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ON THE MOTION OF A COMPRESSIBLE FLUID IN A ROTATING CYLINDER

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL TWENTE, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF. DR. J. KREIKEN, VOLGENS BE- SLUIT VAN HET COLLEGE VAN DEKANEN IN HET OPEN- BAAR TE VERDEDIGEN OP DONDERDAG 24 JUNI 1976 TE 16.00 UUR.

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PROF. DR. J. LOS (Universiteit van Amsterdam)
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1. INTRODUCTION

In the last 20 years, the dynamics of rotating fluids have become of considerable interest, especially in meteorological, oceanographic and geomagnetic questions involving the earth's rotation. Another interesting use of rotating fluids is in the gas centrifuge used for the separation of uranium isotopes. A new object of study in this field is the effect of compressibility. Since the quantitative experimental investigation of a compressible gas in a high-speed cylinder is an exceedingly difficult problem, considerable emphasis has been placed on theoretical efforts to study this problem.

The first obvious step in a theoretical approach is the assumption that the motion consists of a small perturbation on the isothermal state of rigid body rotation. In that case the equations describing the secondary flow can be linearised and, of course, the analytical handling is made much easier. Under this assumption two kinds of solutions for the gas flow in a rotating cylinder are known.

The first kind deals with a closed cylinder of finite length and has been investigated by Matsuda, Hashimoto & Takeda (1976), Matsuda, Sakurai & Takeda (1975), Matsuda (1975), Mikami (1973a, 1973b), Nakayama & Usui (1974), Sakurai & Matsuda (1974) and Sakurai (1975). Assuming a small Ekman number for the primary rotation, the flow consists of an inviscid interior and boundary layers near the walls, viz. Ekman layers near the end caps and two layers of the type first discussed by Stewartson (1957) near the cylinder wall. In fact the approach is a direct extension of the boundary layer methods often applied to incompressible fluids in rotating systems (see Greenspan, 1968). Investigations in this field have been primarily motivated by geophysical questions stemming from models of atmospheric and oceanic currents.
The second kind of solution deals with a cylinder of semi-infinite length. This approach stems from models of gas centrifuges, whose length is considerably larger than their diameter. Solutions for this problem were given by Dirac (1940), Ging (1962a, 1962b), Parker & Mayo (1963), Parker (1963) and Steenbeck (1958). The basic assumption consists of a small axial variation of the flow variables compared to the radial variation. In this case the flow can be described by a set of radial eigenfunctions which decay exponentially in the axial direction. Apart from this treatment Berman (1963) and Soubbaramayer (1961) discussed the case of a constant axial flow driven by respectively a radial and an axial temperature gradient. The calculated radial shape of the flow was approximately equal to the shape of the first eigenfunction.

The two kinds of approach led to different descriptions for the flow field. In the first kind the end cap conditions have the predominating influence. Corresponding to the Taylor–Proudman theorem the flow in an inviscid interior is constant along the axial coordinate and is controlled by the pumping of the Ekman layers at the top and bottom. In the second kind it is assumed that, due to radial diffusive processes end effects decay rather rapidly, so that at a reasonable distance, the flow pattern is practically independent of the end conditions. On the other hand, in the first kind of approach the influence of a strong density gradient, typical in the present day gas centrifuges, was not considered. In the case of isothermal rigid body rotation the density increases exponentially with the radial distance from the axis. Due to the high rotational speed and the large molecular weight of the gas, ratios of $10^7$ between the density at the periphery and at the axis are quite normal today. In the case of increasing the density gradient Parker and Ging observed that the main part of the mass flow is concentrated at the periphery. At the same time the axial decay of the eigenfunctions increases considerably, which indicates that the control of the Ekman layers on an inviscid interior no longer applies.

In order to know whether or not the diffusive processes describing the flow in the main section of the cylinder are important, it is necessary to apply an order of magnitude consideration to the various terms in the Navier–Stokes equations. Here, the influence of (i) the length-to-radius ratio of the cylinder and (ii) the radial density gradient is of central
importance. At first the length effect is studied separately by considering an incompressible fluid in a rotating cylinder whose length-to-radius ratio L increases from unit magnitude to infinity. The Navier-Stokes equations are linearised with respect to rigid body rotation. The Ekman number E, based on the radius, is taken to be small. Three types of flow corresponding to the ranges $E^{1/2} \ll L \ll E^{-1/2}$, $E^{-1/2} \ll L \ll E^{-1}$ and $E^{-1} \ll L$ are found. In the first range we identify a geostrophic flow in the interior extended by Ekman layers near the end caps and an inner and outer Stewartson layer near the cylinder wall (the well-known "geophysical" flow). When L exceeds the upper limit of the first L-range the outer Stewartson layer expands to the interior. For $L \geq E^{-1}$ the inner layer also fills the whole cylinder. Then, the radial diffusion of azimuthal and axial momentum is important in the entire cylinder, a typical situation for studies on flows in semi-infinite cylinders. However, since $E$ is very small most configurations will fall into the first L-range.

In the case of a perfect gas in a rotating cylinder, special attention is focused on strong radial density gradients. The modified Ekman number, $E_{m}$, based on the density at the periphery and based on the radial density scale height (instead of the radius) is taken to be small. Increasing the ratio of the length of the cylinder to the radial density scale height, $L_{m}$, from unit magnitude to infinity three types of flow are identified. These correspond to the ranges $E_{m}^{1/2} \ll L_{m} \ll E_{m}^{-1/2}$, $E_{m}^{-1/2} \ll L_{m} \ll E_{m}^{-1}$ and $E_{m}^{-1} \ll L_{m}$. However, two essential differences are found: (i) An inviscid flow typical of the first type is only observed in a region of limited thickness near the cylinder wall. Due to the decrease of the density with distance from the wall, viscosity and conduction are important in the core of the cylinder. (ii) A change of the flow type appears when both Stewartson layers expand over the small density scale height. Simultaneously the corresponding diffusive regions in the core come up and join. As a result, a change of the flow type appears at relatively lower values of the length-to-radius ratio. All three types are of real practical interest.

Finally, explicit solutions for the velocity distribution in the various parameter ranges of the rapidly rotating heavy gas are calculated. Here, we apply perturbation techniques and the method of matched asymptotic expansions as given by Van Dyke (1964). The boundary conditions indu-
cing the secondary flow are: differential rotation of the end caps, axial injection and removal of fluid at the end caps and temperature perturbations along the end caps and along the cylinder wall. Thus, the flow of a heavy gas in a rapidly rotating cylinder, e.g., in a gas centrifuge, is qualitatively and quantitatively well understood.
2. THE SCALING ANALYSIS

2.1. The incompressible fluid

The motion of an incompressible Newtonian fluid of constant material properties is considered. The fluid is confined in a circular cylinder which rotates about its vertical symmetry axis. The plane horizontal top and bottom caps are assumed to rotate with an angular speed which differs slightly. Furthermore it is assumed that these end caps are permeable and that fluid is injected and withdrawn vertically. However, we restrict ourselves to the case that the total axial net transport is zero.

The equations describing the velocity and pressure field of an incompressible fluid are given by the conservation of mass and of momentum. We confine ourselves to stationary flows. Both equations, written in a rotating frame fixed with respect to the cylinder, are (Greenspan, 1968):

\[ \nabla \cdot \dot{q} = 0 \]  \hspace{1cm} (2.1.1)

\[ (\dot{q} \cdot \nabla) \dot{q} - \frac{1}{\rho} \Omega^2 \dot{q} \nabla |\dot{k} \times \dot{r}|^2 + 2\Omega \dot{k} \times \dot{q} = -\frac{1}{\rho} \nabla P + \dot{F} + \nu \Delta \dot{q} \] \hspace{1cm} (2.1.2)

Here \( \dot{q} \) is the particle velocity and \( \dot{r} \) the position vector in the rotating frame. \( \Omega \) is the angular speed of the cylinder and \( \dot{k} \) is the unit vector along the rotation axis. \( P, \rho \) and \( \nu \) denote respectively pressure, density and kinematic viscosity.

The terms on the left hand side of the momentum equation (2.1.2) represent successively the inertia, centrifugal and Coriolis forces per unit mass. The terms on the right hand side are the pressure, external and shear forces per unit mass. The external body force is assumed to be conservative e.g.

\[ \dot{F} = \nabla f \] \hspace{1cm} (2.1.3)
A static equilibrium is obtained if the velocities in the rotating frame are equal to zero. Setting $\dot{q} = 0$ in (2.1.2) the pressure distribution at rigid body rotation becomes

$$P_e^\text{r} = \rho f + \frac{1}{2\rho n^2} |\dot{k}\dot{x}^\text{r}|^2$$  \hspace{1cm} (2.1.4)

This solution is used to define a relative pressure

$$p = P - P_e^\text{r}$$  \hspace{1cm} (2.1.5)

by which (2.1.2) simplifies to

$$\dot{q} \cdot \dot{q} + 2\ddot{k}\dot{x} = -\frac{1}{\rho} \dot{v}_p + v\dot{\delta}$$  \hspace{1cm} (2.1.6)

The following dimensionless variables are defined

$$\dot{r} = a\dot{x}^* \hat{q} = U\dot{q}^* \quad p = \rho\Omega U p^*$$  \hspace{1cm} (2.1.7)

where $a$ is the radius of the cylinder and where $U$ characterises the typical particle velocity measured in the rotating frame. Putting (2.1.7) into (2.1.1) and (2.1.6), and dropping the asterisks, the dimensionless equations of motion become

$$\nabla \cdot \dot{q} = 0$$  \hspace{1cm} (2.1.8)

$$\epsilon(\dot{q} \cdot \dot{q}) + 2\ddot{k}\dot{x} = -\dot{v}_p + E\dot{\delta}$$  \hspace{1cm} (2.1.9)

The dimensionless parameters now entering into the problem are the Rossby number $\epsilon$ and the Ekman number $E$, defined by

$$\epsilon = U/\Omega a \quad E = v/\Omega a^2$$  \hspace{1cm} (2.1.10)

The Ekman number, based on the radius $a$, is a measure of the magnitude of the viscous forces relative to the Coriolis acceleration. The Rossby number is a measure of the relative importance of the non-linear convective terms with respect to the Coriolis acceleration. Assuming that the velocities measured in the rotating frame are much smaller than the circumferential speed of the cylinder, the non-linear
terms in (2.1.9) can be neglected. In the limit of $\varepsilon \to 0$, (2.1.9) reduces to

$$2\vec{k} \times \vec{q} = -\nabla P + E\Delta \vec{q} \quad (2.1.11)$$

Since we are concerned with a cylindrical configuration, it is convenient to apply cylindrical coordinates $(r, \phi, Lz)$ with the origin at the centre of the cylinder and with corresponding velocity components $(u, v, w)$.

The aspect ratio $L$ is the ratio of the length of the cylinder to its radius,

$$L = l/a \quad (2.1.12)$$

and, at this stage, the axial coordinate has been scaled by the length of the cylinder. Because of axial symmetry of the boundary conditions at the end caps it is reasonable to assume an axial symmetric distribution of the flow. In this case, the continuity equation can be used to define a stream-function $\psi$

$$u = -L^{-1}r(\partial / \partial r)\psi \quad w = r^{-1}(\partial / \partial r)r^2\psi \quad (2.1.13)$$

Furthermore we define an angular velocity $\omega(r, z)$ given by

$$v = \omega r \quad (2.1.14)$$

Applying (2.1.13), (2.1.14) and cross-differentiating the radial and axial component of the momentum equation (2.1.11) to eliminate $P$, the equations of motion become

$$L \varepsilon \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \right] r^2 + \frac{1}{L^2} \frac{\partial^2}{\partial z^2} \right\} \omega = -2 \frac{\partial \psi}{\partial z} \quad (2.1.15)$$

$$L \varepsilon \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \right] r^2 + \frac{1}{L^2} \frac{\partial^2}{\partial z^2} \right\} \psi = +2 \frac{\partial \omega}{\partial z} \quad (2.1.16)$$

where (2.1.15) is the azimuthal component of the momentum equation.

The boundary conditions are given by

$$\omega = \psi = (\partial / \partial r)\psi = 0 \quad \text{at} \quad r = 1 \quad (2.1.17)$$

$$\omega = \omega_b, \quad \psi = \psi_b, \quad (\partial / \partial z)\psi = 0 \quad \text{at} \quad z = 0 \quad (2.1.18)$$
where
\[
\psi_b = -2 \int_0^r w_b r dr, \quad \psi_t = -2 \int_0^r w_t r dr
\]
(2.1.20)

and
\[
\psi_b = \psi_t = 0 \quad \text{at } r = 1
\]
(2.1.21)

Here, \( \omega_b, \omega_t, w_b, \) and \( w_t \) are arbitrary functions of \( r \). The functions \( \omega_b \) and \( \omega_t \) represent the dimensionless different rotation rate of bottom and top end caps and \( w_b \) and \( w_t \) are the imposed dimensionless vertical velocities. Condition (2.1.21) implies that the axial net transport is zero. The no-slip and no-normal-flow condition at the cylinder wall is expressed by (2.1.17).

Differential equations (2.1.15) and (2.1.16) with boundary conditions (2.1.17) - (2.1.21) are the basis for our further investigation. The basic assumption is a small Ekman number, \( E << 1 \), a condition satisfied in many practical situations. The aspect ratio is varied from unit magnitude to infinity.

Consider now the case that \( L \sim 1 \). Since \( E << 1 \) the terms on the left hand side of (2.1.15) and (2.1.16) are small and can be neglected. However, since the terms with the highest derivatives, stemming from the viscous forces, are dropped, the order of the equations is reduced. These simplified equations do not allow all boundary conditions to be met and one may speak of a singular perturbation problem (Van Dyke, 1964). As is well-known since the times of Prandtl, this problem is overcome by introducing boundary layers. These layers are located along the bounding surfaces and are the narrow regions within which the velocity components are adjusted by the viscosity, to the prescribed conditions at the walls. A formal method to develop the boundary layer equations is the stretching of (i) the coordinate normal to the surface and (ii) the dependent variables with a power of the small parameter \( E \) such that for \( E \to 0 \) the highest normal derivatives are also significant, which in turn preserves the applicability of the boundary conditions at the surface. The requirement that at the outer edge of the layer the boundary layer variables attain prescribed values appropriate to the inviscid domain is known as the matching principle. It connects the various flow regions and leads to a consistent description of the flow field.
A treatise on perturbation techniques and the method of matched asymptotic expansions is given by Van Dyke (1964), to which the reader is referred.

Consider now a basic **inviscid** flow. In the limit of \( E \to 0 \) equations (2.1.15) and (2.1.16) reduce to

\[
\begin{align*}
(\partial / \partial z) \psi &= 0 \\
(\partial / \partial z) \omega &= 0
\end{align*}
\]

(2.1.22)

This result, however, leaves the magnitude of \( \psi \) and \( \omega \) with respect to \( L \) and \( E \) unspecified. The answer to this question is found from boundary layer considerations. The **inviscid** solution does not allow all boundary conditions to be met at \( z = 0 \) and \( z = 1 \). Boundary layers near both end caps will adjust the flow variables to the prescribed conditions. In the **inviscid** region \( \psi \) and \( \omega \) are constant with respect to \( z \). Therefore we can expect that these layers will control the flow in the interior; i.e. the scaling magnitude of \( \psi \) and \( \omega \) in the interior will be the same as in both layers. Anticipating the outcome of the boundary layer analyses we scale, for the **inviscid** domain

\[
\psi = E^{1/2} \psi_0 , \quad \omega = \omega_0
\]

(2.1.23)

Putting (2.1.23) into (2.1.15) one sees that the terms on the left hand side are of magnitude \( LE^{1/2} \) and \( L^{-1}E^{1/2} \) compared to those on the right. The terms on the left of (2.1.16) are of magnitude \( LE^{3/2} \), \( L^{-1}E^{3/2} \) and \( L^{-3}E^{3/2} \) compared to those on the right. All these terms are small provided that

\[
E^{1/2} \ll L \ll E^{-1/2}
\]

(2.1.24)

Therefore, a basic **inviscid** flow is only observed when the aspect ratio falls in the range (2.1.24)!

As already stated, the **inviscid** flow does not allow all boundary conditions at \( z = 0 \) and \( z = 1 \) to be satisfied. We are forced to introduce boundary layers near the end caps. A balance in the limit of \( E \to 0 \) between the highest \( z \)-derivatives on the left hand side of (2.1.15) and (2.1.16) and the terms on the right hand side is obtained by introducing a stretched boundary layer coordinate and scaled variables according to the rules

\[
\begin{align*}
z &= L^{-1}E^{1/2} y \\
\psi &= E^{1/2} \psi_0 \\
\omega &= \omega_0
\end{align*}
\]

(2.1.25)
Putting (2.1.25) into (2.1.15) - (2.1.16) and dropping terms \( \propto \) \( E \) directly compared to unit magnitude, the boundary layer equations become

\[
\frac{\partial^2 \tilde{\omega}}{\partial y^2} = -2 \frac{\partial \tilde{\theta}}{\partial y} \quad (2.1.26)
\]

\[
\frac{\partial^4 \tilde{\psi}}{\partial y^4} = +2 \frac{\partial \tilde{\omega}}{\partial y} \quad (2.1.27)
\]

The above layer near the bottom cap is the well-known Ekman layer first described by Ekman (1905). Solving equations (2.1.26) - (2.1.27) and applying the boundary conditions at \( z = 0 \), a compatibility condition for the flow outside the Ekman layers can be derived by matching the velocities to those of the outside flow. This compatibility condition is often referred to as the Ekman suction condition and allows the description of the outside flow without giving explicit solutions for the Ekman layer itself. The suction conditions at both end caps are (Greenspan 1968)

\[
\psi - \frac{1}{2} E^{1/2} \omega = -\frac{1}{2} E^{1/2} F_t \quad \text{at } z = 0 \quad (2.1.28)
\]

\[
\psi + \frac{1}{2} E^{1/2} \omega = +\frac{1}{2} E^{1/2} F_t \quad \text{at } z = 1 \quad (2.1.29)
\]

where the functions \( F_b \) and \( F_t \) are given by

\[
F_b = \omega_b - 2 E^{-1/2} \psi_b \quad (2.1.30)
\]

\[
F_t = \omega_t + 2 E^{-1/2} \psi_t \quad (2.1.31)
\]

Note that a combination of the imposed \( \omega \) and \( \psi \) at the end caps, as expressed in (2.1.30) and (2.1.31), allows a description of the outside flow without making a distinction between both types of boundary conditions.

Applying the suction conditions on the inviscid flow one obtains

\[
\psi_0 = \frac{1}{2} \{ F_t - F_b \}, \quad \omega_0 = \frac{1}{2} \{ F_t + F_b \} \quad (2.1.32)
\]

In the inviscid region \( \psi \) and \( \omega \) are constant with respect to \( z \), a result known as the Taylor–Proudman theorem. Since the axial velocity is constant,
the outflow from the Ekman layer at the bottom is equal to the inflow at the top. There is no interior radial motion and vertical flow is solely re-turned within the Ekman layers. The radial shape of the variables in the interior depends directly upon the radial shape of the imposed conditions at both caps.

Generally, the solution (2.1.32) does not apply near the cylinder wall. Consider, for example, the case of disks at \( z = 0 \) and \( z = 1 \) which rotate at an angular speed slightly different from the surrounding cylinder. Then \( F_b \) and \( F_c \) are constant with respect to \( r \) and solution (2.1.32) does not satisfy the condition that \( \psi \) and \( \omega \) are equal to zero at \( r = 1 \). To overcome this problem we are forced to introduce boundary layers near the cylinder wall.

Putting

\[
(1-r) = L^{1/3} E^{1/3} \zeta_1, \quad \psi = E^{1/2} \psi_1, \quad \omega = L^{-1/3} E^{1/6} \omega_1
\]  

(2.1.33)

in (2.1.15) - (2.1.16) and dropping terms \( \sim L^{-1} E^{1/2} \) and \( \sim L E \) directly compared to unit magnitude (which allows \( L \) to fall in the range \( E^{1/2} \ll L \ll E^{-1} \)) one obtains

\[
\frac{\partial^2 \tilde{\omega}}{\partial \zeta_1^2} = -2 \frac{\partial \psi_1}{\partial \zeta_1}
\]

(2.1.34)

\[
\frac{\partial^4 \psi_1}{\partial \zeta_1^4} = +2 \frac{\partial \tilde{\omega}_1}{\partial \zeta_1}
\]

(2.1.35)

The above boundary layer equations allow all conditions to be met at \( \zeta_1 = 0 \). However, since for this layer \( \omega \sim L^{-1/3} E^{1/6} \) its solution cannot be matched to the interior where \( \omega \sim 1 \). Therefore, a second layer is needed which adjusts the angular speed perturbation of the interior to its no slip condition at the wall. Consistency concerning the matching of the interior and both layers requires that \( \psi \sim E^{1/2} \). As a consequence we scale

\[
\psi = E^{1/2} \psi_2, \quad \omega = \tilde{\omega}_2
\]

(2.1.36)

Introducing (2.1.36) and

\[
(1-r) = L^{1/2} E^{1/4} \zeta_2
\]

(2.1.37)
into (2.1.15) - (2.1.16) and dropping terms $\propto L^{-1}E^{1/2}$ and $\propto LE^{1/2}$ directly compared to unit magnitude it follows that

$$\frac{\partial^2 \bar{a}_2}{\partial \zeta_2^2} = -2 \frac{\partial \bar{\psi}_2}{\partial \zeta} , \frac{\partial \bar{a}_2}{\partial \zeta} = 0 \quad (2.1.38)$$

The above layers near the cylinder wall are the well-known Stewartson layers first described by Stewartson (1957). The outer layer of thickness $L^{1/2}E^{1/4}$ adjusts the angular speed perturbation of the interior to the no-slip condition at the wall. The inner layer of thickness $L^{1/3}E^{1/3}$ brings $\psi$ to zero at $r = 1$, which implies that the axial flow in the interior and in the outer layer returns in the inner layer.

The boundary conditions at $z = 0$ and $z = 1$ for both Stewartson layer problems are obtained from the Ekman suction conditions. In the Ekman layer equations we neglected the highest radial derivatives. These terms will be significant when the radial variations take place on a distance scale that is $\propto E^{1/2}$.

This thickness scale is small compared to the one of both Stewartson layers if $L \gg E^{1/2}$, a condition which, according to (2.1.24), is satisfied.

Therefore, the velocity components in both Stewartson layers are adjusted to the end caps by the Ekman layers. A diagram of the various flow regions in the range $E^{1/2} \ll L << E^{-1/2}$ is given in figure 1.

A flow as outlined above has been discussed by many investigators, for example, Carrier (1966), Greenspan (1968) and Stewartson (1957, 1966). A description in terms of an inviscid flow extended by Ekman layers and Stewartson layers is only valid when the aspect ratio falls in the range $E^{1/2} \ll L << E^{-1/2}$. We are particularly interested in the flow behaviour when $L$ exceeds the upper limit of this range. For this purpose it is convenient to consider the thickness scales of the Ekman layer $L^{-1}E^{1/2}$, the inner Stewartson layer $L^{1/3}E^{1/3}$ and the outer Stewartson layer $L^{1/2}E^{1/4}$.

One can see that for $L \sim E^{-1/2}$ the outer Stewartson layer expands to the centre of the cylinder. For $L \sim E^{-1}$ the inner Stewartson layer also fills the whole cylinder. We can anticipate that in the range $E^{-1/2} \sim L << E^{-1}$ the radial derivatives on the left hand side of (2.1.15) become important describing the flow in the main section. The same applies in the range $E^{-1} \sim L$ for the radial derivatives in (2.1.16). The Ekman layer retains its small thickness scale and is still the narrow layer within which the velocity components are adjusted at $z = 0$ and $z = 1$. 

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Figure 1: Diagram of the flow regions for $E^{1/2} \ll L \ll E^{-1/2}$
At first the flow in the range $E^{-1/2} \ll L \ll E^{-1}$ is considered. Radial diffusion of azimuthal momentum becomes important by putting

$$\psi = E^{1/2} \psi_1, \quad \omega = L^{-1} E^{-1/2} \omega_1$$  \hspace{1cm} (2.1.39)

Here, the absolute scaling magnitude of $\psi$ has been chosen such that it corresponds to the induced flux of the Ekman layers given in (2.1.28) and (2.1.29).

Substituting (2.1.39) into (2.1.15) - (2.1.16) and dropping terms $\sim L^{-2}$, $\sim L^2 E^2$, $\sim E^2$ and $\sim L^{-2} E^2$ directly compared to unit magnitude, we obtain

$$\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} r^2 \omega_1 \right) = - 2 \frac{\partial \psi_1}{\partial z}$$  \hspace{1cm} (2.1.40)

$$\frac{\partial \omega_1}{\partial z} = 0$$  \hspace{1cm} (2.1.41)

Applying the Ekman suction conditions it is easily verified from (2.1.40) and (2.1.41) that

$$\psi_1 = \frac{1}{4} \{ F_t - F_b \} + \frac{1}{4} \{ F_t + F_b \} (z - 1/2) - L^{-1} E^{-1/2} \omega_1 (z - 1/2)$$  \hspace{1cm} (2.1.42)

The problem for $\omega_1$ is given by

$$\left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} r^2 - 2L^{-1} E^{-1/2} \right) \omega_1 = - \{ F_t + F_b \}$$  \hspace{1cm} (2.1.43)

with the boundary condition

$$\omega_1 = 0 \quad \text{at} \quad r = 1$$  \hspace{1cm} (2.1.44)

and the requirement that $\omega_1$ is finite at $z = 0$.

One sees from (2.1.42) and (2.1.43) - (2.1.44) that the velocity distribution in the interior is influenced by the no-slip condition for $\omega$ at the cylinder wall. Furthermore we notice from (2.1.42) that a linear change of the axial flow along the z-coordinate is possible. The Ekman layers no longer dominate.
Figure 2: Diagram of the flow regions for $F^{-1/2} \sim L \ll F^{-1}$. 
In the particular case that the boundary conditions at the end caps are antisymmetric with respect to the mid-plane \( z = 1/2 \), \( F_t = - F_b \), it follows from (2.1.43) - (2.1.44) that \( \omega_1 = 0 \). Putting this result into (2.1.42) one sees that the interior flow is equal to the inviscid one given by (2.1.32). One would expect this situation, of course, since when \( F_t = - F_b \), \( \omega_0 = 0 \) and therefore an outer Stewartson layer is needed and expansion of this layer is irrelevant.

Equations (2.1.40) and (2.1.41) do not allow the boundary conditions for \( \psi \) at the cylinder wall to be met. Therefore the inner Stewartson layer is needed. Its principal function is again the rechanneling of the axial flow. The scaling rules and equations are equal to those given in (2.1.33), (2.1.34) and (2.1.35). A diagram of the flow regions for \( \varepsilon^{-1/2} \propto L \ll \varepsilon^{-1} \) is given in figure 2.

Finally we shall discuss the flow in the range \( \varepsilon^{-1} \propto L \). Introducing the scaled variables

\[
\psi = \varepsilon^{1/2} \psi_2 \quad , \quad \omega = \varepsilon^{1/2} \omega_2
\]

(2.1.45)

in (2.1.15) - (2.1.16) and neglecting the terms \( \sim L^{-2} \) directly compared to unit magnitude, the equations describing the flow in the main section become

\[
L^2 \left( \frac{1}{r^2} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right) \omega_2 = - 2 \frac{\partial \psi_2}{\partial z} \quad \text{(2.1.46)}
\]

\[
L^2 \left( \frac{1}{r^2} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right)^2 \psi_2 = + 2 \frac{\partial \omega_2}{\partial z} \quad \text{(2.1.47)}
\]

These equations allow all boundary conditions to be satisfied at \( r = 1 \).

The conditions at the end caps are again provided by the Ekman layers. The leading order of the Ekman suction conditions becomes

\[
\psi_2 = - \frac{1}{2} F_b \quad \text{at} \quad z = 0 \quad , \quad \psi_2 = + \frac{1}{2} F_t \quad \text{at} \quad z = 1 \quad \text{(2.1.48)}
\]

Eliminating \( \omega_2 \) from (2.1.47) by means of (2.1.46) it follows that

\[
L^2 \varepsilon^2 \left( \frac{1}{r^2} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right)^3 \psi_2 + 4 \frac{\partial^2}{\partial z^2} \psi_2 = 0 \quad \text{(2.1.49)}
\]
Figure 3: Diagram of the flow regions for $L \sim E^{-1}$. 

*eigenfunctions*
\[ \psi_r = \sum_{k=1}^{\infty} \left\{ a_k \exp(-\lambda_k L \varepsilon r) + b_k \exp(+\lambda_k L \varepsilon r) \right\} f_k(r) \]  

(2.1.50)

where \( f_k \) satisfies the differential equation

\[ \left\{ \frac{1}{r} \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} r^2 f_k \right) \right\} = 4 \lambda_k^2 f_k = 0 \]  

(2.1.51)

with the boundary conditions

\[ f_k = \frac{df_k}{dr} = \left\{ \frac{1}{r} \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} r^2 f_k \right) \right\} = 0 \quad \text{at} \quad r = 0 \]  

(2.1.52)

and the requirement that \( f_k \) is finite at \( r = 0 \). The solution of (2.1.51) - (2.1.52) is then an infinite series of eigenfunctions \( f_k \) with positive real eigenvalues \( \lambda_k \) (\( k = 1, 2, \ldots, \infty \)), where the eigenvalues are ordered such that \( \lambda_{k-1} < \lambda_k < \lambda_{k+1} \)). Expanding \( F_d \) and \( F_t \) in the eigenfunctions \( f_k \), the constants \( a_k \) and \( b_k \) in (2.1.50) can be determined from (2.1.48). A diagram of the flow regions for \( L \sim \varepsilon^{-1} \) is given in figure 3.

