underline{k}'\underline{k}'') (\underline{k}\underline{k}'')}{k^2 k''^2} \right], \quad (IX.69b)$$

with

$$\underline{k}'' = \underline{k} - \underline{k}'. \quad (IX.70)$$

Compared with the simple one-dimensional model, it is very difficult to gain some physical insight in such complex equations like (IX.68). However, a cascade process, represented by a part of the second term in (IX.68), also occurs in this system. Indeed, the denominator of the integrand in question is a positive quantity for $0 < k' < k \cos \theta$ (the angle θ being defined by $\underline{k}\underline{k}' = kk' \cos \theta$). Further, it is easily proved that for $k' = 0$, $a(\underline{k}, 0) = \frac{1}{2}(1 - \cos^2 \theta) \geq 0$ and for $k' = k \cos \theta$, $a(\underline{k}, \underline{k}') = \frac{1}{2}[1 - 2 \cos^2 \theta (1 - \cos^2 \theta)] > 0$. Quite generally, we found that, at least for that just mentioned inequality relation ($0 < k' < k \cos \theta$) the whole integrand remained positive.

F. Discussion of the theory of incompressible homogeneous turbulence

From our theory of weak turbulence, we could recognize various current concepts of hydrodynamical turbulence, an explicit analytical expression for the cascade process being the most important one. Nevertheless, scepticism has been raised during the last years against the hierarchy method, as developed in chapter IV for this kind of turbulence. We shall presently discuss this point in more details.

For the one-dimensional Burgers' case, the two first equations of this hierarchy (see eqs. (IV.8a) and (IV.8b), where we shall neglect the terms with F) yield, in view of (IX.34) and (IX.35):

$$\frac{\partial G(k)}{\partial t} = -i \int_{-\infty}^{+\infty} dk' \frac{k}{2} H(k', -k) e^{(-\nu k'^2 - \nu(k-k')^2 + \nu k^2)t} + \{k \leftrightarrow -k\}, \quad (\text{IX.71a})$$

$$\begin{aligned} \frac{\partial H(k', -k)}{\partial t} = & -ik' e^{(-\nu k'^2 - \nu(k-k')^2 + \nu k'^2)t} G(k)G(k-k') + \\ & + ik e^{(-\nu k'^2 - \nu(k-k')^2 + \nu k^2)t} G(k')G(k-k') - \\ & - i(k-k') e^{(-\nu k^2 - \nu k'^2 + \nu(k-k')^2)t} G(k)G(k'). \end{aligned} \quad (\text{IX.71b})$$

Introducing $E(k) = G(k)e^{-2\nu k^2 t}$ we arrive, by eliminating the function H, at the same equations as given in Reid's theory³⁷⁾ viz.

$$\frac{\partial E(k)}{\partial t} = -2\nu k^2 E(k) + T(k, t), \quad (\text{IX.72a})$$

in which

$$T(k, t) = \int_{-\infty}^{+\infty} dk' \Phi(k, k'; t),$$

while Φ should be solved from the differential equation

$$\frac{\partial \Phi(k, k'; t)}{\partial t} = -v[k^2 + k'^2 + (k - k')^2] \Phi(k, k'; t) +$$

$$+ \left\{ k^2 E(k') E(k - k') - k k' E(k - k') E(k) - k(k - k') E(k') E(k) \right\} \cdot$$

(IX.72b)

The neglect of the fourth-order wave-correlation functions F (Millionshtchikov's hypothesis) allows us, as already mentioned in chapter IV, to close the hierarchy. Indeed, as we see from eqs. (IX.71a,b) or from (IX.72a,b), in each case we have to do with a consistent set of two equations. From the first set, we here again could derive, with our Bogoliubov expansion method, a kinetic equation for $E(k, t)$. To solve this equation, we previously only needed the single initial function $E(k, t = 0)$.

However, in order to "solve" the last two coupled equations one needs two initial conditions! Exactly these equations were solved numerically by D. Jeng et al.⁴⁸⁾, by choosing the following initial conditions:

$$\frac{E_1(\kappa, \tau=0)}{E_0} = e^{-\kappa^2} \quad \text{and} \quad \Phi(\kappa, \kappa'; \tau=0) = 0. \quad (\text{IX.73})$$

However, soon after $\tau=0$ their result showed an energy-spectrum function $E(\kappa)$ which admitted negative values in a certain k -region. Of course, this is not acceptable physically because $E(\kappa)$ had to remain positive for all κ -values at all times ($\tau > 0$).

For the three-dimensional case, too, one can set up two equations like (IX.72a) and (IX.72b). These were numerically computed by Ogura⁴⁹⁾ with the initial conditions

$$\left\{ \frac{E_1(\kappa, \tau=0)}{E_0} \right\}_{3-\dim} = \kappa^4 e^{-\kappa^2} \text{ and } \Phi'_{3-\dim}(\kappa, \kappa'; \tau=0) = 0. \quad (\text{IX.74})$$

As soon as a cascade picture arised with these calculations, they also got negative energy spectra. On account of these bad results, criticism has been expressed against the hierarchy technique and, in particular, against Millionshtchikov's hypothesis. From our point of view, however, this criticism is not quite justified. In fact the mentioned computer calculations start right from a nonphysical initial condition, since they neglect the energy transfer at time $\tau = 0$, although at this time its contribution proved to be more important than later on ($\tau > 0$). Physically this means that the real character of turbulence has been neglected completely at $\tau = 0$.

In our ansatz, on the contrary, we took into account the dependence of all higher-order (≥ 3) wave-correlation functions on the second-order one (the energy-spectrum function), following the concepts of Bogoliubov. This enabled us to describe fully developed turbulence, while looking for a relaxation process in the sense of the kinetic theory of statistical mechanics. In view of the obtained results, it seemed that the use of Millionshtchikov's hypothesis was a good assumption, at least in lowest order. For a Reynolds' number increasing to large values, one should have to take into account more equations of the hierarchy if ϵ were kept constant. However, the number of equations does not need to be increased if ϵ would diminish sufficiently during the increase of the Reynolds' number. It is then very likely that small deviations of the velocity field from the Gaussian profile should become important, in particular in the region where $E(k)$ attains maximum values.

Of course, many hypothesis can be set up, but we may not forget that the Burgers' equation merely constitutes a model of turbulence, while in practice one has to deal not only with three-dimensional phenomena, but in most cases

also with inhomogeneous turbulence (see also the next section!). It is even an open question whether the equations of ordinary hydrodynamics will hold for turbulent fluids!

G. Nonlinear interaction between sound waves in a turbulent fluid

The basic equations for a compressible fluid with the restriction that the density fluctuations should be very small, prove to read as follows, remembering (IX.1), (IX.6), (IX.9), and (IX.10):

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v} = -f'(\rho_0) \underline{\nabla} \eta + \nu \nabla^2 \underline{v}, \quad (\text{IX.75})$$

$$\frac{\partial \eta}{\partial t} + \underline{v} \cdot \underline{\nabla} \eta + \underline{\nabla} \cdot (\eta \underline{v}) = 0, \quad (\text{IX.76})$$

where

$$\rho = \rho_0 (1 + \eta), \quad |\eta| \ll 1. \quad (\text{IX.77})$$

First, we shall restrict our analysis to the case neglecting the viscosity term, because the nonlinear terms are in particular important in turbulent problems; it means that the Reynolds' number has to be large enough for undertaining turbulence. Later on, we shall discuss this point in more detail.

The linearized version of eqs. (IX.75) and (IX.76) yields for $\nu = 0$:

$$\frac{\partial \underline{v}_L}{\partial t} = -f'(\rho_0) \underline{\nabla} \eta_L \quad (\text{IX.78})$$

and

$$\frac{\partial \eta_L}{\partial t} + \underline{\nabla} \cdot \underline{v}_L = 0. \quad (\text{IX.79})$$

Introducing again the Fourier transforms of the relevant quantities, viz.

$$\underline{v}_L(\underline{r}, t) = \int_{-\infty}^{+\infty} \underline{v}_L(\underline{k}, t) e^{i \underline{k} \underline{r}} d\underline{k}, \quad (\text{IX.80a})$$

$$\eta_L(\underline{r}, t) = \int_{-\infty}^{+\infty} \eta_L(\underline{k}, t) e^{i \underline{k} \underline{r}} d\underline{k}, \quad (\text{IX.81a})$$

with the properties

$$\underline{v}_L(\underline{k}) = \underline{v}_L^*(-\underline{k}), \quad (\text{IX.80b})$$

$$\eta_L(\underline{k}) = \eta_L^*(-\underline{k}), \quad (\text{IX.81b})$$

we obtain from eqs. (IX.78) and (IX.79) the following relations in $\underline{k}$ -space

$$\frac{\partial \underline{v}_L(\underline{k})}{\partial t} = -i f'(\rho_0) \underline{k} \eta_L(\underline{k}), \quad (\text{IX.82})$$

$$\frac{\partial \eta_L(\underline{k})}{\partial t} + i \underline{k} \underline{v}_L(\underline{k}) = 0, \quad (\text{IX.83})$$

From these equations one observes that only the longitudinal velocity fluctuations are nontrivial solutions, due to the absence of a viscosity term. Therefore, to stress this longitudinal character, we define a scalar y_L by:

$$\underline{v}_L(\underline{k}) = \underline{k} y_L(\underline{k}). \quad (\text{IX.84})$$

It then follows from eq. (IX.80b) that

$$y_L(\underline{k}) = -y_L^*(-\underline{k}). \quad (\text{IX.85})$$

The equations (IX.82) and (IX.83) can then be combined in vectorial form by

$$i \frac{\partial}{\partial t} \begin{bmatrix} y_L(\underline{k}) \\ \eta_L(\underline{k}) \end{bmatrix} = \begin{bmatrix} 0 & f'(\rho_0) \\ k^2 & 0 \end{bmatrix} \begin{bmatrix} y_L(\underline{k}) \\ \eta_L(\underline{k}) \end{bmatrix}. \quad (\text{IX.86})$$

As usual, the time dependence of the corresponding amplitudes evolves like $e^{-i\omega t}$; the corresponding dispersion relation results in $\omega = \omega_R$ with

$$\omega_R^2(k) = k^2 f'(\rho_0) = k^2 v_s^2, \text{ say,} \quad (\text{IX.87})$$

and thus contains two branches:

$$\omega_R(k) = \pm |\underline{k}| v_s = \pm k v_s, \quad (\text{IX.88})$$

where v_s constitutes the well-known sound velocity.

Hence, we are confronted with the situation discussed quite generally in section VII.A, but here with $\gamma(k) = 0$ for all k . It turns out that the (linear) solution of eq. (IX.86) can be represented by

$$y_L(\underline{k}) = B^-(\underline{k}) e^{-ikv_s t} - B^+(\underline{k}) e^{+ikv_s t} \quad (\text{IX.89})$$

and

$$\eta_L(\underline{k}) = \frac{k}{v_s} \left[B^-(\underline{k}) e^{-ikv_s t} + B^+(\underline{k}) e^{+ikv_s t} \right], \quad (\text{IX.90})$$

where $k = |\underline{k}|$.

