Barriers in the transition to global chaos in collisionless magnetic reconnection. I. Ridges of the finite time Lyapunov exponent field

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The transitional phase from local to global chaos in the magnetic field of a reconnecting current layer is investigated. Regions where the magnetic field is stochastic exist next to regions where the field is more regular. In regions between stochastic layers and between a stochastic layer and an island structure, the field of the finite time Lyapunov exponent (FTLE) shows a structure with ridges. These ridges, which are special gradient lines that are transverse to the direction of minimum curvature of this field, are approximate Lagrangian coherent structures (LCS) that act as barriers for the transport of field lines. © 2011 American Institute of Physics. [doi:10.1063/1.3647339]

I. INTRODUCTION

Characterizing the nonlocal properties of a 3-dimensional divergence-free vector field is a difficult task. It requires developing and testing of tools that are appropriate to describe in a concise and effective way the behavior of field lines and to do this not only asymptotically but on all the different spatial scales of interest for the specific problem under consideration. In such a context, magnetic field lines in a plasma are of special importance since particle and energy diffusion in and out of the plasma configuration is related to the field line behavior under rather general conditions of adiabatic motion of particles. In addition magnetic field lines in a plasma provide the opportunity to test such tools not only on simplified “toy model” configurations but also on complex field structures such as those obtained from 3-dimensional numerical simulations of the nonlinear development of magnetic reconnection. In this case, it is possible to test directly what is the level of information that such tools can provide in a realistic case in comparison with other methods and approaches.

The development of magnetic field line reconnection in three-dimensional configurations is an intrinsically chaotic process. Indeed, in the presence of magnetic perturbations with skew spatial orientations, e.g., in the presence of perturbations with different “helicities” in the case of doubly periodic configurations, the magnetic field line Hamiltonian is no longer integrable and the magnetic configuration becomes chaotic for sufficiently large magnetic perturbations. The analysis of a chaotic magnetic field topology is quite complex and, in the case of periodic configurations, relies heavily on mapping techniques such as Poincaré plots. For a given magnetic field line Hamiltonian, the Poincaré plot gives a detailed picture of the chaotic domains of the magnetic field, provided one integrates the Hamilton equations starting from a sufficiently large number of initial conditions and for sufficiently large values of the parameter (position) along the field lines that plays the role of an effective time. In this paper, we will call this effective time the field-line-time or just time. Equivalently, when we speak about motion of a field line, we mean motion in field-line-time not in dynamic time. When we refer to real or dynamic time, we will explicitly say so. Such a Poincaré plot provides information on the location of the Kolmogorov-Arnold-Moser (KAM) surfaces that may be present and delimit the domains of regular “motion” of the magnetic field lines from the chaotic domains. However, Poincaré maps do not provide information on the position along the field lines, i.e., on the interval of the field-line-time, it takes a field line to move around in the plane of the Poincaré section. For instance, Poincaré maps do not predict how fast magnetic field lines migrate and if certain surfaces may act as barriers to this magnetic field line penetration on a given field-line-time scale. This kind of information turns out to be particularly useful when the transition from local to global chaos, that occurs as the magnetic configuration evolves in real, dynamic time, is investigated. During this phase, regions of regular and irregular magnetic fields coexist and coherent structures that form barriers to the transport of the magnetic field line may exist. It is the aim of this paper to investigate on which field-line-time scale these barriers form and how robust they are, i.e., how long it takes a magnetic trajectory to cross such a barrier. These structures are generally referred to as Lagrangian coherent structures (LCSs).

In the case of magnetic reconnection in a fully periodic three-dimensional (3D) configuration, the transition from a regular to a chaotic magnetic field structure has been studied in Ref. 2 in terms of stable and unstable manifolds associated with uniformly hyperbolic trajectories, which are also called distinguished hyperbolic trajectories. These hyperbolic lines are the generalisations in phase space $(x,y,z)$ of X-lines in a two-dimensional geometry. The associated manifolds intersect at the hyperbolic trajectory and form invariant surfaces. Field lines on the stable manifold approach the hyperbolic line exponentially in forward time and field lines on the unstable manifold do so in backward time.
No transport of magnetic field lines will take place across invariant surfaces.