In the case of a cylinder of semi-infinite length \( h \), must be set equal to zero. Then we retain eigenfunctions decaying only in the axial direction. This is exactly the situation considered by Steenbeck (1958), who calculated numerically for the first eigenvalue \( \lambda_1 = 41.44 \) and for the second one \( \lambda_2 = 199.2 \).

Summarising, three types of flow can be distinguished when the aspect ratio increases from unit magnitude to infinity. These types of flow correspond to the parameter ranges \( E^{1/2} \ll L < E^{-1/2}, \epsilon^{-1/2} \ll L < \epsilon^{-1} \) and \( E^{-1} \ll L \). In the first \( L \)-range the flow consists of an inviscid interior extended by Ekman layers near the end caps and two Stewartson layers near the cylinder wall. In the inviscid interior the angular speed and the axial velocity are constant along the axial coordinate, a result in agreement with the Taylor-Proudman theorem. The radial velocity is zero and the in- and outflow from the Ekman layers controls the vertical flow. Its radial shape directly depends upon the radial shape of the imposed boundary conditions at the end caps. The outer Stewartson layer adjusts the angular speed perturbation of the interior at the cylinder wall. The axial flow in the interior and outer Stewartson layer returns in the inner Stewartson layer.
In the second L-range radial diffusion of azimuthal momentum is not restricted to a small region near the cylinder wall (the outer Stewartson layer) but is important in the main part of the cylinder. The radial shape of the flow variables is influenced by the no-slip of the angular velocity at the wall. The Ekman layers no longer dominate, as is typical in the first L-range, and a linear change of the axial flow along the axial coordinate is possible.

In the third L-range radial diffusion of azimuthal and axial momentum are important in the entire cylinder and the flow is strongly influenced by the conditions at the cylinder wall. The velocities can be written as a set of eigenfunctions which decay exponentially from the end caps. Higher eigenfunctions decay faster than lower ones and oscillate more frequently along the radial coordinate. At a reasonable distance from the end caps the flow is mostly described by the first eigenfunction.

Since $E$ is very small most configurations will fall into the first L-range. In other words, the first type of flow is of most practical importance. Explicit solutions for the velocity distribution in the various flow regions are not calculated. Many of the characteristic phenomena also occur in the perfect gas and their solutions will be presented and discussed in chapter 3.
2.2. The perfect gas

2.2.1. The basic equations

In order to investigate the flow of a compressible fluid in a rotating cylinder, a new and more comprehensive formulation is needed. The equations describing the motion and the state are given by the conservation of mass, momentum and energy, together with an equation of state. It is assumed that the fluid has constant material properties such as the dynamic viscosity \( \mu \), the bulk viscosity \( \beta \), the thermal conductivity \( \kappa \) and the specific heat at constant pressure \( c_p \). We confine ourselves to stationary flows. The three conservation equations, written in a rotating frame fixed with respect to the cylinder are (in the absence of body forces; Greenspan 1968):

\[
\nabla \cdot (\rho \hat{q}) = 0 \tag{2.2.1}
\]

\[
\rho \left[ (\hat{q} \cdot \nabla) \hat{q} - \frac{1}{2} \Omega^2 \hat{r} \right] + 2 \Omega \hat{r} \times \hat{q} = - \nabla P - \mu \nabla \times \nabla \hat{q} + \left( \frac{4}{3} \mu + \beta \right) \nabla \cdot \hat{q} \tag{2.2.2}
\]

\[
\rho c_p \hat{q} \cdot \nabla T + \frac{\kappa}{\rho} \frac{\partial T}{\partial \rho} \rho \hat{q} \cdot \nabla P = \kappa \nabla^2 T + \beta (\nabla \cdot \hat{q})^2 + \mu \left[ (\nabla^2 \hat{q} + 2 \nabla \cdot (\nabla \hat{q}) \hat{k} - 2 \hat{k} \cdot \nabla \hat{q} + \nabla \hat{q} \cdot \nabla \hat{q} - \frac{2}{3} (\nabla \cdot \hat{q})^2 \right] \tag{2.2.3}
\]

Here \( \hat{q} \) is the particle velocity and \( \hat{r} \) the position vector in the rotating frame. \( \Omega \) is the angular speed of the cylinder and \( \hat{k} \) is the unit vector along the rotation axis. \( P, \rho \) and \( T \) denote respectively pressure, density and temperature. Relaxation processes are not considered. Therefore the bulk viscosity is set equal to zero: \( \beta = 0 \).
A relation between the state variables $P$, $\rho$ and $T$ is given by the equation of state. For dilute gases a good approximation is the perfect gas equation

$$
\rho V = \rho R T , \quad R = \frac{R_0}{M}
$$

(2.2.4)

Here $R_0$ is the universal gas constant: $R_0 = 8.314 \text{ [kgm}^2/\text{s}^2\text{ Kmol]}$, $M$ is the molecular weight of the gas.

A static equilibrium is obtained if the velocities in the rotating frame are equal to zero: $\ddot{\mathbf{q}} = \mathbf{0}$. Assuming a uniform constant temperature $T_0$ the conservation equations reduce to

$$
\nabla P = \frac{1}{2} \rho c^2 \nabla |\mathbf{k} \times \mathbf{r}|^2
$$

(2.2.5)

Applying the equation of state

$$
\rho c = \rho_0 \exp\left\{\frac{1}{2} \Omega^2 |\mathbf{k} \times \mathbf{r}|^2 / R T_0 \right\}
$$

(2.2.6)

it follows from (2.2.5) that

$$
P = P_0 \exp\left\{\frac{1}{2} \Omega^2 |\mathbf{k} \times \mathbf{r}|^2 / R T_0 \right\}
$$

(2.2.7)

where $P_0$ is the pressure at $\mathbf{y} = \mathbf{0}$.

The solution for the state quantities at isothermal rigid body rotation is used to define a dimensionless relative pressure $p$, density $\tau$ and temperature $\theta$

$$
P = P_0 (1 + \varepsilon_1 P) , \quad \rho = \rho_0 (1 + \varepsilon_2 \tau) , \quad T = T_0 (1 + \varepsilon_3 \theta)
$$

(2.2.8)

Here $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are dimensionless parameters which scale the perturbed state quantities $\rho$, $\tau$ and $\theta$.

The position vector and the particle velocity are made dimensionless as follows

$$
\mathbf{r} = \mathbf{r}^* , \quad \dot{\mathbf{q}} = \mathbf{Uq}^*
$$

(2.2.9)
Note that $\tau$ does not appear in the conservation equations (2.2.20) - (2.2.22). In other words, the linearised equation of state (2.2.19) is an extra equation for the extra variable $\tau$.

Comparing the conservation equations here to those of the incompressible fluid given by (2.1.8) and (2.1.11), we observe (i) an exponential function $\exp\{A\left[kx^2 - 1\right]\}$ which stems from the density distribution at isothermal rigid body rotation of the gas, (ii) the extra term $\frac{1}{3} \varepsilon_3 \beta \nabla |kx|^2$ in the momentum equation (2.2.21) which represents the perturbation of the centrifugal force caused by temperature and (iii) an extra equation (2.2.22), the energy equation, necessary to describe the temperature field. The term on the left hand side of the energy equation stems from the work done by compression $(\mu (\partial/\partial r)p)$. The term on the right represents the heat conduction.

A more concrete opinion concerning the validity of the linearisation, the criterion for which is given by (2.2.18), can be formed when $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are known as a function of $\varepsilon$. Relations between these scale parameters are obtained from balance considerations of the various terms in the linearised equations. It is our intention to consider the flow processes for a small Ekman number. This means that the viscous forces in (2.2.21) can be neglected and hence the pressure gradient must be balanced by the inertia terms. Furthermore since a balance between the Coriolis acceleration and the centrifugal acceleration causes the motion, we put

$$\varepsilon \sim \varepsilon_3 \sim \varepsilon_1/A$$  \hspace{1cm} (2.2.23)
A relation between $\varepsilon_2$ and $\varepsilon$ is obtained from the linearised equation of state. Putting (2.2.23) into (2.2.19) it follows that

$$\varepsilon_2 \tau = \varepsilon Ap - \varepsilon \theta$$

(2.2.24)

The relation between $\varepsilon_2$ and $\varepsilon$ depends on the magnitude of $A$. Let us first consider the case that $A$ is very small, a condition fulfilled in many applications:

(i) $A \ll 1$.

From (2.2.24) we conclude

$$\varepsilon_2 \sim \varepsilon , \tau \sim \theta$$

(2.2.25)

This implies that fluctuations of the density result principally from thermal effects, which is a basis of the Boussinesq approximation (see Spiegel & Veronis, 1960). Substituting the relations between $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3$ and $\varepsilon$ given by (2.2.23) and (2.2.25) in (2.2.18) the linearisation criterion becomes

$$\varepsilon \ll 1 , \varepsilon \sigma_p / B \tau \ll 1$$

(2.2.26)

With (2.2.17) the latter criterion becomes

$$\gamma \varepsilon / A (\gamma - 1) \ll 1$$

(2.2.27)

Because $A$ is very small and $(\gamma - 1)/\gamma$ is of unit magnitude or smaller, condition (2.2.27) is in most cases not satisfied. The parameter $\gamma \varepsilon / A (\gamma - 1)$ scales the heat convection relative to the work done by compression in the energy equation. In many applications this term is relatively large. In that case we must include the heat convection term instead of the compression term and the linearised version of the energy equation applied here is invalid. In fact convection and conduction in the energy equation together with application of the Boussinesq approximation is the basis of many theoretical investigation on stratification in rotating fluids: see Barcilon & Pedlosky (1967a, 1967b, 1967c), Greenspan (1968), Homsy & Hudson (1969, 1971a, 1971b).
We are concerned with gas centrifuges. In this application of rotating fluids the speed parameter is certainly not small:

(ii) \( A \geq 1 \).

It is found from (2.2.24) that

\[ \varepsilon_2 \sim \varepsilon A , \quad \tau = p - \theta/A \]  

(2.2.28)

Substituting the relations between \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \varepsilon \) given by (2.2.23) and (2.2.28) in (2.2.18) the linearisation criterion becomes

\[ \varepsilon A, \gamma \varepsilon / A (\gamma - 1) \ll 1 \]  

(2.2.29)

It is clear that for \( A \sim 1 \) and \( \gamma / A (\gamma - 1) \sim 1 \), the criterion (2.2.29) is satisfied provided that the velocities measured in the rotating frame are small compared to the peripheral speed. On the other hand, for \( A >> 1 \) we must require

\[ \varepsilon A \ll 1 \]  

(2.2.30)

Here \( \varepsilon A \) scales the perturbation of the pressure and density at isothermal rigid body rotation. This can be verified from the pressure distribution at isothermal rigid body rotation given by (2.2.7).

Introducing a perturbation in the angular speed, it is found that the perturbation of \( P \) is a factor \( A \) larger. Nevertheless, we assume that the Rossby number is so small that (2.2.30) is satisfied. The linearised conservation equations (2.2.20) - (2.2.22) together with the relations between \( \varepsilon, \varepsilon_1, \varepsilon_2 \) given by (2.2.23) are the basis of our following investigation.

As we are concerned with a cylindrical configuration it is appropriate to apply cylindrical coordinates \((r, \varphi, Lz)\) with the origin at the centre of the cylinder. As before the aspect ratio \( L \) is the ratio of the length of the cylinder to its radius: i.e.

\[ L = L/a \]  

(2.2.31)
Because of the symmetry of the imposed boundary conditions, an axially symmetric distribution of the flow is assumed. The continuity equation (2.2.20) can then be used to define a streamfunction \( \psi \),

\[
u = -L^{-1}r(\partial / \partial z)\psi, \quad \omega = e^{-A(r^2-1)}r^{-1}(\partial / \partial r)r^2e^{A(r^2-1)}\psi
\]  (2.2.32)

where \( \nu \) is the radial velocity and \( \omega \) the axial velocity. Furthermore we introduce an angular velocity \( \omega(r, z) \) replacing the azimuthal velocity \( \nu \),

\[
\nu = \omega r
\]  (2.2.33)

Cross-differentiating the radial equation (2.2.21) to eliminate \( p \) we get

\[
\begin{align*}
\frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} e^{-A(r^2-1)} \frac{\partial}{\partial r} r^2 + \frac{1}{L^2} e^{-A(r^2-1)} \frac{\partial^2}{\partial z^2} \right) \right] - \frac{4}{3} \left( 2 \frac{A}{L} e^{-A(r^2-1)} \frac{\partial}{\partial z} \right) \psi e^{A(r^2-1)} = \frac{3}{L^2}(2\omega - \theta)
\end{align*}
\]  (2.2.34)

\[
\begin{align*}
L^2 e^{-A(r^2-1)} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} r^2 + \frac{1}{L^2} \frac{\partial^2}{\partial z^2} \right) \right\} \omega = -2 \frac{\partial \psi}{\partial z}
\end{align*}
\]  (2.2.35)

\[
L^2 e^{-A(r^2-1)} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} r^2 + \frac{1}{L^2} \frac{\partial^2}{\partial z^2} \right) \right\} \theta = r^2 \frac{\partial r}{\partial z} \frac{\partial \psi}{\partial z}
\]  (2.2.36)

where (2.2.35) is the tangential component of the momentum equation and where (2.2.36) is the energy equation.

Just as in the incompressible problem it is assumed that the horizontal end caps of the cylinder rotate with an angular speed which differs slightly and that fluid is injected and withdrawn vertically. In addition, we assume temperature distributions along the cylinder wall and along the end caps. The boundary conditions are now such that the velocities and temperatures coincide with those of the walls: i.e.

\[
\omega = \psi = (\partial / \partial r)\psi = 0, \quad \theta = \theta_w \text{ at } r = 1
\]  (2.2.37)

\[
\omega = \omega_b, \quad \psi = \psi_b, \quad \theta = \theta_b, \quad (\partial / \partial z)\psi = 0 \text{ at } z = 0
\]  (2.2.38)

\[
\omega = \omega_t, \quad \psi = \psi_t, \quad \theta = \theta_t, \quad (\partial / \partial z)\psi = 0 \text{ at } z = 1
\]  (2.2.39)
where

\[ \psi_b = r^{-2} e^{-A(r^2 - 1)} \int_0^r \omega_b r e^{A(r^2 - 1)} dr \]  \hspace{1cm} (2.2.40) \]

\[ \psi_t = r^{-2} e^{-A(r^2 - 1)} \int_0^r \omega_t r e^{A(r^2 - 1)} dr \]  \hspace{1cm} (2.2.41) \]

and

\[ \psi_b = \psi_t = 0 \quad \text{at } r = 1 \]  \hspace{1cm} (2.2.42) \]

Here \( \omega_b, \omega_t, \nu_b, \nu_t, \theta_b \), and \( \theta_t \) are arbitrary functions of \( r \) and \( \theta \), \( \psi_b \) is an arbitrary function of \( z \). The functions \( \omega_b \) and \( \omega_t \) represent the dimensionless different rotation rates of the bottom and top end caps. The functions \( \nu_b \) and \( \nu_t \) represent the imposed dimensionless vertical velocities at both end caps. The functions \( \theta_b \) and \( \theta_t \) are the dimensionless temperature distributions along the end caps and \( \theta_w \) is the dimensionless temperature distribution along the cylinder wall. Condition (2.2.42) implies that the axial net transport is zero. The no-slip and no-normal flow condition at the cylinder wall is given in (2.2.37).

Consider now the equations (2.2.34), (2.2.35) and (2.2.36). In equation (2.2.34) the two variables \( \psi \) and \( \omega^1 \theta \) are present, the latter one further denoted as \( \chi \). Multiplying equation (2.2.36) with \( 1/2 \) and subtracting this from (2.2.35) a second equation for \( \psi \) and \( \chi \) is obtained. We are now in the fortunate position that only these two equations have to be considered in order to obtain a solution for \( \chi \) and \( \psi \). However, the procedure is not completely successful because the lower radial derivatives on the left hand side of (2.2.35) and (2.2.36) are not the same. On the other hand, since \( \nu^2 \) is small we expect a basic incompressible flow, where the terms on the left can simply be neglected, extended by boundary layers, where only the highest derivatives are significant. In other words we expect that the mathematical benefits of the foregoing manipulation are still applicable.

A third equation, necessary for the description of \( \omega \) and \( \theta \), is obtained by eliminating \( (\partial/\partial z)\theta \) from (2.2.35) and (2.2.36). As a result we get the following set of equations

\[
LE\left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{3}{L^3} e^{-A(r^2 - 1)} \frac{3}{\partial r^2} r^2 + \frac{1}{L^2} e^{-A(r^2 - 1)} \frac{3^2}{\partial z^2} \right\} + \]

\[ - \frac{A}{3} e^{-A(r^2 - 1)} \frac{3}{\partial r^2} \frac{2}{\partial z^2} \right\} e^{A(r^2 - 1)} \psi = 2 \frac{3 \chi}{\partial z} \]  \hspace{1cm} (2.2.43) \]
Equation (2.2.45) underlies a balance between the viscous forces in the azimuthal momentum equation and the heat conduction in the energy equation, terms which were \( \propto E \) in the original equations of motion. There we dropped terms \( \propto E \) directly compared to unit magnitude whereas in (2.2.45) a part of these non-linear terms compares to \( E \) so that for \( E \propto E \) the neglect is invalid. Therefore, the terms in question, which stem from convection \( \psi (\partial/\partial \theta) v, \psi (\partial/\partial \theta) T \), have been incorporated into (2.2.45).

The boundary conditions for (2.2.43) and (2.2.44) are

\[
\psi = (\partial/\partial r) \psi = 0, \quad \chi = -\tfrac{1}{2} \theta, \quad \text{at } r = 1
\]

\[
\psi = \psi_b, \quad \chi = \omega_b - \tfrac{1}{2} \theta_b, \quad (\partial/\partial \theta) \psi = 0 \quad \text{at } z = 0
\]

\[
\psi = \psi_t, \quad \chi = \omega_t - \tfrac{1}{2} \theta_t, \quad (\partial/\partial \theta) \psi = 0 \quad \text{at } z = 1
\]

The boundary conditions for (2.2.45) are

\[
\phi = \theta_w \quad \text{at } r = 1
\]

\[
\phi = \theta_b + \tfrac{1}{2} r^2 B_r \omega_b \quad \text{at } z = 0
\]

\[
\phi = \theta_t + \tfrac{1}{2} r^2 B_r \omega_t \quad \text{at } z = 1
\]
The coupling between the equations for $\psi$ and $\chi$, \((2.2.43)\) and \((2.2.44)\), and the equation for $\phi$, \((2.2.45)\), is given by the term in $\phi$ on the left hand side of \((2.2.44)\). As has already been stated, one hopes that this term is not of leading order in the various balances of terms to be considered. This is especially advantageous with respect to gas centrifuge problems where one is mainly interested in the distribution of the axial and radial velocity so that only \((2.2.43)\) and \((2.2.44)\) have to be considered. The small perturbations of the angular speed and temperature are often of secondary importance so that \((2.2.45)\) does not need to be solved. Nevertheless, in the next treatment the full set of equations is considered and for each case the conditions on which a decoupling is allowed will be discussed.

Compare now equations \((2.2.43)\) and \((2.2.44)\) to those for the incompressible fluid given by \((2.1.15)\) and \((2.1.16)\). Replacing $\psi$ by $\omega$ one sees that, apart from the term in $\phi$ and some lower order derivatives with respect to $\gamma$ and $z$, both problems are quite similar provided that the speed parameter and the Brinkman number are of unit magnitude or smaller. For $A \leq 1$ the exponential function stemming from the density distribution at isothermal rigid body rotation is of unit magnitude and is insignificant in "an order of magnitude discussion". The same applies to the terms formed with the Brinkman number, as long as $B\nu \leq 1$. In this case a similar procedure as has been applied to the incompressible problem seems appropriate.

Consider now a basic inviscid flow. In the limit of $E \to 0$ equations \((2.2.43)\) and \((2.2.44)\) reduce to

\[
(\partial/\partial z)\chi = 0, \quad (\partial/\partial z)\psi = 0
\]  

\((2.2.53)\)

which implies that the axial velocity and a combination of the angular speed and temperature given by $\omega^{-1/2}$, are constant with respect to $z$, a result that may be referred to as the compressible Taylor-Proudman theorem.

The inviscid solution does not allow all boundary conditions to be met at the walls and similar boundary layers as in the incompressible case are needed. The conditions at the end caps are provided by layers of the Ekman type. In these layers of thickness $L^{-1} R^{1/2}$ the highest $z$-derivatives on the left hand side of \((2.2.43)\) and \((2.2.44)\) are comparable in magnitude.
with the terms on the right. An inner and outer Stewartson layer, of thickness $L^{1/3}E^{1/3}$ and $L^{1/2}E^{1/4}$ respectively, adjust the inviscid flow to the cylinder wall. In the inner layer the highest radial derivatives on the left hand side of (2.2.43) and (2.2.44) are comparable in magnitude to the terms on the right, whereas in the outer layer only the highest radial derivatives of $\chi$ are significant. Replacing $w$ by $\psi$ the scaling rules for the various flow regions are equal to those given for the incompressible case.

The thicknesses of the boundary layers are small when the aspect ratio falls in the range $E^{1/2} << L << E^{-1/2}$. Since $E$ is very small most configurations will fall in this range. When, however, the aspect ratio exceeds the upper limit of this range, both Stewartson layers successively expand to the interior. Just as in the incompressible case, radial diffusion becomes important in the main section of the cylinder.

The distributions of $w$ and $\theta$ respectively can be determined by solving the equation for $\phi$ resulting from the differential equation (2.2.45) and the boundary conditions (2.2.50) - (2.2.52). The coupling between the problems for $\chi$ and $\psi$ on one hand and for $\psi$ on the other is given by the term in $\phi$ on the left hand side of (2.2.44). As long as the magnitude of $\phi$ is not greater than that of $\chi$ in question is not significant. Then in the inviscid interior the entire left hand side of (2.2.44) can be neglected and in the boundary layers the term is still negligible since $\frac{\partial \phi}{\partial x} \sim \frac{\partial \psi}{\partial x} \sim \phi$. However, to satisfy the condition $O(\phi) \ll O(\psi)$ the necessary requirements are (i) the magnitude of the imposed boundary conditions for $\psi$ is not greater than those for $\chi$ and (ii) the magnitude of the non-linear terms on the right hand side of (2.2.45) is not greater than the magnitude of the terms on the left. The necessity of condition (i) is illustrated by considering the particular case that $\theta = \frac{1}{2} \theta$, $\theta = \psi = \psi = 0$. Omitting the term in $\phi$ one would conclude from the boundary conditions (2.2.47) - (2.2.49) and equations (2.2.43) - (2.2.44) that $\psi$ and $\chi$ are zero whereas the term in $\phi$ is now the mechanism by which a secondary flow is generated. Of course, the magnitude of this induced flow is small but, nevertheless, the present approach did not consider this situation. If condition (ii) is not satisfied (the magnitude of the non-linear terms is significantly large) a new scaling analysis would be needed, at this stage, incalculable consequences for the flow field. Anyhow introducing the scaling magnitudes of $\chi$ and $\psi$ of the inviscid inte-
rior, \( \chi \sim 1 \) and \( \psi \sim E^{1/2} \) (similar to the incompressible case) the non-linear terms no longer dominate if \( \varepsilon \sigma_p/LE^{1/2} \leq 1 \) and \( \varepsilon Br(1-\sigma_p)/LE^{1/2} \leq 1 \).

A basic inviscid flow for \( \psi \) and \( \omega = \frac{1}{2} \theta \), extended by Ekman layers and Stewartson layers has been investigated in a number of recent Japanese publications: Matsuda, Sakurai & Takeda (1975), Matsuda (1975), Mikami (1973a, 1973b), Nakayama & Usui (1974), Sakurai & Matsuda (1974) and Sakurai (1975). These authors, however, did not consider the effect of an aspect ratio which increases from unit magnitude to infinity, nor the effect of large radial density gradients. In present day centrifuges \( A \) is quite large. In this case the exponential density function \( \exp\{A(r^2-1)\} \) in equations (2.2.43) - (2.2.45) varies rapidly in magnitude along the radial coordinate and a new scaling analysis appears to be necessary. As will be seen, a fundamental change in the flow phenomena occurs. One also finds that the effect of an increasing aspect ratio becomes more important.

### 2.2.2. The rapidly rotating heavy gas

Due to the high peripheral speed and the relatively large molecular weight of the gas, the speed parameter is quite large in present day centrifuges. Values of 17 for \( A \), or more, are today quite normal. Inspection of (2.2.43) and (2.2.44) shows that the density function \( \exp\{A(1-r^2)\} \) forms, with the Ekman number, a local Ekman number: viz. \( E'_{\ell} = E \exp\{A(1-r^2)\} \). Although \( E'_{\ell} \) is small at \( r = 1 \) it increases very rapidly with distance from the cylinder wall when \( A \) is large: e.g. for \( A = 17 \), \( \exp\{A(1-r^2)\} \) is approximately 25 at \( r = 0.9 \) and \( 2.4 \times 10^7 \) at \( r = 0 \). Furthermore, the presence of the density function in equations (2.2.43) and (2.2.44) requires that all variables depend on \( \exp\{A(1-r^2)\} \): derivatives with respect to \( r \) will not be of unit magnitude but, more explicitly, \( \partial/\partial r \propto A \). In order to perform an adequate scaling analysis it is necessary to introduce a coordinate measured from the cylinder wall which corresponds to a change of unit magnitude of the density function: i.e.

\[
x = A(1-r^2) , \quad 0 \leq x \leq A
\]

(2.2.54)
Furthermore we define the streamfunction $\psi^*$ by

$$\psi^* = A\psi$$  \hspace{1cm} (2.2.55)

Putting (2.2.54) and (2.2.55) into (2.2.43) - (2.2.45) and dropping the asterix we get

$$L_m e^x \left\{ 4 \frac{\partial}{\partial x} e^x \frac{\partial}{\partial x} \left( 1- \frac{x}{A} \right) + \frac{1}{r^2_m} e^x \frac{\partial^2}{\partial y^2} \right\}^2 +$$

$$- \frac{4}{3} \left( 1- \frac{x}{A} \right)^2 \left( \frac{2}{L_m} e^x \frac{\partial}{\partial y} \right)^2 \right\} e^{-x} \psi = 2 \frac{3x}{3x}$$  \hspace{1cm} (2.2.56)

with boundary conditions

$$\psi = \frac{\partial}{\partial x} \psi = 0, \quad x = - \frac{1}{2} \theta_w$$ at $x = 0$

$$\psi = \psi_B, \quad x = \omega_B - \frac{1}{2} \theta_B, \quad (\partial / \partial x) \psi = 0$$ at $z = 0$  \hspace{1cm} (2.2.58)

$$\psi = \psi_t, \quad x = \omega_t - \frac{1}{2} \theta_t, \quad (\partial / \partial x) \psi = 0$$ at $z = 0$

and

$$\left[ 4 \frac{\partial}{\partial x} \left( 1+1 Br \left( 1- \frac{x}{A} \right)^{\frac{3}{3x}} \right) \left( 1+1 Br \left( 1- \frac{x}{A} \right)^{-1} \left( 1- \frac{x}{A} \right) + \frac{1}{L_m^2} \frac{\partial^2}{\partial x^2} \right) \right] \phi =$$

$$- \frac{2}{A} Br \left( 1- \frac{x}{A} \right)^{-1} \left( 1- \frac{x}{A} \right) \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} (1- \frac{x}{A}) e^{-x} \psi \right) \frac{\partial^2}{\partial y^2}$$  \hspace{1cm} (2.2.59)

with boundary conditions

$$\phi = \theta_w$$ at $x = 0$

$$\phi = \theta_B + \frac{1}{4} (1- \frac{x}{A}) Br \theta_B$$ at $z = 0$  \hspace{1cm} (2.2.60)

$$\phi = \theta_t + \frac{1}{4} (1- \frac{x}{A}) Br \theta_t$$ at $z = 1$
Note that the second part of the non-linear term in (2.2.45) has been neglected in (2.2.59) which is valid when $\sigma_0 = 1$. The modified Ekman number $E_m$ and the modified aspect ratio $L_m$ are based on the distance scale of the density decrease instead of the radius, and are related to $E$ and $L$ by

$$E_m = EA^2, \quad L_m = LA$$  \hspace{1cm} (2.2.61)

Although $A^2$ is large we assume that $E_m \ll 1$. Furthermore we take $E_2 \sim 1$. The modified aspect ratio is assumed to vary from unit magnitude to infinity.