The reality condition for $y_L(\underline{k})$ (see eq. (IX.85)) and $\eta_L(\underline{k})$ amounts to:

$$B^s(\underline{k}) = B^{*-s}(-\underline{k}) \text{ for } s = +, -. \quad (\text{IX.91})$$

In order to arrive at the complete nonlinear equations in $\underline{k}$ -space we retain the Fourier representations (IX.80a), (IX.81a), and also the equation (IX.84), removing the subscript L throughout. The equations (IX.75) (with $v = 0$) and (IX.76) are then easily Fourier transformed, but in the remaining integral terms each integral is replaced by half the sum of its original expression and of the one obtained after substituting $\underline{k}' \rightarrow \underline{k} - \underline{k}'$

$$\frac{\partial y(\underline{k})}{\partial t} = -i v_s^2 \eta(\underline{k}) - i \int_{-\infty}^{+\infty} \left[\frac{\underline{k}'(\underline{k}-\underline{k}')}{2} \right] y(\underline{k}-\underline{k}') y(\underline{k}') d\underline{k}', \quad (\text{IX.92})$$

$$\begin{aligned} \frac{\partial \eta(\underline{k})}{\partial t} = & -i k^2 y(\underline{k}) - \frac{i}{2} \int_{-\infty}^{+\infty} \left[\underline{k}(\underline{k}-\underline{k}') \right] y(\underline{k}-\underline{k}') \eta(\underline{k}') d\underline{k}' - \\ & - \frac{i}{2} \int_{-\infty}^{+\infty} \left[\underline{k} \underline{k}' \right] y(\underline{k}') \eta(\underline{k}-\underline{k}') d\underline{k}'. \end{aligned} \quad (\text{IX.93})$$

We pass again to the representation of (IX.89) and (IX.90) without L, while considering B^- and B^+ also time-dependent. Substitution of these expressions into (IX.92) and (IX.93) yields:

$$\begin{aligned} & \frac{\partial B^-(\underline{k})}{\partial t} e^{-i k v_s t} - \frac{\partial B^+(\underline{k})}{\partial t} e^{+i k v_s t} = \\ & = -\frac{i}{2} \int_{-\infty}^{+\infty} \underline{k}' \underline{k}'' \left\{ B^-(\underline{k}'') B^-(\underline{k}') e^{-i(k'+k'') v_s t} - \right. \\ & \quad - B^-(\underline{k}'') B^+(\underline{k}') e^{+i(k'-k'') v_s t} - \\ & \quad - B^+(\underline{k}'') B^-(\underline{k}') e^{-i(k'-k'') v_s t} + \\ & \quad \left. + B^+(\underline{k}'') B^+(\underline{k}') e^{+i(k'+k'') v_s t} \right\} d\underline{k}', \end{aligned} \quad (\text{IX.94})$$

$$\begin{aligned} & \frac{\partial B^-(\underline{k}, t)}{\partial t} e^{-i k v_s t} + \frac{\partial B^+(\underline{k}, t)}{\partial t} e^{+i k v_s t} = \\ & = -\frac{i}{2} \int_{-\infty}^{+\infty} \left\{ \left[(\underline{k} \underline{k}'') \frac{\underline{k}'}{k} + (\underline{k} \underline{k}') \frac{\underline{k}''}{k} \right] \left[B^-(\underline{k}'') B^-(\underline{k}') e^{-i(k'+k'') v_s t} \right] + \right. \\ & \quad + \left[(\underline{k} \underline{k}'') \frac{\underline{k}'}{k} - (\underline{k} \underline{k}') \frac{\underline{k}''}{k} \right] \left[B^-(\underline{k}'') B^+(\underline{k}') e^{+i(k'-k'') v_s t} \right] - \\ & \quad \left. - \left[-(\underline{k} \underline{k}') \frac{\underline{k}''}{k} + (\underline{k} \underline{k}'') \frac{\underline{k}'}{k} \right] \left[B^+(\underline{k}'') B^-(\underline{k}') e^{-i(k'-k'') v_s t} \right] - \right. \\ & \quad \left. + \left[(\underline{k} \underline{k}'') \frac{\underline{k}'}{k} + (\underline{k} \underline{k}') \frac{\underline{k}''}{k} \right] \left[B^+(\underline{k}'') B^+(\underline{k}') e^{+i(k'+k'') v_s t} \right] \right\} d\underline{k}', \end{aligned}$$

$$- \left[(\underline{k}\underline{k}'') \frac{k'}{k} + (\underline{k}\underline{k}') \frac{k''}{k} \right] \left[B^+(k'') B^+(k') e^{+i(k'+k'')v_s t} \right] \} d\underline{k}'. \quad (\text{IX.95})$$

From these last equations one easily calculates a dynamical equation for $\partial B^+(\underline{k}, t)/\partial t$ and $\partial B^-(\underline{k}, t)/\partial t$ separately, by taking the sum and the difference of the last relations; we then get forms similar to our second-model equation (VII.7), but with $\gamma(k) = 0$ for all k . The only difference lies in the fact that we now have to deal with a three-dimensional $\underline{k}$ -space. In the present case the model equations for the individual B -quantities can be represented by

$$\begin{aligned} \frac{\partial B^S(\underline{k}, t)}{\partial t} = & -i \int_{s', s''}^{+\infty} d\underline{k}' v^{s', s'', s''}(\underline{k}; \underline{k}', \underline{k}'') B^{s'}(\underline{k}') B^{s''}(\underline{k}'') \times \\ & \times e^{i[s' \omega_R(k') + s'' \omega_R(k'') - s \omega_R(k)]t}, \end{aligned} \quad (\text{IX.96})$$

if the different interaction strengths are defined as follows:

$$v^{-, -, -}(\underline{k}; \underline{k}', \underline{k}'') = \frac{1}{4} \left[\underline{k}' \underline{k}'' + \frac{k}{k} (\underline{k}'' k' + \underline{k}' k'') \right], \quad (\text{IX.97a})$$

$$v^{-, +, -}(\underline{k}; \underline{k}', \underline{k}'') = \frac{1}{4} \left[-\underline{k}' \underline{k}'' + \frac{k}{k} (\underline{k}'' k' - \underline{k}' k'') \right], \quad (\text{IX.97b})$$

$$v^{-, -, +}(\underline{k}; \underline{k}', \underline{k}'') = \frac{1}{4} \left[-\underline{k}' \underline{k}'' + \frac{k}{k} (\underline{k}' k'' - k'' k') \right], \quad (\text{IX.97c})$$

$$v^{-, +, +}(\underline{k}; \underline{k}', \underline{k}'') = \frac{1}{4} \left[\underline{k}' \underline{k}'' - \frac{k}{k} (\underline{k}' k'' + \underline{k}'' k') \right]. \quad (\text{IX.97d})$$

All these four equations are comprised in the single one:

$$v^{-, s', s''}(\underline{k}; \underline{k}', \underline{k}'') = \frac{1}{4} \left[s' \underline{k}' \cdot s'' \underline{k}'' - \frac{k}{k} (s'' \underline{k}'' k' + s' \underline{k}' k'') \right]. \quad (\text{IX.98})$$

The corresponding interaction strengths $v^{+, s', s''}$ ($s', s'' = +, -$) are then easily calculated with the aid of the relation

(VII.8b) which also holds here; V being real, we thus find

$$v^{s,s',s''}(\underline{k};\underline{k}',\underline{k}'') = -v^{-s,-s',-s''}(-\underline{k};-\underline{k}',-\underline{k}''). \quad (\text{IX.99})$$

Before using the results of the general analysis of chapter VII we first have to define the energy-spectrum functions. On the analogy of eq. (VII.13), we first assume the relation corresponding to (VII.13), viz.

$$\langle B^{s_1}(\underline{k}_1) B^{s_2}(\underline{k}_2) \rangle = G^{s_1}(\underline{k}_1) \delta(\underline{k}_1 + \underline{k}_2) \delta_{s_1, -s_2}. \quad (\text{IX.100a})$$

Hence, using eqs. (IX.84) and (IX.89) (without subscript L)

$$\begin{aligned} E(\underline{k}_1, t) \delta(\underline{k}_1 + \underline{k}_2) &= \langle v(\underline{k}_1, t) v(\underline{k}_2, t) \rangle = -k_1^2 \langle y(\underline{k}_1) y(\underline{k}_2) \rangle = \\ &= k_1^2 \left[\langle B^-(\underline{k}_1) B^+(\underline{k}_2) \rangle + \langle B^+(\underline{k}_1) B^-(\underline{k}_2) \rangle \right]. \end{aligned} \quad (\text{IX.100b})$$

We then introduce the further averages

$$\langle B^+(\underline{k}_1) B^-(\underline{k}_2) \rangle = D^+(\underline{k}_1) \delta(\underline{k}_1 + \underline{k}_2), \quad (\text{IX.101a})$$

and

$$\langle B^-(\underline{k}_1) B^+(\underline{k}_2) \rangle = D^-(\underline{k}_1) \delta(\underline{k}_1 + \underline{k}_2), \quad (\text{IX.101b})$$

with the well-known properties (see eq. (VII.14a))

$$D^-(\underline{k}_1) = D^+(-\underline{k}_1) = D^+(\underline{k}_2). \quad (\text{IX.102})$$

From eq. (VII.12) it is seen that the energy-spectrum functions $E^s(\underline{k})$, describing the energy of the $\underline{k}$ - (velocity) modes, travelling in opposite directions ($s = +, -$) are related as follows with the "new" energy-spectrum functions $D^s(\underline{k})$

$$E^+(\underline{k}) = k^2 D^+(\underline{k}), \quad (\text{IX.103a})$$

$$E^-(\underline{k}) = k^2 D^-(\underline{k}), \quad (\text{IX.103b})$$

while

$$E(\underline{k}) = k^2 \{D^+(\underline{k}) + D^-(\underline{k})\} \equiv k^2 D(\underline{k}). \quad (\text{IX.104})$$

With these conditions and relations in mind, one can immediately make use of the final kinetic equation for this case (see eq. (VII.35)), replacing the E-functions by D-functions. In fact, comparing (IX.96) with (VII.7) we infer that the quantities B^s in the present case correspond to A^s in chapter VII. Moreover, eq. (VII.12) is taken over by (IX.101), so that $D^\pm$ corresponds to E^s in chapter VII. Therefore, we have to replace the E-functions by $D^\pm$ when working out (VII.35) for the present situation. For $s = -$ we then obtain:

$$\begin{aligned} \frac{\partial D^-(\underline{k})}{\partial \tau} = & 4\pi \int_{-\infty}^{+\infty} d\underline{k}' \left\{ \left[|V^{-,-,-}(\underline{k}; \underline{k}', \underline{k}'')|^2 D^-(\underline{k}') D^-(\underline{k}'') - \right. \right. \\ & \left. \left. - 2V^{-,-,-}(\underline{k}; \underline{k}', \underline{k}'') V^{-,+,+}(\underline{k}''; -\underline{k}', \underline{k}) D^-(\underline{k}) D^-(\underline{k}') \right] \times \right. \\ & \times \delta(v_s(k-k'-k'')) + \left[|V^{-,+,+}(\underline{k}; \underline{k}', \underline{k}'')|^2 D^+(\underline{k}') D^-(\underline{k}'') - \right. \\ & \left. - 2V^{-,+,+}(\underline{k}; \underline{k}', \underline{k}'') V^{-,-,-}(\underline{k}''; -\underline{k}', \underline{k}) D^-(\underline{k}) D^+(\underline{k}') \right] \times \\ & \times \delta(v_s(k+k'-k'')) + \left[|V^{-,-,+}(\underline{k}; \underline{k}', \underline{k}'')|^2 D^-(\underline{k}') D^+(\underline{k}'') - \right. \\ & \left. - 2V^{-,-,+}(\underline{k}; \underline{k}', \underline{k}'') V^{+,+,-}(\underline{k}''; -\underline{k}', \underline{k}) D^-(\underline{k}) D^-(\underline{k}') \right] \times \\ & \left. \times \delta(v_s(k-k'+k'')) \right\}. \quad (\text{IX.105}) \end{aligned}$$

The contribution from the fourth term (for which $s' = s'' = +1$) has disappeared because it contains the δ -function $\delta(v_s(k+k'+k''))$ the positive argument of which can never vanish. A similar equation can be derived for $s = +$, in which the con-

tribution of the term with $s' = s'' = -1$ will vanish.

