A network theory for variable epicyclic gear trains

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A NETWORK THEORY FOR VARIABLE EPICYCLIC GEAR TRAINS

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At the top of each page in the outer margin appear chapter and section number. In addition, each item (definition, theorem, equation, etc.) is given a number that appears in the left-hand margin. Such numbers are numbered consecutively within each section. When we refer in a section to an item within the same section, only the item number is given.
CHAPTER 1
INTRODUCTION

1.1. Scope

This study deals with the transmission of power by rotating elements, such as shafts, in certain combinations of epicyclic gear trains with continuously variable transmissions.

Such combinations achieve in principle the same as variable transmissions themselves, viz. the transformation of an angular velocity into another and simultaneously of a torque into another, while the ratio between the angular velocities can be changed continuously. The product of the original angular velocity and torque, called input power, decreased by a dissipative power (absolute value) equals the output power (absolute value). An example is provided by a variable shunt, also called split-power, divided-power, bifurcated, shunt, or differential transmission; see Fig. 1.1.1.

Fig. 1.1.1 A variable shunt. Speed increasing if shaft end B is input shaft and shaft end A is output shaft.

A rotating shaft performs a power path. Other elements allow a power path to branch into more power paths, and two or more power paths to combine into one. It is well known that an epicyclic gear train performs such a branching or combination; see Fig. 1.1.2. A similar branching or combination of power paths is performed by connecting three power transmitting shafts to one transmission; see Fig. 1.1.3. In a systematic description this involves that three co-axial shaft ends are interconnected. A fixed interconnection of three shaft ends is called a 'node'.
Although nodes play a part in the description of the systems to be considered here, they have so far been overlooked in literature to the detriment of lack of generality. Nodes may be found in several designs; see Figs. 1.1.1, 1.1.4, 1.1.5.

Fig. 1.1.2 An epicyclic gear train combined with a transmission; in actual design (a); separated from the transmission in an equivalent design with co-axial shafts A, C, E (b); schematically (c)

Fig. 1.1.3 A node combined with a transmission in actual design (a); separated from the transmission in an equivalent design with power transmitting co-axial shaft ends B, D, E (b); schematically (c)
This study is confined to combinations of epicyclic gear trains with continuously variable transmissions which as a whole act as a variable transmission and therefore are called 'variator networks'. Later it will be proved that a variator network has one input shaft, one output shaft, and that the numbers of epicyclic gear trains and nodes must be equal. Combinations with different degrees of freedom like those of Figs. 1.1.4 and 1.1.5 are left out of consideration.

Fig. 1.1.4 Combination of power paths through a node. Two gear drives, each with its own input shaft have a main gear and an output shaft in common. Actual design (a); schematically (b)

Fig. 1.1.5 Split power gear drive of the double-reduction locked-train type. The distribution of power depends on special properties of the lay-out. Actual design (a); schematically (b)

A variator network has one or more of the following advantages:
- the range of the ratio of a variable transmission in a variator network may be utilised to a larger extent;
- the variable transmission need not transmit all the network power;
- the efficiency of a variator network may be better than that of its variable transmission;
- in an emergency, a variator network may transmit power even if its variable transmission fails.
For describing and studying the performance of varistor networks without undue effort it is convenient to develop a consistent analytical formulation, systematically applicable to any design used in actual practice. It is far less convenient to develop an analytical formulation on the basis of any group of particular designs. The validity of such a specialised approach would be limited to the designs chosen and it would be difficult to deduce a general theory from it. The consistent application of the analytical formulation, called 'mathematical model', satisfies all requirements for basic design in a most economic way, viz.

- the analysis of a varistor network of given type;
- the synthesis of a varistor network satisfying prescribed requirements;
- the synthesis of a varistor network optimal in one or more respects.

1.2. Rotating shaft

1 A rotating shaft is considered, as usual, to be one single element. It has two shaft ends, each marked with a letter. The quantities

\[ \omega \] angular velocity
\[ T \] torque
\[ P \] shaft power

are assigned to a shaft end and further specified by one single suffix corresponding with the letter of the shaft end.

2 A common engineering practice for simple problems is to work exclusively with positive values and to fit signs in formulae to the case. The lack of general conventions and the limitations of the approach in existing literature results in a multitude of formulae. Such a practice is very ineffective in more complicated cases. It is necessary to develop only one set of formulae in a mathematical model operative for any design conceivable. Henceforth, a wider significance will be assigned to the value of a quantity than it commonly has, while a wider range of values will be covered (for instance, in chapter 7 imaginary values are used).

3 For the interpretation of a value the conventions for the signs are very important. The sign of a numerical value indicates a direction of the quantity.

4 One direction of rotation of a shaft will be chosen as the positive direction; see Fig. 1.2.1.

A positive direction once chosen is maintained in the analysis of the system independently of the position of the observer.
An angular velocity is considered positive if the shaft rotates in the positive direction.

A torque exerted on a rotating shaft is considered positive if it would tend to drive the shaft in the positive direction.

The above convention for the torque is the most recommendable one to avoid misunderstanding in the interpretation of torques and reaction torques. Linking the convention for angular velocities to that for torques is important to the interpretation of the sign of shaft powers.

A shaft power is the product of the angular velocity of a shaft end and the torque exerted on that shaft end. Its sign is governed, like in algebra, by the signs of its factors.

\[ P_A = \omega_A T_A \]

A positive shaft power means an input of energy toward the shaft through the shaft end considered. A negative shaft power means an output of energy from the shaft through the shaft end considered.

The dynamic properties of a rotating shaft are described in accordance with common engineering mechanics with the aid of moments of inertia lumped to a number of inertial elements, and stiffnesses lumped to elastic elements of the shaft; see Fig. 1.2.2.

![Fig. 1.2.1 Positive direction of rotation. Directions for positive values of angular velocity \( \omega \) and torque \( T \)](image)

![Fig. 1.2.2 Schematic representation of a shaft end (a), an inertial element (b), and an elastic element (c) of a rotating shaft. An example of a rotating shaft (d) with shaft ends marked A, B, moments of inertia \( J_1, J_2, J_3 \), and stiffnesses \( S_1, S_2 \)](image)
In chapter 3 it will be shown that moments of inertia and stiffnesses assigned to other elements (planet gears) may be transposed to the shafts. In a stationary situation a 'rotating' shaft has one fixed value for the torque transmitted and one for its angular velocity.

1.3. Concept of a three-pole

Epicyclic gear trains and nodes have in common that they interconnect three power transmitting shaft ends, thus constituting 'three-poles' in a variator network. No other entities than three-poles are required to perform and schematically represent the interconnections of rotating shafts. If more than three power path meet, they can be described by a sequence of three-poles. See Fig. 1.3.1.

Fig. 1.3.1 Example of a variator network. The triangles in (a) represent three-poles, the dotted lines contain variable transmissions.
In (b) the three-poles are distinguished in epicyclic gear trains and nodes, indicated by circles and dots, respectively.

2 In most other networks that carry power, Kirchhoff's laws are applicable, for instance on the voltages and currents of electrical networks. In a variator network Kirchhoff's laws, putting it simply, and with adequate sign conventions, may be interpreted for a node:
- the angular velocities of the three shaft ends are mutually equal (two linear equations);
- the sum of the torques is zero (one linear equation).

3 The above interpretation of Kirchhoff's laws for a node does not satisfy for an epicyclic gear train. Contrary to nodes, epicyclic gear trains are characterised by one linear equation for angular velocities, and two for torques. Besides, these equations contain certain parameters.

4 As to both Kirchhoff's laws, epicyclic gear trains and nodes have different properties. Hence, the conception of three-poles without further specification will scarcely be used and an important aspect of
the network theory given hereafter is the distinction of two types of three-poles, viz. epicyclic gear trains and nodes. Therefore, no far-reaching analogue to other specialised network theories can be set up, neither to the kinetic network theory (lay-out of mechanisms), nor to the electrical network theory (except for the analogue to be put forward in chapter 2).