Consider an inviscid flow. In the limit of $E_m \to 0$ equations (2.2.56) and (2.2.57) reduce to: $(\partial/\partial z)\chi = 0$, $(\partial/\partial z)\psi = 0$. These solutions, however, do not specify the magnitude of $\chi$ and $\psi$ and thus the condition on which the terms on the left hand sides of (2.2.56) and (2.2.57) can be neglected. Since $\psi$ and $\chi$ are constant with respect to $z$ we expect that boundary layers near the end caps will determine the magnitude of $\chi$ and $\psi$ in the inviscid domain. Anticipating the outcome of the boundary layer analysis we scale

$$\chi = \chi_0, \quad \psi = (E_m e^x)^{1/2} \psi_0$$  \hspace{1cm} (2.2.62)

and the inviscid flow is described by

$$(\partial/\partial z)\chi_0 = 0, \quad (\partial/\partial z)\psi_0 = 0$$  \hspace{1cm} (2.2.63)

Putting (2.2.62) into (2.2.56) and (2.2.57) one sees that the terms on the left hand side are of magnitude $L_m (E_m e^x)^{3/2}$, $E_m^{-1} (E_m e^x)^{3/2}$, $L_m^{-3} (E_m e^x)^{3/2}$, $L_m (E_m e^x)^{1/2}$ and $L_m^{-1} (E_m e^x)^{1/2}$ compared to those on the right. For $x \sim 1$ these terms can be neglected provided that

$$E_m^{1/2} \ll L_m \ll E_m^{-1/2}$$  \hspace{1cm} (2.2.64)

On the other hand, at larger distances from the wall $x$ increases to $A$ which is supposed to be large. Consequently $e^x$ increases considerably so that the terms on the left of (2.2.56) and (2.2.57) can become significant. Consider the case $L_m \sim 1$. Then all terms on the left will balance.
for $E_m e^x \sim 1$, which implies that an inviscid region is only observed in a limited region near the cylinder wall given by $x \ll \ln E_m^{-1}$. Due to the strong decrease of the density the diffusive terms are necessary to describe the flow in the core of the cylinder. Of course, since $x$ is, at most $A$, a viscous core does not occur when $\ln E_m^{-1} \gg A$.

The inviscid solution fails at the end caps. To overcome this non-uniformity we must introduce boundary layers near $z = 0$ and $z = 1$. The scaling rules for the layer at the bottom cap are

$$x = L_m^{-1} (E_m e^{x_0})^{1/2} y, \quad x = x_0, \quad \psi = (E_m e^{x_0})^{1/2} \psi_0$$

Putting (2.2.65) into (2.2.56) - (2.2.57) and dropping terms $\sim (E_m e^{x_0})^{1/2}$ compared to unit magnitude we obtain

$$\frac{\partial^4 \psi_0}{\partial y^4} - 2 \frac{\partial \chi_0}{\partial y} \quad (2.2.66)$$

$$\frac{\partial^2 \chi_0}{\partial y^2} = -2 \{1 + \frac{1}{2} \eta e(1 - e) \} \frac{\partial \psi_0}{\partial y} \quad (2.2.67)$$

The above layer of thickness $L_m^{-1} (E_m e^{x_0})^{1/2}$ is of the Ekman type and increases in thickness with distance from the cylinder wall. Equations (2.2.66) and (2.2.67) are only applicable for $x \ll \ln E_m^{-1}$. At $x \sim \ln E_m^{-1}$ all terms of the original equations of motion become important, and for $L_m \sim 1$ the layer will expand and join the previously discussed viscous core.

Generally, the inviscid solution does not apply near the cylinder wall, and must be amended by further boundary layer analysis. Two layers of the type discussed by Stewartson (1957) will adjust $\psi$ and $x$ at $x = 0$. The scaling rules for the inner layer are

$$x = (L_m E_m)^{1/3} \zeta_1, \quad x = L_m^{-1/3} E_m^{1/6} \zeta_1, \quad \psi = E_m^{1/2} \psi_1 \quad (2.2.68)$$

Putting (2.2.68) into (2.2.56) - (2.2.57) and letting $L_m^{-1} E_m^{1/2} \rightarrow 0$ and
\( \frac{L_mE_m}{E} \rightarrow 0 \) (which allows \( L_m \) to fall in \( \frac{E_m}{E}^{1/2} \ll L_m \ll E^{-1} \)) the equations become
\[
\frac{\partial^2 \chi_1}{\partial \xi^2} = \frac{3 \chi_1}{3 \xi} \\
2 \frac{\partial^2 \chi_1}{\partial \xi^2} = - (1 + \frac{1}{4} B) \frac{\partial \psi_1}{\partial \xi}
\]  
(2.2.69)

The scaling rules for the outer layer are
\[
\alpha = \frac{L_m^{1/2} E_m^{1/4} \xi}{\eta}, \quad \gamma = \frac{E_m^{1/2} \psi_2}{\eta}
\]  
(2.2.71)

Substituting (2.2.71) into (2.2.56) - (2.2.57) and letting \( L_m^{-1/2} E_m^{1/2} \rightarrow 0 \) and \( L_m E_m^{1/2} \rightarrow 0 \) (which allows \( L_m \) to fall in \( \frac{E_m}{E}^{1/2} \ll L_m \ll E^{-1/2} \)) we get
\[
\frac{\partial \chi_2}{\partial \xi} = 0, \quad 2 \frac{\partial^2 \chi_2}{\partial \xi^2} = - (1 + \frac{1}{4} B) \frac{\partial \psi_2}{\partial \xi}
\]  
(2.2.72)

The outer Stewartson layer brings \( \chi = 0 \) to zero at the cylinder wall.
The inner layer provides the remaining conditions at \( x = 0 \). Both layers are terminated at the top and bottom by the Ekman layer. In figure 4 a diagram of the various flow regions of \( \chi \) and \( \psi \) is given for \( L_m \ll 1 \).

The distributions of \( w \) and \( \theta \) are found from (2.2.59). Substituting the solutions for \( \chi \) and \( \psi \), a solution for \( \phi \) can be deduced from the differential equation (2.2.59) and boundary conditions (2.2.60). Until now it has been assumed that the term in \( \phi \) on the left hand side of (2.2.57) is not significant to leading order in the description of \( \chi \) and \( \psi \). It is therefore necessary to require that the magnitude of \( A^{-1} \phi \) is at most equal to one of \( \chi \). This condition is satisfied when the magnitude of the imposed conditions for \( \phi \) given in (2.2.60) is not greater than \( A \) times the magnitude of the conditions for \( \chi \) given in (2.2.58). Furthermore, the present approach is only applicable if the non-linear terms on the right hand side of (2.2.59) are at most of unit magnitude compared to the terms on the left: i.e., \( \epsilon A_\infty / L_m E_m^{1/2} \ll 1 \).

In the discussion of the flow in the core of the cylinder at \( x \approx \frac{L_m E_m}{E} \) we took \( L_m \ll 1 \). Gas centrifuges have an aspect ratio which is of unit magnitude or larger. Consequently, since \( A \gg 1 \) the modified aspect ratio
Figure 4: Diagram of the flow regions of $\chi$ and $\psi$ for $L_m \sim 1$. 
is large. Therefore, we shall reinvestigate the viscous flow in the core for the \( \ell_\text{m} \) range \( 1 << \ell_\text{m} << \ell_\text{m}^{-1/2} \). Note that in this range the previously discussed boundary layers and inviscid region are still applicable. Since \( \ell_\text{m}^{-1} \) is small the \( z \)-derivations of \( \chi \) and \( \psi \) on the left of (2.2.56) and (2.2.57) can be neglected compared to the \( x \)-derivatives. By the same arguments as used before the term in \( \phi \) on the left of (2.2.57) is neglected. Introducing a contracted coordinate and scaled variables according to the rules

\[
x = \eta_1 + ln(L_{m}/\ell_\text{m})^{-1}, \quad \psi = \ell_\text{m}^{-1/2} \psi_1, \quad \chi = \ell_\text{m}^{-1/2} \chi_1
\]

(2.2.73)

equations (2.2.56) and (2.2.57) become

\[
8 \alpha_1^2 \left( \frac{\partial}{\partial \eta_1} e^{\eta_1 \frac{\partial}{\partial \eta_1}} \right)^2 e^{-\eta_1 \psi_1} = \frac{\partial \chi_1}{\partial z}
\]

(2.2.74)

\[
2 \alpha_1 e^{\eta_1 \frac{\partial^2 \chi_1}{\partial \eta_1^2}} = - \left( 1 + \beta \text{Re}_1 \right) \frac{\partial \psi_1}{\partial z}
\]

(2.2.75)

where terms \( \alpha_1 A^{-1} \alpha_1^{-1} \), are neglected compared to unit magnitude. It is therefore necessary that

\[
\alpha_1 \gg A^{-1}
\]

(2.2.76)

where the constant \( a \), is given by

\[
\alpha_1 = 1 - A^{-1} ln(L_{m}/\ell_\text{m})^{-1}
\]

(2.2.77)

Condition (2.2.76) follows from requiring that

\[
x^2 = 1 - x/A = \alpha_1 - A^{-1} \alpha_1 \sim \alpha_1
\]

(2.2.78)

which implies that the distance scale of the above deduced core, measured from the rotation axis \( \sqrt{\alpha_1} \), must be large compared to \( A^{-1/2} \).

The exponential increase of the terms on the left of (2.2.56) and (2.2.57) is responsible for the balance at \( x \sim ln(L_{m}/\ell_\text{m})^{-1} \). Similarly, for \( x >> ln(L_{m}/\ell_\text{m})^{-1} \) the terms on the left will become much larger than those
on the right and in the limit of $n_1 \rightarrow \infty$ the diffusive terms will dominate: i.e.

$$\left( \frac{3}{2} \frac{\partial}{\partial r} e^r \frac{3}{2} (1 - \frac{x}{A})^2 e^{-2x} \psi_v = 0 \right) \quad \text{(2.2.79)}$$

$$\Delta \frac{3}{3x} \left\{ 1 + \frac{1}{2} Br(1 - \frac{x}{A}) \right\} \frac{3}{3x} \left\{ 1 + \frac{1}{2} Br(1 - \frac{x}{A}) \right\}^{-1} (1 - \frac{x}{A}) \chi_v +$$

$$- \frac{2}{3} \frac{3}{3x} \left\{ 1 + \frac{1}{2} Br(1 - \frac{x}{A}) \right\}^{-1} \phi_v = 0 \quad \text{(2.2.80)}$$

Equation (2.2.59) becomes

$$2 \frac{3}{3x} \left\{ 1 + \frac{1}{2} Br(1 - \frac{x}{A}) \right\} (1 - \frac{x}{A}) \frac{3}{3x} \left\{ 1 + \frac{1}{2} Br(1 - \frac{x}{A}) \right\}^{-1} \chi_v +$$

$$+ \frac{Br}{A} \frac{3}{3x} \left\{ 1 + \frac{1}{2} Br(1 - \frac{x}{A}) \right\}^{-1} (1 - \frac{x}{A}) \chi_v = 0 \quad \text{(2.2.81)}$$

Expanding (2.2.80) - (2.2.81) into the co-ordinate $r$ and integrating with respect to $r$ we obtain

$$\omega_v = \sigma_1 + \frac{\sigma_2}{r^2}, \quad \theta_v = \sigma_3 \ln r + \sigma_4 \quad \text{(2.2.82)}$$

Requiring that $\omega_v$ and $\theta_v$ are finite at $r = 0$ it follows that $\sigma_2 = \sigma_4 = 0$. The solution for $\chi_v$ is then given by

$$\chi_v = \sigma_1 - \frac{1}{4} \sigma_4 \quad \text{(2.2.83)}$$

From (2.2.79) it follows that

$$\omega_v = \sigma_5 \int \frac{e^A(r^2-1)}{r} \, dr + \sigma_6 \ln r + \sigma_7 \quad \text{(2.2.84)}$$

Since $\omega_v$ must be finite at $r = 0$, $\sigma_5$ and $\sigma_6$ must be set equal to zero. Further integrating (2.2.84) with respect $r$ we obtain

$$r^2 e^A(r^2-1) \psi_v = \frac{1}{4} \frac{\sigma_7}{A} e^A(r^2-1) + \sigma_8 \quad \text{(2.2.85)}$$

Requiring that the radial mass flow is zero at $r = 0$, it follows that $\sigma_8 = - \frac{\sigma_7 e^{-A}}{2A}$ and $\psi_v$ becomes

$$\psi_v = \sigma_7 (1 - e^{-Ar^2}) / 2Ar^2 \quad \text{(2.2.86)}$$
The unknown constants \( \sigma_1, \ldots, \sigma_4 \) and \( \sigma_7 \) are determined by matching \((2.2.83)\) and \((2.2.86)\) to the outside flow. Expanding \((2.2.83)\) and \((2.2.86)\) into the coordinate \( \eta_1 \) it follows that \( \ddot{\chi}_1 \) and \( \ddot{\psi}_1 \) must be constant with respect to \( \eta_1 \), provided that \((2.2.76)\) is satisfied. So the boundary conditions for \((2.2.74)\) and \((2.2.75)\) are

\[
(\partial/\partial \eta_1) \ddot{\chi}_1 = (\partial/\partial \eta_1) \ddot{\psi}_1 = 0 \quad \text{as } \eta_1 \to \infty \quad (2.2.87)
\]

The Ekman layer equations given by \((2.2.66)\) and \((2.2.67)\) are only applicable for \( \delta \ll \ln E^{-1} \). The distance scale of the above deduced core, falls in this range provided that \( L_m > 1 \), which is the case according to our parameter choice. In other words, the core is terminated at the top and bottom by the Ekman layers.

Introducing \( x \sim \ln (L_m E^{-1}) \) into \((2.2.62)\) one sees that the scaling magnitude of \( \psi \) in the inviscid region is equal to the one of the above deduced core. The magnitude of \( \chi \), however, is an order smaller and hence the core cannot be matched to the interior. Therefore a second core is needed. Introducing the contracted coordinate and scaled variables

\[
x = \eta_2 + \ln (L_m^2 E^{-1})^{-1}, \quad \chi = \tilde{\chi}_2, \quad \psi = L_m^{-1} \tilde{\psi}_2
\]

the equations become, to leading order

\[
(2.2.88)
\]

\[
(2.2.89)
\]

\[
(2.2.90)
\]

provided that

\[
(2.2.91)
\]

where the constant \( \alpha_2 \) is given by

\[
(2.2.92)
\]
Condition (2.2.91) implies that the distance scale, measured from the rotation axis, $\sqrt{\alpha_2}$, must be large compared to $A^{-1/2}$. For $\eta_2 \to \infty$, it must match to its pure viscous value: $(\partial/\partial \eta_2) \chi_2 = 0$. The scaling magnitudes are equal to those of the inviscid interior so that $\chi_2$ and $\psi_\gamma$ can match smoothly to $\chi_0$ and $\psi_0$ as $\eta_2 \to -\infty$. As before, the core is terminated at the top and bottom by the Ekman layers.

The above discussed cores are denoted as the inner core at $x \sim \ln(L_m E_m)^{-1}$ and the outer core at $x \sim \ln(L_m^2 E_m)^{-1}$. In the outer core $\chi$ is adjusted to its pure viscous value. The inner core brings $\psi$ of the inviscid region and of the outer core to its pure viscous value at the axis. Just as in the inner Stewartson layer the $x$-derivatives of $\chi$ and $\psi$ are significant in the core while, as in the outer Stewartson layer, only the $x$-derivatives of $\chi$ are important in the outer core. On the other hand, in both Stewartson layers the exponential density gradient is negligible whereas in both cores this gradient is the mechanism by which the inviscid flow is coupled to a pure viscous flow at the rotation axis.

Conditions (2.2.76) and (2.2.91) are needed in order that a distinction is possible between a region where diffusion dominates and a region where diffusion and inertia are both important (both cores). If these conditions are not satisfied, either the "region of competition" is located in the immediate vicinity of the rotation axis $r \sim A^{-1/2}$ or it does not appear at all. In the latter case one observes an inviscid region up to the rotation axis.

As already stated, in the Ekman layer all terms become important at $x \sim \ln E_m^{-1}$, where the thickness of the layer is $\sim L^{-1}$. So the complicated "all terms regions" identified in the core for $L_m \sim 1$ has contracted to a layer of thickness $L_m^{-1}$ near the end caps. It will become clear, in the range $1 \ll L_m \ll E_m^{-1/2}$, referred to as the unit cylinder, more analytical progress can be made to describe the flow outside the Ekman layers.

Figure 5 illustrates the various flow regions of $\chi$ and $\psi$ in the unit cylinder.

A description in terms of an inviscid flow, boundary layers and viscous cores is only valid when $L_m$ falls in the range $1 \ll L_m \ll E_m^{-1/2}$. Since $L_m$ can become large we are primarily interested in the flow behaviour when $L_m$ exceeds the upper limit of this range. For this purpose we consider the distance scales of both cores and the scale heights of both
Figure 5: Diagram of the flow regions of \( \chi \) and \( \psi \) in the unit \( \sigma y Zinder. \)
Stewartson layers. For \( L_m^{-1/2} \sim E_m^{-1} \) the outer Stewartson layer expands over the density scale height (\( x \sim 1 \)). Simultaneously the outer core comes up and joins the layer. For \( L_m^{-1} \sim E_m^{-1} \) the inner layer and inner core expand, respectively, come up and join. One can expect that for \( E_m^{-1} / L_m^{-1} \sim \ll E_m^{-1} \) the \( x \)-derivatives of \( \chi \) in (2.2.57) become important terms describing the flow in the main section. For \( L_m^{-1} \sim E_m^{-1} \) the \( x \)-derivatives of \( \psi \) in (2.2.56) will do the same. The Ekman layer retains its small scale height and is still the narrow region within which the flow is adjusted to the end cap.

At first the flow in the range \( E_m^{-1/2} \sim L_m \ll E_m^{-1} \), referred to as the semi-long cylinder, is discussed. The \( x \)-derivatives of \( \chi \) become important by scaling

\[
\psi = (E_m x)^{1/2} \psi_1, \quad \chi = L_m^{-1} E_m^{-1/2} \chi_1
\]

(2.2.93)

Here, the absolute scaling magnitude of \( \psi \) has been chosen such that it corresponds to the induced flux of the Ekman layers. In order to show that the term in \( \phi \) in (2.2.57) can be neglected, we first consider the problem for \( \phi \) given by (2.2.59) and (2.2.60). Substitution of (2.2.93) into (2.2.59) shows that the non-linear terms can be dropped provided that \( \epsilon A \ll 1 \), according to (2.2.30) satisfied. The \( z \)-derivatives can be neglected since \( L_m^{-2} \) is small and we obtain

\[
4 \frac{\partial}{\partial x} \left\{ 1 + \frac{1}{4} Br \left( 1 - \frac{x}{A} \right) \left( 1 - \frac{x}{A} \right) \right\} \frac{\partial}{\partial x} \left\{ 1 + \frac{1}{4} Br \left( 1 - \frac{x}{A} \right) \right\}^{-1} \phi =
\]

\[
= -2 \frac{Bn}{A} L_m^{-1} E_m^{-1/2} \frac{\partial}{\partial x} \left\{ 1 + \frac{1}{4} Br \left( 1 - \frac{x}{A} \right) \right\}^{-1} \left( 1 - \frac{x}{A} \right) \chi_1
\]

(2.2.94)

This simplified problem does not allow the boundary conditions for \( \phi \) at \( z = 0 \) and \( z = 1 \) to be satisfied. Therefore layers of thickness \( L_m^{-1} \) near the end caps are needed.

Integrating (2.2.94) with respect to \( x \) and applying the condition that \( \psi \) must be finite at \( x = A \) (\( r = 0 \)), we get an equation which is used to eliminate the term in \( \phi \) on the left of (2.2.57). Further introducing (2.2.93) into (2.2.56) and (2.2.57) it is seen that the term in \( \phi \) is \( \sim A^{-2} \) compared to unit magnitude and can thus be dropped. The leading order of (2.2.56) and (2.2.57) becomes

\[
\frac{\partial^2 \chi_1}{\partial z^2} = 0
\]

(2.2.95)
The neglected terms are \( \sim \frac{L}{L_m} \), \( \sim \frac{L^2}{L_m} \) and \( \sim A^{-1} \) compared to unit magnitude, which is valid in our parameter setting. One sees that (2.2.95) and (2.2.96) allow the no-slip condition for \( \chi \) at \( r = 1 \). For \( x \to \infty \), \( \chi_1 \) must tend to its pure viscous value: \( \frac{\partial \chi_1}{\partial x} = 0 \). The remaining conditions at the cylinder wall and the pure viscous value for \( \psi \) at the axis, however, are not applicable. Therefore the inner Stewartson layer and the inner core are needed. The scaling rules and equations are equal to those given before. The Ekman layers provide the conditions at the end caps. In figure 6 we have given a diagram of the flow regions for \( \chi \) and \( \psi \) in the semi-long cylinder.

Finally the flow in the range \( J \sim \frac{E_m^{-1}}{E} \), referred to as the long cylinder, is discussed. In this range also the inner core and the inner Stewartson layer come up, respectively, expand and join at \( x \sim 1 \). Introducing the scaled variables

\[
\bar{\psi} = \frac{E_m^{1/2}}{E_m} \psi_2, \quad \bar{x} = \frac{E_m^{1/2}}{E} \chi_2
\]

equations (2.2.56) and (2.2.57) become, to leading order

\[
3 \frac{L E_m \left( \frac{\partial}{\partial x} \right) + \frac{1}{2} \bar{x} \frac{\partial}{\partial x} \right) \bar{\psi}_2 = \frac{\partial \chi_2}{\partial \bar{x}} \tag{2.2.98}
\]

\[
2 \frac{L E_m \left( \frac{\partial}{\partial x} \right)^2 \bar{x} \bar{\psi}_2 = -\left(1 + \frac{1}{2} \frac{E}{E_m} \right) \frac{\partial \bar{\psi}_2}{\partial \bar{x}} \tag{2.2.99}
\]

The neglected terms are \( \sim \frac{L^2}{L_m} \) and \( \sim A^{-1} \). As before, the term in \( \bar{\psi} \) on the left of (2.2.57) can be dropped, which is clear when one performs the same procedure as before with respect to \( E_m^{-1/2} \sim \frac{L}{L_m} \). Equations (2.2.98) and (2.2.99) allow all boundary conditions to be met at the cylinder wall.

For \( x \to \infty \), \( \chi_2 \) and \( \psi_2 \) must tend to their pure viscous values at the axis: \( \frac{\partial \chi_2}{\partial x} = \frac{\partial \psi_2}{\partial x} = 0 \). The conditions at the end caps are provided by the Ekman layers.
Figure 6: Diagram of the flow regions of $\chi$ and $\psi$ in the semi-long cylinder.
2.3. Discussion

In the present chapter we investigated the secondary flow of an incompressible fluid and of a perfect gas in a rotating cylinder, applying a linearised analysis to a small perturbation on the (isothermal) state of rigid body rotation. For the incompressible problem we identified three types of flow, when we increased the length-to-radius ratio $L$ from unit magnitude to infinity. These types correspond to the ranges $E^{1/2} \ll L \ll E^{-1/2}$, $E^{-1/2} \ll L \ll E^{-1}$, $E^{-1} \ll L$, where $E$ is the Ekman number based on the radius and taken to be small. In the first range the flow consists of an inviscid region, where the motion is governed by the Taylor-Proudman theorem, extended by Ekman layers near the end caps and an inner and outer Stewartson layer near the cylinder wall. In fact this flow type is well-known from literature and is of relevance to geophysical problems stemming from models of oceanic and atmospheric currents. In the second $L$-range the outer Stewartson layer expands to the interior and radial diffusion of azimuthal momentum becomes important in the main section of the cylinder. In the third range also the inner Stewartson layer fills the entire cylinder. As a result, radial diffusion of azimuthal and axial momentum are both significant, a situation that is also characteristic in studies on flows in semi-infinite cylinders. Since $E$ is very small most configurations will fall in the first range. The first type of flow is of most practical importance.

For the problem of a perfect gas in a rotating cylinder two extra dimensionless parameters enter into the linearized conservation equations. These are: the speed parameter $A$ and the Brinkman number $Br$. The speed parameter appears in the exponent of the exponential density distribution at isothermal rigid body rotation: i.e.

$$\frac{\rho}{\rho_0} = \exp\{A(r^2-1)\}$$  \hspace{1cm} (2.3.1)
where $A$ is the square of the ratio of the circumferential speed to the
most probable molecular speed. In gas centrifuges $A$ is at least of unit
magnitude. The Brinkman number appears in the linearised energy equation
where $Br/E$ scales the work done by compression $\mu(\partial/vp)$ relative to the
heat conduction. In gas centrifuges $Br$ is of unit magnitude.
The Ekman number $E$ based on the radius and on the density at the cylinder
wall is taken to be small. The length-to-radius ratio $L$ is varied from
unit magnitude to infinity.

In the case of non-diffusive motion, $E = 0$, the azimuthal component
of the momentum equation and the energy equation both lead to the same
result, namely a zero radial velocity and thus an axial velocity which is
constant along $z$: i.e.

$$ \frac{\partial \psi}{\partial z} = 0 \quad (2.3.2) $$

where $\psi$ is the streamfunction. Eliminating the pressure from the radial
and axial momentum equation it follows that

$$ \frac{\partial \omega}{\partial z}(\omega^{-1} \theta) = 0 \quad (2.3.3) $$

where $\omega$ is the angular speed perturbation and $\theta$ the temperature pertur-
bation. The foregoing result is referred to as the compressible Taylor-
Proudman theorem. The distributions of $\omega$ and $\theta$ individually are found
from a lower order balance with respect to $E$, viz. a balance between the
viscous forces in the azimuthal momentum equation and the heat conduction
in the energy equation. In fact, it is possible to separate two equations
for the streamfunction $\psi$ and the composite variable $\omega^{-1/2} \theta (= \chi)$, which can
be treated independently, to leading order, from the equation determining
$\omega$ and $\theta$ individually. This is especially advantageous with respect to gas
centrifuges where one is mainly interested in the distribution of the
axial and radial velocity so that only the problems for $\psi$ and $\chi$ have to
be considered.

Taking a speed parameter of unit magnitude and replacing $\chi$ by $\omega$ the
problems for $\chi$ and $\psi$ are quite analogous to those for $\omega$ and $\psi$ in the in-
compressible case. For $E^{1/2} \ll L \ll F^{-1/2}$ the flow consists of an inviscid
Taking $A \gg 1$, the density decreases strongly with distance from the cylinder wall and this is in fact the underlying reason for a drastic change of the flow phenomena. The parameters now specifying the motion are the modified Ekman number $E_m$ which is based on the scale height of the density decrease instead of the radius and the modified aspect ratio $L_m$ which is the ratio of the length to the density scale height. These modified parameters we related to $E$ and $L$ by: $E_m = EA^2$, $L_m = AL$. Taking $E_m \ll 1$ four types of flow for $\chi$ and $\psi$ can be distinguished when $L_m$ increases from unit magnitude to infinity. For $L_m \sim 1$ we identify a basic inviscid flow governed by the compressible Taylor-Proudman theorem in a limited region near the cylinder wall given by $x < \ln E_m^{-1}$ where $\omega = A(1-r^2)$. Two Stewartson layers of thickness $E_m^{1/3}$ and $E_m^{1/4}$ near the cylinder wall and Ekman layers of thickness $(E_m^2 r^2)^{1/2}$ near the end caps adjust the inviscid flow to the boundaries (see figure 4). Due to the exponential decrease of the density the Ekman layers expand away from the end caps with distance from the cylinder wall and form a viscous core at $x \sim \ln E_m^{-1}$, where the full equations are needed to describe the flow.

For $1 \ll L_m \ll R_m^{-1/2}$ the "all terms" core contracts to a layer of thickness $L_m^{-1}$ near the end caps (see figure 5). On the other hand, in the core of the cylinder we can distinguish two regions where only a part of the viscous forces is important. In an outer core at $x \sim \ln (L_m^2 E_m^{-1})^{-1}$ the $r$-derivatives of $\chi$ balance with the inertia terms just as in the outer Stewartson layer of thickness $L_m^{1/2} E_m^{1/4}$. In an inner core at $x \sim \ln (L_m^2 E_m^{-1})^{-1}$ also the $x$-derivatives of $\psi$ balance just as in the inner Stewartson layer of thickness $L_m^{1/3} E_m^{1/3}$. Provided that $A-\ln (L_m^2 E_m^{-1})^{-1} \gg 1$ and $A-\ln (L_m^2 E_m^{-1})^{-1} \gg 1$ the viscous terms of the outer core and of the inner core, respectively, will dominate compared to the inertia terms at the rotation axis, a result of which is that $\psi$ and $\chi$ are constant with respect to $x$. The outer core adjusts $\chi$ to the pure viscous value at the axis and its scaling magnitude is such that it matches to the inviscid region. The inner core adjusts $\psi$ of the inviscid region and of the outer core to its pure viscous value at the axis.
For $E^{-1/2}_m \sim L_m \ll E^{-1}_m$ the outer Stewartson layer expands over the small density scale height. Simultaneously the outer core comes up from the interior and joins the outer Stewartson layer at $x \approx 1$, so that the viscous $x$-derivatives of $\chi$ are important to describe the flow field in the main section (see figure 6). For $E^{-1}_m \sim L_m$ the inner core and the inner Stewartson layer come up, respectively, expand and join, so that the viscous $x$-derivatives of $\psi$ are also important. As a result, we obtain a viscous flow in the main section which is strongly influenced by the cylinder wall, a situation that is also typical in studies on flows in semi-infinite gas centrifuges.