The actual magnetic system that we will consider is 3D periodic. However, the field perturbations are sufficiently localized in the radial direction such that the radial periodicity does not play a role. Hence, our double periodic system is topologically equivalent with that of a toroidal magnetic field with radial shear. In this geometry, the special hyperbolic lines discussed above will form closed field lines at which the reconnection process will take place.5

When heteroclinic intersections, i.e., intersections between stable and unstable manifolds belonging to different distinguished hyperbolic field lines, are generated, large-scale transport is set up. The transition between regular and chaotic configurations was studied in Ref. 2 in terms of the field-line-time it takes for heteroclinic intersections to appear.

Although the invariant manifolds contain essential information on the transport properties of the system, a drawback of this method is that these invariant surfaces become so densely folded that they are impossible to trace for sufficiently long times such that their intersections become numerically visible. A consequence is that it is impossible to quantify transport on this basis. Therefore, it might be rewarding to settle for a less exact method and to define transport on the basis of approximate, asymptotic properties of the system.

In the present paper, we focus on the geometrical properties of the field of the finite time Lyapunov exponent (FTLE) in order to investigate chaotic magnetic fields by reconsidering the magnetic configuration investigated in Ref. 2. The largest positive FTLE measures the exponential separation between two neighboring field lines after a given interval of field-line-time, and is defined by

$$\sigma(z, z_0, x(z_0)) = \frac{1}{|z - z_0|} \max \frac{\|d\mathbf{x}(z)\|}{\|d\mathbf{x}(z_0)\|}. \quad (1)$$

The standard, infinite timeLyapunov exponent is obtained in the limit $|z - z_0| \to \infty$.

Within the context of FTLE theory, approximate Lagrangian coherent structures may be defined as second-derivative ridges of the scalar FTLE-field. Following Ref. 7 we define a ridge as a curve in the $(x, y)$-plane such that the gradient in the FTLE-field is along the curve, and such that the second order derivative

$$\Sigma = \frac{d^2 \sigma(x)}{dx^2} \quad (2)$$

in the direction perpendicular to the curve is minimal. Here, $\sigma$ is the largest positive FTLE. Inside a stochastic region, the FTLE field tends to be constant and uniform so that no clear ridges will be found.

A problem that is inherent to the concept of the FTLE is the choices of the length of the time-interval $|z - z_0|$ and of the initial position $z_0$ that occur in its definition. In our system that is partly and non-uniformly hyperbolic, stochastic and regular magnetic regions coexist. One might say that the system locally looks like magnetic islands dispersed in stochastic swamps. Trajectories in a partially stochastic region take a long time to get through such a region. A field line might stick for some time near a regular structure before it wanders off. The field line will be caught by the homoclinic tangle which is associated with this island structure. If this tangle has heteroclinic intersections, the field line will wander off eventually and, at a later time, will stick to some other structure at a different radial position. This means that transients of long duration are observed in the FTLE and that the value of the FTLE depends on the position and the width of the time window where it is calculated.

In order to analyse this system in a companion paper,6 we will make use of field line spectroscopy. The frequency spectrum of the motion of a field line in a particular time window will be shown to exhibit the frequencies belonging to the specific regular system where it lingers during that time.

As stated above, in this and in the companion paper, we consider the plasma configuration treated in Refs. 1 and 2 consisting of a collisionless plasma with a current layer embedded in an unstable sheared magnetic field. The magnetic configuration evolves in dynamic time because of the onset of reconnection instabilities at different resonant surfaces. When the nonlinear phase is entered, chaos develops initially around the rational surfaces and then spreads over the whole domain. In these papers, we will not deal with the dynamic evolution of the magnetic field, as already extensively discussed in Ref. 1, but we will focus on investigating the structure of the magnetic field configuration at two particular points in real time, one at the onset of the transition to global chaos and another when the stochasticity is fully developed. A similar approach has been adopted in Ref. 2 when considering the stable and unstable manifolds generated by the reconnection process.

In Secs. II and III, we present some background material on the FTLE and on the LCS as ridges of the FTLE field and we illustrate the adopted numerical technique. In Sec. IV, we describe the reconnecting discharge; in Sec. V, we report the numerical results. The conclusions are drawn in Sec. VI.

II. FTLE

As already stated, we apply the FTLE method to reveal the magnetic field structures produced by magnetic reconnection events. The reconnection instability is responsible for magnetic field topology variations, whose spatial distribution is governed by magnetic field line equations.