In view of the calculations performed in appendix D for the different δ -functions appearing in the kinetic equations, the interaction strengths $V^{s,s',s''}$ are only needed for special values of the variable $x = \cos \theta$, θ being the angle between the vectors $\underline{k}$ and $\underline{k}'$. The relevant values then to be derived from the expressions (IX.97) and (IX.99), are listed below:

$$\left. \begin{aligned} V^{-,-,-}(\underline{k}; \underline{k}', \underline{k}'') &= \frac{3}{4} k' (k - k') \\ V^{-,+,-}(\underline{k}; -\underline{k}', \underline{k}) &= \frac{3}{4} k k' \end{aligned} \right\} \text{ for } \cos \theta = 1, 0 < k' < k; \quad (\text{IX.106})$$

$$\left. \begin{aligned} V^{-,+,-}(\underline{k}; \underline{k}', \underline{k}'') &= \frac{3}{4} k' (k + k'), \\ V^{-,-,-}(\underline{k}; -\underline{k}', \underline{k}) &= \frac{3}{4} k k' \end{aligned} \right\} \text{ for } \cos \theta = -1, 0 < k' < \infty; \quad (\text{IX.107})$$

$$\left. \begin{aligned} V^{-,-,+}(\underline{k}; \underline{k}', \underline{k}'') &= \frac{3}{4} k' (k' - k) \\ V^{+,-,-}(\underline{k}; -\underline{k}', \underline{k}) &= -\frac{3}{4} k k' \end{aligned} \right\} \text{ for } \cos \theta = 1, k < k' < \infty. \quad (\text{IX.108})$$

Substituting also the relevant expressions ((D.7) - (D.9) and (IX.106) - (IX.108)) into the kinetic equation (IX.105) we arrive, taking into account the integrals $\int_{-1}^{+1} dx \delta(x-1) = \int_{-1}^{+1} dx \delta(x+1) = \frac{1}{2}$, at the following explicit dynamical equation:

$$\begin{aligned} \frac{\partial D^{-}(\underline{k})}{\partial t} = & \frac{9\pi^2}{4kvs} \left\{ \int_0^k [\underline{k}'^3 (k - k')^3 D^{-}(\underline{k}') D^{-}(\underline{k} - \underline{k}') - \right. \\ & - 2kk'^3 (k - k')^2 D^{-}(\underline{k}) D^{-}(\underline{k}')] dk' + \int_0^{\infty} [\underline{k}'^3 (k + k')^3 D^{+}(\underline{k}') D^{-}(\underline{k} - \underline{k}') - \\ & - 2kk'^3 (k + k')^2 D^{-}(\underline{k}) D^{+}(\underline{k}')] dk' + \int_k^{\infty} [\underline{k}'^3 (k' - k)^3 D^{-}(\underline{k}') D^{-}(\underline{k}' - \underline{k}) + \\ & \left. + 2kk'^3 (k' - k)^2 D^{-}(\underline{k}) D^{-}(\underline{k}')] dk' \right\}. \quad (\text{IX.109}) \end{aligned}$$

Similarly, the corresponding kinetic equation for $D^+(\underline{k})$ yields:

$$\begin{aligned} \frac{\partial D^+(\underline{k})}{\partial t} = & \frac{9\pi^2}{4kv_s} \left\{ \int_0^k [\underline{k}'^3 (\underline{k}-\underline{k}')^3 D^+(\underline{k}') D^+(\underline{k}-\underline{k}') - \right. \\ & - 2kk'^3 (\underline{k}-\underline{k}')^2 D^+(\underline{k}) D^+(\underline{k}')] d\underline{k}' + \int_0^\infty [\underline{k}'^3 (\underline{k}+\underline{k}')^3 D^-(\underline{k}') D^+(\underline{k}-\underline{k}') - \\ & - 2kk'^3 (\underline{k}+\underline{k}')^2 D^+(\underline{k}) D^-(\underline{k}')] d\underline{k}' + \int_k^\infty [\underline{k}'^3 (\underline{k}'-\underline{k})^3 D^+(\underline{k}') D^+(\underline{k}'-\underline{k}) + \\ & \left. + 2kk'^3 (\underline{k}'-\underline{k})^2 D^+(\underline{k}) D^+(\underline{k}')] d\underline{k}' \right\}. \quad (\text{IX.110}) \end{aligned}$$

For convenience, we maintained the vector character for the energy-spectrum functions $D^s(\underline{k})$ ($s = +, -$).

Observing the relations (IX.106) to (IX.108) for the admitted values of $\cos \theta$ and k' , we deduce that a given $\underline{k}$ -wave can only interact with other sound waves which travel either in the same or in the direction opposite to $\underline{k}$. Further, the three integrals in eqs. (IX.109) and (IX.110) imply different kinds of interaction, depending on the value of the wavenumber k' . In view of the just mentioned relations for $\cos \theta$ and k' , we can distinguish the following cases.

In the first integral, the vector $\underline{k}'$ has the same direction as $\underline{k}$, with $0 < k' < k$. The first term of this integral depends on the coupling of two sound waves travelling in the same direction as $\underline{k}$ does, with both wavenumbers satisfying $0 < k' < k$.

As to the second integral, $\underline{k}'$ has a direction opposite to that of $\underline{k}$, while $0 < k' < \infty$. The first term here concerns a coupling of a sound wave travelling in this opposite direction, with another one travelling in the same direction as $\underline{k}$, while having a wavenumber $|\underline{k}-\underline{k}'| = \underline{k}+\underline{k}'$ larger than k .

Finally, in the third integral, $\underline{k}'$ points in the same direction as $\underline{k}$, while $k < k' < \infty$. The first term of this inte-

gral is associated with a coupling between two sound waves travelling in the same direction as $\underline{k}$, one of which having a wavenumber between k and ∞ and the other one between 0 and ∞ .

We remark that the mentioned first parts of the three integrals in (IX.109) and (IX.110) constitute positive contributions (the energy-spectrum functions are also positive quantities). The remaining terms, all of which are proportional to $D^s(\underline{k})$ ($s = +, -$), represent partly a nonlinear dissipation (the last terms in the two first integrals) and partly a nonlinear growth (the last term in the last integral). Obviously, the same properties hold for the actual energy-spectrum functions $E^s(\underline{k})$ defined by eqs. (IX.103a) and (IX.103b).

H. Sound waves with isotropic energy spectra

Let us introduce in the theory of the preceding section the additional assumption of isotropy in $\underline{k}$ -space so that

$$D^s(\underline{k}) = D^s(-\underline{k}) = D^s(k); \quad (\text{IX.111})$$

as a consequence, the energy-spectrum function $D(k)$ becomes independent of the direction of $\underline{k}$.

From eq. (IX.104) we then obtain:

$$\frac{1}{2} D(k) = D^+(\underline{k}) = D^-(\underline{k}). \quad (\text{IX.112})$$

Using the last relation we arrive at the following special form of the kinetic equation (IX.109) or (IX.110) for $D(k)$:

$$\begin{aligned} \frac{\partial D(k)}{\partial t} = \\ = \frac{9\pi^2}{8v_s} \left(\int_0^k \left[\frac{k'^3 (k-k')^3}{k} D(k') D(k-k') - 2k'^3 (k-k')^2 D(k) D(k') \right] dk' + \right. \end{aligned}$$

$$\begin{aligned}
& + \int_0^{\infty} \left[\frac{k'^3 (k+k')^3}{k} D(k') D(k+k') - 2k'^3 (k+k')^2 D(k) D(k') \right] dk' + \\
& + \int_k^{\infty} \left[\frac{k'^3 (k'-k)^3}{k} D(k') D(k'-k) + 2k'^3 (k'-k)^2 D(k) D(k') \right] dk' \Big\} .
\end{aligned}
\tag{IX.113}$$

Although $k' = |k'|$ constitutes a positive quantity, we introduce a new variable k' , running between $-\infty$ and $+\infty$. Doing so, the second term in the second integral can be transformed into

$$+ \int_{-\infty}^0 2k'^3 (k-k')^2 D(k) D(k') dk' .$$

Hence, all integrals proportional to the energy-spectrum function $D(k)$ can be added to:

$$\begin{aligned}
& \int_{-\infty}^{+\infty} 2k'^3 (k-k')^2 D(k) D(k') dk' - 4 \int_0^k k'^3 (k-k')^2 D(k) D(k') dk' = \\
& = -4 \int_{-\infty}^{+\infty} k k' D(k) D(k') dk' - 4 \int_0^k k'^3 (k-k')^2 D(k) D(k') dk' .
\end{aligned}
\tag{IX.114}$$

Thus, in this case of isotropy this contribution amounts to a dissipation in this sound-wave spectrum.

Of course, the contribution of the actual mode-mode coupling terms of eq. (IX.113) (the first term in each integral) remains positive. If the energy-spectrum function $D^S(k)$ vanishes more rapidly than k^{-3} for $k \rightarrow \infty$, for large k -values the first term becomes the most important one of these three terms. Hence, again, a cascade process is established due to the nonlinear interaction of sound waves!

I. Sound waves influenced by viscosity

The above discussed problem of sound waves travelling in a nondissipative fluid can easily be generalized to include the usual viscosity term. For this case, too, we shall only admit longitudinal fluctuations and, consequently, define:

$$\underline{v}(\underline{k}) = \underline{k}V(\underline{k}). \quad (\text{IX.115})$$

The linearized Fourier transformed set of equations (IX.75) and (IX.76) then becomes, replacing (IX.78) and (IX.79),

$$\frac{\partial}{\partial t} V_L(\underline{k}) = -i\bar{f}'(\rho_0)\eta_L(\underline{k}) - vk^2V_L(\underline{k}), \quad (\text{IX.116})$$

$$\frac{\partial}{\partial t} \eta_L(\underline{k}) = -ik^2v_L(\underline{k}), \quad (\text{IX.117})$$

or, in vector formulation (compare eq. (IX.86)),

$$i \frac{\partial}{\partial t} \begin{pmatrix} V_L(\underline{k}) \\ \eta_L(\underline{k}) \end{pmatrix} = \begin{pmatrix} -ivk^2 & v_s^2 \\ k^2 & 0 \end{pmatrix} \begin{pmatrix} V_L(\underline{k}) \\ \eta_L(\underline{k}) \end{pmatrix}. \quad (\text{IX.118})$$

The physical quantities on the left-hand side of the latter equation evolve as $e^{-i\omega t}$. The dispersion relation (see eq. (II.16)) is then found as

$$\text{Det} \begin{vmatrix} \omega + ivk^2 & -v_s^2 \\ -k^2 & \omega \end{vmatrix} = 0 = \omega^2 + ivk^2\omega - k^2v_s^2. \quad (\text{IX.119})$$

The solution yields

$$\omega = \frac{1}{2} \left\{ -ivk^2 \pm \left[4k^2v_s^2 - (vk^2)^2 \right]^{\frac{1}{2}} \right\} = \omega_R(k) - i\gamma(k). \quad (\text{IX.120})$$

Depending on the k -values and the Reynolds' number

($\sim 1/v$), four different physical regions can be distinguished according to different parameter ranges. For $v \rightarrow 0$ we again get the sound-wave problem and, leaving out of consideration the continuity equation, for $v_s \rightarrow 0$ one would find again Burgers' equation. From the two intermediate regions, we intend to treat only that one in which $4v_s^2 > v^2 k^2$. We then have, in view of (IX.120)

$$\omega_R(k) = \left(k^2 v_s^2 - \left(\frac{vk^2}{2} \right)^2 \right)^{\frac{1}{2}} \quad (\text{IX.121a})$$

and

$$\gamma(k) = \frac{vk^2}{2} . \quad (\text{IX.121b})$$

The analysis as developed in section IX.G can be repeated, replacing the corresponding value of $\omega_R(k)$ and including the damping term. The theory of chapter VII can be used to find the final kinetic equation (see eq. (VII.34)). Nevertheless, concerning the energy spectra it is hard to say anything about the energy gain or loss because now no δ -functions occur which could reduce the problem. However, we shall consider in detail the first nonlinear term of eq. (VII.34), viz.

$$\begin{aligned} \frac{\partial E^S(\underline{k})}{\partial t} = & -2\gamma(k)E^S(\underline{k}) + \\ & + 4 \sum_{s', s''} \int_{-\infty}^{+\infty} d\underline{k}' |v^{s, s', s''}(\underline{k}; \underline{k}', \underline{k}-\underline{k}')|^2 E^{s'}(\underline{k}') E^{s''}(\underline{k}-\underline{k}') \times \\ & \times \frac{[\gamma(k) - \gamma(|\underline{k}-\underline{k}'|) - \gamma(k')]}{[\gamma(k) - \gamma(|\underline{k}-\underline{k}'|) - \gamma(k')]^2 + [s\omega_R(k) - s'\omega_R(k') - s''\omega_R(|\underline{k}-\underline{k}'|)]^2} + \\ & + \text{other terms.} \end{aligned} \quad (\text{IX.122})$$

In view of eq. (IX.121b) we have

$$\gamma(k) - \gamma(|\underline{k}-\underline{k}'|) - \gamma(k') = vk'(k \cos \theta - k') \quad (\text{IX.123})$$

if $\underline{k}k' = kk' \cos \theta$.