The length-to-radius ratio of a gas centrifuge is of unit magnitude or larger. Since in the present day centrifuges $A >> 1$, we can take $L_m >> 1$, which means that the flow type corresponding to $L_m \sim 1$ is not of practical interest. On the other hand, the types corresponding to $1 << L_m << E^{-1/2}_m$, referred to as the unit cylinder, $E^{-1/2}_m \sim L_m \ll E^{-1}_m$, referred to as the semi-long cylinder and $E^{-1}_m \sim L_m$ referred to as the long cylinder, are all of importance and, at the same time, the whole scope of centrifuge flows is covered. In the next chapter we will present explicit solutions for the flow field in these three ranges.

A table of the flow regions outside the Ekman layers for the most important parameter ranges is given on the next page.
<table>
<thead>
<tr>
<th></th>
<th>pure viscous region</th>
<th>inner core</th>
<th>outer core</th>
<th>inviscid region</th>
<th>outer Stewartson layer</th>
<th>inner layer</th>
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<td>no</td>
<td>yes</td>
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<tr>
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<td>yes</td>
<td>yes</td>
<td>yes</td>
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<tr>
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<tr>
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<tr>
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<td>no</td>
<td>no</td>
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<tr>
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</tr>
</tbody>
</table>

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3. THE RAPIDLY ROTATING HEAVY GAS

3.1. The problem statement

3.1.1. Introduction

During the last decade a lot of emphasis has been placed on efforts to develop countercurrent gas centrifuges used for the separation of uranium isotopes. The configuration of a countercurrent gas centrifuge consists of a circular cylinder with plane horizontal end caps and rotating about its vertical symmetry axis. The working fluid in the cylinder is a mixture of gaseous uranium hexafluoride ($\text{UF}_6$) of molecular weight 349 and 352. Due to the rotation of the mixture a radial separation of both components appears. An axial countercurrent flow superimposed on the primary rotation causes a vertical concentration gradient. The input and output of $\text{UF}_6$ at various positions allows a continuous utilization of the separation process.

From a fluid dynamical point of view the physical constants of both components in the mixture are practically equal which allows a homogeneous approach to the flow problem. A different rotation, a different temperature and an imposed axial flow at the end caps, and an axial temperature gradient along the rotor wall are the possible means by which the countercurrent is established. The above mentioned induced perturbations on the state of isothermal rigid body rotation are often small, at least in those regions where the real separation occurs. Therefore a linear treatment as applied in the previous chapter appears reasonable.

The most probable molecular speed $\sqrt{2R_0 T_0/M}$ of $\text{UF}_6$ is approximately equal to 120 [m/s] at room temperature. This quantity is relatively small due to the large molecular weight of $\text{UF}_6$ (air: $\sqrt{2R_0 T_0/M} \approx 480$[m/s]). The separative performance of a centrifuge increases with the circumferential speed (Cohen 1951, Los 1963) and speeds of 500 [m/s] and more are today quite normal. Taking $\alpha \Omega = 500$ [m/s] the speed parameter $\lambda$ becomes:
The viscosity of $U_F$ is equal to $1.694 \times 10^{-5}$ [kg/ms]. Assuming a radius of $10^{-1}$ [m] and a wall pressure of $8000$ [N/m$^2$], the modified Ekman number becomes $E_m \approx 0.92 \times 10^{-4}$. For $E_m$, the heat conductivity is approximately equal to $6.68 \times 10^{-3}$ [W/mK] and the Brinkman number becomes $Br \approx 2$. As a consequence the assumptions $A > 1$, $E_m < 1$ and $Br \approx 1$ are reasonable and a treatment of the secondary flow as discussed in 2.2.2 is applicable.

The governing equations for the variables $\chi$, $\psi$ and $\phi$ are given by (2.2.56), (2.2.57) and (2.2.59). As already seen, equations (2.2.56) and (2.2.57) contain only one term in $\phi$, namely on the left hand side of (2.2.57). But this term is not of leading order in the force balances under consideration and can therefore be omitted. As a result, equations (2.2.56) and (2.2.57) form a complete set determining $\chi$ and $\psi$. Equation (2.2.59) may be considered as an extra equation for the extra variable $\phi$. In gas centrifuge problems one is mainly interested in the distribution of the radial and axial velocity determined by $\psi$. Therefore we restrict ourselves to the equations (2.2.56) and (2.2.57). Omitting the term in $\phi$ these are

\begin{align*}
L_m E_m \left[ 4 \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} (1 - \frac{x}{A}) + \frac{1}{L_m^2} \varphi \frac{\partial^2}{\partial x^2} \right] + \\
\frac{4}{3} (1 - \frac{x}{A}) \left[ \frac{2}{L_m} \varphi \frac{\partial}{\partial x} \right] e^{-\varphi} = 2 \frac{\partial^2}{\partial x} \psi
\end{align*}

(3.1.1)

\begin{align*}
L_m E_m \left[ 4 \frac{\partial}{\partial x} \varphi \frac{\partial}{\partial x} \right] [1 + \frac{1}{2} Br (1 - \frac{x}{A})] \frac{\partial}{\partial x} \left[ 1 + \frac{1}{2} Br (1 - \frac{x}{A}) \right] - (1 - \frac{x}{A}) + \\
+ \frac{1}{L_m^2} \frac{\partial^2}{\partial x^2} \right] \chi = - 2 \left[ 1 + \frac{1}{2} Br (1 - \frac{x}{A}) \right] \frac{\partial^2}{\partial x^2} \psi
\end{align*}

(3.1.2)

with boundary conditions

\begin{align*}
\psi = \frac{\partial}{\partial x} \psi = 0, \quad \chi = - \frac{1}{2} \theta \varphi & \quad \text{at } x = 0 \quad (3.1.3) \\
\psi = \psi_B, \quad \chi = \omega_B - \frac{1}{2} \theta_B & \quad (\partial / \partial x) \psi = 0 \quad \text{at } z = 0 \quad (3.1.4) \\
\psi = \psi_b, \quad \chi = \omega_b - \frac{1}{2} \theta_b & \quad (\partial / \partial x) \psi = 0 \quad \text{at } z = 1 \quad (3.1.5)
\end{align*}
where

\[ \chi = \omega - \frac{1}{2} \theta \quad \text{(3.1.6)} \]

\[ u/r = - \frac{L_m^{-1}}{A} \frac{\partial \psi}{\partial Z}, \quad \omega = - 2e^{\omega} \frac{\partial}{\partial x}(1 - \frac{x}{A})e^{-x}\psi \quad \text{(3.1.7)} \]

\[ x = A(1-r^2) \quad \text{(3.1.8)} \]

\[ \psi_b = \frac{-\omega}{2(1-\frac{x}{A})} \int_0^A e^{-x}\psi_b \, dx, \quad \psi_t = \frac{-\omega}{2(1-\frac{x}{A})} \int_0^A e^{-x}\psi_t \, dx \quad \text{(3.1.9)} \]

\[ \psi_b = \psi_t = 0 \quad \text{at} \quad x = 0 \quad \text{(3.1.10)} \]

Here, \( \omega_t \) and \( \omega_b \) represent the different rotation rates of the top and the bottom. The functions \( \omega_b \) and \( \omega_t \) represent the vertical velocities and \( \theta_b \) and \( \theta_t \) the temperature distributions at the bottom and at the top. According to (3.1.10) we restrict ourselves to the case that the axial net transport is zero. The function \( \theta_{\psi} \) is the temperature distribution along the cylinder wall. The no-slip and no-normal-flow conditions at the cylinder wall are expressed in (3.1.3).

The aspect ratio of a gas centrifuge is at least of unit magnitude. Since \( A >> 1 \) the modified aspect ratio is large. For \( L_m >> 1 \) three types of flow can be distinguished corresponding to the ranges \( 1 << L_m << E_m^{-1/2} \), the unit cylinder, \( E_m^{-1/2} \sim L_m << E_m^{-1} \), the semi-long cylinder and \( E_m^{-1} \sim L_m \), the long cylinder. In all three cases Ekman layers form near the end caps. In the next subsection we will investigate the flow in these layers. In particular, the influence of the imposed boundary conditions at \( \zeta = 0 \) and \( \zeta = 1 \) on the flow outside the Ekman layers i.e. the Ekman suction. In sections 3.2, 3.3 and 3.4 we will successively investigate the flow outside the Ekman layers in the unit cylinder, the semi-long cylinder and the long cylinder.
6.1.2. The Ekman suction

The Ekman layer equations near the bottom cap are obtained by introducing the stretched coordinate

\[ y = L_m(g_m e^x)^{-1/2}z \]  
(3.1.11)

and the perturbation expansion

\[ \chi = \chi_0 + F_m e^x \chi_1 + \ldots \]  
(3.1.12)

\[ \psi = (F_m e^x)^{1/2} \left[ \Psi_0 + E_m e^x \Psi_1 + \ldots \right] \]  
(3.1.13)

Putting (3.1.11) - (3.1.13) into (3.1.1) - (3.1.2) and letting \( E_m e^x \to 0 \) we obtain

\[ \frac{\partial^4 \Psi_0}{\partial y^4} = 2 \frac{\partial^2 \chi_0}{\partial y^2} \]  
(3.1.14)

\[ \frac{\partial^2 \chi_0}{\partial y^2} = -2 \{1 + \frac{1}{4} \beta \Pi \left(1 - \frac{\alpha}{\Lambda} \right)\} \frac{\partial \Psi_0}{\partial y} \]  
(3.1.15)

The terms neglected in these equations are \( \sim E_m e^x \). As a consequence we must require: \( x \ll 2nB_m^{-1} \). At \( x \sim 2nB_m^{-1} \), where the layer is of thickness \( L_m^{-1} \), all terms of the original equations become important. Furthermore, the \( x \)-derivatives are neglected which indicates that the dependent variables may not vary too fast along \( x \). This can be illustrated near \( x = 0 \) by introducing the coordinate \( \zeta = e^{-1/2}e^x \). Then we obtain a cross-section at the corner of the cylinder where both the highest \( x \)- and \( z \)-derivatives are significant. In general, we can conclude that the local \( x \)-variations of the dependent variables \( \Psi_0 \) and \( \chi_0 \) must take place on a distance scale that is large compared to \( (e^{-1/2} e^x)^{1/2} \).

Differentiating (3.1.14) with respect to \( y \) and eliminating \( (\partial^2/\partial y^2) \chi_0 \) by means of (3.1.15) we get

\[ \left( \frac{\partial^4}{\partial y^4} + 4 \{1 + \frac{1}{4} \beta \Pi \left(1 - \frac{\alpha}{\Lambda} \right)\} \frac{\partial \Psi_0}{\partial y} \right) = 0 \]  
(3.1.16)
the solution of which is given by

\[ \frac{\partial \tilde{\psi}_0}{\partial y} = -e^{\pm y_1} \cos y_1 e^{\pm y_1 \sin y_1} \]  \hspace{1cm} (3.1.17)

where

\[ y_1 = y_1 \left(1 + \frac{1}{2} Br(1 - \frac{2}{A})\right)^{1/4} \]  \hspace{1cm} (3.1.18)

Here, \((\partial/\partial y) \tilde{\psi}_0\) may not tend to infinity as \(y \to \infty\), where \(y \to \infty\) means for from the bottom cap on the small thickness scale of the Ekman layer but still within the cylinder. As a result, (3.1.17) can be simplified to

\[ \frac{\partial \tilde{\psi}_0}{\partial y_1} = e^{-y_1} \sin y_1 e^{-y_1 \cos y_1} \]  \hspace{1cm} (3.1.19)

The boundary condition for the radial velocity at the bottom cap is given by

\[ (\partial/\partial y_1) \tilde{\psi}_0 = 0 \]  \hspace{1cm} at \(y_1 = 0\)  \hspace{1cm} (3.1.20)

so that (3.1.19) becomes

\[ \frac{\partial \tilde{\psi}_0}{\partial y_1} = \int e^{-y_1} \sin y_1 \]  \hspace{1cm} (3.1.21)

where the integration constant \(f\) is an arbitrary function of \(x\), except that \((\partial/\partial x)f \ll f(e^{x})^{-1/2}\). Integrating (3.1.21) with respect to \(y_1\) and applying the condition that

\[ \tilde{\psi}_0 = (e^{x})^{-1/2} \psi_b \]  \hspace{1cm} at \(y_1 = 0\)  \hspace{1cm} (3.1.22)

it is found that

\[ \tilde{\psi}_0 = -\frac{1}{2} \int e^{-y_1} (\sin y_1 + \cos y_1) + \frac{1}{2} f + (e^{x})^{-1/2} \psi_b \]  \hspace{1cm} (3.1.23)
Integration of \( (3.1.14) \) with respect to \( y \) and application of the condition that

\[
\tilde{\chi}_0 = \omega_B - \frac{i}{2} \theta_B \quad \text{at} \quad y_1 = 0
\]  

(3.1.24)
yields

\[
\tilde{\chi}_0 = f(1 + i B r (1 - \frac{x}{A}))^{3/4} (1 - e^{-2 I_0 \cos y_1}) + \omega_B - \frac{i}{2} \theta_B
\]  

(3.1.25)

The unknown function \( f \) in the solutions \( (3.1.21), (3.1.23) \) and \( (3.1.25) \) is determined by matching the velocities of the Ekman layers and the outside flow. Here, outside flow means the internal flow in the cylinder outside the Ekman layer. The dependent variables of the outside flow are denoted by the sign \( ^\wedge \). They vary along the \( z \)-coordinate on a scale that is large compared to \( \tilde{r}_m^{-1} (E_m e^x)^{1/2} \), the thickness scale of the Ekman layer. The match conditions are given by

\[
\tilde{\chi}(z = 0) = \tilde{\chi}_0(y \to \infty) + E_m e^x \tilde{\chi}_0(y_1 \to \infty) + \ldots.
\]  

(3.1.26)

\[
\tilde{\psi}(z = 0) = (E_m e^x)^{1/2} \{ \tilde{\psi}_0(y_1 \to \infty) + E_m e^x \tilde{\psi}_0(y_1 \to \infty) + \ldots \}
\]  

(3.1.27)

\[
\frac{\partial \tilde{\psi}}{\partial z} (z = 0) = \{1 + i B r (1 - \frac{x}{A})\}^{1/4} \tilde{r}_m \frac{\partial \tilde{\psi}_0}{\partial y_1} (y_1 \to \infty) +
\]

\[
+ E_m e^x \frac{\partial \tilde{\psi}_0}{\partial y_1} (y_1 \to \infty) + \ldots
\]  

(3.1.28)

Substituting \( (3.1.25) \) into \( (3.1.26) \) and letting \( E_m e^x \to 0 \) we obtain

\[
f = [1 + i B r (1 - \frac{x}{A})]^{3/4} \{ \tilde{\psi}(z = 0) - \omega_B + \frac{i}{2} \theta_B \}
\]  

(3.1.29)

Putting \( (3.1.21) \) into \( (3.1.28) \) one sees that an outside radial velocity can only be adjusted to the end cap by the second order contribution of the Ekman layer extensions. As follows, we conclude that

\[
\tilde{\psi}_0 \sim E_m^{-1} \tilde{r}_m^{-1} (z/\tilde{r}_m) \tilde{\psi}(z = 0)
\]  

(3.1.30)

Putting this in \( (3.1.27) \) we observe that the second order contributions are small compared to the left hand side if

\[
(\tilde{\partial}/\tilde{\partial z}) \tilde{\psi} \ll (E_m e^x)^{-1/2} \tilde{r}_m \tilde{\psi}
\]  

(3.1.31)
which is the case as long as the variations of the outside flow take place on an axial distance scale that is large compared to the one of the Ekman layer. In other words, the adjustment of the radial velocity to its no-slip condition induces a secondary flow which is an order of magnitude smaller than the one induced by the coupling of $\Phi$ and $\hat{x}$ to their proper values at the end cap and can therefore be omitted.

Substituting (3.1.23) into (3.1.27) and applying (3.1.29) we obtain, to leading order

$$\frac{1}{2} (E_m e^x)^{1/2} (\hat{x} - \hat{\omega}_b + \frac{1}{2} \hat{\theta}_b) =$$

$$= \{1 + \frac{1}{2} Br(1 - \frac{x}{A})^{3/4} (\Phi - \psi_b) \} \text{ at } z = 0 \quad (3.1.32)$$

The compatibility condition (3.1.32) for the outside flow is referred to as the Ekman suction condition and allows us to describe the outside flow without an explicit discussion of the Ekman layers themselves. The first order representation of the Ekman layer solutions can now be written as

$$\hat{x}_0 = \{ \hat{x}(z=0) - \omega_b + \frac{1}{2} \theta_b \} (1 - e^{-y_1^1 \cos y_1^1}) + \omega_b - \frac{1}{2} \theta_b \quad (3.1.33)$$

$$\psi_0 = (E_m e^x)^{-1/2} \{ \psi(z=0) - \psi_b \} (1 - e^{-y_1^1 (\sin y_1^1 + \cos y_1^1)}) + \psi_b \quad (3.1.34)$$

Analogous to the above procedure we can derive a suction condition at the top cap. Without further proof we put

$$\frac{1}{2} (E_m e^x)^{1/2} (\hat{x} - \hat{\omega}_t + \frac{1}{2} \hat{\theta}_t) =$$

$$= \{1 + \frac{1}{2} Br(1 - \frac{x}{A} - 1, a - 1, a + \frac{1}{2} t_{10} \} \} \text{ at } z = 1 \quad (3.1.35)$$

The first order representation of the Ekman layer solutions near $z = 1$ is given by

$$\hat{x}_0 = - \{ \hat{x}(z=1) - \omega_t + \frac{1}{2} \theta_t \} (e^{-y_1^t \cos y_1^t} - 1) + \omega_t - \frac{1}{2} \theta_t \quad (3.1.36)$$

$$\psi_0 = (E_m e^x)^{-1/2} \{ \psi(z=1) - \psi_t \} (1 - e^{-y_1^t (\sin y_1^t + \cos y_1^t)}) + \psi_t \quad (3.1.37)$$
We are primarily interested in the flow outside the Ekman layers. For this purpose it is convenient to join the imposed boundary conditions at the end caps as follows

\[
F_b = \omega_b - \frac{1}{2} \theta_b - 2(E_m e^\omega)^{-1/2} (1+i Br (1- \frac{x}{\lambda}))^{3/4} \psi_b \\
F_t = \omega_t - \frac{1}{2} \theta_t + 2(E_m e^\omega)^{-1/2} (1+i Br (1- \frac{x}{\lambda}))^{3/4} \psi_t
\]  

The suction conditions then simplify to

\[
\{1+i Br (1- \frac{x}{\lambda})\}^{3/4} \Phi - \frac{1}{2} (E_m e^\omega)^{1/2} \chi = - \frac{1}{2} (E_m e^\omega)^{1/2} F_b \quad \text{at } z = 0 \]  

\[
\{1+i Br (1- \frac{x}{\lambda})\}^{3/4} \Phi + \frac{1}{2} (E_m e^\omega)^{1/2} \chi = + \frac{1}{2} (E_m e^\omega)^{1/2} F_t \quad \text{at } z = 1
\]

One sees that a mathematical description of the outside flow is possible without making a distinction between the imposed different rotation, different temperature and axial flow at both ends. Only the combination of these three boundary conditions as expressed by (3.1.39) and (3.1.40) is essential. The governing equations for the flow outside the Ekman layers are obtained by dropping the \(z\)-derivatives on the left hand side of (3.1.1) and (3.1.2)

\[
\frac{L_m E_m}{m} \{4 \frac{\partial}{\partial x} e^\omega \frac{\partial}{\partial x} (1- \frac{x}{\lambda}) \}^2 \Phi e^{-\chi} = 2 \frac{\partial \chi}{\partial z} \]  

\[
4 \frac{L_m E_m}{m} e^\chi \frac{\partial}{\partial x} \{1+i Br (1- \frac{x}{\lambda})\} \frac{\partial}{\partial x} (1+i Br (1- \frac{x}{\lambda}))^{-1} (1- \frac{x}{\lambda}) \chi = - 2 \{1+i Br (1- \frac{x}{\lambda})\} \frac{\partial \Phi}{\partial z}
\]

The boundary conditions are given by the suction conditions and

\[
\Phi = (\partial/\partial x) \Phi = 0, \quad \chi = - \frac{1}{2} \theta_{yz}(z) \quad \text{at } x = 0.
\]

Equations (3.1.43) - (3.1.44) with boundary conditions (3.1.41), (3.1.42) and (3.1.45) describe the flow outside the Ekman layers in the unit, semi-long and long cylinder.
3.2. The unit cylinder

In the present section we investigate the flow outside the Ekman layers in the unit cylinder whose \( \mathcal{L}_m \)-range is given by

\[
1 \ll \mathcal{L}_m \ll \mathcal{E}_m^{-1/2}
\]

(3.2.1)

As was seen in section 2.2.2 five flow regions can be distinguished. These are: the inner and the outer Stewartson layer, the inviscid region and the inner and the outer core (see figure 5). Firstly, we discuss the flow in the inviscid region.

In the inviscid region \( \chi \) and \( \psi \) are constant along \( z \). Therefore the Ekman layers at the top and the bottom will control the flow, which means that the scaling magnitudes of the dependent variables correspond to the induced flux expressed in the Ekman suction conditions. As a consequence we scale for the inviscid flow

\[
\psi = (\mathcal{E}_m \mathcal{E}_m^{1/2})^{1/2} \psi_0, \quad \chi = \chi_0
\]

(3.2.2)

Substituting (3.2.2) into (3.1.43) and (3.1.44) we obtain

\[
L_m \mathcal{E}_m^{3/2} 4 \frac{\partial}{\partial x} e^x \frac{\partial}{\partial x} \left[ 1 - \frac{x}{\mathcal{A}} \right]^2 \psi_0 e^{-1/2x} = 2 \frac{\partial \psi}{\partial z}
\]

(3.2.3)

\[
4L_m \mathcal{E}_m^{1/2} e^{1/2x} \frac{\partial}{\partial x} \left[ 1 + \frac{1}{4} Br \left( 1 - \frac{x}{\mathcal{A}} \right) \frac{3}{\partial x} \left( 1 + \frac{1}{4} Br \left( 1 - \frac{x}{\mathcal{A}} \right) \right) \right] \psi_0 e^{-1/2x} = -2 \left[ 1 + \frac{1}{4} Br \left( 1 - \frac{x}{\mathcal{A}} \right) \right] \frac{\partial \psi}{\partial z}
\]

(3.2.4)
The terms on the left hand side of (3.2.3) and (3.2.4) can be neglected if, in the first place, $L_m$ falls in the range (3.2.1). Secondly, the terms in question increase with a power of $e^X$ by which we must add the condition that $x << \ln(L_m^2 F_m)^{-1}$. For $x \approx \ln(L_m^2 F_m)^{-1}$ the terms on the left of (3.2.4) become of leading order and the resulting balance is one of the outer core. In the inviscid region we get

$$\frac{\partial x_0}{\partial s} = 0, \frac{\partial \psi_0}{\partial s} = 0 \quad (3.2.5)$$

Applying the suction conditions (3.1.41) and (3.1.42), the solutions for $x_0$ and $\psi_0$ become

$$x_0 = \frac{4}{3} (F_t + F_b) \quad (3.2.6)$$

$$\psi_0 = \frac{4}{3} (1 + \frac{1}{4} Br(1 - \frac{x}{A})^{-3/4} (F_t - F_b)) \quad (3.2.7)$$

Application of (3.1.7) gives for the radial and axial flow

$$\rho_0 \omega = 0, 2\rho_0 \omega = \frac{\partial}{\partial s} \left( E_m^2 e^{\frac{X}{A}} \right)^{1/2} e^{-x} \left( 1 - \frac{x}{A} \right) \left( 1 + \frac{1}{4} Br(1 - \frac{x}{A})^{-3/4} (F_b - F) \right) \quad (3.2.8)$$

where $\rho_0 (= e^{-\frac{x}{A}})$ is the dimensionless density distribution at isothermal rigid body rotation. One sees that in the inviscid region the axial flow is constant along $z$. The outflow from the Ekman layer at the bottom is equal to the inflow at the top. There is no interior radial motion and the axial flow is only returned within the Ekman layers. Both layers are in direct communication and control the flow in the interior, a consequence of the compressible Taylor-Proudman theorem. Once again we must point out that such a flow is only observed in a limited region near the cylinder wall given by $x << \ln(L_m^2 F_m)^{-1}$.

Generally, the solutions (3.2.6) and (3.2.7) do not apply at the cylinder wall. To overcome this problem we must introduce the Stewartson layers. At first we consider the outer Stewartson layer of thickness $L_m^1 F_m^{-1/4}$. Introducing the stretched coordinate

$$z_2 = L_m^{-1/2} F_m^{-1/4} x \quad (3.2.9)$$
and the perturbation expansion
\[
\hat{x} = \bar{x}_2 + L_m^{1/2} E_m^{1/4} \hat{x}_2 + \ldots .
\]
\[
\hat{\psi} = \frac{1}{2} E_m^{1/2} (\hat{\psi}_2 + L_m^{1/2} E_m^{1/4} \hat{\psi}_2 + \ldots )
\]

into (3.1.43) - (3.1.44) and letting \( L_m^{-1} E_m^{1/2} \to 0 \) and \( L_m E_m^{1/2} \to 0 \) we get
\[
\frac{\partial \hat{x}_2}{\partial z} = 0 \quad \text{(3.2.12)}
\]
\[
2 \frac{\partial^2 \hat{x}_2}{\partial z^2} = - (1 + \frac{1}{2} Br) \frac{\partial \hat{\psi}_2}{\partial z} \quad \text{(3.2.13)}
\]

Differentiating (3.2.13) with respect to \( z \) and applying (3.2.12) we get
\[
\frac{\partial^2 \hat{\psi}_2}{\partial z^2} = 0 \quad \text{(3.2.14)}
\]

The boundary conditions at \( z = 0 \) and \( z = 1 \) are obtained from the Ekman suction. In section 3.1.2 we pointed out that the x-variations of the dependent variables in the Ekman layers must take place on a distance scale that is large compared to \( E_m^{1/2} \). The distance scales of the inner and the outer Stewartson layer are respectively \( L_m^{1/3} E_m^{1/3} \) and \( L_m^{1/2} E_m^{1/4} \). These thickness scales are large compared to \( E_m^{1/2} \) as \( L_m >> E_m^{1/2} \), a condition satisfied in the unit cylinder. Both Stewartson layers are terminated at the top and the bottom by the Ekman layers.

Substituting (3.2.9) - (3.2.11) into (3.1.41) - (3.1.42) and letting
\( L_m^{1/2} E_m^{1/4} \to 0 \) we obtain
\[
(1 + \frac{1}{2} Br)^{3/4} \hat{\psi}_2 - \frac{1}{2} \bar{x}_2 = - \frac{1}{2} F_B(0+) \quad \text{at } z = 0 \quad \text{(3.2.15)}
\]
\[
(1 + \frac{1}{2} Br)^{3/4} \hat{\psi}_2 + \frac{1}{2} \bar{x}_2 = + \frac{1}{2} F_c(0+) \quad \text{at } z = 1 \quad \text{(3.2.16)}
\]

Generally, the imposed boundary conditions at both end caps are discontinuous at the corners of the cylinder \((x = 0, z = 0, 1)\). On the other hand, for \( x > 0 \) \( F_B \) and \( F_c \) are smooth with respect to \( x \). Therefore
\[
F_B (\zeta_2 L_m^{1/2} E_m^{1/4}) = F_B(0+) \quad \text{(3.2.17)}
\]
\[
F_c (\zeta_2 L_m^{1/2} E_m^{1/4}) = F_c(0+) \quad \text{(3.2.18)}
\]
in the limit of \( L_m^{1/2} F_m^{1/4} \to 0 \). In other words, the outer Stewartson layer does not see the discontinuity at the corner, in contrast to the inner Stewartson layer, as will be seen later in this section.

Having in mind that \( \chi_2 \) is constant with respect to \( z \) it follows from (3.2.14), (3.2.15) and (3.2.16) that

\[
\Psi_2 = \frac{i}{4} (1 + \frac{1}{2} \frac{L_m^{1/2}}{F_m^{1/4}} + \frac{1}{2} \frac{L_m^{1/2}}{F_m^{1/4}}) \left[ \frac{\chi}{2} - F_t(0+) + \frac{1}{2} F_t(0+) - \frac{1}{2} \chi_2 \right] \quad (3.2.19)
\]

Substitution of (3.2.19) into (3.2.13) gives

\[
\left\{ 2 \frac{d^2}{d \zeta_2^2} - \frac{1}{2} (1 + \frac{1}{2} \frac{L_m^{1/2}}{F_m^{1/4}}) \right\} \chi_2 = - \frac{i}{4} (1 + \frac{1}{2} \frac{L_m^{1/2}}{F_m^{1/4}}) \left[ F_t(0+) + F_t(0+) \right] \quad (3.2.20)
\]

At the outer edge we must require that \( \chi_2 \) matches the inviscid solution: i.e.

\[
\chi_2(\zeta_2 \to \infty) = \chi_0(x \to 0) \quad (3.2.21)
\]

The solution of (3.2.20) is now given by

\[
\chi_2 = a e^{\left( \frac{1}{8} \frac{L_m^{1/2}}{F_m^{1/4}} \right) \zeta_2^2/2} + \frac{1}{2} \left[ F_t(0+) + F_t(0+) \right] \quad (3.2.22)
\]

where the unknown constant \( a \) follows from the matching to the inner Stewartson layer. Putting (3.2.22) into (3.2.19) the solution of \( \Psi_2 \) becomes

\[
\Psi_2 = \frac{i}{4} (1 + \frac{1}{2} \frac{L_m^{1/2}}{F_m^{1/4}})^{-3/4} \left[ \frac{1}{2} (F_t(0+) - F_b(0+)) + a(1 - 2\pi) \right] e^{\left( \frac{1}{8} \frac{L_m^{1/2}}{F_m^{1/4}} \right) \zeta_2^2/2} \quad (3.2.23)
\]

a solution that also matches the inviscid solution.