In our analysis, we assume the following magnetic field representation

$$\mathbf{B} = B_0 \mathbf{e}_z + \mathbf{e}_x \times \nabla \Psi, \quad (3)$$

where $B_0$ is constant and normalized to unity, and $\Psi = \Psi(x, y, z, t)$ is the poloidal magnetic flux function, which varies in time under the reconnection process.

At each time $t$, the trajectory $x(z; z_0, x_0)$ of the magnetic field that passes through $x(z_0) = x_0$, obeys the Hamiltonian system

$$\frac{dx}{dz} = -\frac{\partial \Psi}{\partial y}, \quad \frac{dy}{dz} = \frac{\partial \Psi}{\partial x}. \quad (4)$$
The system (4) describes in field line time what has been coined in fluid dynamics as chaotic advection, the magnetic field playing the role of the advecting velocity field. This system is Hamiltonian irrespective of the character of the plasma which may be either dissipative or ideal. In Sec. IV, we will adopt a model for the Hamiltonian $\Psi$ that we take from a numerical experiment on collisionless reconnection.

According to Eq. (3), the time evolution of the distance between two neighboring field lines, $\delta x = x(z, z_0, x_0 + \delta x_0) - x(z, z_0, x_0)$, satisfies the linearized equation

$$\frac{d\delta x}{dz} = F(x(z), z)\delta x,$$

where the Jacobian matrix $F$ is given by

$$F(z) = \begin{pmatrix} -\Psi_{xy} & -\Psi_{yy} \\ \Psi_{xx} & \Psi_{xy} \end{pmatrix}.$$}

Since the magnetic field is incompressible, we have $\text{trace } F = 0$.

We write $\delta x(z) = X(z, z_0, x_0)\delta x(z_0)$, where the solution matrix $X(z, z_0, x_0)$ satisfies

$$\frac{dX}{dz} = FX, \quad X(z_0, z_0, x_0) = I,$$

$I$ being the unit matrix. Hyperbolicity is a property of this linear system.

The linearized distance between the two trajectories is measured by the Euclidean norm $\|\delta x(z)\|^2 = \delta x^T(z)\delta x(z)$, where the index $T$ denotes the transpose, so that

$$\|\delta x(z)\|^2 = \delta x^T(z_0)X^T X \delta x(z_0).$$

The matrix $X^T X$ is real symmetric and has real eigenvalues. Here, we assume that the largest eigenvalue $\lambda$ is greater than one. In our case of incompressible fields, two eigenvalues $\lambda_{1,2}$ with $\lambda_{1,2} = 1$ exist.

Aligning the initial separation $\delta x(z_0)$ with the eigenvector belonging to this largest eigenvalue, we have

$$\max \|\delta x(z)\| = \sqrt{\lambda} \|\delta x(z_0)\|.$$

The square root of the largest positive eigenvalue is the factor by which a perturbation is maximally stretched. This expression can be written as

$$\max \|\delta x(z)\| = e^{\sigma(z - 0, x_0)} \|\delta x(z_0)\|.$$

Here, $\sigma$ is the FTLE which is defined by

$$\sigma(z, z_0, x_0) = \frac{1}{|z - z_0|} \ln \sqrt{\lambda} (z, z_0, x_0),$$

$|z - z_0|$ being the length of the effective time the FTLE is computed. This FTLE depends on the initial position $z_0$ and on the length of the interval $z - z_0$. Its values calculated either forward or backward in time will in general not be equal.

Since our system is area preserving, the infinite time limit $|z - z_0| \to \infty$ exists almost everywhere, i.e., exist for almost all field lines. In this limit, the standard Lyapunov coefficient is obtained. In Ref. 7, it is shown that

$$\frac{d\sigma(z, z_0, x_0)}{dz_0} = \mathcal{O}\left(\frac{1}{|z - z_0|}\right),$$

so that at long field-line-times, the FTLE becomes a constant. Note that, while this standard Lyapunov coefficient is constant along each field line, the FTLE is not.

III. LCS

A definition of a LCS of the magnetic field configuration can be based upon the FTLE. Our analysis follows the method discussed in Ref. 7, where the Lagrangian coherent structures of a velocity field are defined as ridges of FTLE field of the corresponding particle trajectories. In particular the “second derivative ridge” definition is assumed, which relies on the Hessian of the FTLE field. In the case of magnetic fields, we define a LCS as a ridge in the FTLE-field of the magnetic configuration.

A ridge is a curve $c(s)$ in the $(x, y)$-plane ($s$ being the parameter along the curve) that satisfies two requirements.