As mentioned above, a node satisfies two linear equations for angular velocities and one for torques, whereas an epicyclic gear train satisfies one linear equation for angular velocities and two for torques. These sets of linear equations underlie the variator network theory in the same way as the common Kirchhoff's laws do for other network theories.

In the mathematical model a three-pole will be specified by means of a set of three linear equations. A further specification is obtained by a few strikingly simple assumptions, given later on.

For the sake of mathematical rigour and generality, in the mathematical model the names of machine elements should not be used. To avoid too abstract a treatment, however, the terms shaft, epicyclic gear train, and other terms will be used for entities in the mathematical model. In an application any such term indicates an element in the actual design. The double sense of a term is not likely to cause erroneous interpretations or undesired limitations.

Summarising, a three-pole with two linear equations for angular velocities and one for torques is called a node. A three-pole with one linear equation for angular velocities and two for torques is called an epicyclic gear train.

1.4. **Node**

An interconnection of three power transmitting shafts is called a 'node'. The symbol for a node is a dot with three lines; see Fig. 1.4.1. It can readily be accepted that the two equations for angular velocities involve mutual equality of that three angular velocities, while the equation for torques formulates the equilibrium

![Node Diagram](Fig. 1.4.1 Node)
between the torques. The set of equations for a node with shaft ends \(A, B, C\) is

\[
2 \quad \omega_A = \omega_B = \omega_C
\]

\[
3 \quad T_A + T_B + T_C = 0
\]

From 2, 3, and 1, 2, 3 it follows for the shaft powers that

\[
4 \quad P_A + P_B + P_C = 0
\]

### 1.5. Epicyclic gear train

The analytical formulation of an epicyclic gear train can be deduced most easily if done for a black-box unit with all three shafts rotating. As mentioned before, an epicyclic gear train will be represented in the mathematical model as a three-pole with one linear equation for angular velocities and two linear equations for torques. That representation will be justified by the fact that any conceivable design satisfies such a set of equations. By adding a few strikingly simple assumptions, the equations will be specified completely. First, we deduce the equation for angular velocities and next the two for torques.

The symbol of an epicyclic gear train is a circle with three lines; see Fig. 1.5.1.

![Fig. 1.5.1 Epicyclic gear train](image)

Let the equation for angular velocities be written in the general form

\[
a\omega_A + b\omega_B + c\omega_C = d
\]

The only assumption then to be made for specifying the coefficients \(a, b, c, d\) is that in case of internal blocking of the black-box unit, the three co-axial shafts will be allowed to have the same arbitrary angular velocity. In mathematical terms, the three angular velocities, in case of mutual equality, have the same arbitrary value. The identity

\[
(a + b + c)\omega_A = d
\]

has to be satisfied for any arbitrary value of \(\omega_A\).

Hence \((a + b + c) = 0\) and \(d \neq 0\).
Since a, b and c are not simultaneously zero, one of them, say a, may be taken \( a \neq 0 \). Hence
\[
\omega_a - \left(-\frac{b}{a}\right)\omega_b + \left(-\frac{b}{a} - 1\right)\omega_c = 0
\]
Instead of \( -\frac{b}{a} \) we write \( i \) and obtain
\[
\omega_a - i\omega_b + (i - 1)\omega_c = 0
\]
in which \( i \) represents a parameter characteristic of the design of the epicyclic gear train.

3 To avoid the chaotic treatment known from literature on the level customary in engineering, the symmetry in the formula will be restored. For this purpose, the parameter is further specified by two suffixes separated by a virgule (\( / \)). The suffixes determine the sequence of two rotating shafts. Moreover, for a reason explained here below (1.5.7), the parameter is provided with a straight bar placed overhead. So, writing \( \tilde{\omega}_{A/B} \) instead of \( i \), we get
\[
\omega_a - \tilde{\omega}_{A/B}\omega_B + (\tilde{\omega}_{A/B} - 1)\omega_c = 0
\]
in which the meaning of \( \tilde{\omega}_{A/B} \) becomes clear by blocking the shaft end
\[
\tilde{\omega}_{A/B} = \frac{\omega_A}{\omega_B} \quad \text{for} \quad \omega_c = 0
\]
5 The parameter \( \tilde{\omega}_{A/B} \) called 'binary ratio' stands for a gear ratio in a situation with one blocked (C) and two rotating shaft ends (A and B).
Although equation 4 may suggest a preference for a certain sequence of the shaft ends A, B, C, yet such a preference does not exist. Any consistent transposition of suffixes in 4 yields another binary ratio dependent on \( \tilde{\omega}_{A/B} \), without disturbance of the relation between \( \omega_a, \omega_B, \) and \( \omega_c \).
6 A consistent transposition of suffixes throughout the formulae is called a 'permutaUon'. Each sequence of suffixes is attainable. An example of a permutation is the interchange of A and B in equation 5, which results in the relation of a parameter with a reciprocal one.
\[
\tilde{\omega}_{B/A} = \frac{1}{\tilde{\omega}_{A/B}}
\]
7 The straight bar placed overhead distinguishes the binary ratio from the quantity \( \tilde{\omega}_{A/B} \) with a broken bar overhead. The latter is called 'ternary ratio' and is defined by
\[
\tilde{\omega}_{A/B} = \frac{\omega_A}{\omega_B} \quad \text{in a situation with three rotating shafts}
\]
For more details, see chapter 2.
To specify the equations for torques two additional assumptions have to be made.

The first assumption formulates the equilibrium of the three torques

$$\tau_a + \tau_b + \tau_c = 0$$

The other assumption is that the three torques may be simultaneously zero. Hence, the second equation for the torques can be written

$$\alpha \tau_a + \beta \tau_b + \gamma \tau_c = 0$$

while $\alpha$, $\beta$, $\gamma$ are generally different. Elimination of $\tau_c$ yields

$$\left(\frac{\alpha - \gamma}{\beta - \gamma}\right) \tau_a + \tau_b = 0$$

The factor $\left(\frac{\alpha - \gamma}{\beta - \gamma}\right)$ is a constant, further on treated as a parameter. In spite of the seemingly cumbersome way of writing we replace $\left(\frac{\alpha - \gamma}{\beta - \gamma}\right)$ by $\eta_{AB} \eta_{BA}$

$$\eta_{AB} \eta_{BA} \tau_a + \tau_b = 0$$

In the product $\eta_{AB} \eta_{BA}$ the parameter $\eta_{AB}$ is the binary ratio defined above. The meaning of $\eta_{BA}$ becomes clear by the deduction

$$\eta_{BA} = \frac{1}{\eta_{AB}} \eta_{AB} \eta_{BA} = \frac{\omega_B}{\omega_A} \left(\frac{\tau_b}{\tau_a}\right) = -\frac{\beta}{\alpha}$$

for $\omega_c = 0$

The parameter $\eta_{BA}$ is called 'binary efficiency'.

The two equations 10 and 12 for the torques can be written

$$\eta_{AB} \eta_{BA} \tau_a + \tau_b = 0$$

$$\left(1 - \eta_{AB} \eta_{BA}\right) \tau_a + \tau_c = 0$$

The straight bar placed overhead distinguishes the binary efficiency from the quantity $\eta_{BA}$ with a broken bar overhead. The latter is called 'ternary efficiency' and is defined by

$$\hat{\eta}_{BA} = -\frac{\beta}{\alpha}$$

in a situation with three rotating shafts

For more details, see chapter 2.

Finally, consider the sum of shaft powers

$$P_a + P_b + P_c = \omega_a \tau_a + \omega_b \tau_b + \omega_c \tau_c = \left(\omega_a - \eta_{AB} \eta_{BA} \omega_b + (\eta_{AB} \eta_{BA} - 1) \omega_c\right) \tau_a = z \left(1 - \hat{\eta}_{BA}\right) \left(\omega_a - \omega_c\right) \tau_a$$
Generally, this sum is unequal to zero. Apart from the three shaft powers a power with another characteristic has to be distinguished in the epicyclic gear train.