The equations and boundary conditions for the inner Stewartson layer problem are obtained by introducing the stretched coordinate

\[
\zeta_1 = \left( \frac{L_m F_m}{L_m F_m} \right)^{-1/3} x \quad (3.2.24)
\]
and the perturbation expansions

\[ \tilde{\chi} = \tilde{\chi}_2 + b(\tilde{\chi}_1 + (L_mE_m)^{1/3}\tilde{\chi}^1_1 + \ldots) \]  
\[ \varphi = E_m^{1/2}\varphi_2 + b(L_mE_m)^{1/3}(\varphi_1 + (L_mE_m)^{1/3}\varphi^1_1 + \ldots) \]

where \( b \) is an unknown constant that fixes the absolute magnitude of the dependent variables. Its value depends upon the function of the layer and will be discussed in the subsequent treatment. Because of algebraic convenience we have also added the outer Stewartson layer contributions in expansions (3.2.25) and (3.2.26). Putting (3.2.24) - (3.2.26) into (3.1.43) - (3.1.44), applying (3.2.22) - (3.2.23) and dropping the terms

\[ \sim h^{-1}E_m^{-1/2}, \sim h^{-1}L_m^{1/6}E_m^{5/12}, \sim L_m^{1/3}E_m^{1/3} \]

compared to unit magnitude, the first order representation of the inner Stewartson layer equations becomes

\[ \frac{\partial^4 \psi_1}{\partial z^4} = \frac{\partial \tilde{\chi}_1}{\partial z} \]  
\[ 2 \frac{\partial^2 \tilde{\chi}_1}{\partial z^2} = - (1 + \frac{1}{3}Br) \frac{\partial \varphi_1}{\partial z} \]

Substituting (3.2.24) - (3.2.26) into the suction conditions (3.1.41) - (3.1.42) and dropping terms \( \sim L_m^{-1/3}E_m^{1/6} \) one obtains

\[ \psi_1 = - b^{-1}(1 + \frac{1}{3}Br)^{-3/4}e_*^{1/3}F_t(0+){H(\zeta_1)}^{-1} \text{ at } z = 0 \]  
\[ \psi_1 = + b^{-1}(1 + \frac{1}{3}Br)^{-3/4}e_*^{1/3}F_t(0+){H(\zeta_1)}^{-1} \text{ at } z = 1 \]

where \( e_* = L_m^{-1}E_m^{1/2} \) and is small.

As already stated, the imposed boundary conditions are discontinuous at the corners of the configuration. To illustrate this behaviour we replaced \( F_t(L_m^{1/3}E_m^{1/3}\zeta_1) \) and \( F_t(L_m^{1/3}E_m^{1/3}\zeta_1) \), \( \zeta_1 \geq 0 \), by respectively \( F_t(0+){H(\zeta_1)} \) and \( F_t(0+){H(\zeta_1)} \), where \( H(\zeta_1) \) is Heaviside's unit function. Of course, a discontinuity in \( F_t \) and \( F_t \) will be smoothed out to thickness \( E_m^{1/2} \) in a cross-section in the Ekman layer. But this thickness scale is much smaller than the one of the inner Stewartson layer. Therefore, the inner layer still "sees" a jump. At the outer edge of the inner Stewartson layer the variables must match to those of the outer Stewartson layer. Due to the
addition of the outer Stewartson layer contributions in the expansions
(3.2.25) - (3.2.26), however, matching is provided if

\[ \bar{\chi}_1 = \bar{\psi}_1 = 0 \quad \text{as} \quad \zeta_1 \to \infty \]  

(3.2.31)

The boundary conditions at the cylinder wall are given by (3.1.45). From
(3.2.25) - (3.2.26) and (3.2.22) - (3.2.23) one finds that

\[ \bar{\chi}_1 = -b^{-1} \left[ \frac{1}{2} \theta_{\psi} + a + \frac{1}{2} (F_b(0^+) + F_t(0^+)) \right] \quad \text{at} \quad \zeta_1 = 0 \]  

(3.2.32)

\[ \bar{\psi}_1 = -b^{-1} \varepsilon_*^{1/3} (1 + 1.5 B r)^{-3/4} \left[-F_b(0^+) + \epsilon \{F_b(0^+) + F_t(0^+)\} + \right. \]

\[ \left. + \{\frac{1}{2} F_b(0^+) + \frac{1}{2} F_t(0^+) + a(1-2\zeta)\} \right] \quad \text{at} \quad \zeta_1 = 0 \]  

(3.2.33)

\[ \frac{\partial \bar{\psi}_1}{\partial \zeta_1} = -b^{-1} \varepsilon_*^{1/2} (1 + 1.5 B r)^{-5/8} (1+2\zeta)/\sqrt{2} \quad \text{at} \quad \zeta_1 = 0 \]  

(3.2.34)

Inspection of the conditions (3.2.29) - (3.2.30) and (3.2.32) - (3.2.34)
shows that the flow induced in the inner Stewartson layer by condition
(3.2.32) is largest with respect to the small parameter \( \varepsilon_* \). Therefore we
put

(i) \( b = 1 \). Letting \( \varepsilon_* \to 0 \) the conditions reduce to

\[ \bar{\psi}_1 = 0 \quad \text{at} \quad \zeta = 0 \quad \text{and} \quad \zeta = 1 \]  

(3.2.35)

\[ (\bar{\psi}_1, (\bar{\psi}_1)_{\zeta} = 0 \quad \text{at} \quad \zeta_1 = 0 \]  

(3.2.36)

\[ \bar{\chi}_1 = -\left\{ \frac{1}{2} \theta_{\psi} + a + \frac{1}{2} F_b(0^+) + \frac{1}{2} F_t(0^+) \right\} \quad \text{at} \quad \zeta_1 = 0 \]  

(3.2.37)

Differentiating (3.2.27) twice with respect to \( \zeta_1 \) and eliminating
\( (\partial^2/\partial \zeta_1^2) \bar{\psi}_1 \) by means of (3.2.28) one gets

\[ 16 \frac{\partial^6 \bar{\psi}_1}{\partial \zeta_1^6} + (1 + 1.5 B r) \frac{\partial^2 \bar{\psi}_1}{\partial \zeta_1^2} = 0 \]  

(3.2.38)

Separation of variables and application of the homogeneous boundary con-
ditions (3.2.35) gives

\[ \bar{\psi}_1 = \sum_{n=1}^{\infty} X_n \sin n \pi z \]  

(3.2.39)
where $X$ satisfies the differential equation

$$
16 \frac{d^6 X_n}{d \xi^6} - (1 + \frac{1}{2} B \nu) n^2 \pi^2 X_n = 0
$$

(3.2.40)

Applying (3.2.31) the solution of (3.2.40) becomes

$$
X_n = A_n e^{-\delta_n \xi_1} + B_n e^{-\delta_n \xi_1} \cos \frac{1}{2} \delta_n \xi_1 + \\
+ C_n e^{-\delta_n \xi_1} \sin \frac{1}{2} \delta_n \xi_1
$$

(3.2.41)

where $\delta_n$ is given by

$$
\delta_n = \left\{ \frac{n \pi}{4} \right\} (1 + \frac{1}{2} B \nu)^{1/2} \frac{1}{3}
$$

(3.2.42)

Applying (3.2.36) it follows that

$$
\tilde{B}_n = -A_n, \quad \tilde{C}_n = \frac{1}{3} / 3 \delta_n
$$

(3.2.43)

From (3.2.27) we deduce

$$
\tilde{\lambda}_1 = -2(1 + \frac{1}{2} B \nu) \sum_{n=1}^{\infty} A_n \delta_n \cos n \pi z \left\{ e^{-\delta_n \xi_1} + \\
+ \frac{1}{3} \delta_n \xi_1 \sin \frac{1}{3} \delta_n \xi_1 + e^{-\delta_n \xi_1} \cos \frac{1}{3} \delta_n \xi_1 \right\}
$$

(3.2.44)

Putting (3.2.44) into the boundary condition (3.2.37) one finds that

$$
\frac{1}{2} \delta w + a + \frac{1}{2} F_D(0^+) + \frac{1}{2} F_T(0^+) = \\
= 4(1 + \frac{1}{2} B \nu) \sum_{n=1}^{\infty} A_n \delta_n \cos n \pi z
$$

(3.2.45)

Along the end caps and along the cylinder wall we took an arbitrary temperature distribution. Up to now we have not defined the uniform constant
temperature $T_0$ at isothermal rigid body rotation. For this purpose we write the temperature along the cylinder wall $T_w$ as

$$T_w = T_0 \{ 1 + \sum_{n=1}^{\infty} \theta_{wn} \cos n\pi z \} \quad (3.2.46)$$

In other words, $T_0$ is the zero order term of the expansion of $T_w$ in $\cos n\pi z$. As a result $\theta_{wn}$ is given by

$$\theta_{wn} = \frac{1}{2} \int_0^1 \theta_w \cos n\pi z \, dz, \quad n = 1, \ldots, \infty \quad (3.2.48)$$

Inspection of the right hand side of (3.2.45) shows that the zero term of the $\cos n\pi z$ expansion of $x_1$, is missing, which is in fact due to the requirement that $x_1$ tends to zero as $t \to \infty$. Therefore the boundary condition (3.2.45) can only be satisfied if

$$a = -\frac{1}{2} \{ F_b(0+) + F_t(0+) \} \quad (3.2.49)$$

As a result, the inner Stewartson layer cannot bring $x_0$ of the inviscid region to zero at the cylinder wall. Therefore the outer Stewartson layer is needed! Substituting (3.2.49) into (3.2.22) and (3.2.23) the outer Stewartson layer solutions become

$$\bar{x}_2 = \frac{1}{2} \{ F_b(0+) + F_t(0+) \} \{ 1 - e^{-(1+\frac{1}{2}Br)^{1/8} \zeta_2^{1/2}} \} \quad (3.2.50)$$

$$\bar{y}_2 = \frac{1}{2} (1+\frac{1}{2}Br)^{-3/4} \left[ \frac{1}{2} \{ F_t(0+) - F_b(0+) \} + \right.$$  
$$\left. - \frac{1}{2} \{ F_b(0+) + F_t(0+) \} (1-2\delta) e^{-(1+\frac{1}{2}Br)^{1/8} \zeta_2^{1/2}} \right] \quad (3.2.51)$$

By means of (3.2.47) - (3.2.49) it follows from boundary condition (3.2.34) that

$$A_n = \frac{1}{4} (1+\frac{1}{2}Br)^{-1} \delta^{-1} \int_0^1 \theta_w \cos n\pi z \, dz \quad (3.2.52)$$
and the inner Stewartson layer solution becomes

\[ \tilde{\chi}_1 = -\frac{1}{4} \sum_{n=1}^{\infty} \left\{ \int_0^1 \cos n\pi z \, dz \right\} \cos n\pi \cdot \left( e^{-\delta_1 \xi_1} + \frac{2}{\sqrt{3}} e^{-\frac{1}{\delta_1} \xi_1 \cos(\frac{1}{3} \delta_1 \xi_1 - \frac{\pi}{6})} \right) \]

\( \Psi_1 = \frac{1}{4} (1 + \frac{1}{4} Br)^{-1} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_0^1 \cos n\pi z \, dz \right\} \sin n\pi \cdot \left( e^{-\delta_1 \xi_1} - \frac{2}{\sqrt{3}} e^{-\frac{1}{\delta_1} \xi_1 \cos(\frac{1}{3} \delta_1 \xi_1 + \frac{\pi}{6})} \right) \) \hspace{1cm} (3.2.53)

The above solution is obtained by putting \( b = 1 \) in conditions (3.2.32) - (3.2.34) and implies that the inner Stewartson layer requires an \( O(1) \) term in \( \chi \) to balance the temperature perturbation at the wall. The induced flux is \( O(L^1/3) \) and represents a closed circulation within the layer. In figure 7 we have plotted \( (1 + \frac{1}{4} Br)^{5/6} \frac{\partial \Psi_1}{\partial \zeta_1} \) versus \( (1 + \frac{1}{4} Br)^{1/6} \zeta_1 \).

![Figure 7: \( (1 + \frac{1}{4} Br)^{5/6} \frac{\partial \Psi_1}{\partial \zeta_1} \) versus \( (1 + \frac{1}{4} Br)^{1/6} \zeta_1 \) at \( z=0.05, z=0.25, z=0.50 \) for \( \xi_{\infty} = -z\).](image_url)
at $z = 0.05$, $z = 0.25$, $z = 0.50$ for a linear temperature gradient along the wall, $\theta = -z + \frac{1}{2}$.

The requirement, however, that the axial flow in the inviscid region and in the outer Stewartson layer returns in the inner Stewartson layer, expressed by (3.2.33), is not get satisfied. Taking $\theta = 0$ and (ii) $b = c_1^{1/3}$, the leading order with respect to $c_1$ of (3.2.29) - (3.2.34) becomes

\[
\varphi_1 = - (1 + \frac{1}{2} \beta r)^{-3/4} F_b(0^+) [H(\eta_1) - 1] \quad \text{at } z = 0 \quad (3.2.55)
\]
\[
\varphi_1 = + (1 + \frac{1}{2} \beta r)^{-3/4} F_t(0^+) [H(\eta_1) - 1] \quad \text{at } z = 1 \quad (3.2.56)
\]
\[
\varphi_1 = - \frac{1}{4} (1 + \frac{1}{2} \beta r)^{-3/4} [- F_b(0^+) + z \{F_b(0^+) + F_t(0^+)] \{H(z)H(1-z) - \frac{1}{2}\} \quad \text{at } \zeta_1 = 0 \quad (3.2.57)
\]
\[
\tilde{x}_1 = (\partial \partial z \eta_1) \varphi_1 = 0 \quad \text{at } \zeta_1 = 0 \quad (3.2.58)
\]

One sees that the right hand sides of (3.2.55) and (3.2.56) are zero when $\zeta_1 > 0$. We therefore replace (3.2.55) - (3.2.57) by

\[
\varphi_1 = 0 \quad \text{at } z = 0 \text{ and } z = 1 \quad (3.2.59)
\]
\[
\varphi_1 = - (1 + \frac{1}{2} \beta r)^{-3/4} [- F_b(0^+) + z \{F_b(0^+) + F_t(0^+)] \{H(z)H(1-z) - \frac{1}{2}\} \quad \text{at } \zeta_1 = 0 \quad (3.2.60)
\]

and the discontinuities at the corners of the cylinder are now mathematically discounted in the axial boundary instead of the radial one. From the equations (3.2.27) - (3.2.28) and the boundary conditions (3.2.58) - (3.2.59) it follows that

\[
\tilde{v}_1 = \sum_{n=1} D_n \sin n\pi z \ e^{-\frac{1}{4} \beta_n \zeta_1} \sin \left(\frac{1}{4} 3\beta_n \zeta_1 + \frac{\pi}{3}\right) \quad (3.2.61)
\]

where the unknown $D_n$ follows from (3.2.60). Therefore we expand the right hand side of (3.2.60) in the eigenfunctions $\sin n\pi z$, $n \geq 1$, according to

\[
2[F_b(0^+) - z\{F_b(0^+) + F_t(0^+)] \{H(z)H(1-z) - \frac{1}{2}\} = \sum_{n=1} \alpha_n \sin n\pi z \quad (3.2.62)
\]
where

\[ \alpha_n = \frac{1}{n\pi} \left\{ 1 - (-)^n \left\{ F_b(0+) - F_t(0+) \right\} + \frac{1}{n\pi} \left\{ 1 + (-)^n \left\{ F_b(0+) + F_t(0+) \right\} \right\} \right\} \quad (3.2.63) \]

The expansion in \( \sin n\pi z \) requires that the left hand side of (3.2.62) satisfies the same boundary conditions at \( z = 0 \) and \( z = 1 \) as \( \sin n\pi z \).

This requirement is provided by both Heaviside's functions. Therefore, the incorporation of the discontinuities in the Ekman suction conditions was necessary. In contrast to the inviscid region and the outer Stewartson layer, the inner layer indeed sees the jumps at the corners, a result also identified by Moore & Saffman (1969). Substituting (3.2.61) into (3.2.60) and applying (3.2.62) it follows that

\[ D_n = \frac{1}{3} \sqrt{3(1 + \frac{1}{4} Br)}^{-3/4} \alpha_n \quad (3.2.64) \]

After some algebraic manipulation the inner Stewartson layer solutions become

\[ \Psi_1 (1 + \frac{1}{4} Br)^{3/4} = \]

\[ = \left\{ F_b(0+) - F_t(0+) \right\} \sum_{n=1}^{\infty} \frac{2(-)^n \cos \pi (2n+1) \left( \frac{z}{2} - \frac{1}{2} \right)}{\pi (2n+1) \sqrt{3}} e^{-\frac{1}{2} \gamma_n \sin \left( \frac{1}{3} \gamma_n + \frac{\pi}{3} \right)} + \]

\[ + \left\{ F_b(0+) + F_t(0+) \right\} \sum_{n=1}^{\infty} \frac{(-)^n \sin \pi (2n+1) \left( \frac{z}{2} - \frac{1}{2} \right)}{n \sqrt{3}} e^{-\frac{1}{2} \beta_n \sin \left( \frac{1}{3} \beta_n + \frac{\pi}{3} \right)} \quad (3.2.65) \]

and

\[ \hat{x}_1 (1 + \frac{1}{4} Br)^{1/12} = \]

\[ = \left\{ F_b(0+) - F_t(0+) \right\} \sum_{n=0}^{\infty} \frac{2^{1/3} (-)^n \sin \pi (2n+1) \left( \frac{z}{2} - \frac{1}{2} \right)}{3^{1/2} (2n+1)^{2/3} \pi^{2/3}} e^{-\frac{1}{2} \gamma_n \sin \left( \frac{1}{3} \gamma_n + \frac{\pi}{3} \right)} + \]

\[ + \left\{ F_b(0+) + F_t(0+) \right\} \sum_{n=0}^{\infty} \frac{2^{1/3} (-)^n \cos \pi (2n+1) \left( \frac{z}{2} - \frac{1}{2} \right)}{3^{1/2} (2n\pi)^{2/3}} e^{-\frac{1}{2} \beta_n \sin \left( \frac{1}{3} \beta_n + \frac{\pi}{3} \right)} \quad (3.2.66) \]
where $\gamma$ and $\beta$ are given by

$$
\gamma_n = \left( \frac{(2n+1)\pi}{4} (1+\frac{1}{4}Br) \right)^{1/2} \frac{1}{\zeta_1}
$$

(3.2.67)

$$
\beta_n = \left( \frac{n\pi}{2} (1+\frac{1}{4}Br) \right)^{1/2} \frac{1}{\zeta_1}
$$

(3.2.68)

The above solution for the inner Stewartson layer represents the rechanneling of the axial flow moving from the bottom to the top Ekman layer. In figure 8 we have plotted $(1+\frac{1}{4}Br)^{7/12} (\delta/\delta \zeta_1) \psi_1$ versus $(1+\frac{1}{4}Br)^{1/6} \zeta_1$ at $z = 0.05$, $z = 0.25$, $z = 0.50$ for $F_b(0^+) = -F_c(0^+) = 1$ and $\psi_w = 0$. Letting $z \to 0$ one sees that the axial flow profile tends to a Dirac function and thus joins the discontinuity at the corner of the cylinder.

Figure 8: $(1+\frac{1}{4}Br)^{7/12} (\delta/\delta \zeta_1) \psi_1$ versus $(1+\frac{1}{4}Br)^{1/6} \zeta_1$ at $z = 0.05$, $z = 0.25$, $z = 0.50$ for $F_b(0^+) = -F_c(0^+) = 1$ and $\psi_w = 0$. 

Summarising, in the outer Stewartson layer $\psi_0$ is brought to zero at the cylinder wall. The inner Stewartson layer serves a dual purpose. First, it requires an $O(1)$ term in $\chi$ to balance the temperature perturbation at the wall. The induced flux is $O(L_m^{1/3} F_m^{1/3})$ and represents a closed circulation within the layer. Secondly, it requires an $O(E_m^{1/2})$ term in $\psi$ in order to return the axial flow moving from the bottom to the top Ekman layer. Whereas the imposed conditions at the end caps induce a secondary flow in the entire cylinder, the temperature perturbation along the cylinder wall only produces an internal circulation within the inner Stewartson layer. A similar situation was observed by Homsy & Hudson (1969), Matsuda (1975), Nakayama & Usui (1974) and Sakurai & Matsuda (1974).

The axial flow behaviour in the inner Stewartson layer is illustrated in figures 7 and 8. In figure 7 we have plotted $(1 + \frac{1}{2} Br) (\partial / \partial x) \psi_1$ versus $(1 + \frac{1}{2} Br)^{1/6} \xi_1$ at $z = 0.05$, $z = 0.25$, $z = 0.50$ for $\theta_w = -z + \frac{1}{2}$ and $F_B(0+) = F_t(0+) = 0$. We clearly observe the closed circulation in the layer. In figure 8 we have plotted $(1 + \frac{1}{2} Br)^{7/12} (\partial / \partial z) \psi_1$ versus $(1 + \frac{1}{2} Br)^{1/6} \xi_1$ at $z = 0.05$, $z = 0.25$, $z = 0.50$ for $F_B(0+) = -F_t(0+) = 1$ and $\theta_w = 0$.

One sees that the inner layer rechannels the vertical flow. Letting $z \to 0$ the axial flow profile tends to a Dirac function and thus joins the discontinuity at the corner of the cylinder.

In the discussion of the suction conditions applied to the outer Stewartson layer we have already noticed that the radial shape of $F_B$ and $F_t$ is subjected to certain restrictions. This is best illustrated by putting the inviscid solutions (3.2.6) and (3.2.7) into the original equations of motion (3.2.3) and (3.2.4). In this case, we observe that the terms on the left hand side can only be neglected if the imposed functions $F_B$ and $F_t$ satisfy the requirement

$$F^{-1} (\partial^2 / \partial x^2) F \ll (L_m^{1/2} e^{1/2x})^{-1}$$  \hspace{1cm} (3.2.69)

Condition (3.2.69) implies that for $x \ll \ln(L_m^2 e_m^{1/4})^{-1}$ the x-variations of $F$ must take place on a distance scale that is large compared to $L_m^{1/2} e_m^{1/4}$, the distance scale of the outer Stewartson layer. In the case of a discontinuity in $F$ at $x \ll \ln(L_m^2 e_m^{1/4})^{-1}$ free vertical Stewartson layers will be formed. In these layers the discontinuity is smoothed out by viscous forces. Such a situation is observed in the split-disk configuration e.g. discussed by Baker (1967), Moore & Saffman (1969) and Stewartson (1957). Of course,
Condition (3.2.69) does not apply to the corners of the cylinder. The jumps at \( x = 0 \) are, in fact, responsible for the appearance of the side wall layers. As already stated, the inner Stewartson layer "sees" the discontinuity: the outer layer and interior do not.

As already noticed, an inviscid flow is only observed in a limited region near the cylinder wall. Due to the exponential decrease of the density viscous forces become important in the core of the cylinder. Two types of viscous cores can be distinguished: viz. an inner core and an outer core. At first we discuss the inner core. Introducing the contracted coordinate

\[
\eta_1 = x - \ln \left( \frac{\nu_m E_m}{\nu_m} \right)^{-1}
\]  

(3.2.70)

and the perturbation expansion

\[
\Phi = \alpha (\overline{\psi}_1 + \mathcal{A}^{-1} \alpha_1^{-1} \overline{\psi}_1 + \ldots \} \]  

(3.2.71)

\[
\overline{\chi}_1 = \alpha (\overline{\chi}_1 + \mathcal{A}^{-1} \alpha_1^{-1} \overline{\chi}_1 + \ldots \} \]  

(3.2.72)

into (3.1.43) - (3.1.44) and letting \( \mathcal{A}^{-1} \alpha_1^{-1} \to 0 \) the first order representation of the inner core equations becomes

\[
8 \alpha_1^2 \left( \frac{\partial}{\partial \eta_1} \right) e^{\eta_1} \frac{\partial}{\partial \eta_1} \right) e^{\eta_1} \frac{\partial}{\partial \eta_1} \overline{\psi}_1 = \frac{\partial \overline{\chi}_1}{\partial z} \]  

(3.2.73)

\[
2 \alpha_1 \eta_1 \frac{\partial^2 \overline{\chi}_1}{\partial \eta_1^2} = - \left( 1 + \frac{1}{\alpha_1} \right) \frac{\partial \overline{\psi}_1}{\partial z} \]  

(3.2.74)

where

\[
\alpha_1 = 1 - \mathcal{A}^{-1} \ln \left( \frac{\nu_m}{E_m} \right)^{-1}
\]  

(3.2.75)

The condition that \( \mathcal{A}^{-1} \alpha_1^{-1} \ll 1 \) implies that the distance scale of inner core measured from the rotation axis must be large compared to \( \mathcal{A}^{-1/2} \); i.e. \( r^2 >> \mathcal{A}^{-1} \).
The unknown constant \( a \) in expansions \((3.2.71)\) and \((3.2.72)\) determines the absolute magnitude of \( \tilde{\chi} \) and \( \tilde{\psi} \). Its value depends upon the function of the inner core and will be discussed in the subsequent treatment.