1. The gradient in the FTLE-field is along the curve. This means that the tangent vectors $c'(s)$ and $\nabla \sigma(c(s))$ have to be parallel.
2. $\Sigma(n, n) < 0$ is minimal, where $n$ is the unit normal vector to the curve $c(s)$ and $\Sigma$ is the Hessian (2), evaluated at the curve.

This latter condition means that if one would walk along the ridge and one would step aside in any direction, one would step to lower values of the FTLE-field. In particular, if one would step in the direction of the normal $n$ to the curve, one would make the largest step down.

In order to extract the ridges in the FTLE field from the magnetic data obtained by the numerical simulations of 3D reconnection processes, we developed a new computational algorithm. This solver starts with the computation of the Lyapunov exponents for a set of magnetic field lines at different finite effective-time $z$ according to the method described in Ref. 10. This method is based on an efficient decomposition of the tangent map $X$ that does not require the typical renormalization or reorthogonalization of the traditional techniques.

When applied to the two-degrees of freedom Hamiltonian system (4), the matrix $X$ can be written as the product $X = QR$ of an orthogonal $2 \times 2$ matrix $Q$ and an upper-triangular $2 \times 2$ matrix $R$ with positive diagonal entries

$$Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad R = \begin{pmatrix} e^{\mu_1} & r \\ 0 & e^{\mu_2} \end{pmatrix},$$

where $\theta$ is an angle variable and $r$ is an independent super-diagonal term, while $\mu_i$ are intimately related to the Lyapunov exponents. It can be shown, in fact, that the asymptotic Lyapunov exponents are equal to $\mu_i/\omega$, when $z \to \infty$. Starting from Eq. (7), by simple algebraic passages, it is possible to derive the evolution equations for $\mu_i$ and $\theta$ in terms of the Jacobian matrix components $F$. 
Equations (14) and (15) are integrated for a grid of magnetic field lines uniformly distributed on a $\chi - \gamma$ Cartesian mesh at the initial time $z_0$. Each magnetic line, starting from a $\chi_0$ position, is advected forward in $\chi$ with a variable-order, variable-step Adams algorithm\(^{14}\) integrating magnetic field data expressed through the fast Fourier transform coefficients. Once the final effective-time $\chi$ is reached, the FTLE at each point of the initial grid is easily obtained as $\sigma(\chi, \gamma, \chi_0) = \mu(\chi, \gamma, \chi_0)/\chi$. This procedure is then repeated for a range of times $\chi$ to provide a time-series of FTLE fields. It is important to note that forward-time integration allows to reveal just the repelling branches of the Lagrangian coherent structures, e.g., the unstable manifolds. In order to locate also the attracting Lagrangian coherent structures, e.g., the stable manifolds, also the backward time integration have to be carried out. In this case, the relevant FTLE is $\sigma(\chi, \gamma, \chi_0) = -\mu(\chi, \gamma, \chi_0)/\chi$.

As a next step, according to second condition in the definition of a “second derivative ridge,” the algorithm computes the Hessian $\Sigma$ of the FTLE field from finite differencing. Once the eigenvectors $\mathbf{n}$ corresponding to the minimum eigenvalue direction of the Hessian are extracted, in order to satisfy the first requirement of the adopted ridge definition, a scalar field is formed by taking the inner product of these eigenvectors with the FTLE gradient field, evaluated as the Hessian, by a finite difference scheme. Then ridges are extracted by looking at the zero-valued level sets of this scalar field.

As stated in Ref. 15 when ridges are computed using the Hessian of the scalar field, noise amplification can become an issue, especially when chaotic “velocity fields” are taken into account, as in the case of the magnetic fields we considered in this paper. A large collection of criteria addressing ridge filtering is available in technical literature.\(^{15}\) Here, in order to remove weak features caused by noise, we have chosen a natural, easy to implement, criterion which prescribes a minimum height of the ridge $s \geq s_{\text{min}}$. It has been shown that in the case of finite Lyapunov exponent ridges, this leads to significant, consistent, and reliable visualizations.\(^{16}\)

### IV. THE RECONNECTION MODEL

Our aim is to study the coherent magnetic structures that emerge during the nonlinear stages of a collisionless, magnetic reconnection process. The dynamics of this process is described in Ref. 1 on the basis of a two-fluid plasma description,\(^{17}\) valid in the presence of an intense, externally imposed, magnetic guide field. The reconnecting field has only components in the $\chi-\gamma$ plane but depends on all three spatial coordinates. This model neglects the magnetic field line curvature and takes the axial magnetic field to be constant. The model retains the contribution coming from the electron temperature, through the ion sound Larmor radius, and from electron inertia, through the electron skin depth. Electron inertia provides the mechanism that breaks the frozen-in condition and allows the rearrangement of the magnetic field topology.