This power is known as the 'dissipative power', or in more popular terms the power loss, indicated by $\varphi$.

$$\varphi + \varrho + \vartheta + \varphi = 0$$

According to the sign conventions a dissipative power is never positive. Dissipative powers are not neglected throughout the study. A neglect would be unacceptable as will be proved later on.

The epicyclic gear train will be discussed extensively in chapter 2.

1.6. Transmission and variator

A common gear drive has two power transmitting shaft ends connected to other units. The torques exerted on these two two shaft ends are not in equilibrium. If the torque on the frame (gear box) is also considered, we do get the equilibrium of three torques. Therefore, three elements will be considered. In this respect a common gear drive is similar to an epicyclic gear train.

In the mathematical model a common gear drive, or more generally, a transmission, will be defined as a particular case of an epicyclic gear train, viz. an epicyclic gear train with one stationary shaft end (angular velocity zero). So, all properties of a transmission to be dealt with in the theory will be taken into account automatically. A transmission deduced from an epicyclic gear train need not have co-axial shafts, for it is not a requirement of the mathematical model. The co-axiality of epicyclic gear trains is only a practical aspect in the design.

Instead of 'gear drive' the more general name 'transmission' is used. A transmission may be any design with two shaft ends ensuring the desired ratios of angular velocities and of torques. The above discussed equilibrium of torques need not in general be considered.

A continuously variable gear drive usually has a design substantially different from that of common gear drives. In our theory the actual design will be disregarded, and the only kind of performance to be considered in principle is the transformation of an angular velocity into another and of a torque into another. Still one aspect remains, namely the one distinguishing a continuously variable gear drive from a common gear drive, i.e. the variation of the transmission ratio.
4 The variation of the transmission ratio is realized by control from outside the variable gear drive, independent of the situation inside.

As to the equations for angular velocities and torques a continuously variable gear drive is similar to a transmission with fixed transmission ratio.

5 A continuously variable gear drive will be called 'variator', mainly to emphasize the possible variation of its transmission ratio. A 'transmission' is considered to have a fixed transmission ratio.

Transmissions and variators are special cases of an epicyclic gear train in which one of the shaft ends is considered permanently blocked. A distinction between binary and ternary parameters need not be made. The angular velocity of the blocked shaft, say \( \omega_a \), is zero. Hence, equation 1.5.4 becomes

\[ \omega_a = \dot{i}_{AB} \omega_B \]

The torque of the blocked shaft end will be ignored and so is equation 1.5.10. Equations 1.5.12 and 1.5.19 become

\[ \dot{i}_{AB} \eta_{BA} \tau_A + \tau_B = 0 \]

\[ \tau_A + \tau_B + \tau_C = 0 \]

The symbol for a transmission is an oval after the example of a driving belt. The mark of the shaft end on the 'pulley' side is the first suffix of the transmission ratio \( i_{AB} \) and the second suffix of the efficiency \( \eta_{BA} \). The inscription \( \eta_{BA} \) is bracketed to enable a distinction from the inscription \( \dot{i}_{AB} \), especially when values are inscribed instead of letter symbols; see Fig. 1.6.1.

The symbol of a variator is that of a transmission, supplemented with an arrow. It is often practical to simplify the notation of the parameters of a variator by introducing such symbols as \( x \) for \( \dot{i}_{AB} \), and \( \eta_x \) for \( \eta_{BA} \); see Fig. 1.6.2.

\[ \text{Fig. 1.6.1 Transmission} \quad \text{Fig. 1.6.2 Variator} \]

1.7 **Power flow**

1 The supply and discharge of energy may be represented by a Sankey-diagram, or more simply by a pattern of arrows.
By convention, a positive power is represented by an arrow towards the unit, a negative power by an arrow away from the unit.

The power flow in a variator network or in an epicyclic gear train may include a closed loop; see Fig. 1.7.2.

DEFINITION A blind power is the smallest power along a closed path of cyclically equally directed powers.

A branched power flow is a power flow that does nowhere include a closed loop with blind power. An example of a branched power flow is given in Fig. 1.7.1.

A remarkable situation arises when in spite of a power supply on at least one shaft end no shaft power can be withdrawn from any part of the variator network. Such a situation, caused by excessive blind power, is called self-locking. See Fig. 1.7.3.

1.8. Foundation of the mathematical model

In the previous sections several definitions were given and assumptions made which underlie the variator network theory. The definitions concern the significance attached, and the names given, to entities in the mathematical model, see remark 1.3.6. The assumptions are self-evident to such an extent that undoubtedly any conceivable design is covered by the theory developed hereafter.

The definitions and assumptions discussed so far are concisely compiled below. For the sake of completeness 7(I), (II), (III) and 9(III), (IV) are added. The discussion of 7 and 9 will follow in chapter 3.

(I) DEFINITION A rotating shaft in the generalised notion is an element that transmits power from one shaft end to another shaft end by no other action than rotation about its axis. It may include a
shaft in the common sense and connecting elements such as couplings
and clutches.

(II) ASSUMPTION A shaft end is characterised by an angle of
rotation and a torque that generally are time-dependent.

(III) DEFINITION The first derivative with respect to time of the
angle of rotation is the angular velocity.

(IV) ASSUMPTION A rotating shaft will be considered an alternating
sequence of elastic and inertial elements.

(V) DEFINITION An elastic element is characterised by a constant
quantity called stiffness.

(VI) ASSUMPTION The torque in an elastic element of a shaft is
equal to the product of the stiffness and the difference of angles of
rotation of the connecting shaft elements.

(VII) DEFINITION An inertial element is characterised by a constant
quantity called moment of inertia.

(VIII) ASSUMPTION The sum of torques acting on an inertial element
is equal to the quotient of the second derivative of the angle of rotation
with respect to time and the moment of inertia.

2 DEFINITION

(I) A three-pole is an element connecting shaft ends of three
rotating shafts;

(II) a shaft end cannot be connected to more than one three-pole;

(III) a three-pole realises a set of three linear equations, either two
for angular velocities and one for torques, or one for angular velocities
and two for torques.

(IV) ASSUMPTION No other elements than three-poles are needed to
describe the connections between rotating shafts.

3 DEFINITION

(I) A three-pole with two equations for angular
velocities and one for torques is called a node.

(II) ASSUMPTION The two equations for the angular velocities
involve mutual equality of these angular velocities.

(III) ASSUMPTION The equation for torques formulates the
equilibrium between the torques.

4 DEFINITION

(I) A three-pole with one equation for angular
velocities and two equations for torques is called an
epicyclic gear train.

(II) ASSUMPTION The equation for the three angular velocities
permits them, in the case of mutual equality, to have the same
arbitrary value.

(III) ASSUMPTION The two equations for torques permit all three
of them to be simultaneously zero.

(IV) ASSUMPTION One of the equations for torques formulates the
equilibrium between the torques.
5 (I) DEFINITION A transmission is an epicyclic gear train with one stationary shaft end.
(II) ASSUMPTION The angular velocity of a stationary shaft end is zero.
(III) ASSUMPTION The torque of a stationary shaft end, as well as the shaft end itself, is left out of consideration.

6 (I) DEFINITION A variator is a transmission of which the ratio between the angular velocities can be varied continuously by control from outside the transmission, independent of the situation inside the transmission.
(II) ASSUMPTION For every ratio between angular velocities there exists one equation for the torques.

7 DEFINITION
(I) A variator network is a coherent system of rotating shafts, epicyclic gear trains, nodes, transmissions and variators;
(II) all angular velocities in a variator network are determined by stating one angular velocity.
(III) all torques in a variator network are determined by stating one torque.

8 (I) DEFINITION A shaft power is the product of the angular velocity of a shaft end and the torque exerted on that shaft end.