Again, a cascade process (towards increasing k -values) appears to be exhibited for k' -values satisfying $0 < k' < k \cos \theta$, like in the case of three-dimensional incompressible turbulence! Of course, the remaining contribution describes a loss of energy.

C H A P T E R X

A MODEL FOR MAGNETOHYDRODYNAMICAL TURBULENCE

A. Introduction

The investigation of magnetohydrodynamical (MHD)-turbulence is not only important for plasma physics, but also for the fields of geophysics and astrophysics. In the latter cases the main item seems to be the "turbulent dynamo" problem: the question whether turbulent magnetic energy will grow to an equilibrium level or will die away. The question then arises whether an equipartition between kinetic and magnetic energy will be reached.

We have noticed that, due to mathematical complications, it is very difficult to get a sufficiently physical insight even in the case of purely hydrodynamical turbulence. Therefore, we shall restrict ourselves again to a one-dimensional model system, based on an extended Burgers' equation. We further assume that no quantities with nonvanishing average are present in the system; thus, we exclude a mean motion and an exterior magnetic field. Hence, we only have to deal with decaying magnetic-field fluctuations and

decaying velocity-field fluctuations, these being due to magnetic diffusivity and viscosity, respectively. These two turbulent fields interact with each other because they have to satisfy two coupled differential equations.

Again, dynamical equations for the energy-spectrum functions will be derived with the aid of our weak-turbulence method, these functions referring to the kinetic and the magnetic energy, respectively. Integrating over all relevant modes, an expression for the total (normalized) kinetic and magnetic energy then results.

B. Derivation of a model system for magnetohydrodynamical turbulence

The usual incompressible MHD-equations read:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} - (\underline{h} \cdot \nabla) \underline{h} = - \nabla(p/\rho) + \nu \nabla^2 \underline{u}, \quad (\text{X.1})$$

$$\frac{\partial \underline{h}}{\partial t} + (\underline{u} \cdot \nabla) \underline{h} - (\underline{h} \cdot \nabla) \underline{u} = \lambda \nabla^2 \underline{h}, \quad (\text{X.2})$$

where $\underline{u}(\underline{r}, t)$ represents the turbulent velocity and $\underline{h}(\underline{r}, t)$ the turbulent, properly normalized, magnetic field; ν is the kinematic viscosity and λ the magnetic diffusivity (proportional to the electric resistivity). It is well known (see, for instance, ref. 50) that in the absence of the two dissipation terms the total energy (kinetic and magnetic) is conserved for this system.

As pointed out by Thomas¹²⁾, this three-dimensional system can be reduced to a suitable one-dimensional form, in accordance with Burgers' equation for vanishing magnetic fluctuations. The proposed system looks like:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - 3h \frac{\partial h}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad (\text{X.3})$$

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} - h \frac{\partial u}{\partial x} - \lambda \frac{\partial^2 h}{\partial x^2} = 0. \quad (X.4)$$

As we shall see later, the factor 3 appearing in eq. (X.3) proves to be necessary in order to satisfy the required conservation of the total energy.

We assume again

$$\langle u(x,t) \rangle = \langle h(x,t) \rangle = 0. \quad (X.5)$$

To analyze the turbulent fluctuations we introduce the Fourier transforms, connected with the modes of the linearized approximation:

$$u(x,t) = \int_{-\infty}^{+\infty} u(k,t) e^{ikx - \nu k^2 t} dk \text{ with } u^*(k,t) = u(-k,t), \quad (X.6)$$

$$h(x,t) = \int_{-\infty}^{+\infty} h(k,t) e^{ikx - \lambda k^2 t} dk \text{ with } h^*(k,t) = h(-k,t). \quad (X.7)$$

Moreover, in view of eq. (X.5) we have

$$u(k=0,t) = h(k=0,t) = 0. \quad (X.8)$$

Next, by Fourier transforming the above equations, we get the following dynamical equations for the fluctuating kinetic and magnetic k-modes respectively:

$$\begin{aligned} \frac{\partial u(k,t)}{\partial t} = & -\frac{ik}{2} \int_{-\infty}^{+\infty} u(k',t) u(k-k',t) e^{-2\nu k'(k'-k)t} dk' + \\ & + \frac{3ik}{2} \int_{-\infty}^{+\infty} h(k',t) h(k-k',t) e^{-(\lambda-\nu)k^2 t - 2\lambda k'(k'-k)t} dk', \end{aligned} \quad (X.9)$$

$$\frac{\partial h(k,t)}{\partial t} = i \int_{-\infty}^{+\infty} (2k'-k) u(k',t) h(k-k',t) e^{-(\lambda+\nu)k'^2 t + 2\lambda k k' t} dk'. \quad (X.10)$$

From the last equation we observe that, in order to generate new magnetic k-modes, it is necessary to have kinetic k-modes! According to the first equation, however, new velocity modes can be excited without the magnetic modes (Burgers' case); in turn, the excited magnetic modes of eq. (X.10) help to generate new velocity modes!

C. Derivation of the kinetic equations

On the analogy of the Burgers' case, we shall introduce the following energy-spectrum functions defined by:

$$E(k) = E(-k) = G(k)e^{-2\nu k^2 t}, \quad (X.11a)$$

$$H(k) = H(-k) = F(k)e^{-2\lambda k^2 t}, \quad (X.11b)$$

to be defined with the aid of the assumptions

$$\langle u(k_1, t) u(k_2, t) \rangle = G(k_1) \delta(k_1 + k_2), \quad (X.12a)$$

$$\langle h(k_1, t) h(k_2, t) \rangle = F(k_1) \delta(k_1 + k_2). \quad (X.12b)$$

Further, the various third-order wave-correlation functions are now to be defined by, i.e.

$$\langle u(k_1) u(k_2) h(k_3) \rangle = S(u, u, h; k_1, k_2, k_3) \delta(k_1 + k_2 + k_3), \quad (X.13)$$

where we omit the t-dependence. Obviously, the first three arguments in the S-function are connected with the three k-values respectively, as one can see from the left-hand side of this equation. In this way one can define quite generally purely kinetic, purely magnetic, and mixed wave correlations.

With the aid of all these definitions, the first equation of the kinetic and magnetic hierarchy can be derived by the method quite similar to that of chapter IV. The results

are:

$$\begin{aligned} \frac{\partial}{\partial t} G(k) = & \left\{ -\frac{ik}{2} \int_{-\infty}^{+\infty} S(u, u, u; k', k-k', -k) e^{-2vk'(k'-k)t} dk' + \right. \\ & + \frac{3ik}{2} \int_{-\infty}^{+\infty} S(u, h, h; -k, k', k-k') e^{-(\lambda-v)k^2 t - 2\lambda k'(k'-k)t} dk' \left. \right\} + \\ & + \{k \rightarrow -k\}, \end{aligned} \quad (X.14)$$

$$\begin{aligned} \frac{\partial}{\partial t} F(k) = & \left\{ i \int_{-\infty}^{+\infty} (2k'-k) S(u, h, h; k', k-k', -k) \times \right. \\ & \times e^{-(\lambda+v)k'^2 t + 2\lambda k k' t} dk' \left. \right\} + \{k \rightarrow -k\}. \end{aligned} \quad (X.15)$$

In the appendix E it will be shown that, starting from these two equations, the total energy is conserved "a priori", apart from the dissipation terms, that is, we have:

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} [E(k) + H(k)] dk = -2 \int_{-\infty}^{+\infty} [vk^2 E(k) + \lambda k^2 H(k)] dk. \quad (X.16)$$

Therefore, in eq. (X.3) the factor 3 had been introduced.

When evaluating the next term of both hierarchies we shall neglect the direct coupling between a kinetic and magnetic mode, that is

$$\langle u(k_2) h(k_1) \rangle = 0 \text{ for all } k_1 \text{ and } k_2. \quad (X.17)$$

This assumption is based on three arguments. First, it prevents $\frac{\partial}{\partial t} \langle h(k, t) \rangle$ from becoming different from zero (see eq. (X.10)). Secondly, if one would derive a closed kinetic equation for this mixed correlation, eq. (X.17) would turn out to be a trivial solution. Finally, Chandrasekhar⁵¹⁾ showed explicitly that such correlations must vanish for the three-dimensional case (eqs. (X.1) and (X.2)).

We need not to specify the fourth-order wave-correlation functions, because they are represented by trivial expressions like eq. (X.13).

In order to calculate the third-order wave-correlation functions appearing in eq. (X.14) and eq. (X.15) we use the basic equations (X.9) and (X.10) and the relations (X.12a,b) and (X.17). An analysis similar to that of chapter IV yields

$$\begin{aligned} \frac{\partial}{\partial t} S(u, u, u; k', k-k', -k) = & -ik' G(k)G(k-k')e^{-2\nu(k^2-kk')t} - \\ & -i(k-k')G(k)G(k')e^{-2\nu kk'^2 t} + \\ & +ik G(k')G(k-k')e^{-2\nu(k'^2-kk')t} + \\ & +(\text{terms with fourth-order wave-correlation functions}), \end{aligned} \quad (\text{X.18a})$$

$$\begin{aligned} \frac{\partial}{\partial t} S(u, h, h; -k, k', k-k') = & i(2k-k')G(k)F(k-k')e^{-(\lambda+\nu)k^2 t+2\lambda kk' t} + \\ & +i(k+k')G(k)F(k')e^{-(\lambda+\nu)k^2 t+2\lambda k(k-k')t} - \\ & -3ik F(k')F(k-k')e^{-(\lambda-\nu)k^2 t-2\lambda k'(k'-k)t} + \\ & +(\dots), \end{aligned} \quad (\text{X.18b})$$

$$\begin{aligned} \frac{\partial}{\partial t} S(u, h, h; k', k-k', -k) = & +3ik' F(k)F(k-k')e^{-(\lambda-\nu)k'^2 t-2\lambda k(k-k')t} - \\ & -i(k+k')G(k')F(k)e^{-(\lambda+\nu)k'^2 t-2\lambda k'(k-k')t} + \\ & +i(k-2k')G(k')F(k-k')e^{-(\lambda+\nu)k'^2 t+2\lambda kk'^2 t} + (\dots). \end{aligned} \quad (\text{X.18c})$$

The first of these three results proves to be identical, as it should, with that for the one-dimensional hydrodynamical case (see eq. (IX.71b)!).