The model equations are solved numerically in a 3D-periodic slab geometry starting from a static equilibrium configuration with a one-dimensional shear magnetic field.

The spontaneous reconnection process is induced by multiple helicity perturbations with high values of the linear stability parameter $\alpha'$.

We consider a configuration with background toroidal and poloidal magnetic fields that carries initially a resonant mode at each of two neighboring surfaces.

The magnetic flux function consists of an equilibrium part $\psi_{eq}(x)$ and a wave-like contribution $\psi(x, y, z; t)$:

$\psi(x, y, z, t) = \psi_{eq}(x) + \psi(x, y, z; t), \quad (16)$

where $\psi$ may be written as a sum over Fourier modes $\psi(x, y, z; t) = \sum \psi_i(x, k_y y + k_z z, t)$ with $k_y = 2\pi n_i/L_y$, $k_z = 2\pi n_i/L_z$, where $m_i$ and $n_i$ are the poloidal and toroidal mode numbers, respectively.

The surfaces $\pm x_{s_i}$ where the modes are resonant are characterized by $B_{eq} \cdot \nabla \psi = 0$, which yields

$$
\frac{d\psi_{eq}(x)}{dx} = -\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_i}{\partial y} = -\frac{k_{y_i}}{k_{z_i}}, \quad (17)
$$

The numerical simulations were carried out in a triple-periodic slab, with amplitude $L_x = 2\pi$, $L_y = 4\pi$, $L_z = 32\pi$, starting from an equilibrium configuration with magnetic flux function $\psi_{eq} = 0.19 \cos(x)$. The initial perturbation consists of two unstable contributions, $\psi_1$ and $\psi_2$, with different helicities

$\psi(x, y, z; t) = \tilde{\psi}_1(x, t) \exp(ik_{y_1} y + ik_{z_1} z) + \tilde{\psi}_2(x, t) \exp(ik_{y_2} y + ik_{z_2} z). \quad (18)$

The functions $\tilde{\psi}_{1,2}(x, t)$ approximate the analytic solutions of the linearized dynamical equations. The wave numbers ($m_i$, $n_i$) of the two components of the perturbation are $(1, 0)$, for $i = 1$, and $(1, 1)$, for $i = 2$. The amplitude $\tilde{\psi}_1$ is of order $10^{-4}$ and is ten times bigger than $\tilde{\psi}_2$. Note that the equations of motion (3) and the initial conditions (18) are invariant under $(y \rightarrow -y, z \rightarrow -z)$.

In the small amplitude linear phase, when the two helicities evolve independently from each other, each mode induces a magnetic island chain around its resonant surfaces, $x = x_{s_i}$, where $B_{eq} \cdot \nabla \psi = 0$. For the case, we present here $x_{s_1} = 0$, $\pi$ and $x_{s_2} = 0.71$, $\pi - 0.71$. Since resonant surfaces with $x_L > \pi/2$ are simply due to the periodicity of the magnetic equilibrium $\psi_{eq}$, they will be omitted and we will focus on the magnetic field structure in the reduced interval $-\pi/2 < x < \pi/2$. The corresponding linear X-points are $(0, 2\pi n)$ and $(0.71, 0)$.

When the magnetic islands are sufficiently large to interact with each other, the nonlinear phase of the process enters.
Modes with different helicity and higher order modes of the same helicities of the initial perturbation are generated. At this stage, the magnetic field topology exhibits regions where field lines are stochastic and whose area tends to spread during the evolution of the reconnection process. The modes remain sufficiently localized in the radial direction so that the radial periodicity of the equilibrium configuration never becomes of any importance.