9 (I) DEFINITION Dissipative powers occur in epicyclic gear trains, transmissions, as well as variators.
(II) ASSUMPTION A dissipative power has a value less than or equal to zero.
(III) ASSUMPTION For an arbitrarily dissected part of a variator network the sum of shaft powers of all shaft ends and the dissipative powers is zero.
(IV) ASSUMPTION A dissipative power does not influence any relation between angular velocities.
Although the literature of the last half century discusses the single epicyclic gear train in detail, there is still reason to start all over again. A highly consistent mathematical model has to be developed in order to investigate complicated varactor networks. Retroactively, new aspects are found for the analysis of simple systems.

All well-known formulae are deduced with minimum effort; a few new functions are introduced, especially for the determination of the direction of the power flow. A complete summary of possible power flow directions is given. Finally, a very convenient scheme is proposed for design calculations.

The detailed analysis of the single epicyclic gear train in this chapter may be skipped by the reader who is interested in the 'varactor network theory' proper, which starts in chapter 3.

2.1 Angular velocities and transmission ratios

Equation 1.5.4 holds for every sequence of the shaft ends A, B, C. Therefore, the complete equation for angular velocities is

$$w_A - i_A B w_B + (i_A B - 1) w_C = 0 \quad \text{A,B,C permutable}$$

The binary relations $i_{A B}$, $i_{B C}$, ..., all determined if one of them is given. The first relation between binary ratios is obtained by permutation of letters in 2, resulting in

$$i_{B A} = \frac{1}{i_{A B}} \quad \text{A,B,C permutable}$$

The second relation between binary ratios results from a transposition of the letters B and C, and rearrangement of terms in 1

$$w_A + (i_{A C} - 1) w_B - i_{A C} w_C = 0$$
and subtraction of 1 from 4

\[(\tau_{AB} + \tau_{AC} - 1)(\omega_B - \omega_C) = 0\]

Since there is but one relation for the angular velocities, equation 5 must be valid for arbitrary values of \(\omega_B\) and \(\omega_C\). Hence

\[\tau_{AC} = 1 - \tau_{AB} \quad \text{A,B,C permutable}\]

For the illustration of the interdependency of the six binary ratios and several other relations, diagrams with special scales will be used. The scales contract the range of numbers from \(-\infty\) to \(+\infty\) to a finite segment. The semi-reciprocal scale is well known for the purpose and will be used in this study later on. See Fig. 2.1.1.

In the theory of epicyclic gear trains three intervals of equal importance are to be distinguished, viz from \(-\infty\) to 0, from 0 to +1 and from +1 to \(+\infty\). Accordingly, we introduce a scale adapted to this division, here called the 'triple-interval scale'. The positive part of the scale is identical with that of the semi-reciprocal scale. The negative part is condensed in one interval. See Fig. 2.1.2.

Fig. 2.1.1 Semi-reciprocal scale
- distance to the origin (-2, \(-\frac{1}{x}\)) for inscribed values \(x \leq 0\)
- distance to the origin \(x\) for inscribed values \(-1 \leq x \leq 1\)
- distance to the origin \((+2, \frac{1}{x})\) for inscribed values \(x > 1\)

Fig. 2.1.2 Triple-interval scale
- distance to the origin \(\frac{x}{1-x}\) for inscribed values \(x \leq 0\)
- distance to the origin \(x\) for inscribed values \(0 \leq x \leq 1\)
- distance to the origin \((+2, \frac{1}{x})\) for inscribed values \(x > 1\)

The six binary ratios are represented in Fig. 2.1.3 with triple-interval scales.

In many considerations not the angular velocities themselves, but the differences between them play a part. A relation for these differences is equation 1 rewritten as

\[(\omega_A - \omega_C) - \tau_{AB}(\omega_B - \omega_C) = 0 \quad \text{A,B,C permutable}\]

or as an extended proportion

\[\frac{(\omega_A - \omega_C)}{(\omega_B - \omega_C)} = \frac{(\omega_C - \omega_A)}{(\omega_B - \omega_A)} = \frac{(1 - \tau_{AB})}{(-1)(\tau_{AB})} \quad \text{A,B,C permutable}\]
The product of three binary ratios (parameters) with a cyclic suffix sequence

\[ \frac{r_{BA} r_{BC} r_{CA}}{r_{AB}} = \frac{r_{BA}}{r_{AB}} = -1 \]

must not be confused with that of three ternary ratios (instantaneous ratios)

\[ \frac{r_{BA} r_{BC} r_{CA}}{r_{AB}} = \frac{\omega_A \omega_B \omega_C}{\omega_B \omega_C \omega_A} = +1 \]

### 2.2. Torques, powers and efficiencies

Equations 1.5.12 and 1.5.15 hold for every sequence of the shaft ends A, B, C. Therefore, the complete equations are

1. \( \bar{T}_A + \bar{r}_{BA} = 0 \) \hspace{1cm} A, B, C permutable
2. \( (1-\bar{r}_{BA})Y_A + \bar{T}_C = 0 \) \hspace{1cm} A, B, C permutable

Similarly equation 1.2.8 is written

3. \( P_A = \omega_A \bar{r}_A \) \hspace{1cm} A, B, C permutable

4. COROLLARY \( \bar{r}_{BA} = -\frac{P_B}{P_A} \) for \( \omega_C = 0 \) \hspace{1cm} A, B, C permutable

There are six binary efficiencies \( \bar{r}_{BA}, \bar{r}_{AC}, \ldots \), all determined if one of them is given. The first relation between binary efficiencies is obtained by interchanging letters in 4, resulting in
The second relation between binary efficiencies, deduced from a transposition of letters in \( \bar{A} \):
\[
\overline{\eta}_{AB} = \frac{1}{\overline{\eta}_{BA}} \quad \text{A,B,C permutable}
\]

by comparison of 6 with 2 and by means of 2, 1, 6 is

\[
\overline{\eta}_{CA} = \frac{1 - \overline{\eta}_{AB} \overline{\eta}_{BA}}{1 - \overline{\eta}_{AB}} \quad \text{A,B,C permutable}
\]

The product of three binary efficiencies (parameters) with a cyclic suffix sequence

\[
\overline{\eta}_{BA} \overline{\eta}_{AC} \overline{\eta}_{CB} = \overline{\eta}_{BA} \frac{1 - \overline{\eta}_{AB} \overline{\eta}_{BA}}{1 - \overline{\eta}_{AB}} \cdot \frac{1 - \overline{\eta}_{BA} \overline{\eta}_{AB}}{1 - \overline{\eta}_{BA}} = \pm 1
\]

must not be confused with that of three ternary efficiencies (instantaneous quotients)

\[
\overline{\eta}_{BA} \overline{\eta}_{AC} \overline{\eta}_{CB} = \left( -\frac{B}{A} \right) \left( -\frac{C}{B} \right) \left( -\frac{A}{C} \right) = -1
\]

The product \( \overline{\eta}_{AB} \overline{\eta}_{BA} \) is a constant as appears from

\[
\overline{\eta}_{AB} \overline{\eta}_{BA} = \frac{\omega_B}{\omega_B} \left( \frac{\omega_B T_B}{\omega_A T_A} \right) = -\frac{T_B}{T_A} = \overline{\eta}_{AB} \overline{\eta}_{BA}
\]

Finally, 1, 3, 7 and 2, 1, 6 lead via

\[
P_A + \frac{P_B}{\overline{\eta}_{BA}} + \frac{P_C}{\overline{\eta}_{CA}} = 0 \quad \text{A,B,C permutable}
\]

to

12 LEMMA \( \mathcal{P} = 0 \) for \( \overline{\eta}_{BA} = 1 \)

and to the inverse

13 LEMMA \( \overline{\eta}_{BA} = 1 \) for \( \mathcal{P} = 0 \)

Proof Elimination of \( \mathcal{P} \) in 11 and 5, 19 for \( \mathcal{P} = 0 \) gives

\[
(\overline{\eta}_{BA} - 1) \left( \frac{P_B}{\overline{\eta}_{BA}} + \frac{P_C}{1 - \overline{\eta}_{AB} \overline{\eta}_{BA}} \right) = 0 \quad \text{for } \mathcal{P} = 0
\]

Since 14 must be valid for arbitrary values of \( \mathcal{P} \) and \( \mathcal{P} \),

\( (\overline{\eta}_{BA} - 1) = 0 \) for \( \mathcal{P} = 0 \).