For \( \eta_1 \to \infty \) \( \tilde{\chi}_1 \) and \( \tilde{\psi}_1 \) must tend to their pure viscous values, given by

\[
\left( \frac{\partial}{\partial \eta_1} \right) \tilde{\chi}_1 = \left( \frac{\partial}{\partial \eta_1} \right) \tilde{\psi}_1 = 0, \quad \text{as } \eta_1 \to \infty
\]

The boundary conditions at the end caps are obtained from the Ekman suction conditions. Putting \((3.2.70) - (3.2.72)\) into \((3.1.41) - (3.1.42)\) and dropping terms \( \sim L_m^{-1/2} \) and \( \sim A^{-1} \alpha^{-1} \) compared to unit magnitude we obtain

\[
\tilde{\psi}_1 = -\frac{1}{2}(1+4\alpha) -3/4 \alpha^{-1} L_m^{-1/2} e^{1/4 \eta_1} F_b, \quad \text{at } z = 0
\]

\[
\tilde{\psi}_1 = +\frac{1}{2}(1+4\alpha) -3/4 \alpha^{-1} L_m^{-1/2} e^{1/4 \eta_1} F_t, \quad \text{at } z = 1
\]

Provided that

\[
a \gg I_m^{-1/2} e^{1/4 \eta_1} F_b, \quad I_m^{-1/2} e^{1/4 \eta_1} F_t
\]

boundary conditions \((3.2.77)\) and \((3.2.78)\) reduce to

\[
\tilde{\psi}_1 = 0, \quad \text{at } z = 0 \text{ and } z = 1
\]

Now integrate \((3.2.74)\) with respect to \( z \) from 0 to 1 and apply \((3.2.80)\)

\[
\frac{z^2}{\eta_1^2} \int_0^1 \tilde{\chi}_1 \, dz = 0
\]

Integrating \((3.2.81)\) with respect to \( \eta_1 \) it follows that

\[
\left( \frac{\partial}{\partial \eta_1} \right) \int_0^1 \tilde{\chi}_1 \, ds
\]

is constant with respect to \( \eta_1 \) and, consequently, cannot tend to two different finite values as \( \eta_1 \to \pm \infty \). As a result, the inner core cannot adjust \( (\partial/\partial x) \chi \) which is an order of magnitude larger than \( E_m^{1/2} e^{1/2 x} F_b \)

\( E_m^{1/2} e^{1/2 x} F_t \). Since \( \chi \sim F_b, F_t \) in the inviscid region a second core, the outer core, is needed to adjust \( (\partial/\partial x) \chi \) to its pure viscous value.
Introducing the contracted coordinate

\[ \eta_2 = \chi - \ln(L_m^2 k_m^{-1})^{-1} \]  

and the scaled dependent variables

\[ \chi = \chi_2, \quad \psi = i_m^{-1} \psi_2 \]  

into (3.1.43) - (3.1.44) and letting \( L_m^{-2} \to 0 \), the outer core equations become

\[ \frac{\partial \chi_2}{\partial z} = 0 \]  

\[ 2e^{\eta_2} \frac{\partial}{\partial \eta_2} \left( 1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A) \right) \frac{\partial}{\partial \eta_2} \left( 1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A) \right)^{-1} (\alpha_2 - \eta_2/A) \chi_2 = \]  

\[ = - \left( 1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A) \right) \frac{\partial \psi_2}{\partial z} \]  

where

\[ \alpha_2 = 1 - A^{-1} \ln(L_m^2 k_m^{-1})^{-1} \]  

Differentiating (3.2.86) with respect to \( z \) and applying (3.2.85) one gets

\[ \frac{\partial^2 \psi_2}{\partial z^2} = 0 \]  

The boundary conditions at \( z = 0 \) and \( z = 1 \) are obtained from the Ekman suction conditions. Putting (3.2.83) - (3.2.84) into (3.1.41) - (3.1.42) one finds that

\[ \{1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A)\}^{3/4} \psi_2 - \frac{1}{4} e^{\frac{1}{2} \eta_2} \chi_2 = - \frac{1}{4} e^{\frac{1}{2} \eta_2} F_B \text{ at } z = 0 \]  

\[ \{1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A)\}^{3/4} \psi_2 + \frac{1}{4} e^{\frac{1}{2} \eta_2} \chi_2 = + \frac{1}{4} e^{\frac{1}{2} \eta_2} F_t \text{ at } z = 1 \]  

Having in mind that \( \chi_2 \) is constant with respect to \( z \) it follows from (3.2.88) - (3.2.90) that

\[ \psi_2 = \frac{1}{4} \{1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A)\}^{-3/4} e^{\frac{1}{2} \eta_2} (\chi_2 - F_B + z F_B + F_t - 2 \dot{\chi}_2) \]
Substituting this result into (3.2.86) we get

\[ 2\pi^{-1/4} e^{1/2} \frac{d}{d\eta_2} \Pi_2 \frac{d}{d\eta_2} \left\{ \Pi_2^{-1}(\alpha_2 - \eta_2/A)\chi_2 \right\} - \chi_2 = -\frac{1}{4}(F_b + F_t) , \]

\[ \Pi_2 = 1 + \frac{1}{4} Br(\alpha_2 - \eta_2/A) . \]  

(3.2.92)

For \( \alpha_2 \gg A^{-1} \), which implies that the distance-scale of the outer core, measured from the rotation axis, must be large compared to \( A^{-1/2} \), \( \eta_2 \gg A^{-1} \), equation (3.2.92) can be simplified to

\[ 2\alpha_2 (1 + \frac{1}{4} Br\alpha_2)^{-1/4} e^{1/2} \frac{d^2 \chi_2}{d\eta_2^2} - \chi_2 = -\frac{1}{4}(F_b + F_t) \]  

(3.2.93)

For \( \eta_2 \to -\infty \), \( \chi_2 \) must match the inviscid solution: i.e.

\[ \eta_2 \to -\infty \quad \lim_{\eta_2} \chi_2 = \frac{1}{4}(F_b + F_t) \]  

(3.2.94)

For \( \eta_2 \to \infty \), \( (\partial/\partial\eta_2)\chi_2 \) must match the inner core solution. However, as we have previously shown, the inner core is not capable adjusting \( (\partial/\partial x)\chi \) to its pure viscous value at the axis and, consequently, the outer core directly brings \( (\partial/\partial x)\chi \) to zero: i.e.

\[ \eta_2 \to \infty \quad \lim_{\eta_2} \frac{\partial \chi_2}{\partial \eta_2} = 0 \]  

(3.2.95)

Equation (3.2.93) with boundary conditions (3.2.94) and (3.2.95) appears more familiar when we introduce the coordinate

\[ \nu_2 = \sqrt{8} \alpha_2^{-1/2} (1 + \frac{1}{4} Br\alpha_2)^{1/8} e^{1/2} \eta_2 \]  

(3.2.96)

Putting (3.2.96) into (3.2.93) - (3.2.95) one finds

\[ \left\{ \frac{1}{\nu_2} \frac{d}{d\nu_2} \nu_2 \frac{d}{d\nu_2} - 1 \right\} \chi_2 = -\frac{1}{4}(F_b + F_t) \]  

(3.2.97)

\[ \nu_2 \frac{d\chi_2}{d\nu_2} = 0 \quad \text{at} \quad \nu_2 = 0 \]  

(3.2.98)

\[ \chi_2 \sim \frac{1}{4}(F_b + F_t) \quad \text{as} \quad \nu_2 \to 0 \]  

(3.2.99)
This problem can be solved by means of the Hankel transform of zero order: see Erdélyi et al. (1954) and Sneddon (1972)

\[ \tilde{x}_{2k} = \int_{0}^{\infty} \nu_2 \tilde{x}_2 J_0(k \nu_2) \, dk \]  
\[ \tilde{x}_2 = \int_{0}^{\infty} k \tilde{x}_{2k} J_0(k \nu_2) \, dk \]  
\[ F_k = \frac{1}{2} \int_{0}^{\infty} \nu_2 (F_t + F_b) J_0(k \nu_2) \, dk \]  
\[ \frac{1}{2}(F_t + F_b) = \int_{0}^{\infty} k F_k J_0(k \nu_2) \, dk \]

where \( J_0 \) is the zero order Bessel function. The solution of (3.2.97) - (3.2.99) is given by

\[ \tilde{x}_2 = \int_{0}^{\infty} \frac{k F_k}{k^2 + 1} J_0(k \nu_2) \, dk \]  

provided that

\[ \nu_2^{1/2} (d/\nu_2) \tilde{x}_2 = \nu_2^{-1/2} \tilde{x}_2 = 0 \quad \text{as} \quad \nu_2 \to \infty \]  
\[ \nu_2 \tilde{x}_2 = 0 \quad \text{as} \quad \nu_2 \to 0 \]

Consider the case: \( \frac{1}{2}(F_t + F_b) = \nu_2^{-1} \). Then \( F_k = \nu_2^{-1} \) and the solution for \( \tilde{x}_2 \) becomes

\[ \tilde{x}_2 = \frac{1}{2\pi} \sum_{m=0}^{\infty} \left[ \frac{1/2 \nu_2}{(m!)^2} - \frac{(1/2 \nu_2)^{2m+1}}{\Gamma((m+3)/2)} \right] \]

where \( \Gamma \) is the Gamma function

\[ \Gamma(m + \frac{3}{2}) = \frac{1}{2\pi} \frac{1 \cdot 3 \cdot 5 \cdots (2m+1)}{4^{m+1}} \]

In figure 9 we have plotted \( \tilde{x}_2 \) and \( \nu_2 \tilde{x}_2 \) versus \( \nu_2 \) for \( \frac{1}{2}(F_t + F_b) = \nu_2^{-1} \). Note that for \( \nu_2 \to \infty \) \( \nu_2 \tilde{x}_2 \) tends to 1 which is exactly the inviscid solution.

The outer core adjusts \( \chi_0 \) to its pure viscous value in the rotation axis. Until now, however, \((\partial/\partial \alpha)\psi\) is not brought to zero. Therefore the

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inner core is needed. The scaling rules for this region are obtained by putting \( a = \frac{r}{r_m}^{1/2} \) in expansions (3.2.71) - (3.2.72). The equations are given by (3.2.73) - (3.2.74) and the boundary conditions at \( z = 0 \) and \( z = 1 \), given in (3.2.77) - (3.2.78), now become

\[
\bar{\chi}_2 = - \frac{1}{3} (1 + \frac{1}{2} Br \alpha_1)^{-3/4} e^{\frac{1}{2} \eta_1 F_b} \quad \text{at } z = 0 \tag{3.2.109}
\]

\[
\bar{\psi}_2 = + \frac{1}{3} (1 + \frac{1}{2} Br \alpha_1)^{-3/4} e^{\frac{1}{2} \eta_1 F_t} \quad \text{at } z = 1 \tag{3.2.110}
\]

For \( \eta_1 \to -\infty \) the dependent variables must match those of the outer core. Let us therefore investigate the behaviour of the outer core solutions for \( \eta_2 \) expanded into \( \eta_1 \):

\[
\eta_2 = \eta_1 + \ln \frac{r}{r_m} \tag{3.2.111}
\]

Furthermore it is appropriate to rescale the dependent variables of the
outer core such that their magnitudes correspond to those of the inner core: i.e.

$$\bar{\chi}_2 = L_m^{-1/2} \bar{\chi}_{12}, \quad \bar{\psi}_2 = L_m^{1/2} \bar{\psi}_{12}$$  \hspace{1cm} (3.2.112)

Putting (3.2.111) - (3.2.112) into (3.2.91) - (3.2.92) and dropping terms \( \propto a_1^{-1} \) compared to unit magnitude, we obtain

$$\bar{\psi}_{12} = \frac{1}{2} (1 + \frac{1}{2} Br a_1)^{-3/4} e^{3/4 n_1 \xi} \left( L_m^{-1/2} \bar{\chi}_{12} - F_B^t + 2 L_m^{-1/2} \bar{\psi}_{12} \right)$$  \hspace{1cm} (3.2.115)

$$2 a_1 (1 + \frac{1}{2} Br a_1)^{-1/4} e^{3/4 n_1} \frac{d^2 \bar{\psi}_{12}}{d n_1^2} - L_m^{-1/2} \bar{\chi}_{12} = - \frac{1}{2} (F_B^t + F_B^f)$$  \hspace{1cm} (3.2.116)

Since \( L_m \) is large in the unit cylinder, (3.2.115) - (3.2.116) can be simplified to

$$\bar{\psi}_{12} = \frac{1}{2} (1 + \frac{1}{2} Br a_1)^{-3/4} e^{3/4 n_1 \xi} \left( - F_B^t + z (F_B^f - F_B^t) \right)$$  \hspace{1cm} (3.2.117)

$$\frac{d^2 \bar{\psi}_{12}}{d n_1^2} = - \frac{(1 + \frac{1}{2} Br a_1)^{1/4}}{4 a_1} e^{-3 n_1} (F_B^f + F_B^t)$$  \hspace{1cm} (3.2.118)

As a consequence, the inner core not only must adjust \( \bar{\psi} \) but also \( \left( \frac{\partial^2}{\partial x^2} \right) \bar{\chi} \). Although not shown explicitly in the present work, the same applies to the inner Stewartson layer. Here, \( \bar{\psi} \) and \( \left( \frac{\partial^2}{\partial x^2} \right) \bar{\chi} \) are adjusted: see Stewartson (1966). Integrating (3.2.118) with respect to \( n_1 \) one finds that

$$\frac{d \bar{\psi}_{12}}{d n_1} = - \frac{(1 + \frac{1}{2} Br a_1)^{1/4}}{4 a_1} \int_{+\infty}^{n_1} e^{-3 n_1} (F_B^f + F_B^t) \, d n_1 + \frac{d \bar{\psi}_{12}}{d n_1} (n_1 \to \infty)$$  \hspace{1cm} (3.2.119)

The outer core brings \( \left( \frac{\partial}{\partial x} \right) \bar{\chi} \) to zero in the rotation axis. Therefore the integration constant, \( \frac{d}{d n_1} \bar{\psi}_{12} (n_1 \to \infty) \), must be set equal to zero and the match conditions for the inner core become

$$n_1 \to +\infty \quad \bar{\psi}_1 = \frac{1}{2} (1 + \frac{1}{2} Br a_1)^{-3/4} e^{3/4 n_1 \xi} \left( - F_B^t + z (F_B^f - F_B^t) \right)$$  \hspace{1cm} (3.2.120)

$$n_1 \to +\infty \quad \frac{d \bar{\psi}_1}{d n_1} = - \frac{(1 + \frac{1}{2} Br a_1)^{1/4}}{4 a_1} \int_{+\infty}^{n_1} e^{-3 n_1} (F_B^f + F_B^t) \, d n_1$$  \hspace{1cm} (3.2.121)

The inner core problem is now specified by differential equations

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(3.2.73) – (3.2.74) and boundary conditions (3.2.76), (3.2.109), (3.2.110), (3.2.120) and (3.2.121).

Introducing

$$\eta_1 = \frac{2n}{4\alpha_1^3/2} - Z_n \nu_1$$

and

$$\psi_1 = \frac{1}{4\alpha_1^{3/4} (1+\frac{1}{2} \partial_\alpha_1)} 1/2 \quad \bar{\psi}_1, \quad \bar{x}_1 = \frac{1}{2\alpha_1^{1/4}} \bar{x}_1$$

we obtain

$$\frac{\partial^2 \psi_1}{\partial \nu_1^2} = \frac{\partial x_1}{\partial z}$$

$$\left\{ \frac{\partial}{\partial \nu_1} \nu_1 \frac{\partial}{\partial \nu_1} \psi_1 \right\} = \frac{\partial \psi_1}{\partial z}$$

$$\bar{\psi}_1 = -\nu_1^{-1/2} \bar{F}_b$$ at $z = 0$

$$\bar{\psi}_1 = +\nu_1^{-1/2} \bar{F}_t$$ at $z = 1$

$$\nu_1 \left( \frac{\partial}{\partial \nu_1} \bar{x}_1 \right) = \nu_1 \left( \frac{\partial}{\partial \nu_1} \bar{\psi}_1 \right) = 0$$ at $\nu_1 = 0$

$$\bar{\psi}_1 = \nu_1^{-1/2} \left\{ -\bar{F}_b + \bar{z} (\bar{F}_b + \bar{F}_t) \right\}$$ as $\nu_1 \to \infty$

$$\nu_1 \frac{\partial \bar{x}_1}{\partial \nu_1} = \int_0^{1/\nu_1} \frac{\partial \bar{x}_1}{\partial \nu_1} d\nu_1$$ as $\nu_1 \to \infty$

More analytical progress can be made by the substitution

$$\bar{\psi}_1 = \bar{\psi}_1' + \nu_1^{-1/2} \left\{ -\bar{F}_b + \bar{z} (\bar{F}_b + \bar{F}_t) \right\}$$

$$\nu_1 \frac{\partial \bar{x}_1}{\partial \nu_1} = \nu_1 \frac{\partial \bar{x}_1}{\partial \nu_1} - \int_0^{1/\nu_1} \frac{\partial \bar{x}_1}{\partial \nu_1} \frac{1}{\nu_1^{1/2}} d\nu_1$$
It is quite evident that there should hold

\[
\nu_1 \frac{\partial}{\partial \nu_1} \left( \nu_1 \frac{d^2}{d\nu_1^2} \right)_{\nu_1^2} \left[ \psi_1'' + \nu_1^{-1/2} \left( -F_b + \chi (F_b + F_t) \right) \right] = \delta \frac{\partial}{\partial x} \nu_1 \frac{\partial \chi_1'}{\partial \nu_1},
\]

(3.2.133)

\[
\frac{\partial}{\partial \nu_1} \nu_1 \frac{\partial}{\partial \nu_1} \chi_1'' = -\frac{\partial \psi_1''}{\partial x},
\]

(3.2.134)

\[
\overline{\psi}_1'' = 0 \quad \text{at } x = 0 \text{ and } x = 1 \quad (3.2.135)
\]

\[
\nu_1 \left( \frac{\partial}{\partial \nu_1} \right) \overline{\psi}_1'' = 0 \quad \text{at } \nu_1 = 0 \quad (3.2.136)
\]

\[
\nu_1 \left( \frac{\partial}{\partial \nu_1} \right) \overline{\psi}_1'' = \nu_1 \left( \frac{\partial}{\partial \nu_1} \right) \nu_1^{-1/2} \left( F_b - \chi (F_b + F_t) \right) \quad \text{at } \nu_1 = 0 \quad (3.2.137)
\]

\[
\nu_1 \left( \frac{\partial}{\partial \nu_1} \right) \chi_1'' = \overline{\psi}_1'' = 0 \quad \text{as } \nu_1 \to \infty \quad (3.2.138)
\]

If it is assumed that \( F_b \) and \( F_t \) satisfy the ordinary differential equation

\[
\nu_1 \frac{d}{d\nu_1} \left( \nu_1 \frac{d^2}{d\nu_1^2} \right)_{\nu_1^2} \rho = 0
\]

(3.2.139)

one deduces from (3.2.133) and (3.2.134) that

\[
\frac{\partial}{\partial \nu_1} \nu_1 \frac{\partial}{\partial \nu_1} \left( \nu_1 \frac{d^2}{d\nu_1^2} \right)_{\nu_1^2} \psi_1'' + \frac{\partial^2 \psi_1''}{\partial x^2} = 0
\]

(3.2.140)

the solution of which is given by

\[
\overline{\psi}_1'' = \sum_{n=1}^{\infty} f \sin \pi n x
\]

(3.2.141)

where \( f \) satisfies the differential equation

\[
\frac{d}{d\nu_1} \nu_1 \frac{d}{d\nu_1} \left( \nu_1 \frac{d^2}{d\nu_1^2} \right)_{\nu_1^2} f - n^2 \pi^2 f = 0
\]

(3.2.142)
and boundary conditions

\[ v_1 \left( \frac{d}{dv_1} \right) f^* = \frac{1}{n + 1} \left( 1 + (-)^n \right) v_1 \left( \frac{d}{dv_1} \right) v_1^{-1/2} (F_b - F') + \]
\[ + \frac{1}{n + 1} \left( 1 + (-)^n \right) v_1 \left( \frac{d}{dv_1} \right) v_1^{-1/2} (F_b + F') \quad \text{at } v_1 = 0 \quad (3.2.143) \]

\[ v_1 \frac{d}{dv_1} \left( v_1 \frac{d}{dv_1} \right)^2 v_1 f^* = 0 \quad \text{at } v_1 = 0 \text{ and as } v_1 \to \infty \quad (3.2.144) \]

\[ f^* = 0 \quad \text{as } v_1 \to \infty \quad (3.2.145) \]

The six basic solutions of (3.2.142) are given by

\[ f = \beta_{10} + \beta_{11} v_1^2 + \beta_{12} v_1^4 + \ldots. \quad (3.2.146) \]

\[ = \beta_0 v_1 + \beta_{21} v_1^3 + \beta_{22} v_1^5 + \ldots. \quad (3.2.147) \]

\[ = \beta_{30} v_1^2 l_n v_1 + \beta_{31} v_1^3 l_n v_1 + \beta_{32} v_1^5 l_n v_1 + \ldots. \quad (3.2.148) \]

\[ = \beta_{40} l_n v_1 + \beta_{41} v_1^2 l_n v_1 + \beta_{42} v_1^4 l_n v_1 + \ldots. \quad (3.2.149) \]

\[ = \beta_{50} v_1^2 l_n^2 v_1 + \beta_{51} v_1^3 l_n^2 v_1 + \beta_{52} v_1^5 l_n^2 v_1 \quad (3.2.150) \]

\[ = \frac{1}{v_1} \left( \beta_{60} + \beta_{61} v_1^2 l_n^3 v_1 + \beta_{62} v_1^4 l_n^3 v_1 + \ldots. \right) \quad (3.2.151) \]

From the boundary condition

\[ v_1 \frac{d}{dv_1} \left( v_1 \frac{d}{dv_1} \right)^2 v_1 f^* = 0 \quad \text{at } v_1 = 0 \quad (3.2.152) \]

it follows that \( \beta_{50} \) and \( \beta_{60} \) must be set equal to zero so that the solutions (3.2.150) and (3.2.151) are dropped. In order to satisfy the boundary conditions as \( v_1 \to \infty \) it is necessary to know the rate of convergence of the series. To find asymptotic expansions seems to be a quite formidable task.
The inner core problem is, in general, too difficult for a satisfactory
treatment and no further attempts will be made. At last we shall investi-
gate the restrictions subjected to $F_b$ and $F_t$ as $v_1 \rightarrow 0$. From (3.2.137) one finds that

$$\bar{\psi}'_1 \sim v^{1/2} \left\{ F_0 \pm \bar{\psi}^2 \right\} \quad \text{as} \quad v_1 \rightarrow 0$$

(3.2.153)

Putting this into (3.2.134) and integrating with respect to $v_1$ we get

$$v_1 \frac{\partial \bar{\psi}''_1}{\partial v_1} \sim \int_0^1 (F_b + F_t) v^{1/2} \, dv_1 \quad \text{as} \quad v_1 \rightarrow 0$$

(3.2.154)

In order that condition (3.2.136) is satisfied we must require

$$F_b, F_t \sim v_1^a, \quad a > -\frac{1}{2} \quad \text{as} \quad v_1 \rightarrow 0$$

(3.2.155)

or, in terms of the coordinate $x$,

$$F_b, F_t \sim e^{-\alpha_1 x}, \quad \alpha_1 > -\frac{1}{2} \quad \text{as} \quad x \rightarrow \infty$$

(3.2.156)

In fact the same requirement applies to the outer core. For small $v_2$
(3.2.97) reduces to

$$\frac{1}{v_2} \frac{d}{dv_2} v_2 \frac{d}{dv_2} v_2 - \frac{1}{2} (F_t + F_b)$$

(3.2.157)

Integrating this with respect to $v_2 \, dv_2$ and requiring that the condition
(3.2.98) holds, we find that

$$F_b, F_t \sim v_2^a, \quad a > -2 \quad \text{as} \quad v_2 \rightarrow 0$$

(3.2.158)

which is, in terms of $x$, see (3.2.96) and (3.2.83), exactly the same re-
requirement as given by (3.2.156).
3.3. The semi-long cylinder

In the present section we investigate the flow outside the Ekman layers in the semi-long cylinder whose \( L_m \)-range is given by

\[
E_m^{-1/2} \ll L_m \ll E_m^{-1}
\]  

(3.3.1)

Three flow regions can be distinguished. These are: the main section, the inner Stewartson layer and the inner core (see figure Q. Firstly, the flow in the main section is discussed.

The \( x \)-derivatives of \( \chi \) become significant by scaling \( \Psi \) and \( \hat{\chi} \) as

\[
\Psi = (E_m e^{x_1})^{1/2} \{ \psi_1 + \int m^2 \psi_1 \} + \ldots \}
\]  

(3.3.2)

\[
\hat{\chi} = L_m^{-1} E_m^{-1/2} \{ \chi_1 + L_m^2 \chi_1 \} + \ldots \}
\]  

(3.3.3)

Putting (3.3.2) - (3.3.3) into (3.1.43) - (3.1.44) and dropping terms \( \sim L_m^2 \) compared to unit magnitude we obtain

\[
\frac{\partial \chi_1}{\partial \xi} = 0
\]  

(3.3.4)

\[
2e^{\frac{1}{4} \alpha} \frac{\partial}{\partial \alpha} \left[ 1 + 4 Br (1 - \frac{\xi}{A}) \right] \frac{\partial}{\partial \alpha} \left[ 1 + 4 Br (1 - \frac{\xi}{A}) \right]^{-1} (1 - \frac{\xi}{A}) \chi_1 =
\]

\[
= - \left[ 1 + 4 Br (1 - \frac{\xi}{A}) \right] \frac{\partial \psi_1}{\partial \gamma}
\]  

(3.3.5)

Differentiating (3.3.5) with respect to \( \gamma \) and applying (3.3.4) we get

\[
\frac{\partial^2 \psi_1}{\partial \gamma^2} = 0
\]  

(3.3.6)
The boundary conditions at $z = 0$ and $z = 1$ are obtained from the Ekman suction conditions. Putting $(3.3.2) - (3.3.3)$ into $(3.1.41) - (3.1.42)$ it follows that

$$\{1 + \frac{1}{4} Br(1 - \frac{x}{A})\}^{3/4} \psi_1 - \frac{1}{2} \mu_m E^{-1/2} \chi_1 = - \frac{1}{4} F_b \quad \text{at } z = 0$$

$$\{1 + \frac{1}{4} Br(1 - \frac{x}{A})\}^{3/4} \psi_1 + \frac{1}{2} \mu_m E^{-1/2} \chi_1 = + \frac{1}{4} F_t \quad \text{at } z = 1$$

Having in mind that $\chi_1$ is constant along $z$ we deduce from $(3.3.6) - (3.3.8)$ that

$$\psi_1 = (1 + \frac{1}{4} Br(1 - \frac{x}{A}))^{-3/4} [- \frac{1}{2} F_b + \frac{1}{2} E^{-1} E_m^{-1/2} \chi_1 +

+ x (\frac{1}{4} F_b + \frac{1}{4} F_t - \frac{1}{4} E^{-1} E_m^{-1/2} \chi_1)]$$

Putting $(3.3.9)$ into $(3.3.5)$ one finds that

$$2 \pi^{-1/4} e^{-\frac{1}{4} x} \frac{d}{dx} \pi^{-1} \psi_1 \chi_1 = - \frac{1}{2} (F_b + F_t)$$

$$= \pi^{-1} \chi_1$$

$$= 1 + \frac{1}{4} Br(1 - \frac{x}{A})$$

(3.3.10)

For $x \rightarrow 1$ and since $A^{-1} \ll 1$ differential equation $(3.3.10)$ can be simplified to

$$2 (1 + \frac{1}{4} Br) e^{-\frac{1}{4} x} \frac{d^2 \chi_1}{dx^2} - \frac{1}{2} E^{-1} E_m^{-1/2} \chi_1 = - \frac{1}{4} (F_b + F_t)$$

(3.3.11)

For $x \rightarrow 0$ $\chi_1$ must match the inner Stewartson layer. For $x \rightarrow \infty$ $(\partial/\partial x)\chi_1$ must match the inner core. Just as in the unit cylinder, however, the inner Stewartson layer and the inner core are not capable of adjusting $\chi$ respectively $(\partial/\partial x)\chi$ to zero. As a consequence, $\chi_1$ and $(d/dx)\chi_1$ are brought to zero in the main section itself: i.e.

$$\chi_1 = 0 \quad \text{at } x = 0$$

$$\frac{d}{dx} \chi_1 = 0 \quad \text{as } x \rightarrow \infty$$

(3.3.12)

(3.3.13)
The analytical handling becomes easier by putting

\[ v_3 = \frac{\sqrt{8(1+\frac{1}{4}Br)} L^{1/2} n^{1/4}}{L_m^{1/2} E^{1/4}} e^{1/4x} \]  

Differential equation (3.3.11) and boundary conditions (3.3.12) - (3.3.13) now become

\[ \frac{1}{v_3} \frac{d}{dv_3} v_3 \frac{d}{dv_3} \chi_1 - \chi_1 = -F_1 \]  

(3.3.15)

\[ \chi_1 = 0 \quad \text{at} \quad v_3 = a_3 \]  

(3.3.16)

\[ v_3(d/dv_3)\chi_1 = 0 \quad \text{at} \quad v_3 = 0 \]  

(3.3.17)

where

\[ F_1 = \frac{1}{4} L_m E^{1/2} (F_0 + F_t), \quad a_3 = \sqrt{8(1+\frac{1}{4}Br)} L_m^{1/2} E^{1/4} \]  

(3.3.18)

The solutions of (3.3.15) with (3.3.16) - (3.3.17) can be constructed in terms of Green's function: see Courant & Hilbert (1953). Without further proof we put

\[ \chi_1 = - \left( I_0(v_3) \int_0^{v_3} y K_0(y) F_1(y) dy + K_0(v_3) \int_0^{v_3} y I_0(y) F_1(y) dy \right) + \]  

\[ \left( \frac{K_0(a_3)}{I_0(a_3)} \right) \frac{I_0(v_3)}{v_3} \int_0^{v_3} y I_0(y) F_1(y) dy \]  

(3.3.19)

where \( I_0 \) and \( K_0 \) are the modified zero order Bessel functions: see Abramowitz & Segun (1968). From solution (3.3.19) one deduces that

\[ v_3(d/dv_3)\chi_1 \sim \frac{v_3^3}{y F_1} \text{ as } v_3 \to 0 \]  

(3.3.20)

In order that condition (3.3.17) is satisfied we must require for \( F_1 \)

\[ F_1 \sim v_3^\alpha, \quad \alpha > -2 \quad \text{as} \quad v_3 \to 0 \]  

(3.3.21)
or
\[ F_\ell + F_b \propto e^{-a/4x}, \quad a > -2 \quad \text{as} \ x \to \infty \] (3.3.22)

which is exactly the same requirement as identified for the outer and the inner core in the unit cylinder.

One sees from (3.3.9) and (3.3.19) that the velocity distribution in the main section is influenced by the no-slip condition for \( \chi \) at the cylinder wall. Furthermore we notice from (3.3.9) that a linear change of the axial velocity along the \( z \)-coordinate is possible. The Ekman layers no longer dominate, as is typical in an inviscid domain.

In the particular case that the boundary conditions at the end caps are antisymmetric with respect to the mid-plane \( z = 1/2 \), \( F_\ell = -F_b \), it follows from (3.3.19) that \( \chi_1 = 0 \). Putting this result into (3.3.9) one finds that the interior Slow is equal to the inviscid one given by (3.2.7). One would expect this situation, of course, since when \( F_\ell = -F_b \), \( \chi_0 = 0 \) and therefore no outer Stewartson layer and no outer core are needed and expansion of these regions is irrelevant. The flows in the unit and the semi-long cylinder are the same.

Generally, the previously given solutions do not apply near the cylinder wall. Therefore the inner Stewartson layer extensions are needed. The equations and boundary conditions for this layer are obtained by introducing the stretched coordinate

\[ \zeta_1 = (L_m E_m)^{-1/3} x \] (3.3.23)

and the perturbation expansions

\[ \hat{\chi} = L_m^{-1/2} e_m^{1/2} \chi_1 + b(\bar{\chi}_1 + (\bar{E}_m E_m)^{1/3} \bar{\chi}_1) + \ldots \] (3.3.24)

\[ \Psi = E_m^{1/2} \psi_1 + b(L_m E_m)^{1/3} (\bar{\psi}_1 + (L_m E_m)^{1/3} \bar{\psi}_1) + \ldots \] (3.3.25)

where \( b \) is an unknown constant that determines the absolute magnitude of the dependent variables. Its value depends upon the function of the layer and will be discussed in the subsequent treatment. Because of the convenience in algebraic manipulation we have also added the "main section so-
olutions" in expansions (3.3.24) – (3.3.25). Putting (3.3.23) – (3.3.25) into (3.1.43) – (3.1.44) and dropping the terms \( O^{-1/3} E^{-1/6} \), \( O^{-1/3} E^{-1/3} \) compared to unit magnitude, the first order representation of the inner Stewartson layer equations becomes

\[
\frac{\partial^4 \tilde{\Psi}_1}{\partial \zeta^4} = \frac{\partial \tilde{\chi}_1}{\partial z} \tag{3.3.26}
\]

\[
2 \frac{\partial^2 \tilde{\chi}_1}{\partial \zeta^2} = - (1 + \frac{1}{4} B) \frac{\partial \tilde{\Psi}_1}{\partial z} \tag{3.3.27}
\]

Substituting (3.3.23) – (3.3.25) into the suction conditions (3.1.41) – (3.1.42) and dropping terms \( O^{-1/3} E^{-1/6} \) one obtains

\[
\tilde{\Psi}_1 = - b^{-1} (1 + \frac{1}{4} B)^{-3/4} \epsilon_0 F_B(0+) \{ H(\zeta_1) - 1 \} \quad \text{at } z = 0 \tag{3.3.28}
\]

\[
\tilde{\Psi}_1 = + b^{-1} (1 + \frac{1}{4} B)^{-3/4} \epsilon_0 F_t(0+) \{ H(\zeta_1) - 1 \} \quad \text{at } z = 1 \tag{3.3.29}
\]

Where \( \epsilon_0 = L^{-1} \frac{1}{m} \) and is small.