To solve these three equations we have to introduce the weak turbulence scheme:

$$F = \varepsilon F^{(1)}, G = \varepsilon G^{(1)}, S = \varepsilon^2 S^{(2)} + \dots \quad (X.19)$$

Like in the general analysis in chapter V we again notice that the second-order wave-correlation functions $G^{(1)}(k)$ and $F^{(1)}(k)$ are time-independent in lowest order ($O(\varepsilon)$). Applying then the synchronization hypothesis for the initial conditions of eqs. (X.18a,b,c), one can solve these equations uniquely which yields

$$\begin{aligned} & S^{(2)}(u, u, u; k, k', -k) = \\ & = + \frac{ik'}{2\nu(k^2 - kk')} G^{(1)}(k) G^{(1)}(k-k') e^{-2\nu(k^2 - kk')t} + \\ & + \frac{i(k-k')}{2\nu kk'} G^{(1)}(k) G^{(1)}(k') e^{-2\nu kk't} - \\ & - \frac{ik}{2\nu(k'^2 - kk')} G^{(1)}(k') G^{(1)}(k-k') e^{-2\nu(k'^2 - kk')t}, \quad (X.19a) \end{aligned}$$

$$\begin{aligned} & S^{(2)}(u, h, h; -k, k', k-k') = \\ & = + \frac{i(2k-k')}{2\lambda kk' - (\lambda + \nu)k^2} G^{(1)}(k) F^{(1)}(k-k') e^{-(\lambda + \nu)k^2 t + 2\lambda kk't} + \\ & + \frac{i(k+k')}{2\lambda k(k-k') - (\lambda + \nu)k^2} G^{(1)}(k) F^{(1)}(k') e^{-(\lambda + \nu)k^2 t + 2\lambda k(k-k')t} + \\ & + \frac{3ik}{(\lambda - \nu)k^2 + 2\lambda k'(k'-k)} F^{(1)}(k') F^{(1)}(k-k') \times \\ & \times e^{-(\lambda - \nu)k^2 t - 2\lambda k'(k'-k)t}, \quad (X.19b) \end{aligned}$$

$$\begin{aligned}
& S^{(2)}(u, h, h; k', k-k', -k) = \\
& = - \frac{3ik'}{(\lambda-\nu)k'^2 + 2\lambda k(k-k')} F^{(1)}(k) F^{(1)}(k-k') \times \\
& \times e^{-(\lambda-\nu)k'^2 t - 2\lambda k(k-k')t} - \\
& - \frac{i(k+k')}{2\lambda k'(k'-k) - (\lambda+\nu)k'^2} G^{(1)}(k') F^{(1)}(k) \times \\
& \times e^{-(\lambda+\nu)k'^2 t - 2\lambda k'(k-k')t} + \\
& + \frac{i(k-2k')}{2\lambda k k' - (\lambda+\nu)k'^2} G^{(1)}(k') F^{(1)}(k-k') \times \\
& \times e^{-(\lambda+\nu)k'^2 t + 2\lambda k k' t}. \tag{X.19c}
\end{aligned}$$

Strictly speaking, in order to avoid the appearance of poles a term $+i\varepsilon$ should be added in the denominators of the two last expressions. The limit $\varepsilon \rightarrow 0$ could then be taken afterwards like we already did for the general theory of chapter V in the special section VI.D! For convenience, we have not indicated this term $+i\varepsilon$ because we shall directly apply the result of section VI.D for our final kinetic equations.

The above expressions for the different triple-order wave-correlation functions are now to be substituted into the corresponding dynamical equations (X.14) and (X.15). These equations, which are correct up to $O(\varepsilon^2)$, may be multiplied by $e^{-2\nu k^2 t}$ and $e^{-2\lambda k^2 t}$, respectively. Omitting the indices (1) for $G^{(1)}$ and $F^{(1)}$ we get, with the aid of definitions (X.11a,b):

$$\frac{\partial G(k)}{\partial t} e^{-2\nu k^2 t} =$$

$$\begin{aligned}
&= \left\{ \frac{k}{2} \int_{-\infty}^{+\infty} dk' \left[\frac{k'}{2\nu k(k-k')} E(k)E(k-k') + \frac{(k-k')}{2\nu k k'} E(k)E(k') - \right. \right. \\
&- \frac{k}{2\nu k'(k'-k)} E(k')E(k-k') \left. \right] - \frac{3k}{2} \int_{-\infty}^{+\infty} dk' \left[\frac{(2k-k')}{2\lambda k k' - (\lambda+\nu)k^2} E(k)H(k-k') + \right. \\
&+ \frac{(k+k')}{2\lambda k(k-k') - (\lambda+\nu)k^2} E(k)H(k') + \\
&+ \left. \left. \frac{3k}{(\lambda-\nu)k^2 + 2\lambda k'(k'-k)} H(k')H(k-k') \right] \right\} + \{k \leftrightarrow -k\}, \\
\end{aligned} \tag{X.20a}$$

$$\begin{aligned}
&\frac{\partial F(k)}{\partial t} e^{-2\lambda k^2 t} = \\
&= \left\{ \int_{-\infty}^{+\infty} dk' \left[\frac{3k'(2k'-k)}{(\lambda-\nu)k'^2 + 2\lambda k(k-k')} H(k)H(k-k') + \right. \right. \\
&+ \frac{(2k'-k)(k+k')}{2\lambda k'(k'-k) - (\lambda+\nu)k'^2} E(k')H(k) + \\
&+ \left. \left. \frac{(2k'-k)^2}{2\lambda k k' - (\lambda+\nu)k'^2} E(k')H(k-k') \right] \right\} + \{k \leftrightarrow -k\}. \\
\end{aligned} \tag{X.20b}$$

The term $\{k \leftrightarrow -k\}$ in both equations (X.20a) and (X.20b) turns out to be the same as the sum of the preceding ones written out explicitly. To prove so one simply has to replace the integration variable k' by $-k'$ in this term and one has to make use of the even character of $E(k)$ and $H(k)$!

Next, by changing the variable k' into $k-k'$, the first term on the right-hand side of eq. (X.20a) proves to be identical with the second one and, similarly, the fourth identical with the fifth one. Passing from $G(k)$ to $E(k)$ in the left-hand side, eq. (X.20a) thus leads to the following full kinetic equation for $E(k)$, for $k > 0$:

$$\begin{aligned}
\frac{\partial E(k)}{\partial t} = & -2\nu k^2 E(k) + \\
+ \int_{-\infty}^{+\infty} dk' & \left[-\frac{E(k)E(k')}{\nu} + \frac{k^2}{2\nu k'(k-k')} E(k')E(k-k') \right] + \\
+ P \int_{-\infty}^{+\infty} dk' & \left[-\frac{6(k+k')k \cdot E(k)H(k')}{2\lambda k(k-k') - (\lambda + \nu)k^2} + \frac{9k^2 \cdot H(k')H(k-k')}{2\lambda k'(k-k') - (\lambda - \nu)k^2} \right]. \quad (X.21a)
\end{aligned}$$

Similarly, in eq. (X.20b) we only change the first term by replacing k' by $k-k'$. The full kinetic equation for $H(k)$ then becomes:

$$\begin{aligned}
\frac{\partial H(k)}{\partial t} = & -2\lambda k^2 H(k) + \\
+ P \int_{-\infty}^{+\infty} dk' & \left[\frac{6(k-k')(k-2k') \cdot H(k)H(k')}{(\lambda - \nu)(k-k')^2 + 2\lambda k k'} + \frac{2(2k'-k)(k+k') \cdot E(k')H(k)}{2\lambda k'(k'-k) - (\lambda + \nu)k'^2} + \right. \\
& \left. + \frac{2(2k'-k)^2 \cdot E(k')H(k-k')}{2\lambda k k' - (\lambda + \nu)k'^2} \right]. \quad (X.21b)
\end{aligned}$$

In view of the discussion mentioned in section VI.D, the integrals indicated as such should be considered as principal values in order to remove the difficulties with poles in the integrands.

The first term and the first integral on the right-hand side of eq. (X.21a) represent a purely hydrodynamical contribution, already known from the Burgers' case. These terms are not influenced by the magnetic-field fluctuations and are identical with those in the kinetic equation for hydrodynamical turbulence (see eq. (IX.46)). In the second integral of eq. (X.21a), two different interactions can be recognized, viz. a kinetic-magnetic and a purely magnetic one. The first interaction leads to a nonlinear dissipation or growth, depending on the various parameters k, k', λ, ν . The last one describes a coupling between two different magnetic modes (k' and $k-k'$)

which results in a contribution to a particular kinetic k-mode!

The first term in the second dynamical equation (X.21b) describes the well-known linear dissipation due to the conversion of magnetic-field fluctuations into Joule heat, while the two first terms in the integral represent, in succession, a nonlinear dissipation or growth due to a purely magnetic and due to a mixed (kinetic and magnetic) interaction. The last term of this integral, too, describes a mixed interaction of two different modes (k' and $k-k'$) which contributes to the magnetic-field fluctuations.

With the aid of the above dynamical equations one can derive (see appendix F) the "a posteriori" energy conservation expressed by:

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} [E(k) + H(k)] dk = -2 \int_{-\infty}^{+\infty} [\nu k^2 E(k) + \lambda k^2 H(k)] dk. \quad (X.22)$$

D. Magnetohydrodynamical turbulence in the case of a magnetic diffusivity equal to the kinematic viscosity

It turns out that $\lambda = \nu$ constitutes a very interesting special case of the theory of the preceding section. Indeed, the dynamical equations (X.21a) and (X.21b) then reduce, in view of the property of $E(k)$ and $H(k)$ to be even, to²¹⁾

$$\begin{aligned} \frac{\partial E(k)}{\partial t} = & -2\nu k^2 E(k) + T_{EE}(k, t) + T_{HH}(k, t) - \\ & - \frac{E(k)}{\nu} \int_{-\infty}^{+\infty} [E(k') - 3H(k')] dk', \end{aligned} \quad (X.23a)$$

$$\frac{\partial H(k)}{\partial t} = -2\nu k^2 H(k) + T_{EH}(k, t) - \frac{H(k)}{\nu} \int_{-\infty}^{+\infty} [E(k') + 9H(k')] dk'. \quad (X.23b)$$

In view of the further discussion we express these

integrals as follows in terms of positive integrals:

$$\begin{aligned}
 T_{EE}(k, t) &= \frac{k^2}{2\nu} \int_{-\infty}^{+\infty} \frac{E(k')E(k-k')}{k'(k-k')} dk' = \\
 &= \frac{k^2}{2\nu} \left\{ \int_0^k \frac{E(k')E(k-k')}{k'(k-k')} dk' - \int_k^{\infty} \frac{E(k')E(k-k')}{k'(k'-k)} dk' - \right. \\
 &\quad \left. - \int_0^{\infty} \frac{E(k')E(k+k')}{k'(k+k')} dk' \right\}, \quad (X.24)
 \end{aligned}$$

$$\begin{aligned}
 T_{HH}(k, t) &= \frac{9k^2}{2\nu} \int_{-\infty}^{+\infty} \frac{H(k')H(k-k')}{k'(k-k')} dk' = \\
 &= \frac{9k^2}{2\nu} \left\{ \int_0^k \frac{H(k')H(k-k')}{k'(k-k')} dk' - \int_k^{\infty} \frac{H(k')H(k-k')}{k'(k'-k)} dk' - \right. \\
 &\quad \left. - \int_0^{\infty} \frac{H(k')H(k+k')}{k'(k+k')} dk' \right\}, \quad (X.25)
 \end{aligned}$$

$$\begin{aligned}
 T_{EH}(k, t) &= \frac{1}{\nu} \int_{-\infty}^{+\infty} (k-2k')^2 \frac{E(k')H(k-k')}{k'(k-k')} dk' = \\
 &= \frac{1}{\nu} \left\{ \int_0^k (k-2k')^2 \frac{E(k')H(k-k')}{k'(k-k')} dk' - \right. \\
 &\quad - \int_k^{\infty} (k-2k')^2 \frac{E(k')H(k-k')}{k'(k'-k)} dk' - \\
 &\quad \left. - \int_0^{\infty} (k+2k')^2 \frac{E(k')H(k+k')}{k'(k+k')} dk' \right\}. \quad (X.26)
 \end{aligned}$$

Each first term in the latter three right-hand sides represents the energy supply from all lower modes with $0 < k' < k$ to one particular k -mode. Like in the purely hydrodynamical case, these "cascade terms" are the dominating ones for large

k-values. All these terms in T_{EE} and T_{HH} , originating from purely kinetic and purely magnetic turbulence, respectively, contribute to the kinetic energy spectrum. The other transfer term T_{EH} is characteristic for the interaction between kinetic and magnetic-field fluctuations and contributes to the magnetic-energy spectrum. The remaining integral terms in eqs. (X.23a) and (X.23b) represent a nonlinear dissipation or growth in terms of the total kinetic and magnetic energies.