We are interested in analyzing the structure of the magnetic field when a large number of modes has been generated, and the magnetic field has developed a chaotic behavior on a substantial part of the volume between the initial resonant surfaces. In such a partially stochastic system, island chains will exist at radial surfaces where the rotational transform \( t = -(L_z/L_y)\psi_{eq}^f(x) \) is a rational number. The main chains will be at the surfaces where \( t = 0 \) (\( m = 1, n = 0 \)) and \( t = 1 \) (\( m = 1, n = 1 \)). In between these main structures, island chains are induced by these dominant modes at all radial surfaces. Each chain of islands is embedded in a stochastic region. It can easily be found from Eqs. (4) and (18) that the main frequencies that are associated with the radial motion \( x(z) - x(z_0) \) of a field line near an intermediate rational surface characterized by its value of \( t \) are \( t \) and \( (1 - t) \).

We consider subsequent stages of the dynamics of the reconnection process described above, around the Chirikov regime where the transition to global stochasticity occurs, when the size of the islands becomes such that they overlap.

In particular we focus on the magnetic field behavior at two different dynamical times, \( t = 415\tau_A \) and \( t = 425\tau_A \), as obtained in the numerical simulations reported in Ref. 1. Here, \( \tau_A = 4\pi nm L_z/B_0 \) is the poloidal Alfvén time with \( L_z = 2\pi \) and \( B_0 = 0.19 \). During this dynamic time interval, the invariant manifolds associated with the main hyperbolic points start to intersect, producing heteroclinic tangles.

In order to simplify the numerical analysis, controlling in particular the computational time of the FTLE field, we will use approximate descriptions of the Hamiltonian function \( \psi \). As shown in Ref. 2, just the higher amplitude modes obtained from the Fourier decomposition of the original data need to be considered, in order to well approximate the Hamiltonian. This truncation results in a magnetic flux function with 20 spectral components. For the dynamical evolution times considered here, it turns out that these modes have the helicity of either of the original perturbations. Hence, we may write

\[
\psi(x, y, z, t) = \psi_1(x, y/L_y, t) + \psi_2(x, y/L_y + z/L_z, t),
\]

with \( \psi_1 = \Sigma \psi_{1n}(x) \exp[2\pi nm/L_y] \) and \( \psi_2 = \Sigma \psi_{2n}(x) \exp[2\pi nm(y/L_y + z/L_z)] \).

V. NUMERICAL RESULTS

Figures 1–4 characterize the discharge at an early stage in the nonlinear reconnection process (\( t = 415\tau_A \)) when the system is still only partially chaotic and the main stochastic layers are still unconnected. Equivalently, Figures 5–8 describe the system at a later stage (\( t = 425\tau_A \)) when global stochasticity has set in.

Since the problem under consideration is periodic in \( z \), we have adopted the Poincaré technique for the visualization of the magnetic topology. The black dots in Fig. 1 represent the Poincaré plot obtained by integrating the dynamical equations (3) for magnetic field lines up to 10000\( L_z \) starting from 40 initial conditions distributed at \( y = -2r \) between \( x = 0 \) and \( x = 0.75 \).

The green curves represent the stable and unstable manifolds associated with the main hyperbolic points cut the \( z = 0 \)-plane. The superimposed colored lines represent the stable and unstable manifolds associated with the main hyperbolic points and calculated with a contour dynamics code. These points from where the invariant manifolds emanate are just the points where the special hyperbolic lines (the generalized X-lines) cut the \( z = 0 \)-plane.

The green curves represent the stable and unstable manifolds with the initial conditions centered at (0, \( -2\pi r \)) and the red lines show the manifolds associated with the point...
Both sets of manifolds are taken at \( z = 6L_c \). The light blue, dark blue, magenta, and purple colored curves show the manifolds emanating from the points \((0.617, 0)\), \((0.51827, -1.914)\), \((0.52195, 0)\), and \((0.51, 0)\), respectively. The first two set of manifolds are traced up to \( z = 36L_c \), while the third and the fourth manifolds correspond to \( z = 80L_c \) and \( z = 50L_c \), respectively.

The figure shows that a chaotic region is associated with each set of manifolds. When they are followed for longer times, the manifolds appear to be more and more folded inside their chaotic region. At this stage in real time \((t = 415s)\), heteroclinic intersections are just to appear, but the main stochastic regions are still well-separated.

The FTLE field for this system has been calculated for a set of \( 1.28 \times 10^8 \) magnetic field lines initially distributed at \( z = 0 \) over a uniform \( 8000 \times 16000 \) mesh on the domain \( 0 < x < 0.8, -2\pi < y < 2\pi \). Figure 2 shows the results after 16 iterations along the toroidal direction. This choice is motivated by a comparison of the results for different integration times. In particular, we have carried out simulations up to 20 iterations. Indeed, the essential structures are pretty well represented already after 12 iterations, which corresponds to \( |z - z_0| = 364\pi \).