Also here we have introduced Heaviside's unit function to express explicitly the discontinuities of \( F_B \) and \( F_t \) at the corners of the cylinder.

At the outer edge of the layer the variables must match those of the main section. Due to the addition of the main section contributions in expansions (3.3.24) – (3.3.25) matching is provided if

\[
\tilde{\Psi}_1 = \hat{\chi}_1 = 0 \quad \text{as } \zeta_1 \to \infty \tag{3.3.30}
\]

The boundary conditions at the cylinder wall are given by (3.1.45). From (3.3.24) – (3.3.25) and applying the main section solutions one finds that

\[
\tilde{\chi}_1 = - \frac{1}{3} b^{-1} \theta_0 (z) \quad \text{at } \zeta_1 = 0 \tag{3.3.31}
\]

\[
\tilde{\Psi}_1 = - b^{-1} \epsilon_0 \left( 1 + \frac{3}{4} B \right)^{-3/4} \left[ F_B(0+) + \theta_0(0+) + \theta_1(0+) \right] \quad \text{at } \zeta_1 = 0 \tag{3.3.32}
\]

\[
(\partial / \partial \zeta_1) \tilde{\Psi}_1 = - b^{-1} E^{-1/2} \left( \partial / \partial z \right) \tilde{\Psi}_1 \big|_{z=0} = 0 \quad \text{at } \zeta_1 = 0 \tag{3.3.33}
\]

Just as in the unit cylinder, we assume that the temperature distribution along the cylinder wall can be expanded into \( \cos \frac{n\pi z}{\pi} \) where the uniform
constant temperature at isothermal rigid body rotation corresponds to \( n = 0 \). As a result

\[
\theta_{wn} = \sum_{n=1}^{\infty} \theta_{wn} \cos n \pi z
\]  
\[
\theta_{wn} = 2 \int_{0}^{1} \theta_{w} \cos n \pi z \, dz
\]

Comparing the inner Stewartson layer equations and boundary conditions, here, to those identified in the unit cylinder, one sees that both problems are equal. Putting \( b = 1 \) we obtain the solutions (3.2.53) - (3.2.54). Taking \( b = \varepsilon_{\text{m}}^{1/3} \) we get the solutions (3.2.65) - (3.2.66). In other words, the inner Stewartson layer again serves a dual purpose. Firstly, it balances the temperature perturbation at the cylinder wall, inducing a closed circulation within the layer. Secondly, it requires an \( O(\varepsilon_{\text{m}}^{1/2}) \) term in \( \psi \) in order to return the axial flow moving from the bottom to the top Ekman layer.

Also in the semi-long cylinder the radial shape of \( F_b \) and \( F_t \) is subjected to certain restrictions. It is necessary that

\[
F^{-1}(\frac{d^2}{dz^2})F \ll (Lm E_m^a)^{-2/3}
\]

This condition implies that for \( x \ll \ln(Lm E_m)^{-1} \) the \( i \)-variations of \( F \) must take place on a distance scale that is large compared to \( Lm^{1/3} E_m^{1/3} \), the distance scale of the inner Stewartson layer. In the case of a discontinuity in \( F \) free vertical inner Stewartson layers will be formed, smoothing out the discontinuity. Of course, condition (3.3.36) does not apply to the corner of the cylinder. Again, the inner layer "sees" the jump; the main section does not.

Finally we shall discuss the flow in the inner core. Introducing the contracted coordinate

\[
\eta_1 = x - \ln(Lm E_m)^{-1}
\]

and the perturbation expansions

\[
\psi = L_m^{-1/2} (\bar{\psi}_1 + A^{-1} a_1^{-1} \bar{\psi}_1 + \ldots )
\]  
\[
\chi = L_m^{-1/2} (\bar{\chi}_1 + A^{-1} a_1^{-1} \bar{\chi}_1 + \ldots )
\]
into (3.1.43) - (3.1.44) and letting $A^{-1}a_1^{-1} \to 0$ the inner core equations become

\[ 8 \alpha_1^2 \left( \frac{\partial}{\partial \eta_1} e^{\eta_1 \frac{\partial}{\partial \eta_1}} \right)^2 e^{-\eta_1 \psi_1} = \frac{\partial \chi_1}{\partial z} \]  
\[ 2 \alpha_1 e^{\eta_1 \frac{\partial^2 \chi_1}{\partial \eta_1}} = - (1 + B r) \frac{\partial \psi_1}{\partial z} \]  
\[ (3.3.40) \]
\[ (3.3.41) \]

For $\eta_1 \to \infty$ $\chi_1$ and $\psi_1$ must tend to their pure viscous values, given by

\[ (\partial / \partial \eta_1) \chi_1 = (\partial / \partial \eta_1) \psi_1 = 0 \]  
as $\eta_1 \to \infty \]  
\[ (3.3.42) \]

The boundary conditions at $z = 0$ and $z = 1$ are obtained from the Ekman suction conditions, which are, to leading order

\[ \bar{\psi}_1 = - \frac{1}{2} (1 + \frac{1}{4} B r)^{-3/4} e^{\frac{1}{4} \eta_1 E_b} \]  
at $z = 0 \]  
\[ (3.3.43) \]
\[ \bar{\psi}_1 = + \frac{1}{2} (1 + \frac{1}{4} B r)^{-3/4} e^{\frac{1}{4} \eta_1 E_t} \]  
at $z = 1 \]  
\[ (3.3.44) \]

For $\eta_1 \to - \infty$ the dependent variables must match those of the main section. We therefore investigate the behaviour of the main section for $x$ expanded into $\eta_1$: i.e.

\[ x = \eta_1 + \ln \left( \frac{E_m}{E_m} \right)^{-1} \]  
\[ (3.3.45) \]

Rescaling the dependent variables of the main section such that their magnitudes correspond to those of the inner core, i.e.,

\[ \chi_1 = \frac{1}{2} E_m^{1/2} \eta_1^{1/2} \chi_{12}, \quad \psi_1 = e^{-\frac{1}{4} \eta_1 \psi_{12}} \]  
\[ (3.3.46) \]

one deduces from (3.3.9) - (3.3.10) that

\[ \psi_{12} = (1 + \frac{1}{4} \alpha_1 B r)^{-3/4} e^{\frac{1}{4} \eta_1} \left[ - \frac{1}{2} E_b + \frac{1}{2} E_m^{-1/2} \chi_{12} + \right. \]
\[ + \left. \alpha \left( \frac{1}{2} E_b + \frac{1}{2} E_t - E_m^{-1/2} \chi_{12} \right) \right] \]  
\[ (3.3.47) \]

\[ 2 \alpha_1 (1 + \frac{1}{4} \alpha_1 B r)^{-1/4} e^{\frac{1}{4} \eta_1} \frac{\partial^2 \chi_{12}}{\partial \eta_1^2} - E_m^{-1/2} \chi_{12} = - \frac{1}{2} (E_t + E_b) \]  
\[ (3.3.48) \]
where we dropped terms \( \alpha^{-1} A^{-1} \) compared to unit magnitude. Since \( L_m \) is large (3.3.47) - (3.3.48) can be simplified to

\[
\psi_{12} = \frac{1}{2} (1 + \frac{i}{4} \alpha_1 B r) \frac{-3/4}{\partial} \frac{4 \alpha_1}{\eta_1} \left\{ -F_B + z(F_B + F_L) \right\} \tag{3.3.49}
\]

\[
\frac{\partial^2 \chi_{12}}{\partial \eta_1^2} = - \frac{(1 + \frac{i}{4} \alpha_1 B r)^{1/4}}{4 \alpha_1} \frac{1}{\partial} \frac{4 \alpha_1}{\eta_1} \left( \nu_B + F_L \right) \tag{3.3.50}
\]

Integrating (3.3.50) with respect to \( \eta_1 \) and applying the condition

\[
(\partial / \partial \eta_1) \chi_{12} = 0 \quad \text{as } \eta_1 \to \infty \tag{3.3.51}
\]

since \( (\partial / \partial \eta_1) \chi_{12} \) is brought to zero in the main section, we obtain

\[
\frac{\partial \chi_{12}}{\partial \eta_1} = - \frac{(1 + \frac{i}{4} \alpha_1 B r)^{1/4}}{4 \alpha_1} \int^{\eta_1}_{\eta_0} \frac{-1}{\partial} \frac{4 \alpha_1}{\eta_1} \left( \nu_B + F_L \right) d\eta_1 \tag{3.3.52}
\]

The match conditions for the inner core now become

\[
\left. \eta_1 \to \infty \right\} \psi_1 = \frac{1}{2} (1 + \frac{i}{4} \alpha_1 B r)^{-3/4} \frac{4 \alpha_1}{\eta_1} \left\{ -F_B + z(F_B + F_L) \right\} \tag{3.3.53}
\]

\[
\left. \eta_1 \to \infty \right\} \frac{\partial \chi}{\partial \eta_1} = - \frac{(1 + \frac{i}{4} \alpha_1 B r)^{1/4}}{4 \alpha_1} \int^{\eta_1}_{\eta_0} \frac{-1}{\partial} \frac{4 \alpha_1}{\eta_1} \left( \nu_B + F_L \right) d\eta_1 \tag{3.3.54}
\]

The inner core problem is now specified and appears to be the same as one given for the unit cylinder. Therefore we shall not investigate this problem any further.
3.4. The long cylinder

In the present section we investigate the flow outside the Ekman layers in the long cylinder whose $L_m$-range is given by

$$ E_m^{-1} \sim L_m \quad (3.4.1) $$

Radial diffusion becomes an important factor in the flow in the main section. Let

$$ \Psi = E_m^{1/2} \psi_2, \quad \chi = E_m^{1/2} \chi_2 \quad (3.4.2) $$

Here, the absolute scaling magnitude of $\Psi$ has been chosen such that it corresponds to the induced flux of the Ekman layers. Substituting (3.4.2) into (3.1.43) - (3.1.44) we obtain

$$ L_m E_m \left\{ 4 \frac{\partial}{\partial x} e^x \frac{\partial}{\partial x} (1 - \frac{x}{A})^2 \psi_2 e^{-x} = 2 \frac{\partial \psi_2}{\partial z} \right\} = \quad (3.4.3) $$

$$ 4L_m E_m e^x \frac{\partial}{\partial z} \left( 1 + \frac{1}{2} Br (1 - \frac{x}{A}) \right) \left( 1 + \frac{1}{2} Br (1 - \frac{x}{A}) \right)^{-1} (1 - \frac{x}{A}) \chi_2 = \quad (3.4.4) $$

For $x \sim 1$ the terms $\sim x/A$ can be neglected compared to unit magnitude, leading to

$$ 8L_m E_m \left( \frac{\partial}{\partial x} e^x \frac{\partial}{\partial x} \right)^2 \psi_2 e^{-x} = \frac{\partial \chi_2}{\partial z} \quad (3.4.5) $$

$$ 2L_m E_m e^x \frac{\partial^2 \chi_2}{\partial x^2} = - \left( 1 + \frac{1}{2} Br \right) \frac{\partial \psi_2}{\partial z} \quad (3.4.6) $$
for \( x \to \infty \) the diffusive terms on the left of (3.4.3) and (3.4.4) will dominate. As a result, \( \chi_2 \) and \( \psi_2 \) must tend to their pure viscous values given by

\[
(\partial/\partial x)\psi_2 = (\partial/\partial x)\chi_2 = 0 \quad \text{as} \quad x \to \infty \tag{3.4.7}
\]

The boundary conditions at the cylinder wall are given by (3.1.45). In terms of \( \psi_2 \) and \( \chi_2 \) these are

\[
(\partial/\partial x)\psi_2 = \psi_2 = 0, \quad \chi_2 = -\frac{1}{2}E_m^{-1/2}e^{x} \quad \text{at} \quad x = 0 \tag{3.4.8}
\]

The conditions at the end caps are obtained from the Ekman suction conditions. Putting (3.4.2) into (3.1.41) - (3.1.42) and dropping terms \( \sim E_m^{1/2} \) and \( \sim A^{-1} \) we get

\[
\begin{align*}
\psi_2 &= -\frac{1}{2}(1+1Br)^{-3/4}e^{x}F_{\psi} \quad \text{at} \quad z = 0 \tag{3.4.9} \\
\psi_2 &= +\frac{1}{2}(1+1Br)^{-3/4}e^{x}F_{\psi} \quad \text{at} \quad z = 1 \tag{3.4.10}
\end{align*}
\]

The problem for the main section is now specified by differential equations (3.4.5) - (3.4.6) and boundary conditions (3.4.7) - (3.4.10). Because of the convenience in algebraic manipulation we substitute

\[
\begin{align*}
\chi_2 &= \chi_2' - \frac{1}{2}E_m^{-1/2}e^{x} \tag{3.4.11} \\
\psi_2 &= \psi_2' - \frac{1}{32L_m^{3/2}E_m} (1-e^{-x}e^{-x})(d/dx)\theta \tag{3.4.12}
\end{align*}
\]

Differential equations (3.4.5) - (3.4.6) now become

\[
\begin{align*}
8L_mE_m\frac{\partial}{\partial x} e^{x} \frac{\partial}{\partial x} e^{x} 2e^{-x} \psi_2' &= \frac{\partial \chi_2'}{\partial x} \tag{3.4.13} \\
2L_mE_m e^{x} \frac{\partial^2 \chi_2'}{\partial x^2} &= - \left( \frac{1}{2}Br \right) \frac{\partial \psi_2'}{\partial x} + \left( \frac{1+1Br}{32L_m^{3/2}E_m} \right) (1-e^{-x}e^{-x})(d^2/dx^2)\theta \tag{3.4.14}
\end{align*}
\]
The solution of (3.4.15) can be written as

\[ \psi_2' = \frac{\varphi}{m} \chi_j(x) \chi_k(x) \]  

where the eigenfunctions \( \chi_k \) are solutions of the eigenvalue problem

\[ e^x \frac{d^2}{dx^2} \left( \frac{d}{dx} e^x \frac{d}{dx} \right)^2 e^{-x} \chi_k + \lambda_k \chi_k = 0 \]  

(3.4.19)

\[ (d/d\omega)\chi_k = (d/d\omega)(d/d\omega) e^x (d/d\omega) + \omega^2 \chi_k = 0 \]  

as \( x \to \infty \)  

(3.4.20)

\[ (d/d\omega)\chi_k = \chi_k = (d/d\omega) e^x (d/d\omega) e^{-x} \chi_k = 0 \]  

at \( x = 0 \)  

(3.4.21)

Putting (3.4.18) into (3.4.15), multiplying this with \( e^{-\omega x} \chi_j \), where \( \chi_j \) is the adjoint eigenfunction, integrating this with respect to \( x \) from 0 to \( \infty \) and applying the orthogonality relation

\[ \int_0^\infty e^{-\omega x} \chi_k \overline{\chi}_j dx = 0 \quad \text{for } k \neq j \]  

(3.4.22)
we get for \( Z_k \)

\[
\frac{d^2 Z_k}{dz^2} - 16(1 + \frac{1}{4} B_1 r)^{-1} \frac{L_m E_m}{m m m} \lambda Z_k = 0
\]

\[
= \frac{1}{32 L_m E_m^{3/2}} \int_0^\infty x_k x_k^2 dx
\]

\[
\frac{d^3 \omega}{dz^3}
\]

(3.4.23)

The solution of (3.4.23) is given by

\[
Z_k = a_k e^{-4\sqrt{\lambda} k m m E_m (1 + \frac{1}{4} B_1 r)^{-1/2}} + b_k e^{+4\sqrt{\lambda} k m m E_m (1 + \frac{1}{4} B_1 r)^{-1/2}} + Z_{pk}
\]

(3.4.24)

where \( Z_{pk} \) is a particular solution of the differential equation (3.4.23). In order to determine \( Z_{pk} \) it is necessary to know the axial distribution of \( \omega \). In the subsequent treatment we restrict ourselves to the case that \( \omega \) is a linear function of \( z \). Then \( Z_{pk} \) is equal to zero and the solution for \( \psi_{2} \) is given by

\[
\psi_{2} = \sum_{k=1}^{\infty} \left\{ a_k e^{-4\sqrt{\lambda} k m m E_m (1 + \frac{1}{4} B_1 r)^{-1/2}} + b_k e^{+4\sqrt{\lambda} k m m E_m (1 + \frac{1}{4} B_1 r)^{-1/2}} \right\} X_k
\]

\[- \frac{1}{32 L_m E_m^{3/2}} (1 - e^{-x} - xe^{-x}) (d/dx) \theta \]

(3.4.25)

The six basic solutions of the differential equation (3.4.19) are already given by (3.2.146) - (3.2.151). Therefore we only have to replace \( v \) by \( e^{-x} \). Applying the boundary conditions (3.4.20) we retain

\[
X_k = \sum_{n=0}^{\infty} \beta_{1n} e^{-2nx} + e^{-x} \sum_{n=0}^{\infty} \beta_{2n} e^{-2nx} + x e^{-x} \sum_{n=0}^{\infty} \beta_{3n} e^{-2nx}
\]

(3.4.26)

Putting (3.4.26) into (3.4.19) one gets recurrence relations for the coefficients \( \beta_{1n}, \beta_{2n}, \beta_{3n} \). These relations express \( \beta_{1n}, \beta_{2n}, \beta_{3n}, n \geq 1 \) in terms of the basic coefficients \( \beta_{10}, \beta_{20}, \beta_{30} \). Application of the boundary conditions

\[
(d/dx) X_k = (d/dx) e^{-x}(d/dx) e^{-x} X_k = 0 \quad \text{at} \quad x = 0
\]

(3.4.27)
gives $\beta_{20}$ and $\beta_{30}$ as a function of $\beta_{10}$. As a result the coefficients can be written as follows

\[
\beta_{10} = \delta_{100} + \delta_{101}\lambda_k + \delta_{102}\lambda_k^2 + \ldots
\]

\[
\beta_{20} = \delta_{200} + \delta_{201}\lambda_k + \delta_{202}\lambda_k^2 + \ldots
\]

\[
\beta_{30} = \delta_{300} + \delta_{301}\lambda_k + \delta_{302}\lambda_k^2 + \ldots
\]

\[
\beta_{11} = \delta_{111}\lambda_k + \delta_{112}\lambda_k^2 + \ldots
\]

\[
\beta_{21} = \delta_{211}\lambda_k + \delta_{212}\lambda_k^2 + \ldots
\]

\[
\beta_{31} = \delta_{311}\lambda_k + \delta_{312}\lambda_k^2 + \ldots
\]

\[
\beta_{12} = \delta_{122}\lambda_k^2 + \ldots
\]

\[
\beta_{22} = \delta_{222}\lambda_k^2 + \ldots
\]

\[
\beta_{32} = \delta_{322}\lambda_k^2 + \ldots
\]

(3.4.28)

(3.4.29)

(3.4.30)

<table>
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<th>$k$ = 2</th>
<th>$k$ = 3</th>
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</tr>
<tr>
<td>$\beta_{33}$</td>
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</tr>
</tbody>
</table>

Table 1:

$\sqrt{\lambda}$, $\beta_{1n}$, $\beta_{2n}$, $\beta_{3n}$

$n = 0, 1, 2, 3$

for $k = 1, 2, 3$. 
respect to $x$ from 0 to $\infty$ and applying the orthogonality relation (3.4.22) we get

$$a_k = \frac{\phi_k}{32L_m E_m^{3/2}} \left( e^{+\Pi_4} - 1 \right) - \frac{1}{2} (1+\beta r)^{-3/4} (T_{bk} e^{+\Pi_4} + T_{tk})$$

$$= \frac{1}{32L_m E_m^{3/2}} (1 - e^{-\Pi_4}) + \frac{1}{2} (1+\beta r)^{-3/4} (T_{bk} e^{-\Pi_4} + T_{tk})$$

where

$$\phi_k = (d/dx) \theta_{\beta i} \int_0^\infty (1-e^{-x-xe^{-x}}) e^{-x \bar{x}_k} dx$$

$$T_{bk} = \int_0^\infty F_b e^{-\frac{1}{2} \bar{x}_k} dx$$

$$T_{tk} = \int_0^\infty F_t e^{-\frac{1}{2} \bar{x}_k} dx$$

(3.4.34)

(3.4.35)

(3.4.36)

(3.4.37)

(3.4.38)

Since $\bar{x}_r \sim 1$ as $x \to \infty$, the integrals

$$\int_0^\infty F_b e^{-\frac{1}{2} \bar{x}_k} dx \text{ and } \int_0^\infty F_t e^{-\frac{1}{2} \bar{x}_k} dx$$

are only finite if

$$F_b, F_t \sim e^{-\frac{1}{2} \alpha x}, \alpha > -1, \text{ as } x \to \infty$$

(3.4.39)

(3.4.40)

This is exactly the same requirement as identified for the inner and the outer core!

Summarising, the imposed boundary conditions at the end caps induce a $\chi$ and $\psi$ in the main section which are $\sim E_m^{1/2} F_b$ and $\sim E_m^{1/2} F_t$. The variables

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can be written as a set of radial eigenfunctions which decay exponentially with the distance from the end caps. Higher eigenfunctions decay faster than lower ones and at a reasonable distance from the ends the flow is mainly described by the first eigenfunction.

A linear temperature gradient along the cylinder wall induces a flow

\[ \omega \sim (L_m E_m)^{-1} (d/dz) \theta_n. \]

The radial shape of the streamfunction is almost equal to that of the first eigenfunction. The particular solution for \( \psi_2 \) given in (3.4.25) also indicates that \( \psi \) is constant along \( z \). However, due to the fact that the induced flow is not accepted by the Ekman layers, the decaying eigenfunctions (for the most part the first eigenfunction) must bring \( \psi \) to zero at \( z = 0 \) and \( z = 1 \): see the solutions for \( a_k \) and \( b_k \) given in (3.4.34) - (3.4.36). As a result the rate of decay of the first eigenfunction and thus the magnitude of \( 4 \sqrt{\lambda} \; L_m E_m (1+\frac{1}{2} Br)^{-\frac{1}{2}} \) mainly determines the axial behaviour of \( \psi \). Consider the case that \( L_m \gg E_m^{-1} \). All eigenfunctions now decay rapidly with distance from the ends. A linear temperature gradient induces an axial flow being constant along \( z \). This flow is brought to zero at \( z = 0 \) and \( z = 1 \) by the first eigenfunction in a region of thickness \( L_m E_m^{-1} \). The \( \chi \) induced here is adjusted at the end caps by the Ekman layers.

Parker & Mayo (1963) and Parker (1963) considered a semi-infinite cylinder. The Navier-Stokes equations were linearised taking a small perturbation on the isothermal state of rigid body rotation. Furthermore, they assumed eigenfunction solutions of the type

\[ e^{-\alpha_k z} f_k(r), \; k = 1, 2, \ldots, \infty \]

(3.4.41)

for the dependent variables. Putting (3.4.41) into the complete set of linearised equations, the resulting eigenvalue problem was solved numerically. Since the eigenvalues and the radial shape of the eigenfunctions were calculated for various magnitudes of \( \Lambda \), their results can be compared with our asymptotic solution for large \( \Lambda \). In figure 12 we have plotted \( \chi_1 \) and the numerically calculated smallest eigenfunction for the angular speed perturbation of Parker versus \( \chi \) for \( \Lambda = 9 \) and \( \Lambda = 16 \). Here one should notice that in the long cylinder \( \phi \sim \Lambda^{-1} \chi \) (see equation (2.2.59)) so that from (2.2.46) it follows that

\[ \chi \sim (1+\frac{1}{2} Br) \omega, \; \theta = -\frac{1}{2} Br \omega \]

(3.4.42)
In order to compare the graphs in figure 12 we took $\omega_1 \sim 1$ as $x \to \infty$. One sees that the agreement between the numerical and the asymptotic solution is already good for $A = 16$. Without showing this explicitly, the same conclusion applies to the magnitude of the smallest eigenvalue and the corresponding radial shape of the temperature perturbation, the axial velocity and the radial velocity.

![Figure 12: First eigenfunction of the angular speed perturbation versus $x$.](image)

Ging (1962a) also studied the semi-infinite cylinder applying an asymptotic solution for large $A$, similar to the one used by Dirac (1940), but including effects due to heat conduction. In the present work Ging's result is obtained by setting $\theta_w = 0$ and $b_k = 0$ in the solution (3.4.25).

Ging (1962b) also showed that the differential equation (3.4.19) with the boundary conditions (3.4.20) and (3.4.21) only yields positive real eigenvalues. He calculated for the first eigenvalue: $\sqrt{\lambda_1} = 2.408$. Two remarks must be made, however, concerning Ging's asymptotic solution.

(i) He used the coordinate

$$x_g = 2A(1 - r)$$

(3.4.43)
which is related to $x$ by

$$x = \frac{x}{g} - \frac{1}{2} \frac{x^2}{g}$$  \hspace{1cm} (3.4.44)

The exponential density function was approximated as follows

$$e^{-A(1-x^2)} \approx e^{-\frac{x}{g}} \frac{1}{g} \frac{x^2}{A} e^{-\frac{x}{g}}$$  \hspace{1cm} (3.4.45)

and the resulting equations were equal to those applied in the present work. In the above approximation terms $\sim \frac{x^2}{g}$ are dropped whereas in the present work we neglected terms $\sim \frac{x}{g}$ compared to unit magnitude. We therefore expect that for $x \gg 1$ the stretched coordinate applied here gives a somewhat better approach at relatively smaller values of $A$. An illustration is given in figure 12 where we also plotted $\omega_1$ versus $x$ obtained in the case of Ging's coordinate for $A = 9$. Of course, in the limit of $A \to \infty$ both asymptotic solutions become the same.

(iii) Ging did not investigate the asymptotic behaviour of the equations for large $x$ and assumed that $\omega, \theta, \omega, \omega$ and $\omega$ are constant with respect to $r$ as $r \to 0$. From the solutions (2.2.82), (2.2.84) and (2.2.86) one finds that this assumption applies to $\omega, \theta$ and $\omega$ but not to $\omega$. For example, when $r \sim A^{-1/2}$, given by (2.2.86) and thus $\omega$ deviates considerably from the asymptotic value as $x \to \infty$. Without showing this explicitly, Parker et al obtained this deviating behaviour for $\omega$ as $r \to 0$.

Berman (1963), Lotz (1970), Olander (1972) and Soubbaramayer (1961) considered the case of an axial flow being constant along $z$. The axial component of the momentum equation was used to determine the radial shape of $\psi$, being equal, for large $A$, to the particular solution given in (3.4.25). It is clear that this solution is only applicable as long as

$$L_m \gg n^{-1}.$$
3.5. Discussion

In this chapter we have developed explicit solutions for the stream function \( \psi \) and the composite variable \( \chi (\chi = \omega - \frac{1}{2} \theta \), where \( \omega \) is the angular speed perturbation and \( \theta \) the temperature perturbation). The modified Ekman number was taken to be small, \( E_m << 1 \), the speed parameter to be large, \( A >> 1 \), and the Brinkman number to be of unit magnitude, \( Br \sim 1 \). Furthermore we have considered three successive parameter ranges for the modified aspect ratio \( L_m \) given by \( 1 << L_m << E_m^{-1/2} \), referred to as the unit cylinder, \( E_m^{-1/2} \sim L_m << E_m^{-1} \), referred to as the semi-long cylinder and \( E_m^{-1} \leq L_m \), referred to as the long cylinder. The boundary conditions inducing the secondary flow are: the differential rotation of the end caps, axial injection, removal of fluid at the end caps, temperature perturbations along the end caps and along the cylinder wall.

In all three \( L_m \)-ranges Ekman layers form near the end caps. The influence of the imposed boundary conditions at \( z = 0 \) and \( z = 1 \) on the flow outside the Ekman layers can be expressed in terms of the Ekman suction conditions given by (3.1.41) and (3.1.42). These conditions allow us to describe the outside flow as a function of a certain combination of the imposed boundary conditions at the end caps expressed in terms of the functions \( F_b \) and \( F_t \); see (3.1.39) and (3.1.40).