The binary parameters \( \overline{\eta}_{AB} \) and \( \overline{\eta}_{BA} \) can be added to the symbol for an epicyclic gear train in the same way as the parameters for a transmission in Fig. 1. 6, 1. An adapted 'oval' between the shaft ends...
2.2. Dissipative power, internal powers

The dissipative power of an epicyclic gear train may be written

\[ P = -P_A - P_B - P_C = -\omega_A \tau_A - \omega_B \tau_B + \omega_C (\tau_A + \tau_B) = \\
-\omega_A \tau_A - (\omega_B - \omega_C) \bar{\tau}_{BA} \tau_A = -(1 - \bar{\eta}_{BA}) \omega_A - \omega_C \tau_A \quad a, b, c \text{ permutable} \]

The appearance of a product of a torque with a difference of angular velocities in this formula motivates the definition of an internal power.

2.3. Definition

\[ P_{ab} = (\omega_A - \omega_B) \tau_A \quad a, b, c \text{ permutable in pairs} \]

There are six internal powers, all determined if a binary ratio, a binary efficiency and one of the internal powers are given. The first relation for internal powers results from a transposition of letters in 2 and from 2, 1, 9 and 2, 2, 1.

\[ P_{ba} = -\bar{\eta}_{BA} P_{ab} \quad a, b, c \text{ permutable in pairs} \]

The second relation results from a transposition of letters in 2 and from 2, 1, 10 and 2, 1, 6.

\[ P_{bc} = \bar{\eta}_{BC} P_{bc} \quad a, b, c \text{ permutable in pairs} \]

The dissipative power is proportional to an internal power.

\[ P = -(1 - \bar{\eta}_{BA}) P_{ab} \quad a, b, c \text{ permutable in pairs} \]

Another important relation follows from 3 and 5

\[ P_{ba} + P_{ab} + P = 0 \quad a, b, c \text{ permutable} \]

3. Remark

In an actual design, only two of the six internal powers can be interpreted as planet powers (see 2.7.2), the others have no actual meaning. However, details of design do not concern the formal
analysis, which leads to a simple and rapid calculation method of the moment and power distribution of a given epicyclic gear train. See 2.3.

2.4. Interrelationships between binary efficiencies

The positive product \( 2 \times 2.8 \) and equation \( 2.2.5 \) teach that from the six binary efficiencies either two or all six are positive. An inversion of sign of a binary efficiency occurs if one of the three torques is zero. In the diagrams 2.4.1 to 2.4.6 inclusive this results in three transition lines. These transition lines and the lines \( T_{AVB} = 0, \ T_{AB} = 1, \) and \( T_{VB} = \infty \) divide each diagram into nine fields. The interrelations of these fields are indicated by numbers in the six diagrams.

![Diagram showing interrelationships between binary efficiencies](image)

\[ \text{Figs. 2.4.1 ... 2.4.6 Interrelationships between binary efficiencies. In the diagrams triple-interval scales are used.} \]

The fields 2, 3, 5, 6, 8, 9 constitute the (shaded) area in which only two binary efficiencies are positive. It now depends on the values of the angular velocities whether three shaft powers are non-negative or at least one of the shaft powers is negative.

An epicyclic gear train is 'self-locking' if one shaft power is positive and the others are non-negative.

The fields 1, 4, and 7 constitute an area for which the six binary efficiencies are simultaneously positive. Because of \( 2.2.11 \) at least one of the shaft powers has to be negative. So the epicyclic gear train cannot be 'self-locking' in this area.

2.5. Inequalities for angular velocities and torques

If \( \omega_A = \omega_B = \omega_C \) is excluded, the proportion \( 2.1.10 \) implies that
2.5.

1. either $\omega_c < \omega_b < \omega_a$ or $\omega_a < \omega_c < \omega_b$ for $l_{AB} > 0$

2. either $\omega_c < \omega_b < \omega_a$ or $\omega_b < \omega_c < \omega_a$ for $0 < l_{AB} < 1$

3. either $\omega_A < \omega_B < \omega_C$ or $\omega_C < \omega_A < \omega_B$ for $l_{AB} > 1$

Thus, for a given design it is definitely established which angular velocity stands in the middle of an inequality. For clarity, the shaft concerned may be represented by a double line in the symbolic representation; see table 2.5.1.

The torque with the largest absolute value has a sign different from the others. Hence, the inequalities for torques can be written

4. $\tau_a$ and $\tau_b$ equal sign, $\tau_c$ largest absolute value for $l_{AB} \tilde{r}_{BA} < 0$

5. $\tau_b$ and $\tau_c$ equal sign, $\tau_a$ largest absolute value for $0 < l_{AB} \tilde{r}_{BA} < 1$

6. $\tau_c$ and $\tau_a$ equal sign, $\tau_b$ largest absolute value for $l_{AB} \tilde{r}_{BA} > 1$

For the analysis of the interrelation of the inequalities 1, 2, 3 and 4, 5, 6 it is not sufficient to consider only the quantities mentioned above. The power flow in an epicyclic gear train is not seldom determined by the moments of inertia inherent to the design. These moments of inertia will be incorporated in auxiliary functions. The auxiliary functions are used in the inequalities which determine the direction of the internal power flow. It will suffice here to consider the moments of inertia $I_a$, $I_b$, $I_c$ in the configuration of Fig. 2.5.1 (transferred to the shafts according to a method in chapter B). The torques exerted on the epicyclic gear train are $\tau_a$, $\tau_b$, $\tau_c$, the torques in the shafts are $M_a$, $M_b$, $M_c$. The angles of rotation of the shafts are $\alpha_a$, $\alpha_b$, $\alpha_c$. Assumption 1.8.1 (VIII) yields

7. $M_a - \tau_a = J_a \tilde{\alpha}_a \quad \alpha_{ABC}$ permutable

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Thus

\[
\begin{bmatrix}
\frac{M_A - J_B \phi_B}{M_A - J_A \phi_A} = \frac{T_0}{T_A} = -\phi \bar{\eta}_{BA} \\
\frac{M_C - J_B \phi_B}{M_C - J_A \phi_A} = \frac{T_C}{T_A} = (\bar{\eta}_{CB} \bar{\eta}_{BA} - 1)
\end{bmatrix}
\]

from which \( \phi_A, \phi_B, \phi_C \) are solved in view of a later elimination.

\[
\phi_A = \frac{\{ \bar{\eta}_{AB} \bar{\eta}_{BA} + (\bar{\eta}_{AB} - 1)(\bar{\eta}_{AB} \bar{\eta}_{BA} - 1) \} M_A + \bar{\eta}_{AB} J_B M_B + (1 - \bar{\eta}_{AB}) J_B M_C}{J_B - \bar{\eta}_{AB} \phi_A - J_A - (\bar{\eta}_{AB} - 1) \phi_B}
\]

\[
\phi_B = \frac{\bar{\eta}_{AB} \bar{\eta}_{BA} - J_A M_A + \{ \bar{\eta}_{AB} - 1)(\bar{\eta}_{AB} \bar{\eta}_{BA} - 1)J_B + 1 \} M_B + (\bar{\eta}_{AB} - 1) J_B \bar{\eta}_{Ba} J_B M_C}{J_B - \bar{\eta}_{AB} \phi_A - J_A - (\bar{\eta}_{AB} - 1) \phi_B}
\]

\[
\phi_C = \frac{-\bar{\eta}_{AB} \bar{\eta}_{BA} - J_A M_A + \bar{\eta}_{AB} \{ \bar{\eta}_{AB} \bar{\eta}_{BA} - 1 \} J_B M_B + \{ 1 - \bar{\eta}_{AB} \bar{\eta}_{BA} \} M_C}{J_B - \bar{\eta}_{AB} \phi_A - J_A - (\bar{\eta}_{AB} - 1) \phi_B}
\]

Now two auxiliary functions \( U_A \) and \( g_{BA} \) are defined.