We further observe that the tendency to equipartition (mentioned in refs. 52 and 53 for the three-dimensional case) that is, $u(x,t) = 3^{1/2}h(x,t)$, should amount to $H(k) = \frac{1}{3} E(k)$; however, such a tendency does not appear in our final kinetic equations (X.23a,b) although the model system (X.3)-(X.4) for $\lambda=\nu$ could admit an "equipartition" solution.

We next consider the time dependence of the ensemble averages $\langle u^2 \rangle$ and $\langle h^2 \rangle$ of the properly normalized kinetic and magnetic energies. First of all we mention the purely hydrodynamical relation (compare eq. (IX.49)) $\int_{-\infty}^{+\infty} T_{EE}(k,t)dk = \frac{1}{\nu} \left\{ \int_{-\infty}^{+\infty} E(k)dk \right\}^2$, which can be proved by substituting the expression (X.24) into the left-hand side and by passing in the resulting double integral from k to the new integration variable $k+k'$. From (X.23a) and (X.23b) we next derive the expressions:

$$\begin{aligned} \frac{\partial}{\partial t} \langle u^2(x,t) \rangle &= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} E(k,t)dk = -2\nu \int_{-\infty}^{+\infty} k^2 E(k)dk + \\ &+ \int_{-\infty}^{+\infty} T_{HH}(k,t)dk + \frac{3}{\nu} \int_{-\infty}^{+\infty} E(k)dk \int_{-\infty}^{+\infty} H(k')dk' \end{aligned} \quad (X.27)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \langle h^2(x,t) \rangle &= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} H(k,t)dk = -2\nu \int_{-\infty}^{+\infty} k^2 H(k)dk + \\ &+ \int_{-\infty}^{+\infty} T_{EH}(k,t)dk - \frac{9}{\nu} \int_{-\infty}^{+\infty} H(k)dk \int_{-\infty}^{+\infty} H(k')dk' - \frac{1}{\nu} \int_{-\infty}^{+\infty} E(k')dk' \int_{-\infty}^{+\infty} H(k)dk. \end{aligned} \quad (X.28)$$

The analysis of appendix F leads to the individual vanishing of each of the double integrals of (F.1). In our case $\lambda = \nu$; this involves the relations

$$\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \left\{ \frac{6(k+k')k.E(k)H(k')}{2\nu k k'} - \frac{2(2k'-k)(k+k')E(k')H(k)}{2\nu k k'} + \frac{(2k'-k)^2 E(k')H(k-k')}{\nu k' (k-k')} \right\} = 0, \quad (X.29)$$

$$\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \left\{ \frac{6(k-k')(k-2k')H(k)H(k')}{2\nu k k'} + \frac{9k^2 H(k')H(k-k')}{2\nu k' (k-k')} \right\} = 0. \quad (X.30)$$

In the first relation we interchange k and k' in the middle term which is next added to the first term; a further reduction, to be performed also in the first term of the second relation, is then possible remembering the even character of $E(k)$ and $H(k)$. Moreover, in the remaining integrals, the integrands with respect to k' are recognized to be those of T_{EH} and T_{HH} , respectively, as defined in eqs. (X.25) and (X.26). This leads to the further relations:

$$\int_{-\infty}^{+\infty} T_{EH}(k, t) dk + \frac{2}{\nu} \int_{-\infty}^{+\infty} E(k) dk \int_{-\infty}^{+\infty} H(k') dk' = 0, \quad (X.31)$$

$$\int_{-\infty}^{+\infty} T_{HH}(k, t) dk - \frac{9}{\nu} \int_{-\infty}^{+\infty} H(k) dk \int_{-\infty}^{+\infty} H(k') dk' = 0, \quad (X.32)$$

from which it immediately follows that

$$\int_{-\infty}^{+\infty} T_{EH}(k, t) dk < 0, \quad (X.33)$$

$$\int_{-\infty}^{+\infty} T_{HH}(k, t) dk > 0. \quad (X.34)$$

Hence, all terms in eq. (X.28) are negative and, consequently,

the nonlinear terms in eq. (X.27) are positive. In conclusion, we infer that in this model system which concerns weak turbulence, no amplification of the magnetic energy ("turbulent dynamo" problem) is observed for $\lambda = \nu$. Even for $\lambda = 0$ the same effect is found. As to other values of λ , only large numerical experiments may give information of what happens.

C H A P T E R   X I

C O N C L U S I O N S

We have treated in detail only a small part of the whole field of turbulence. First of all, we considered homogeneous isotropic turbulence and the concept of weak turbulence on a mathematical basis. We restricted ourselves to dynamical systems in which all relevant quantities show fluctuations with vanishing averages or, possibly, also quantities the nonvanishing averages of which are not affected by these fluctuations. Further, no exterior sources or sinks should be present. To get an insight into these phenomena it was necessary to investigate the various quantities and properties in a statistical way.

In order to use the concept of k -modes (k being the wavenumber), we analyzed the random fluctuations in terms of their spatial Fourier transforms. Small-scale fluctuations then correspond to large k -values and, vice versa, large-scale fluctuations to small k -values. The main objective has been to find dynamical (also called kinetic) equations for the energy-spectrum functions, these being connected with the squared fluctuation amplitude of a particular k -mode. These equations then contain, for instance, explicit analytical ex-

pressions for the transfer of energy between the different k-modes; integration over the whole k-space then gives information about the statistical properties of the original quantities (such as the total energy).

The mentioned kinetic equations could be arrived at quite generally by using the Bogoliubov expansion method in the context of weak turbulence. As a matter of fact, problems referring to very different kinds of fluctuations could be treated, no matter how the latter linearly behave in the wavenumber space, viz. as purely damped or growing modes (imaginary frequencies), as undamped stable modes (real frequencies), or as general stable or unstable modes (complex frequencies). The only restriction for the method requires that, according to the final kinetic equations, the weak turbulence should be maintained all the time.

In view of the total-energy conservation considerations we have no doubt about the reliability of the used method to all kinds of weak turbulence problems, even when no conservation laws are known. Earlier results for the well-known case of nonlinear interaction between undamped stable waves were recovered. However, in most cases these latter results are rather poor insofar as they concern turbulence; this does not apply to those which refer to sound waves in a nondissipative fluid (connected with a rather special form of their frequency relation) and Vlasov turbulence (special properties of the medium being relevant there). Indeed, in general only a three-mode coupling proved to occur, while the frequencies had to satisfy an additional condition which even cannot be fulfilled in many cases. On the contrary, we have shown that systems of which the imaginary part of the frequency of the associated modes differs from zero (damped and (or) unstable modes), may actually be called (weak) turbulent, because each of their k-modes interacts with all other ones.

In order to avoid irrelevant mathematical complications our results were derived from three one-dimensional model equations instead of the underlying general dynamical equation.

Obviously, the theory can be extended to other cases, described by similar models, thus admitting the treatment of all kinds of perturbations. However, the attack of problems depending on more branches of the linear dispersion relation, and on more physical quantities, would lead to tedious algebra and very complicated equations as may be clear, for instance, from chapter VII. Very special properties only, inherent to the investigated system, such as symmetry and real interaction strengths, could reduce the equations to quite simpler forms.

The investigated applications, too, required, for simplicity's sake, a restriction to one-dimensional models. Although our method fails to describe strong turbulence in (magneto) hydrodynamics, we could recover the "cascade" process for its modes, viz. the nonlinear transport of energy from the lower k -modes to the higher ones! In fact, this process is also found in more realistic problems such as three-dimensional incompressible fluid turbulence, and in the nonlinear interaction between sound waves.

Finally, we believe that the investigation of the used models has given an insight into the relevant turbulent processes, mathematically as well as physically. Of course, we do not know how good our approximations are, but large computer calculations could be used to examine these problems quite rigorously.

A C K N O W L E D G E M E N T S

The author wishes to thank Professor H. Bremmer for his stimulating interest and very useful remarks, also concerning the English language. He is indebted to Dr. M.P.H. Weenink for his critical comments. For their help with the numerical calculations, he is grateful to Dr. G.J. Dalenoort and Drs. H. Schrijver.

The careful typework of Miss H.A. Scholman and the critical preparing of the manuscript is much appreciated.

This work was performed as part of the research program of the association agreement of Euratom and the "Stichting voor Fundamenteel Onderzoek der Materie" (FOM) with financial support from the "Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek" (ZWO) and Euratom.

APPENDIX A

The fundamental equation (II.1) reads:

$$i \frac{\partial \underline{\psi}(\underline{r}, t)}{\partial t} = L(\underline{V}) \underline{\psi}(\underline{r}, t) + \underline{B}(\underline{V}, \underline{\psi}(\underline{r}, t); \underline{V}, \underline{\psi}(\underline{r}, t)). \quad (\text{A.1})$$

With the aid of the definitions (II.2)-(II.4), namely

$$\begin{aligned} \underline{\psi}(\underline{r}, t) &= \underline{\Phi}(\underline{r}, t) + \underline{\sigma}(\underline{r}, t) = \langle \underline{\psi}(\underline{r}, t) \rangle + \underline{\sigma}(\underline{r}, t); \\ \langle \underline{\sigma}(\underline{r}, t) \rangle &= 0, \end{aligned} \quad (\text{A.2})$$

the first equation (A.1) can be represented as follows, using the property that L is a linear and $\underline{B}$ a bilinear operator:

$$\begin{aligned} i \frac{\partial \langle \underline{\psi} \rangle}{\partial t} + i \frac{\partial \underline{\sigma}}{\partial t} &= L(\underline{V}) \langle \underline{\psi} \rangle + L(\underline{V}) \underline{\sigma} + \underline{B}(\underline{V}, \langle \underline{\psi} \rangle; \underline{V}, \langle \underline{\psi} \rangle) + \\ &+ \underline{B}(\underline{V}, \langle \underline{\psi} \rangle; \underline{V}, \underline{\sigma}) + \underline{B}(\underline{V}, \underline{\sigma}; \underline{V}, \langle \underline{\psi} \rangle) + \underline{B}(\underline{V}, \underline{\sigma}; \underline{V}, \underline{\sigma}). \end{aligned} \quad (\text{A.3})$$

Taking the ensemble average of this equation, we arrive at

$$i \frac{\partial}{\partial t} \langle \underline{\psi} \rangle = L(\underline{V}) \langle \underline{\psi} \rangle + \underline{B}(\underline{V}, \langle \underline{\psi} \rangle; \underline{V}, \langle \underline{\psi} \rangle) + \langle \underline{B}(\underline{V}, \underline{\sigma}; \underline{V}, \underline{\sigma}) \rangle. \quad (\text{A.4})$$

Subtracting this relation from (A.3) we arrive at a dynamical equation for the fluctuating quantity:

$$i \frac{\partial}{\partial t} \underline{\sigma} = L(\underline{\nabla}) \underline{\sigma} + \underline{B}(\underline{\nabla}, \langle \underline{\psi} \rangle; \underline{\nabla}, \underline{\sigma}) + \underline{B}(\underline{\nabla}, \underline{\sigma}; \underline{\nabla}, \langle \underline{\psi} \rangle) + \underline{B}(\underline{\nabla}, \underline{\sigma}; \underline{\nabla}, \underline{\sigma}) - \\ - \langle \underline{B}(\underline{\nabla}, \underline{\sigma}; \underline{\nabla}, \underline{\sigma}) \rangle. \quad (\text{A.5})$$

The last term on the right-hand of eq. (A.4) represents the influence of the turbulence on the time dependence of the averaged quantity $\langle \underline{\psi} \rangle$. The problem to be treated in this thesis is only applicable to systems for which this term vanishes; only in this case one can solve (A.4) as an uncoupled equation for $\langle \underline{\psi} \rangle$ only, the solution of which can be substituted into eq. (A.5) afterwards.