In the unit cylinder we identify a basic inviscid flow in a limited region near the cylinder wall. In this region \( \chi \) and \( \psi \) are given by

\[
\chi = \frac{1}{2}(F_t + F_b), \quad \psi = \frac{1}{2}(E_m e^{x^2})^{1/2}(1+\frac{1}{2}Br(1-x^2))^{-3/4}(F_t - F_b), \quad x > 0 \quad (3.5.1)
\]

where

\[
x = A(1 - x^2) \quad (3.5.2)
\]
The streamfunction and the composite variable are constant along $z$. There is no radial motion and the axial flow is only returned within the Ekman layers, a result referred to as the compressible Taylor-Proudman theorem. Generally the imposed conditions at the end caps are discontinuous at the corners of the cylinder ($z = 0.1; x = 0$). The inviscid solution (3.5.1) does not "see" the discontinuity at the corner so that (3.5.1) does not satisfy the condition that $\psi$ and $\chi$ are zero at the cylinder wall. To overcome this non-uniformity, inner and outer Stewartson layer extensions are needed. The outer layer of thickness $L_m^{1/2}E_m^{1/4}$ brings $\chi$ to zero at $x = 0$. The inner layer of thickness $L_m^{1/3}E_m^{1/3}$ adjusts $\psi$ at the cylinder wall.

Due to the exponential decrease of the density with distance from the cylinder wall, the diffusive terms will become much larger than the inertia terms near the rotation axis, resulting in that $\chi$ and $\psi$ are constant with respect to $x$. Two regions, referred to as the inner and the outer core, adjust the pure viscous flow near the rotation axis and the inviscid flow near the cylinder wall. In the outer core at $x \propto \ln(L^{2E}_m)^{-1}$ the $x$-derivatives of $\chi$ balance the inertia terms just as in the outer Stewartson layer. In the inner core at $x \propto \ln(L^{2E}_m)^{-1}$ the $x$-derivatives of $\psi$ are also important, just as in the inner Stewartson layer. On the other hand, in both Stewartson layers the exponential density gradient is negligible whereas in both cores this gradient is the mechanism by which the inviscid flow and the pure viscous flow are coupled. For both the inner and the outer core problem we must require that

$$F_t = e^{-\frac{1}{2}ax}, \quad a > -1, \quad \text{as } x \to \infty \quad (3.5.3)$$

The inner Stewartson layer also serves a second purpose. It requires an $O(1)$ term in $\chi$ to balance the temperature perturbation at the cylinder wall. The induced $\psi$ is $\propto L_m^{1/3}E_m^{1/3}$ and represents a closed circulation within the layer.

In the semi-long cylinder the outer Stewartson layer expands over the radial density scale height and joins the outer core which comes up from the centre of the cylinder. As a result, radial diffusion of $\chi$ becomes an important factor describing the flow in the main section. The flow field is influenced by the no-slip condition for $\chi$ at the cylinder wall. A linear change of the axial flow along $z$ is possible. The Ekman layers no longer dominate which is typical for an inviscid flow. The inner Stewartson layer of thickness $L_m^{1/3}E_m^{1/3}$ brings $\psi$ of the main section to zero at the
The inner core at $x \propto \ln(\ell_m F_m)^{-1}$ adjusts to its pure viscous value near the rotation axis. As before, the imposed conditions at the end caps must satisfy the requirement (3.5.3). Also in the semi-long cylinder a temperature perturbation along the cylinder wall only induces a closed circulation within the inner Stewartson layer.

Similarly, in the long cylinder the inner Stewartson layer expands over the radial density scale height and joins the inner core which comes up from the centre of the cylinder. As a result, radial diffusive processes strongly influence the flow outside the Ekman layers. The imposed conditions at the end caps induce a flow which can be described by an infinite series of radial eigenfunctions which decay exponentially with distance from the end caps. Higher eigenfunctions decay faster than the lower ones and at a reasonable distance from the ends the flow is mostly described by the first eigenfunction. Also in the long cylinder the imposed conditions must satisfy the requirement (3.5.3).

A linear temperature gradient along the cylinder wall induces a constant axial flow along $z$, whose radial shape is practically equal to that of the first eigenfunction. This flow, however, can only be adjusted to the ends by mainly the first eigenfunction. In the case that $10 L E_m >> 1$ the first eigenfunction decays rapidly. As a result a linear temperature gradient produces an axial flow which is constant along $z$ over the greatest part of the cylinder.

The developed solutions allow us to describe the secondary flow in present day gas centrifuges. A convenient situation for the separation in a countercurrent gas centrifuge is the one in which the axial flow is constant along $z$. The theoretical maximum separative power of the centrifuge $6U_{\text{max}}$ is then given by (Cohen 1951, Los 1963):

$$6U_{\text{max}} = \rho \mathcal{D}(\pi L/2)(\Delta M/M) A^2 \frac{m^2}{1+m^2} E_f$$

where $\rho$ is the density of the mixture, $\mathcal{D}$ the diffusion constant, $L$ the length of the cylinder, $\Delta M$ the difference between the molecular weights of both components, $M$ the molecular weight of the mixture ($\Delta M/M << 1$) and $A$ the speed parameter. To first order the product $\rho \mathcal{D}$ is constant at constant temperature.
The influence of the secondary flow on the separative power is expressed in the flow parameter $m$ and the flow pattern efficiency $E_f$, given by

$$E_f = \frac{4 \left[ \int_0^1 r^3e^A(x^2 - 1) \psi^2 \, dr \right]^2}{\int_0^1 r^3e^{2A}(x^2 - 1) \psi^2 \, dr}$$

$$m = \frac{e \mu}{E \rho D} \left[ 2 \int_0^1 r^3e^A(x^2 - 1) \psi^2 \, dr \right]^{1/2}$$

In terms of the coordinate $x$ appropriate for the rapidly rotating heavy gas and having (2.2.55) in mind, we get

$$E_f = \frac{2 \int_0^A (1 - \frac{x}{A}) \psi e^{-2x} \, dx}{A \int_0^A (1 - \frac{x}{A}) \psi^2 e^{-2x} \, dx}$$

$$m = \frac{e^{1/2}}{E_m} \frac{\mu}{\rho D} \left[ \int_0^A (1 - \frac{x}{A}) \psi^2 e^{-2x} \, dx \right]^{1/2}$$

The flow efficiency $E_f$ depends upon the radial shape of the streamfunction $\psi$. Let us therefore consider the flow in the various parameter ranges. As already stated for optimal separation the flow must be constant along $z$. In the unit cylinder such a situation is observed in the inviscid region. In the semi-long cylinder one can also create a constant axial flow in an inviscid region by taking anti-symmetrical boundary conditions with respect to the mid-plane $s = 1/2$, $F_b = -F_t$. In this special case, only, an outer Stewartson layer and an outer core are missing and the flows in the unit and the semi-long cylinder are equal. As a consequence, for $F_b = -F_t$ and $L_m \ll E_m^{-1}$ we observe an inviscid region up to the inner core. In this region $\chi = 0$ and $\psi$ is described by

$$\psi = -\frac{1}{4} (\kappa_m \sigma_x)^{1/2} \left[ 1 + \frac{1}{4} \beta r (1 - \frac{x}{A}) \right]^{-3/4} F_b$$
provided that
\[ x << \ln(L_m F_m)^{-1} \]  \hspace{1cm} (3.5.10)

At \( x \sim \ln(L_m F_m)^{-1} \), which is the location of the inner core, the inviscid solution no longer applies. For \( x >> \ln(L_m F_m)^{-1} \), the diffusive terms dominate, resulting in that \( \psi \) is constant along \( s \). This implies that \( + e^{-x} \) approaches zero. At the cylinder wall \( \psi \) is brought to zero in the inner Stewartson layer of thickness \( L_m F_m^{1/3} \).

For \( L_m \sim F_m^{-1} \) the flow is mainly described by the first eigenfunction being almost constant along \( z \). A reasonable description for \( \psi \) is then
\[ \psi = a F_m^{1/2} (1 - e^{-x} - x e^{-x}) \]  \hspace{1cm} (3.5.11)

where \( a \) determines the magnitude of \( + \). A good approximation for \( a \) is given by
\[ a \sim \int_0^\infty F_b e^{-\frac{1}{2}x} (1 - e^{-x} - x e^{-x}) dx \]  \hspace{1cm} (3.5.12)

The imposed conditions at the end caps only determine the magnitude of the countercurrent flow; the radial shape is not influenced.

For \( L_m \gg F_m^{-1} \) a constant axial flow along \( z \) can be created by imposing a linear temperature gradient along the cylinder wall. In this case \( \psi \) is described by
\[ \psi = -\frac{1}{32 L_m F_m} \frac{1}{L_m} (1 - e^{-x} - x e^{-x}) \left( \frac{d}{dz} \frac{\theta}{\bar{\theta}} \right) \]  \hspace{1cm} (3.5.13)

The radial shape is again that of the first eigenfunction.

Concerning the efficiency three parameter ranges can now be distinguished.

(i) \( L_m \leq e^{-\frac{1}{4}A} \). The inner core is located in the immediate vicinity of the rotation axis, \( \rho \sim A^{-1/2} \), or it does not exists at all. One observes an inviscid region almost up to the rotation axis. As a result the most optimal radial shape for \( \psi \) yielding the highest efficiency can be generated. From Los and Kistemaker (1958) it is known that
\[ e^{-x} \psi \sim H(x) - \frac{1}{2} \]  \hspace{1cm} (3.5.14)
gives the highest $E_f$ equal to 1. Of course, Heaviside's unit function $H$ is smoothed out to thickness $L_m E_m^{-1/3} E_m^{-1/3}$ in the inner Stewartson layer, but since $L_m << E_m^{-1}$ the net result for $E_f$ is almost unaffected.

Another important profile in present day gas centrifuges is known as the two-shell profile:

$$e^{-x}\psi \sim (1-\frac{x}{A})^{-1}H(b-x)\{H(x) - \frac{1}{2}\} , \quad 0 < b < A \quad (3.5.15)$$

In this case the axial mass flow $(e^{-x}\psi)$ consists of two Dirac pulses, the inner one at $x = b$ and the outer one near the cylinder wall and rechanneling the axial flow. The efficiency is now given by

$$E_f = -2b^2/A^2 \ln(1-\frac{b}{A}) \quad (3.5.16)$$

which has a maximum equal to 0.814 for $b = 0.7A$. In other words, the most optimal position of the inner shell is $r = 0.55$. For $b < 0.7A$ $E_f$ decreases with decreasing $b$. Of course the inner and outer pulse are smoothed out to thickness $L_m E_m^{-1/3}$ by respectively a free vertical inner Stewartson layer at $x = b$ and an inner Stewartson layer at $x = 0$.

(ii) $e^{-A} << L_m E_m \ll 1$. The inner core is located between the rotation axis and the cylinder wall. For $x << \ln(L_m E_m)^{-1}$ the most optimal radial shape for $\psi e^{-x}$ can be generated. On the other hand, for $x >> \ln(L_m E_m)^{-1}$ $\psi e^{-x}$ approaches zero which means that the pure diffusive region does not contribute to $E_f$! If the inner core is located at $r \geq 0.55$ an optimum $E_f$ is given by the two shell profile. The minimum radial position of the inner shell, thus yielding the highest $E_f$, is now the location of the inner core. The corresponding $E_f$ is obtained by putting

$$b = \ln(b_1 (L_m E_m)^{-1}) \quad (3.5.17)$$

into (3.5.16). Since the distribution of $\psi$ in the inner core is not solved the value of the constant number $b_1$ is unknown.

(iii) $L_m E_m \gg 1$. In this case one can generate an almost constant $\psi$ along $z$ whose radial shape is equal to that of the first eigenfunction: $1 - e^{-\frac{x}{A} - x e^{-x}}$. The corresponding $E_f$ is equal to

$$E_f = \frac{36}{5A} \quad (3.5.18)$$
An estimation for the contribution of the $O(A^{-1})$ terms in (3.5.18) cannot be given since the $O(A^{-1})$ solutions for $\Psi$ are not presented. On the other hand, Parker & Mayo (1963) calculated numerically, $E_\ell^f$ for the first eigenfunction and obtained: $E_\ell^f = 0.67$ for $A = 9$ (asymptotic $E_\ell^f = 0.80$), $E_\ell^f = 0.42$ for $A = 16$ (asymptotic $E_\ell^f = 0.45$) and $E_\ell^f = 0.29$ for $A = 25$ (asymptotic $E_\ell^f = 0.29$). One sees that for $A \geq 16$ the relative deviation of $E_\ell^f$ given by (3.5.18) is already less than 7%.

In the range (i) $E_\ell^f$ can be a maximum dependent upon the radial shape of $F_b$. Due to the appearance of the inner core, in the range (ii) $E_\ell^f$ decreases with increasing $Lm$ and $A$. In the range (iii) $E_\ell^f$ is a minimum and proportional to $A^{-1}$.

In the case of the two-shell profile a continuous description of $E_\ell^f$ for the three parameter ranges can be approximated as follows:

\[
(i) \quad 5 L_m E_\ell (1 + \frac{1}{4} B r) - 1/2 \leq e^{-0.7A + 18/5}, \quad b = 0.7A \quad (3.5.19)
\]

\[
(ii) \quad e^{-0.7A + 18/5} \leq 5 L_m E_\ell (1 + \frac{1}{4} B r) - 1/2 \leq 1,
\]

\[
b = 18/5 + ln(5 L_m E_\ell (1 + \frac{1}{4} B r) - 1/2) - 1 \quad (3.5.20)
\]

\[
(iii) \quad 1 \leq 5 L_m E_\ell (1 + \frac{1}{4} B r) - 1/2, \quad b = 18/5 \quad (3.5.21)
\]

where

\[
E_\ell^f = - \frac{2 b^2 A^2}{2 A^2 \ln (1 - \frac{b}{A})} \quad (3.5.22)
\]

Although $E_\ell^f$ is small, most present day centrifuges will fall into the ranges (ii) and (iii), which is due to the large value of $e^{0.7A}$ and $L_m$.

As a result, efficiencies considerably lower than the maximum one obtained in the range (i), $E_\ell^f = 0.814$, can be expected.

The dependence of $6U_{\max}^\infty$ on the magnitude of the flow occurs through $m$. Inspection of (3.5.4) shows that $6U_{\max}^\infty$ reaches its largest value as $m \to \infty$. However, the limiting value is approached very rapidly: when $m = 3$,
\[ \frac{m^2}{1 + m^2} = 0.9 \] and when \( m = 5 \), \( \frac{m^2}{1 + m^2} = 0.96 \). Therefore in actual counter-current gas centrifuges \( m \) is of unit magnitude and from (3.5.8) one can test the validity of a linear treatment. For \( L \leq m \leq 1 \) the optimal flow is generated at the end caps. In a inviscid region one requires that \( \psi \approx e^x \).

In the pure diffusive region \( \psi \approx e^{-x} \) approaches zero. As a result, the maximum perturbation of \( \psi \) occurs in the inner core at \( x \approx \ln(L_m E_m)^{-1} \). In order to induce a \( \psi \approx e^{-x} \) that is constant along \( x \) in the inviscid region, it is necessary that \( F_p \sim L_m^{1/2} E_m^{1/2} e^{1/2} \); see (3.5.9). Here we scaled \( F_p \) such that at \( x \approx \ln(L_m E_m)^{-1} \), where the perturbed \( \psi \) is a maximum, \( F_p \sim 1 \). Taking \( m \sim 1 \) one finds from (3.5.8)

\[
\varepsilon \sim \frac{\rho D}{\mu} \frac{1}{L_m^{1/2} A^{1/2} \ln(L_m E_m)^{-1}} \tag{3.5.23}
\]

For the working fluid in a gas centrifuge \((\rho F_\theta)\): \( \rho D/\mu \sim 1 \). Since \( L_m \) and \( A \) are large and since \( \ln(L_m E_m)^{-1} \) varies in magnitude from \( A \) (the inner core near the rotation axis) to \( 1 \) (the inner core near the cylinder wall) \( \varepsilon \) is small and a linear treatment seems reasonable. Of course, a more concrete opinion concerning the overall validity of the linearisation can be formed by considering the magnitude of the non-linear terms in the various flow regions. For instance, it is well-known that in the Stewartson layers a stronger linearisation criterion applies, due to the first order derivatives with respect to \( \varphi \) in the non-linear convective terms. Nevertheless, one hopes that a linear approach is still applicable for an \( F_p \) consideration.

For \( L \leq m \gg 1 \) a constant axial flow is induced by a linear temperature gradient along the cylinder wall. Taking \( (\partial \theta/\partial z)_w \sim 1 \) and \( m \sim 1 \), and applying (3.5.13) one finds from (3.5.8)

\[
\varepsilon \sim \frac{\rho D}{\mu} \frac{L_m F_m}{A^{1/2}} \tag{3.5.24}
\]

Although \( F_m / A^{1/2} \) is small, \( \varepsilon \) can become of unit magnitude or larger, since \( L_m F_m \) can reach considerable magnitudes and the \( \Delta T \) necessary to drive the secondary flow becomes to large for a linear approach.

Due to the rapid decrease of the density with distance from the cylinder wall, the density near the rotation axis can be quite low with corre-
spondingly long mean free paths. Thus, in many practical centrifuge applications, the flow at the centre of the cylinder will not be continuum flow.

An opinion about the applicability of continuum flow theory can be formed by determining the region where the Knudsen number $Kn$ is of unit magnitude. Here, $Kn$ is the ratio of the mean free path $\lambda$ to the characteristic length scale of the gas flow. The mean free path is given by

$$\lambda = \frac{\mu}{\rho \sqrt{R_0 T / M}} \quad (3.5.25)$$

Near the rotation axis two regions can be distinguished (see figures 5 and 6). These are: the pure diffusive region at the inner side of the inner core and the Ekman layers of thickness $r_m^{-1}$ near the end caps. The characteristic radial length scale in the pure diffusive region is that of the density variation: $\alpha / A$. The characteristic axial length scale in the Ekman layers is $L_m^{-1}$, which is equal to $\alpha / A$. As a result, for both regions the Knudsen number can be approximated by

$$Kn = \frac{E \lambda^{3/2} e^x}{E_m^{1/2} \alpha} \quad (3.5.26)$$

One sees that rarefied gas flow occurs at $x \gtrsim \ln(E_m / A^{1/2})^{-1}$. For the unit, the semi-long and the long cylinder this region appears within the pure diffusive region, a region that does not contribute to the separation. In other words, rarefied considerations are not needed.
LIST OF FREQUENTLY USED SYMBOLS

\( A \)  
speed parameter \( \Omega^2 a^2 M/2 R_0 T_0 \)

\( a \) 
radius of the cylinder

\( Br \)  
Brinkman number \( \mu \Omega^2 a^2 / \kappa T_0 \)

\( \alpha_p \)  
specific heat at constant pressure

\( D \)  
diffusion constant

\( E \)  
Ekman number \( v/\Omega a^2 \), \( \nu/\Omega a^2 \rho \)

\( \mathcal{E}_p \)  
flow pattern efficiency

\( \mathcal{E}_m \)  
modified Ekman number \( E \mathcal{A}^2 \)

\( I_B \)  
combination of the imposed boundary conditions at the bottom cap

\( I_T \)  
combination of the imposed boundary conditions at the top cap

\( \mathcal{E} \)  
Heaviside's unit function

\( \hat{k} \)  
unit vector along the rotation axis

\( L \)  
aspect ratio, length-to-radius ratio

\( L_m \)  
modified aspect ratio \( \mathcal{A} \mathcal{L} \)

\( L \)  
length of the cylinder

\( Ma \)  
Mach number of the primary rotation

\( M \)  
molecular weight

\( \Delta M \)  
difference between molecular weights

\( m \)  
flow parameter

\( P \)  
pressure

\( P_e \)  
equilibrium pressure at isothermal rigid body rotation

\( P_0 \)  
pressure at \( \hat{r} = 0 \)

\( P_w \)  
wall pressure

\( \rho \)  
pressure perturbation

\( \hat{q} \)  
particle velocity in the rotating frame

\( R_0 \)  
universal gas constant

\( \hat{r} \)  
position vector in the rotating frame

\( r \)  
radial coordinate

\( T \)  
temperature
uniform constant temperature

$T_0$

temperature along the cylinder wall

$T_w$

typical particle velocity in the rotating frame

$U$

radial velocity

$u$

perturbed azimuthal velocity

$v$

axial velocity

$w$

imposed axial velocity at the bottom cap

$\omega_b$

imposed axial velocity at the top cap

$\omega_t$

radial eigenfunction

adjoint radial eigenfunction

$X_K$

radial density coordinate

$X_K$

stretched axial boundary layer coordinate

$y$

axial coordinate

$s$

axial coordinate

$\alpha_1$

square of the distance between inner core and rotation axis

$\alpha_2$

square of the distance between outer core and rotation axis

$\beta$

bulk viscosity

$\Gamma$

Gamma function

$\gamma$

ratio of specific heats at constant pressure and volume respectively

$\delta U_{max}$

maximum separative power

$\varepsilon$

Rossby number $U/\Omega a$

$\varepsilon_1$

dimensionless parameter scaling the pressure perturbation

$\varepsilon_2$

dimensionless parameter scaling the density perturbation

$\varepsilon_3$

dimensionless parameter scaling the temperature perturbation

$\zeta$

stretched radial boundary layer coordinate

$\eta$

contracted radial core coordinate

$\theta$

temperature perturbation along the bottom cap

$\theta_b$

temperature perturbation along the top cap

$\theta_t$

temperature perturbation along the cylinder wall

$\theta_w$

dynamic viscosity

$\nu$

kinematic viscosity

$\kappa$

heat conductivity

$\lambda_k$

eigenvalue

$\rho$

density

$\rho_e$

equilibrium density at isothermal rigid body rotation

$\rho_w$

density at the cylinder wall

$\alpha_p$

Prandtl number $\mu c_p/\kappa$
density perturbation
\[ \rho \text{ } + \frac{1}{2} \rho u^2 B \]
\[ \chi \text{ } \omega = \frac{1}{2} \theta \]
streamfunction
\[ \psi_p \text{ } \text{imposed streamfunction at the bottom cap} \]
\[ \psi_t \text{ } \text{imposed streamfunction at the top cap} \]
\[ \Omega \text{ } \text{angular speed of the cylinder} \]
\[ \omega \text{ } \text{angular speed perturbation} \]
\[ \omega_b \text{ } \text{imposed angular speed perturbation at the bottom cap} \]
\[ \omega_t \text{ } \text{imposed angular speed perturbation at the top cap} \]
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Summary

The secondary flow of an incompressible fluid and of a perfect gas in a rotating cylinder is considered, by applying a linearised analysis to a small perturbation on the isothermal state of rigid body rotation.

In chapter 2 we apply an order of magnitude consideration to the linearised dimensionless conservation equations. In section 2.1 the incompressible fluid is treated assuming a small Ekman number $E$ based on the radius of the cylinder. The importance of the various viscous terms is investigated increasing the length-to-radius ratio $L$ from unit magnitude to infinity. As a result, three types of flow can be distinguished which correspond to the $L$-ranges: $E^{1/2} \ll L \ll E^{-1/2}$, $E^{-1/2} \ll L \ll E^{-1}$ and $E^{-1} \ll L$. In the first $L$-range we identify a flow which underlies a balance between the Coriolis forces and the pressure gradients. This inviscid flow is adjusted to the boundaries by Ekman layers near the end caps and two Stewartson layers near the cylinder wall. In fact, this flow type is well-known from the literature and is of importance to models of oceanic and atmospheric currents. In the second $L$-range the outer Stewartson layer expands to the interior which means that radial diffusion of azimuthal momentum becomes an important parameter describing the flow in the main section of the cylinder. In the third $L$-range also, the inner Stewartson layer fills the entire cylinder. Radial diffusion of azimuthal and axial momentum are both important, a situation that characterises flows in semi-infinite cylinders.

In section 2.2 the perfect gas is considered, special attention being focussed on rapidly rotating heavy gases. In this case the density decreases strongly with the distance from the cylinder wall, a situation that occurs in present day gas centrifuges. The modified Ekman number $E_m$, based on the density at the cylinder wall and on the density scale height
assumed to be of unit magnitude. The secondary flow is investigated for the case that the ratio of the length to the radial density scale height $L_m$ increases from unit magnitude to infinity. Again three types of flow can be distinguished which correspond to the $L_m$-ranges $R_m^{1/2} \ll L_m \ll E_m^{-1/2}$, $R_m^{-1/2} \sim L_m << E_m^{-1}$ and $E_m^{-1} \ll L_m$. However, compared to the incompressible fluid two essential differences are found.

(i) An **inviscid** flow characteristic for the first range is only observed in a limited region near the cylinder wall. Due to the strong density decrease diffusive processes are important in the core of the cylinder.

(ii) A change of the flow type appears when both Stewartson layers successively expand over the small radial density scale height. Simultaneously corresponding diffusive regions come up from the centre of the cylinder and join. As a result, a change of the flow type appears at relatively small values of the length-to-radius ratio. In contrast to the incompressible fluid where the first type is of most practical importance, here all three types are of strong interest.

In chapter 3 explicit solutions for the flow in the various regions of the rapidly rotating heavy gas are presented. Here we apply perturbation techniques and the method of matched asymptotic expansions. The imposed boundary conditions are: temperature differences along the end caps and along the cylinder wall, the differential rotation of the end caps and axial injection and removal of fluid at the end caps.
Samenvatting

De secundaire stroming van een onsamendrukbare vloeistof en van een ideaal gas in een roterende cylinder wordt beschouwd, waarbij de Navier-Stokes vergelijkingen zijn gelineariseerd ten opzichte van isotherme starre rotatie.

In hoofdstuk 2 wordt een orde-grootte beschouwing toegepast op de gelineariseerde dimensieloze behoudswetten. In paragraaf 2.1. behandelen we de onsamendrukbare vloeistof en nemen aan dat het Ekman getal \(E\), gebaseerd op de straal van de cylinder, klein is. Vervolgens onderzoeken we de belangrijkheid van de verschillende viskeuze termen wanneer de lengte-straal verhouding van de cylinder \(L\) toeneemt van eenheidsgrootte tot oneindig. Het blijkt dan dat drie typen stromingen onderscheiden kunnen worden welke overeenkomen met de \(L\)-gebieden \(E^{1/2} \ll L \ll E^{-1/2}, E^{-1/2} \ll L \ll E^{-1}\) en \(E^{-1} \ll L\). In het eerste gebied vinden we een stroming waaraan een evenwicht tussen de Coriolis krachten en de drukgradienten ten grondslag ligt. Deze niet-viskeuze stroming wordt aangepast aan de randen van de cylinder door Ekman lagen bij de einddeksels en twee Stewartson lagen langs de cylinder wand. Dit type stroming is erg bekend uit de literatuur en is van belang voor modellen van stromingen in de atmosfeer en in de oceaan. In het tweede \(L\)-gebied expandeert de buitenste Stewartson laag naar binnen hetgeen betekent dat radiale diffusie van azimuthale impuls belangrijk wordt voor de beschrijving van de stroming in het hoofdgedeelte van de cylinder. In het derde \(L\)-gebied vult ook de binnenste Stewartson laag de hele cylinder. Radiale diffusie van azimuthale en axiale impuls zijn beide belangrijk, een situatie die stromingen in half-oneindige cylinders karakteriseert.

In paragraaf 2.2. behandelen we het ideale gas waarbij we vooral onze aandacht richten op snel roterende zware gassen. Hierbij neemt de dichtheid sterk exponentieel af met afstand van de cylinderwand, een situatie
Welke voorloopt in de huidige gas centrifuges. We nemen aan dat het gemon-dificeerde Ekman getal \( E_m \), dat gebaseerd is op de dichtheid aan de wand en op de radiale dichtheidsschaal (in plaats van de straal) klein is en dat het Brinkman getal \( B_r \) van eenheidsgrootte is. We onderzoeken de secundaire stroming voor het geval dat de verhouding van de cylinderlengte tot de radiale dichtheidsschaal \( L_m \) toeneemt van eenheidsgrootte tot oneindig. We nemen weer drie typen stromingen waar die overeenkomen met de \( L_m \)-gebieden \( E_m^{1/2} \ll L_m \ll E_m^{-1/2}, \ E_m^{1/2} \sim L_m \ll E_m^{-1} \) en \( E_m^{-1} \sim L_m \). Echter, vergeleken met de onsamendrukbare vloeistof vinden we twee essentiële verschillen.

(i) Een niet-viskeuze hoofdstroming karakteristiek voor het eerste \( L_m \)-gebied treedt alleen op in een gebied van beperkte dikte langs de cylinderwand. Tengevolge van de sterke dichtheidafname zijn diffusie processen belangrijk in het hart van de cylinder.

(ii) Een verandering van het type stroming vindt plaats wanneer beide Stewartson lagen achtereenvolgens over de radiale dichtheidsschaal expanderen. Tegelijkertijd komen overeenkomstige gebieden uit het centrum van de cylinder op. Hieruit voortvloeiend zal een verandering van het type stroming optreden bij relatief kleine waarden van de lengte-straalverhouding. In tegenstelling tot de onsamendrukbare vloeistof waar het eerste type van het meeste praktische belang is, zijn nu alle drie de typen belangrijk.

In hoofdstuk 3 presenteren we expliciete oplossingen voor de stromingen in de verschillende gebieden van het snel roterende zware gas. Hierbij maken we gebruik van een mathematisch formeel proces, de z.g. singuliëre storingsrekening en "the method of matched asymptotic expansions". De opgelegde randvoorwaarden die de secundaire stroming veroorzaken zijn: temperatuurverschillen over de einddeksels en langs de cylinderwand, verschillende rotatiesnelheden van onder- en bovendeksel en axiale inspuiting en extractie van gas bij de deksels.