13 DEFINITION \( U_A = \frac{T_A}{T_A} - \bar{\eta}_{AB} \frac{T_B}{T_B} + (\bar{\eta}_{AB} - 1) \frac{T_C}{T_C} \) A, B, C permutable

which, because of 7 and 10 can also be written as

15 DEFINITION \( g_{BA} = -\frac{1}{\bar{\eta}_{AB}} + \frac{1}{J_B} + \frac{1}{J_C} \) A, B, C permutable

There are six auxiliary functions \( g_{BA}, g_{AC}, \ldots \), between which the relations

17 \( g_{AB} = \frac{1}{g_{BA}} \) A, B, C permutable

18 \( g_{CA} = \frac{1 - \bar{\eta}_{AB} g_{BA}}{1 - \bar{\eta}_{AB}} \) A, B, C permutable

are similar to the relations 2.2.5 and 2.2.7.

The torque \( \tau_A \) via 7 and 11 expressed in the auxiliary functions is

19 \( \tau_A = \frac{U_A}{\bar{\eta}_{AB} - \bar{\eta}_{AB} \bar{\eta}_{BA} - (\bar{\eta}_{BA} - 1) \phi_B} \) A, B, C permutable
The inequalities for epicyclic gear trains, determining the direction of the power flow.

<table>
<thead>
<tr>
<th>$\frac{J_C}{J_A} &lt; \frac{J_B}{J_A}$</th>
<th>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</th>
<th>$\frac{1}{\bar{n}<em>{BA}} &lt; \bar{n}</em>{BA} &lt; \bar{n}_{BA}$</th>
<th>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}} &lt; 0$</th>
<th>$0 &lt; \bar{n}_{BA} &lt; 1$</th>
<th>$1 &lt; \bar{n}_{BA}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</td>
<td>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</td>
<td>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</td>
<td>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</td>
<td>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</td>
<td>$\bar{n}<em>{BA} &lt; \frac{1}{\bar{n}</em>{BA}}$</td>
</tr>
</tbody>
</table>

$\bar{n}_{BA} < \bar{n}_{BA}$

$0 < \frac{J_C}{J_A} < \frac{J_B}{J_A}$

$\frac{J_C}{J_A} < \frac{1}{\bar{n}_{BA}} < 1$

$\frac{1}{\bar{n}_{BA}} < \bar{n}_{BA} < \frac{1}{\bar{n}_{BA}}$

$\bar{n}_{BA} < \frac{1}{\bar{n}_{BA}}$
The dissipative power expressed in these functions is

\[ P = \frac{(\omega_A - \omega_B) U_A (\bar{\eta}_{BA} - 1)}{\bar{\eta}_{BA} (\bar{\eta}_{BA} - 1)} \text{ A, B, C permutable} \]

Assumption 1.3.9 (II) requires \( P \leq 0 \); hence

\[ \text{CONDITION} \quad \frac{(\omega_A - \omega_B) U_A (\bar{\eta}_{BA} - 1)}{\bar{\eta}_{BA} (\bar{\eta}_{BA} - 1)} \leq 0 \text{ A, B, C permutable} \]

This condition and the inequalities 1 to 6 inclusive have been worked out in table 2.5.1 and presented in diagrams 2.5.2 to 2.5.6 inclusive. The only impossible combinations are those for which simultaneously \( \frac{T_A}{U_A} < 0 \), \( \frac{T_B}{U_B} < 0 \), \( \frac{T_C}{U_C} < 0 \).

\[ \begin{array}{c|c|c}
\bar{\eta}_{BA} & \xi > 0 & \bar{\eta}_{BA} \\
\hline
(ACB) & (BAC) & (ABC) \\
(BCA) & (CAB) & (CBA) \\
(BAC) & (CAB) & (CBA) \\
(BCA) & (CAB) & (CBA) \\
\end{array} \]

\[ \begin{array}{c|c|c}
\bar{\eta}_{BA} & \xi > 0 & \bar{\eta}_{BA} \\
\hline
(ACB) & (BAC) & (ABC) \\
(BCA) & (BAC) & (ABC) \\
(ACB) & (BAC) & (ABC) \\
(BCA) & (CAB) & (CBA) \\
\end{array} \]

Figs. 2.5.2 ... 2.5.4 Inequalities for angular velocities when the sign of a torque is known. The abbreviation \( \{ABC\} \) means either \( \omega_A < \omega_B < \omega_C \) or \( \omega_C < \omega_B < \omega_A \).

Fig. 2.5.5 The torque with the largest absolute value
22. The internal powers are zero in the unlikely cases

\[ \bar{\eta}_{BA} = \infty \quad \text{i.e. permanently} \quad \tau_A = 0 \]
\[ \bar{\eta}_{BA} = 0 \quad \text{i.e. permanently} \quad \tau_B = 0 \]
\[ \bar{\eta}_{BA} = \frac{1}{l_{AB}} \quad \text{i.e. permanently} \quad \tau_C = 0 \]
\[ \bar{\eta}_{BA} = 1 \quad \text{i.e. permanently} \quad \omega_A = \omega_B = \omega_C \]

23. The internal powers are zero in the compatible case

\[ \bar{\eta}_{BA} = g_{BA} \]

from which follows \( U_A = U_B = U_C = 0 \) and by virtue of 11, 12, 13:

\[ M_A = J_A \ddot{\theta}_A, \quad M_B = J_B \ddot{\theta}_B, \quad M_C = J_C \ddot{\theta}_C, \]

i.e. each torque causes only the angular acceleration of the shaft end concerned, \( \ddot{\theta}_A = \ddot{\theta}_B = \ddot{\theta}_C = 0 \).

The inequalities for angular velocities and torques in table 2.5.1 were deduced by elimination of the angular accelerations, thus describing only a momentary situation; they determine the direction of the internal power flow in any design. To a certain extent they are decisive as to which cases are impossible, but they need not necessarily imply that a certain supposition is realisable. A further limitation of the possible cases will be found in the equilibrium conditions discussed in chapter 0. 

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2.6. Design of a single epicyclic gear train

1. The component parts of a single epicyclic gear train are two sun gears A, B, one planet carrier C, and planet gears a, b. The shaft ends of A, B, and C are co-axial. Planet gear a meshes sun gear A, likewise b meshes B. A transmission in the planet carrier connects a and b. When the planet gears have a common shaft, or when they are identical, their transmission ratio is \( i_{ab} = +1 \). A planet gear a, a planet gear b, and the transmission between them form a planet group. There are \( h \) planet groups, \( h \geq 1 \).

2. All that will be said about epicyclic gear trains with cylindrical gears on parallel shafts can easily be transferred to epicyclic gear trains with other types of gears. Therefore, the representation in Fig. 2.6.1 with only cylindrical gears on parallel shafts will suffice here.

![Fig. 2.6.1 General design of a single epicyclic gear train with parallel shafts. For clarity, one of the planet groups is omitted and the planet carrier is indicated schematically. An numerical example is added.](image-url)

If the design of the planet groups is completely identical, then there is a condition for the number of planet groups which depends on the number of teeth \( z_a, z_b, z_2, z_3, \) and the ratio \( i_{ab} \).
CONDITION for identical planet groups, equally distributed in the planet carrier

\[
\begin{align*}
&\text{Condition} \ 4. \quad |z_\alpha z_\beta^* - z_\delta z_\gamma^*| \text{ divisible by the number of planet groups} \\
&\text{Condition} \ 5. \quad \frac{z_\beta^*}{z_\delta} = \frac{z_\alpha}{z_\beta} \\
&\text{Condition} \ 6. \quad z_\alpha^* \text{ and } z_\delta^* \text{ mutually indivisible}
\end{align*}
\]

**Proof**  Planet gear \( a \), moving to the place of \( a \) in the next planet group, rotates over \( \frac{z_\alpha}{k} \) pitches and may be shifted \( p \) pitches. With respect to the planet carrier, \( a \) rotates over an angle \( \frac{1}{z_\beta} \left( \frac{z_\alpha}{k} + p \right) \), and \( b \) likewise rotates over an angle \( \frac{1}{z_\delta} \left( \frac{z_\alpha}{k} + q \right) \). Hence

\[
\frac{1}{z_\alpha} \left( \frac{z_\alpha}{k} + p \right) = i a \frac{1}{z_\beta} \left( \frac{z_\alpha}{k} + q \right)
\]

\( p \) and \( q \) arbitrary integers

which is equivalent to

\[
k(qz_\alpha^* - pz_\delta^*) = z_\alpha z_\beta^* - z_\delta z_\alpha^*
\]

\( \rho \) and \( q \) arbitrary integers, \( z_\alpha^* \) and \( z_\delta^* \) defined by 5 and 6.