APPENDIX B

The second-order correlation function has been defined as:

$$R(\xi, t) = \langle \psi(x, t) \psi(x + \xi, t) \rangle = \int_{-\infty}^{+\infty} E(k, t) e^{ik\xi} dk, \quad (\text{B.1})$$

the inverse Fourier transform of which reads

$$E(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle \psi(x, t) \psi(x + \xi, t) \rangle e^{-ik\xi} d\xi. \quad (\text{B.2})$$

Also, we introduced

$$\psi(x, t) = \int_{-\infty}^{+\infty} \psi(k, t) e^{ikx} dk = \int_{-\infty}^{+\infty} A(k, t) e^{ikx - i\omega(k)t} dk, \quad (\text{B.3})$$

so that

$$\psi(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(x, t) e^{-ikx} dx. \quad (\text{B.4})$$

Considering eqs. (B.1) and (B.3), we expect a possible connection between two $\psi(k)$'s and $E(k)$. Therefore, we start with the second-order wave correlation

$$\begin{aligned} \langle B \rangle &\equiv \langle \psi(k_1, t) \psi(k_2, t) \rangle = \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle \psi(x_1, t) \psi(x, t) \rangle e^{-ik_1 x_1 - ik_2 x} dx dx_1. \end{aligned} \quad (B.5)$$

Introducing $x_1 = x + \xi$ we get

$$\langle B \rangle = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle \psi(x, t) \psi(x + \xi, t) \rangle e^{-i(k_1 + k_2)x - ik_1 \xi} dx d\xi. \quad (B.6)$$

In view of the definition of homogeneous turbulence, the quantity $\langle \psi(x, t) \psi(x + \xi, t) \rangle$ should be independent of x so that we can carry out the integration over x -space. With the aid of the formula

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(k_1 + k_2)x} dx = \delta(k_1 + k_2), \quad (B.7)$$

we obtain

$$\begin{aligned} \langle B \rangle &\equiv \langle \psi(k_1, t) \psi(k_2, t) \rangle = \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \langle \psi(x, t) \psi(x + \xi, t) \rangle e^{-ik_1 \xi} \delta(k_1 + k_2) d\xi. \end{aligned} \quad (B.8)$$

Hence,

$$E(k_1, t) \delta(k_1 + k_2) = \langle \psi(k_1, t) \psi(k_2, t) \rangle. \quad (B.9)$$

The appearance of δ -functions in k -space for all wave correlations of different order is a direct consequence of the definitions of homogeneous turbulence. We notice that the expression (B.9) is also known as the random phase approximation (refs. 1,16).

Finally, we notice that the analysis of this appendix is quite analogous to that in the theory of statistical mechanics related with the Wiener-Khinchin theorem.

APPENDIX C

Our choice of the functional dependence of the wave-correlation functions $H(k_1, k_2)$, $F(k_1, k_2, k_3) \dots$ on $E(k_i)$ only at time $t = 0$ seems rather artificial with respect to Bogoliubov's theory⁹). In the latter, a synchronization was established for all times after a certain transition time. However, we shall prove in this appendix that, assuming the just mentioned initial conditions at $t = 0$, the wave-correlation functions $S(k_1, k_2)$, $L(k_1, k_2, k_3), \dots$, as defined in chapter III, are synchronized with the second-order wave-correlation function $E(k_i)$ for all times ($t \geq 0$)!

Remembering eq. (III.16a), we find:

$$S(k', -k) = H(k', -k) e^{-i\{\omega(k') + \omega(-k) + \omega(k-k')\}t} \quad (C.1)$$

by taking

$$k_1 = k', \quad k_2 = -k, \quad \text{and} \quad k_3 = k - k'. \quad (C.2)$$

The function $S(k_1, k_2)$ has been introduced as the Fourier transform of the third-order correlation function $\langle \psi(x, t) \psi(x + \xi, t) \psi(x + \xi', t) \rangle$. Substitution of this result for $H^{(2)}(k', -k)$ (see eq. (V.14)) into (C.1), yields (up to $O(\epsilon^2)$), with the aid of eqs. (II.29) and (II.31b,c):

$$\begin{aligned} S(k', -k) \Big|_{O(\epsilon^2)} = & \\ = & 2V(k'; k, k'-k)E(k)E(k-k') \frac{1}{\omega(k) + \omega(k'-k) - \omega(k')} + \\ & + 2V(-k; -k', k'-k)E(k')E(k-k') \frac{1}{\omega(-k') + \omega(k'-k) - \omega(-k)} + \end{aligned}$$

$$+ 2V(k-k'; k, -k') E(k) E(k') \frac{1}{\omega(k) + \omega(-k') - \omega(k-k')} . \quad (C.3)$$

Hence, the time dependence of the third-order wave-correlation function $S(k', -k)$ is completely determined by the second-order wave correlation function $E(k)$. Symbolically, this result can be represented by

$$S(k', -k, t) = S(k', -k; E(k)). \quad (C.4)$$

One could also find the corresponding result for the higher-order wave-correlation functions.

APPENDIX D

We shall consider the following δ -function, appearing in the first term of eq. (IX.105), viz.

$$\begin{aligned} \delta(v_s(k-k'-k'')) &= \\ &= \delta(v_s(|\underline{k}| - |\underline{k}'| - |\underline{k}-\underline{k}'|)) = \frac{1}{v_s} \delta(k-k' - |k^2+k'^2-2kk'x|^{\frac{1}{2}}); \end{aligned} \quad (D.1)$$

the parameter x introduced here is defined by

$$\underline{k}\underline{k}' = kk' \cos \theta = kk'x. \quad (D.2)$$

We next observe that the quantity

$$b(x) \equiv k-k' - |k^2+k'^2-2kk'x|^{\frac{1}{2}} \quad (D.3)$$

only vanishes if $x = 1$ provided that, moreover, $k' \leq k$. We shall apply the well-known formula

$$\delta(b(x)) = \sum_{\alpha} |b'(a_{\alpha})|^{-1} \delta(x-a_{\alpha}), \quad (D.4)$$

the summation of which extends over all zeros $x = a_{\alpha}$ of $b(x)$,

assuming the absence of multiple zeros, that is

$$b'(a_\alpha) \neq 0. \quad (D.5)$$

Introducing spherical coordinates for the integration over the entire $\underline{k}'$ -space:

$$\int_{-\infty}^{+\infty} d\underline{k}' f(\underline{k}') = \int_{-1}^{+1} dx \int_0^{2\pi} d\phi \int_0^{\infty} k'^2 f(\underline{k}') dk' \quad (D.6)$$

we then arrive at

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta(v_s(k-k'-k'')) f(\underline{k}') d\underline{k}' &= \frac{2\pi}{v_s} \int_{-1}^{+1} dx \int_0^k dk' f(\underline{k}') k'^2 \frac{(k-k')}{kk'} \delta(x-1) = \\ &= \frac{2\pi}{kv_s} \int_{-1}^{+1} dx \delta(x-1) \int_0^k dk' k'(k-k') f(\underline{k}'). \end{aligned} \quad (D.7)$$

The other relevant integrals over the whole $\underline{k}'$ -space can be calculated in a similar way. We find

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta(v_s(k+k'-k'')) f(\underline{k}') d\underline{k}' &= \\ = \frac{2\pi}{kv_s} \int_{-1}^{+1} dx \delta(x+1) \int_0^{\infty} dk' k'(k+k') f(\underline{k}') \end{aligned} \quad (D.8)$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} \delta(v_s(k-k'+k'')) f(\underline{k}') d\underline{k}' &= \\ = \frac{2\pi}{kv_s} \int_{-1}^{+1} dx \delta(x-1) \int_k^{\infty} dk' k'(k'-k) f(\underline{k}'). \end{aligned} \quad (D.9)$$

APPENDIX E

We want to prove here that total energy is conserved for the model system (X.3) and (X.4), including the curious factor 3. Hence, with the aid of eqs. (X.14) and (X.15)

$$\begin{aligned}
 & \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \left[G(k) e^{-2vk^2 t} + F(k) e^{-2\lambda k^2 t} \right] dk = \\
 & = -2 \int_{-\infty}^{+\infty} \left[vk^2 G(k) e^{-2vk^2 t} + \lambda k^2 F(k) e^{-2\lambda k^2 t} \right] dk + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk dk' \times \\
 & \times \left[\frac{3ik}{2} S(u, h, h; -k, k', k-k') e^{-(\lambda+v)k^2 t - 2\lambda k'(k'-k)t} - \right. \\
 & - \frac{3ik}{2} S(u, h, h; k, k', -k-k') e^{-(\lambda+v)k^2 t - 2\lambda k'(k'+k)t} + \\
 & + i(2k'-k) S(u, h, h; k', k-k', -k) e^{-(\lambda+v)k'^2 t + 2\lambda k k' t - 2\lambda k^2 t} + \\
 & \left. + i(2k'+k) S(u, h, h; k', -k-k', k) e^{-(\lambda+v)k'^2 t - 2\lambda k k' t - 2\lambda k^2 t} \right] \\
 & \hspace{15em} (E.1)
 \end{aligned}$$

because the purely hydrodynamical term of eq. (X.14) (the first one of the right-hand side) vanishes immediately, as had been proven in connection with the Burgers' case (see eq. (IX.49)).

In order to indicate further reductions we mark the four successive contributions between the brackets [ ] in the double integral of (E.1) as (1), (2), (3), and (4). This integral remains unchanged when the following substitutions are performed. In (2) we replace k' by $-k'$; then we add half the expression obtained from (3) by replacing k by $k'-k$ and half the original expression (3) and next interchange k and k' in the resulting quantity; finally, we add half of (4), with k replaced by $-k-k'$, to half the original expression (4), and then interchange k and $-k'$. The three new terms,

thus obtained, are added to the unchanged quantity (1). It then turns out that the new fourth expression is cancelled by the first one, and the new third expression by the second one. Hence, the double integral of (E.1) completely vanishes, which proves the conservation of energy apart from the dissipation terms constituting the single-integral contribution in the right-hand side of (E.1).

APPENDIX F

The left-hand side of eq. (X.22) can be worked out as follows with the aid of (X.21a) and (X.21b):

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} [E(k) + H(k)] dk &= -2 \int_{-\infty}^{+\infty} [vk^2 E(k) + \lambda k^2 H(k)] dk + \\ &+ \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \left[-\frac{6(k+k')k \cdot E(k)H(k')}{2\lambda k(k-k') - (\lambda + \nu)k^2} + \right. \\ &+ \left. \frac{2(2k'-k)(k+k') \cdot E(k')H(k)}{2\lambda k'(k'-k) - (\lambda + \nu)k'^2} + \frac{2(2k'-k)^2 \cdot E(k')H(k-k')}{2\lambda kk' - (\lambda + \nu)k'^2} \right] + \\ &+ \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \left[\frac{6(k-k')(k-2k') \cdot H(k)H(k')}{(\lambda - \nu)(k-k')^2 + 2\lambda kk'} + \right. \\ &\left. + \frac{9k^2 \cdot H(k')H(k-k')}{2\lambda k'(k-k') - (\lambda - \nu)k^2} \right]; \end{aligned} \quad (F.1)$$

in the derivation of which we have used the identity

$$\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \left\{ -E(k)E(k') + \frac{k^2}{2k'(k-k')} E(k')E(k-k') \right\} = 0,$$

which is based on the even character of $E(k)$ (see section IX.C).