Ridges of the FTLE field are clearly visible in Fig. 2. A highly crumpled up distribution of the field, with extremely sharp gradients, is localized in the area enclosed between the right boundary of the island pattern related with the hyperbolic point \((0.0, -2\pi)\) and the left border of the island structures related with the hyperbolic point \((0.71, 0)\).

Between these regions and those where the FTLE has a quite smooth behavior and is very small lie regions where no ridges are found, but where the FTLE is rather spiky. These
are the areas where the magnetic field is most stochastic. When the number of iterations is increased, the spiky regions tend to spread all over the stochastic area, where the ridges of the FTLE turn out to be embedded. In such a region, the infinite time LE would tend to be a uniform constant. However, the FTLE is not an invariant and will depend quite irregularly and strongly on the length \( |z - z_0| \) of the time interval and on the initial point \( z_0 \) in the definition (11).

The second derivative ridges in this FTLE field are extracted according to the algorithm described in Sec. III and are shown in Fig. 3 (black curves) together with the manifolds (colored curves) that belong to the five dominant hyperbolic points. Among the ridges, it is possible to recognize some rather regular curves, enclosing finite area regions. The figure shows that the ridges tend to be aligned with and practically coincide with the branches of the corresponding stable and unstable manifolds computed for a few \( L_p \) periods. Since magnetic field lines cannot cross these manifolds, it is expected that these FTLE ridges form barriers with respect to magnetic field line transport.

In Fig. 4, we have overplotted the FTLE ridges on the Poincaré map on the \( z = 0 \) plane at \( t = 415 \tau_A \). The map has been produced by the numerical integration of the magnetic field line equations (4) after \( 5 \times 10^2 \) toroidal iterations for two patches of \( 5 \times 10^2 \) initial conditions distributed around the hyperbolic points at \( (0, -2\pi) \) (green region) and \( (0.71, 0) \) (red region).

This figure demonstrates that inside regular (island) structures of Fig. 1, the FTLE field vanishes and no ridges exist. Ridges in the FTLE field do not exist inside the stochastic regions where the LE field tends to be uniform. Figure 4 also shows that ridges do exist between a stochastic region and a regular structure and in between the stochastic areas confining the green and red regions. In particular, these latter ridges will have our attention as it is expected that they will act as the final barriers to field line transport before global stochastization takes place. The ridge that forms the right-hand boundary of the green stochastic region in Fig. 4 is formed by the purple colored curves in Fig. 1. It will be shown that this ridge is associated with the rotational transform \( \nu = 3/5 \) and is the last surviving FTLE-ridge in the process towards global stochastization.

Analogous results for the topology of the magnetic field and the ridges of the corresponding field line FTLE at a later dynamic time in the reconnection process, \( t = 425 \tau_A \), are shown in Figs. 5–8. The Poincaré plot in Fig. 5, taken on the section \( z = 0 \), is obtained by integrating the equations (4) for a set of 35 magnetic field lines initially distributed along the axis \( y = -2\pi \) after 1000 periods along the toroidal direction. The map shows the stochastic area of the magnetic field in the portion of the computational domain we are taking into account, \( 0 < x < 0.8, -2\pi < y < 2\pi \). The FTLE field in Fig. 6 refers to magnetic field lines, with the same initial distribution adopted for Fig. 2 but now after 12 toroidal iterations.

Upon comparing Figs. 2 and 3 with Figs. 6 and 7, it is seen that most ridges between the red and green stochastic areas have become very weak or have disappeared and that the associated stochastic areas have merged. A last strong ridge is present in Fig. 8. This ridge is associated with the manifolds belonging to the hyperbolic point at \( (0.52576, 0) \) and corresponds to the purple colored manifolds of Fig. 1. This point is located at the intersection of two ridges in Fig. 7. These manifolds have heteroclinic intersections with the manifolds belonging to the original main hyperbolic points at \( (0, -2\pi) \) and \( (0.71, 0) \).