The term \((qz_\alpha^* - pz_\delta^*)\) may represent any integer, hence condition 4.

**Condition** 3 is unnecessary for planet groups so designed that planet gears can individually be assembled for correct meshing.

**Condition** 3 may be omitted when the equal distribution of the planet groups in the planet carrier is replaced by a distribution in which the angle between two planet groups is a multiple of \( \frac{360^\circ}{|z_\alpha z_\beta^* - z_\delta z_\alpha^*|} \).

**Figs. 2.6.2 to 2.6.8 inclusive** give a few examples of single epicyclic gear trains.

![Diagram](image-url)

**Fig. 2.6.2** Epicyclic gear train with a negative sun ratio
Fig. 2.6.3 Differential gear

Fig. 2.6.4 Epicyclic gear train with a large transmission ratio

Fig. 2.6.5 Epicyclic gear train with a degenerated sun gear

Fig. 2.6.6 Epicyclic gear train with a non-linear degenerated sun gear. $\phi$ is the angle between a vertical plane and the plane of $\mathcal{S}$.

\[ \frac{p_a}{p_b} = \arctan (\cos \phi \tan \psi) \]
The angular velocity of a planet gear $a$ relative to the planet carrier $C$ is

$$\omega_a = r_{aA}(\omega_A - \omega_T)$$

while

$$r_{aA} = \frac{Z_A}{Z_a}$$
2.7. Power flow in a single epicyclic gear train

A shaft power is positive when energy is supplied to the epicyclic gear train and negative when energy is discharged from the epicyclic gear train. See Fig. 2.7.1.

Table 2.7.1 Representation of the power flow when $0 < \eta_{BA} < 1$

<table>
<thead>
<tr>
<th>$\zeta_{AB} &lt; 0 &lt; \eta_{BA}$</th>
<th>(\eta_{BA} &lt; 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>Y_A</td>
<td>Y_B</td>
</tr>
</tbody>
</table>

The arrows of blind powers are marked with a dot. Transposition of A and B gives the power flow for $\eta_{BA} > 1$.
Fig. 2.7.1 Powers in a single epicyclic gear train indicated by arrows in positive directions, when \( C \) is planet carrier.

2. If \( C \) is planet carrier, the internal powers \( P_{ab} \) and \( P_{ba} \) are called planet powers. According to 2.3.6 the dissipative power \( P \) can be related to these planet powers. The dissipative power is negative.

3. If \( C \) is planet carrier, the products \( w_C T_b \) and \( w_C T_a \) are called carrier powers. They are not related to the dissipative power.

The sign of a power reverses when the sign of the angular velocity reverses. The signs of the angular velocities can be described by adding a weight between inequality signs in the inequality of the angular velocities (in four places) or by adding an equalisation to zero (in three places). For instance, \( \omega_b < \omega_c < \omega_a \) becomes

\[
\omega_b = 0 < \omega_c < \omega_a \quad \omega_b < \omega_c = 0 < \omega_a \quad \omega_b < \omega_c < \omega_a = 0
\]

\[
0 < \omega_b < \omega_c < \omega_a \quad \omega_b < 0 < \omega_c < \omega_a \quad \omega_b < \omega_c < 0 < \omega_a \quad \omega_b < \omega_c < \omega_a < 0
\]

This makes possible the diagrammatic representation of the power flows in table 2.7.1.

2.8. Sun ratio and sun efficiency

The coefficients in the equations for angular velocities 2.1.1 and for torques 2.2.1 and 2.2.2 are determined by one binary ratio and one binary efficiency. In practical applications it is convenient to select the parameters \( i_{A/B} \) and \( i_{B/A} \) for the sun gears \( A, B \); these parameters are called sun ratio and sun efficiency.

DEFINITION The sun ratio is a binary ratio of the sun gears

\[
i_{A/B} = \frac{Z_a Z_B}{Z_a Z_b}
\]

where \( C \) is planet carrier.
2. DEFINITION The sun efficiency is $\eta_{BA}$ if $\eta_{A/B}$ is sun ratio.

3. The reciprocal quantity $\eta_{BA}$ is a sun ratio as well, likewise $\eta_{A/B}$ is a sun efficiency. However, in actual applications only one sun ratio and one sun efficiency will suffice. The suffixes of the sun efficiency are the interchanged suffixes of the sun ratio.

4. According to 2.3.3 the sun efficiency $\eta_{BA}$ is the negative ratio of the planet powers $P_a$ and $P_b$. These planet powers depend on the torques and on the relative angular velocities $(\omega_A - \omega_C)$ and $(\omega_B - \omega_C)$. The sun efficiency is equal to the efficiency of the transmission between A and B while the planet carrier is blocked and the same torques and the same relative angular velocities are operative. Now, something can be said about the probable values of the sun efficiency.

5. SUPPOSITION For a given design under given circumstances, for each combination of torque $\tau_A$ and relative angular velocity $(\omega_A - \omega_C)$, if the planet power $P_b > 0$, the other planet power $P_a$ will have only one value (suffixes $a, b$ permutable in pairs).

6. The planet powers cannot be both negative, at least one of them is positive. Hence, supposition 5 leads to

7. COROLLARY For a given design under given circumstances the sun efficiency has a value above 1 or negative for one direction of the internal power flow, and a value below 1 (possibly negative) for the other direction of the internal power flow.

8. For designs of practical importance negative sun efficiencies are improbable. Then the sun efficiency may have two values, one $0 < \eta_{BA} < 0$ and the other $\eta_{BA} > 1$.

9. For most designs in common operative conditions the values of the sun efficiency vary between rather narrow limits near to 1. In the following chapters the sun efficiency is considered constant, either below 1, or (for reversed power flow) above 1. The variations of the sun efficiency, dependent on momentary conditions or due to the influence of the design, should be taken into account in actual applications.

2.9. Rapid calculation scheme

1. Of an epicyclic gear train are supposed to be known:
   - a binary ratio,
   - which shaft end belongs to the planet carrier,
   - the sun efficiencies for both directions of power flow,
   - one torque,
   - two angular velocities.
2. The remaining angular velocity results from equation 2.1.1. The calculation of the remaining torques and the powers is done in a simple table, via internal powers; see table 2.2.1.

Suppose the planet carrier is A and the known torque is $T_C$. Then the internal power is $P_b = (\omega_C - \omega_b)T_C$

If $P_b > 0$, then $\bar{\eta}_{CB} > 1$; if $P_b < 0$, then $\bar{\eta}_{CB} < 1$, which decides the choice $\bar{\eta}_{CB} = 1/\bar{\eta}_b$ or $\bar{\eta}_{CB} = \bar{\eta}_b$. Next, $P_c = -R_{CB}P_b$,

$$P_c = -R_{BC} - P_b, \quad T_0 = \frac{P_b}{(\omega_b - \omega_a)}, \quad T_a = -T_b - T_c$$

After calculating $P_a, P_b, P_c$, a check will be found in $P_a + P_b + P_c + P_T = 0$

3. If the known torque is the torque on the planet carrier, the first internal powers to be determined cannot be the planet powers, hence other binary efficiencies have to be calculated with 2.2.1 to find the relation between the internal powers.