Investigating the first double integral with the mixed

(kinetic-magnetic) interaction we interchange k and k' in its first term which can then be added to the second one. This results in a new double integral with integrand

$$-\frac{2(k+k')^2 E(k') H(k)}{2\lambda k'(k'-k) - (\lambda + \nu) k'^2} . \quad (F.2)$$

Introducing next $k_0 = k' - k$ as a new integration variable replacing k , we find a contribution which is cancelled by the third term of the first double integral in (F.1); hence, the first double integral vanishes altogether.

The second double integral of (F.1) may be evaluated as follows. After inverting the order of integration in its last term, we first replace k by $k_0 = k - k'$ and then in the new double integral k' by $-k'$. The addition to the first term of the second double integral of (F.1) then gives

$$-3 \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \frac{(k^2 - k'^2) H(k) H(k')}{(\lambda - \nu)(k - k')^2 + 2\lambda k k'} , \quad (F.3)$$

which expression has to vanish since it changes its sign when interchanging k and k' . Therefore, both double integrals in (F.1) disappear which completes the proof of the relation (X.22).

REFERENCES

1. Kadomtsev, B.B., Plasma Turbulence, Academic Press (New York, 1965).
2. Heisenberg, W., Phys.Today 20 (1967) 27.
3. Zabusky, N.J., in Nonlinear Partial Differential Equations, ed. by Ames, W.F., Academic Press (New York, 1967).
4. Galeev, A.A. and Karpman, V.I., Soviet Phys.-JETP 17 (1963) 403.
5. Benney, D.J. and Saffman, P.G., Proc.Roy.Soc. (London) A289 (1966) 301.
6. Coppi, B., Rosenbluth, M.N., and Sudan, R.N., Ann.Phys. 55 (1969) 207.
7. Davidson, R.C., MATT Report 496, Princeton (1966).
8. Sandri, G., Ann.Phys. 24 (1963) 332.
9. Bogoliubov, N.N., in Studies in Statistical Mechanics, vol. I, eds. J. de Boer and G.E. Uhlenbeck, North-Holland Publ.Co. (Amsterdam, 1962).
10. Burgers, J.M., Acad.Sci.Amsterdam 53 (1950) 247.
11. Millionshtchikov, M.D., Compt.Rend.Acad.Sci. URSS 32 (1941) 615.
12. Thomas, J.H., Phys.Fluids 11 (1968) 1245.
13. Davidson, R.C., J.Plasma Phys. 1 (1967) 341.
14. Galeev, A.A. and Sagdeev, R.Z., Nonlinear Plasma Theory, W.A. Benjamin Inc. (New York, 1969).

15. Karpman, V.I., Soviet Phys.-Doklady 8 (1964) 919.
16. Tsytovich, V.N., Nonlinear Effects in Plasma, Plenum Press (New York, 1970).
17. Batchelor, G.K., The Theory of Homogeneous Turbulence, Cambridge University Press (London, 1956).
18. Proudman, I. and Reid, W.H., Phil.Trans.Roy.Soc. London A247 (1954) 163.
19. Wu, T.Y., Kinetic Equations of Gases and Plasmas, Addison-Wesley Publ.Co. (New York, 1966).
20. Malfliet, W.P.M., Physica 45 (1969) 257.
21. Malfliet, W.P.M., Proc. 9th International Conference on Phenomena in Ionized Gases, Bucharest (1969) 391.
22. Malfliet, W.P.M., Phys. Letters 31A (1970) 138.
23. Hasselmann, K., J.Fluid Mech. 12 (1962) 481.
24. Busch, N.E., Radio Sci. 4 (1970) 1377.
25. Bretherton, F.P., Radio Sci. 4 (1970) 1279.
26. Rebhan, E. and Wimmel, H., Plasma Phys. 10 (1969) 869.
27. Aamodt, R.E. and Sloan, M.L., Phys.Rev.Letters 19 (1967) 1227.
28. Davidson, R.C. and Kaufman, A.N., J.Plasma Phys. 3 (1969) 97.
29. Davidson, R.C., Phys.Fluids 12 (1969) 149.
30. Kovrizhnykh, L.M., EUR-CEA-FC-258 Report (1964).
31. Frieman, E.A., MATT Report 106, Princeton (1961).
32. Balescu, R. and Senatorski, A., Ann.Phys. 58 (1970) 587.
33. Saffman, P.G., in Topics in Nonlinear Physics, ed. by N.J. Zabusky, Springer-Verlag (Berlin, 1968).
34. Tatsumi, T., Proc.Roy.Soc. London A239 (1957) 16.
35. Kahng, W.H. and Siegel, A., J.Fluid Mech. 41 (1970) 593.
36. Kawahara, T., J.Phys.Soc. Japan 25 (1968) 892.
37. Reid, W.H., Appl.Sci.Res. A6 (1956) 85.
38. Hosokawa, I. and Yamamoto, K., Phys. Fluids 13 (1970) 1683.
39. Meecham, W.C. and Siegel, A., Phys. Fluids 7 (1964) 1178.
40. Onsager, L., Phys.Rev. 68 (1945) 286.

41. Hinze, J.O., Turbulence, McGraw-Hill Book Comp. (New York, 1959).
42. Ling, S.C. and Huang, T.T., Phys. Fluids 13 (1970) 2912.
43. Uberoi, M.S. and Freymuth, P., Phys. Fluids 12 (1969) 1359.
44. Uberoi, M.S. and Freymuth, P., Phys. Fluids 13 (1970) 2205.
45. Kraichnan, R.H., Phys. Fluids 7 (1964) 1030.
46. Kraichnan, R.H., J.Fluid Mech. 5 (1959) 497.
47. Wyld Jr., H.W., Ann.Phys. 14 (1961) 143.
48. Jeng, D.T., Foerster, R., Haaland, S., and Meecham, W.C., Phys. Fluids 9 (1966) 2114.
49. Ogura, Y., J.Fluid Mech. 16 (1963) 33.
50. Kraichnan, R.H. and Nagarajan, S., Phys. Fluids 10 (1967) 859.
51. Chandrasekhar, S., Proc.Roy.Soc. London A233 (1955) 322, 330.
52. Batchelor, G.K., Proc.Roy.Soc. London A201 (1950) 405.
53. Parker, E.N., Astrophys.J. 160 (1970) 383.

Curriculum vitae

Na het behalen van het diploma van de humaniora, afdeling Latijn-Wiskunde, aan het St.-Jozef Klein Seminarie te St.-Niklaas (België), liet de auteur zich eind september 1961 inschrijven aan de Katholieke Universiteit te Leuven. Daar verkreeg hij het kandidaatsdiploma in de wetenschappen, voor de groep der Natuurkundige en voor de groep der Wiskundige wetenschappen.

Vervolgens begon hij in september 1963 de studie aan de Rijksuniversiteit te Utrecht. Als onderdeel van deze studie heeft hij in het FOM-Instituut voor Plasma-Fysica te Jutphaas relativistische berekeningen uitgevoerd van energieën en zwevingstijden van elektronen bij cyclotronresonantie. Het afstudeerwerk o.l.v. Prof.dr. N.G. van Kampen handelde over de tijdschalenmethode in de statistische mechanica.

Na het doctoraalexamen, hoofdvak theoretische fysica, in juni 1967, kwam hij per 1 augustus 1967 in dienst van de Stichting F.O.M. bij de werkgroep TN-1 (theoretische plasma-fysica) te Jutphaas. Het onderzoek aldaar verricht resulteerde in dit proefschrift.

S T E L L I N G E N

1. De argumentatie van Davidson voor het weglaten van de beginvoorwaarde voor de golf-correlatiefunctie van de derde orde is niet van toepassing op het geval van stabiele ongedempte golven.

Davidson, R.C., MATT 496, Princeton (1966)
pagina 63 e.v.

2. Het is merkwaardig te moeten vaststellen dat men problemen behandelt met dynamische vergelijkingen, analoog aan onze eerste modelvergelijking, hoewel deze problemen klaarblijkelijk aan onze tweede modelvergelijking beantwoorden.

- a) Einaudi, F. en Sudan, R.N., Plasma Physics 11 (1969) 359.
- b) Davidson, R.C., Phys. Rev. 176 (1968) 344.
- c) Zakharov, V.E. en Sagdeev, R.Z., Sov. Phys. Doklady 15 (1970) 439.

3. De opmerking van Galeev en Sagdeev dat er voor hydrodynamische turbulentie geen kinetische vergelijking bestaat, is onjuist.

Galeev, A.A. en Sagdeev, R.Z., Nonlinear Plasma Theory, W.A. Benjamin Inc. (New York, 1969)
zie pagina 25.

4. De dynamische vergelijking voor het energiespectrum in het Burgers' model, door Meecham en Siegel afgeleid met behulp van de Cameron-Martin-Wiener-methode, kan op een triviale wijze rechtstreeks uit de twee eerste vergelijkingen van de overeenkomstige hiërarchie van correlatiefuncties worden gevonden.

- a) Meecham, W.C. en Siegel, A., Phys. Fluids 7 (1964) 1178.
b) Dit proefschrift, paragraaf IX.F.

5. Het is wenselijk om de numerieke berekeningen van Ogura en Jeng (en anderen), respectievelijk voor driedimensionale en één dimensionale onsamendrukbare vloeistofturbulentie, te herhalen met een initieel reeds van nul verschillende niet-lineaire interactieterm.

- a) Ogura, Y., J. Fluid Mech. 16 (1963) 33.
b) Jeng, D. e.a., Phys. Fluids 9 (1966) 2114.
c) Dit proefschrift, pagina 115.

6. De essentie van de tijdschalenmethode, zoals deze in de statistische mechanica gebruikt wordt voor het afleiden van kinetische vergelijkingen, gaat verloren in het geval van inhomogene media.

Sandri, G., Ann. Phys. 24 (1963) 381.

7. Het begrip turbulentie wordt in verschillende publicaties en leerboeken te onduidelijk vastgelegd.

8. Voor een krachtvrij magneetveld ($\underline{j} = \alpha(\underline{r}, t)\underline{B}$), dat bovendien nog rotatiesymmetrisch is, geldt de eigenschap dat op elk magnetisch oppervlak α constant is.
9. De bewering van R. Malfliet en Th. Ruygrok, dat de door hen afgeleide vergelijkingen voor niet-relativistische verstrooiing van één deeltje aan een gebonden toestand van twee andere, een éénduidige oplossing hebben, is onjuist.

Malfliet, R. en Ruygrok, Th., Physica 33
(1967) 607.

10. De maximale relativistische energie van een elektron in een draaiend elektrisch veld, met loodrecht hierop een homogeen magneetveld, kan analytisch worden onderzocht.
11. Het zou aanbeveling verdienen om in België het aggregaat van het hoger onderwijs af te schaffen en de graad van doctor in de wetenschappen wettelijk aan te passen aan het niveau van dat aggregaat.
12. De taak van de tweede promotor bij de nieuwe eisen voor het behalen van de graad van doctor in de technische wetenschappen is onduidelijk.
13. Voor de verkorting van de studieduur, i.h.b. voor het doctoraal examen in de natuurkunde, dient men te zoeken naar een systeem dat ligt tussen dat van het Belgische en Nederlandse universitair onderwijs.
14. Bij het gebruik van stortings-acceptgirokaarten voor het verlengen van abonnementen e.d. zou de mogelijkheid van annulering ingebouwd moeten worden.