Due to these intersections, this ridge is not an exact LCS. This can be seen in Figure 8 where magnetic field lines from the stochastic area in red start to cross the ridge and to penetrate into the green stochastic region and field lines from the green region are penetrating into the red region. Thus, the magnetic field topology reveals that the stochastic layers around the island structures at the original resonant surfaces \( \lambda_1 \) and \( \lambda_2 \) are merging and going to form just one single area, which confirms the transition of the system to the global chaotic state at later times. At those times, the last ridge becomes also very weak and no strong barrier preventing the magnetic field line transport between the original resonant surfaces exists anymore.
VI. CONCLUSIONS

The identification in a magnetic configuration of the LCSs, defined as “ridges” of the distribution in space of the FTLEs, provides a tool to analyze partly stochastic systems. In this approach, finite time Lyapunov exponents are not seen as approximations to the infinite time Lyapunov exponents that determine the asymptotic divergence of neighboring field lines but as quantities that characterize their behavior on shorter distances along field lines, with the idea that in a number of real cases, this piece of information is more relevant than their asymptotic value. Such a case is provided e.g., in the problem of the diffusion of charged particles in a magnetic configuration when the distance a particle travels along a field line during the characteristic evolution time of the system is much smaller than the characteristic length of a field line to connect to its asymptotic destination.

In this approach, no periodicity of the system is required, but analogous considerations hold for periodic field configurations where the field line behavior is usually characterized in terms of Poincaré maps. These give a detailed picture of the chaotic domains of the magnetic field and of the location of the KAM surfaces that may be present and delimit regular domains from the chaotic domains but do not provide information on the length it takes a field line to move around in the plane of the Poincaré section. For instance, Poincaré maps do not predict if certain surfaces may act as “temporary” barriers to magnetic field lines in the sense that the field lines may stick to these barriers for a long length, i.e., for many turns in the periodicity direction, before being able to pass them.

FTLEs, in particular LCSs, and similar finite time schemes have been used for an exceptionally wide range of scientific investigations from, to name a few, oceanography and geophysical flows to MHD fields, to biophysics.

In the present paper, we have addressed the problem of characterizing the “degree of stochasticity,” in the sense discussed above in terms of finite field line lengths, in a 3-dimensional magnetic field configuration in a plasma where magnetic reconnection is taking place. We consider in particular a configuration where magnetic field lines are mostly aligned in the z direction and have variations in the perpendicular plane that are fast with respect to those along z. Such a configuration has been obtained in the numerical simulations reported in Ref. 1 as the result of the onset of a magnetic reconnection instability involving two different spatial inclinations (helicities) of the perturbed magnetic field. In this case, the Hamiltonian that describes the field lines is not integrable and regions of regular (on invariant tori) and of irregular (chaotic) magnetic fields coexist separated by KAM surfaces. As the instability evolves in time, the KAM surfaces are destroyed and the configuration becomes fully stochastic.

The main point of interest of the present paper is to address the field line transport in the phase of the instability when the magnetic configuration is changing from local to global chaoticity. As mentioned in the Introduction, an investigation based on the explicit construction and representation of the stable and unstable manifolds emanating from the distinguished hyperbolic lines, that take the role of the separatrices in a non-integrable configuration, is hindered by the difficulty of following numerically their complex folding, as would be needed in order to determine whether heteroclinic intersections have occurred. In such a situation, the use of an approximate tool such as determining the ridges in the spatial distribution of the FTLEs may provide a simpler method of characterizing field line transport. We remind that, in this context of magnetic field line transport, finite time means finite field line time, i.e., finite distance along a field line, as distinct from finite dynamical time.

An important result of this paper is to show that transport undergoes a qualitative change in the transition phase and in the fully stochastic configuration.

We have analyzed the spatial distribution of the ridges of the finite time Lyapunov exponents at \( t = 415 \tau_A \) (partial stochasticity) and at \( t = 425 \tau_A \) (full stochasticity) and have shown how these ridges disappear at the latter time.

In order to verify that the ridges actually act as barriers, in an accompanying paper, by selecting specific examples taken from the behavior of sample field lines inside intervals in field line time, we show that these field lines remain temporarily bound around a specific magnetic island structure before moving to a neighboring one by crossing a ridge. The identification of the magnetic island structure that temporarily traps the field line is done by inspection of the Poincaré map and by adopting the technique of field line spectroscopy that allows us to compare the frequencies of the field line motion along the periodicity direction to the corresponding frequencies of the distinguished hyperbolic line.

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14 See www.nag.co.uk/numeric/fl/manual/pdf/D02/d02cjf.pdf for a description of the first-order ordinary differential equation solver we adopted for the integration of Eqs. (14) and (15).