4. In the ALGOL 60 procedure below shaft ends are marked by suffix numbers instead of letters. The procedure operates in principle in the same way as the hand-written calculation scheme.

Table 2.2.1 Three examples of a calculation started from binary ratio $\omega_B = -4$

sun efficiency 0.960 or 0.966, torque $T_C = 1200$ N.m, angular velocities $\omega_B = 3$ rad/s, $\omega_C = 4$ rad/s

<table>
<thead>
<tr>
<th>$\omega_b$</th>
<th>$T_b$</th>
<th>$P_b$</th>
<th>$\omega_c$</th>
<th>$T_c$</th>
<th>$P_c$</th>
<th>$\omega_a$</th>
<th>$T_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A + 8</td>
<td>-200</td>
<td>-1600</td>
<td>B + 3</td>
<td>1000</td>
<td>-3000</td>
<td>C + 4</td>
<td>1200</td>
</tr>
<tr>
<td>B + 3</td>
<td>1000</td>
<td>-3000</td>
<td>A + 8</td>
<td>-230.4</td>
<td>-1843</td>
<td>B + 3</td>
<td>-969.6</td>
</tr>
<tr>
<td>C + 4</td>
<td>1200</td>
<td>4800</td>
<td>C + 4</td>
<td>1200</td>
<td>4800</td>
<td>C + 4</td>
<td>1200</td>
</tr>
<tr>
<td>-5</td>
<td>-1000</td>
<td>5000</td>
<td>ac + 5</td>
<td>-230.4</td>
<td>-1152</td>
<td>ca + 1</td>
<td>1200</td>
</tr>
<tr>
<td>-4</td>
<td>1200</td>
<td>-4800</td>
<td>ca + 1</td>
<td>1200</td>
<td>1200</td>
<td>ca + 1</td>
<td>1200</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>V</td>
<td>-48</td>
<td></td>
<td>V</td>
<td>-38</td>
</tr>
</tbody>
</table>

$\bar{\eta}_{CB} = 0.960$ $\bar{\eta}_{AC} = 0.960$

$T_0 = 0.960$ $T_0 = 0.966$
procedure epicyclic gear train with shaft ends (h1, h2, h3)
   bin ratio;(n1, n2, ratio) carrier:(n3) sun off:(off)
   ang vel:(k1, vel1) ang vel:(k2, vel2) torque:(k3, tor3);

value k1, k2, k3, n1, n2, n3;
integer h1, h2, h3, k1, k2, k3, n1, n2, n3;
real off, ratio, tor3, vel1, vel2;

comment formal parameters:
   h1, h2, h3 suffixes of the shaft ends of an epicyclic gear train
   n1/n2 suffixes of a binary ratio
   ratio value of this binary ratio
   n3 suffix of the planet carrier
   eff value of the sun efficiency, depending on the direction
       of the power flow, in the calculation either maintained
       or replaced by 1/eff
   k1, k2 suffixes of angular velocities
   vel1, vel2 values of these angular velocities
   k3 suffix of a torque
   tor3 value of this torque

The procedure determines the quantities below, the identifiers of which
are not declared in the procedure body:
   sun1, sun2 suffixes of the sun gears in the direction of the planet
   power flow from sun1 to sun2
   omega[h] angular velocity of shaft end h1, h2, h3
   torque[h] torque of shaft end h1, h2, h3
   power[h] power on shaft end h1, h2, h3

The procedure assigns the booleans below, the identifiers of which are
not declared in the procedure body:

second solution true if internal power flow is possible in both
   directions for the same input power on shaft
   end k3 = n3. The quantities of only one solution
   are determined with suffixes
   either n3 = h1, sun1 = h2, sun2 = h3,
   or   n3 = h2, sun1 = h3, sun2 = h1,
   or   n3 = h3, sun1 = h1, sun2 = h2 for a sun
       efficiency entered as a value less than 1. The
       suffixes sun1 and sun2 interchange for a sun
       efficiency above 1. The other solution can be
       obtained by repeating the whole procedure for a
       reciprocal value of the sun efficiency
       false in all other cases;

no solution true if no internal power flow is possible for the given
   input power on shaft end k3 = n3. The angular velocities
   are taken zero, the torques are calculated for sun
   efficiency = 1
       false in all other cases;
begin integer jj, j1, j2, j3; real dispower, uu;
real array u[1:2, 2:3, 1:3], vel, tor[1:3]; boolean determined;
integer procedure N(n); integer n;
N := if n = k1 then 1 else if n = k2 then 2 else 3;
procedure supplution;
begin j3 := 6 - j1 - j2; u[jj, j1, j2] := uu; u[jj, j1, j3] := 1 - uu;
  u[2, 3, 3] := -1; k1 := N(k1); k2 := N(k2); k3 := N(k3);
j1 := n1 := N(n1); j2 := n2 := N(n2); n3 := N(n3); j := 1: uu := ratio;
supplution: vel[k1] := vel1; vel[k2] := vel2; j3 := 6 - k1 - k2;
vel[j3] := u[1, j3, k1] X vel1 + u[1, j3, k2] X vel2; j1 := n3 + 1;
if j1 = 4 then j1 := 1; j2 := n3 + 2; if j2 = 3 then j2 := j2 - 3;
sun1 := j1; sum2 := j2; jj := 2; tor[k3] := tor3;
second solution := true; no solution := false; determined := false;
internal: uu := u[1, 1, 2] X eff; supplution: k1 := k3 + 1;
if k1 = 4 then k1 := 1; k2 := 6 - k1 - k3;
dispower := - tor3 X (vel[k3] - vel[k2]) X (1 - u[2, k3, k1] / u[1, k3, k1]);
if determined then
begin if dispower > 0 then second solution := false; goto out end;
begin second solution := false; eff := 1 / eff; goto internal end;
if dispower > 0 then second solution then
begin no solution := true; jj := 1; vel1 := vel2 := vel3 := 0;
dispower := 0
end; tor[sun1] := - tor3 X u[jj, k3, sun1];
  tor[sun2] := - tor3 X u[jj, k3, sun2];
  tor[sun1] X vel[sun1] = vel[n3] < 0 then
begin sun1 := sum2; sum2 := 6 - n3 - sun1 end;
sun1 := if sum1 = 1 then h1 else if sum1 = 2 then h2 else h3;
sun2 := if sum2 = 1 then h1 else if sum2 = 2 then h2 else h3;
omega[h1] := vel1; omega[h2] := vel2; omega[h3] := vel3;
power[h1] := omega[h1] X torque[h1];
power[h2] := omega[h2] X torque[h2];
power[h3] := omega[h3] X torque[h3]; determined := true;
if second solution \\^ k3 = n3 then begin eff := 1 / eff; goto internal end;
if second solution \\^ k3 \neq n3 then second solution := false;
out:
end;
CHAPTER 3
INTRODUCTORY CONSIDERATIONS ON THE NETWORK THEORY

In this chapter the 'variator network theory' proper starts with schematising the elements mentioned in chapter 1. These elements are considered as interconnections of either three shaft ends or two shaft ends. Their properties are governed by equations between angular velocities and equations between torques. These equations are the basis of the variator network theory.

3.1. Components of a variator network

Special attention was given in chapter 1 to elements that interconnect three shaft ends, generally called three-poles (1.8.2), and divided into nodes (1.8.3) and epicyclic gear trains (1.8.4).

In contrast, elements interconnecting two shaft ends were the rotating shafts (1.8.1), transmissions (1.8.5), and variators (1.8.6). A general term for elements interconnecting two shaft ends is 'branch'.

1 DEFINITION A branch is an interconnection of two shaft ends of different three-poles.

A branch may be realised by a rotating shaft, a transmission, a variator, or by a sequence of these elements connected in series. The angular velocities of the two shaft ends have a certain ratio, and likewise the torques.

2 When three-poles are interconnected by branches, they constitute a 'network'. Of the numerous configurations only those will be considered that act as a variator and, therefore, will be called 'variator network'. See Fig. 3.1.1. To achieve the limitation, two restrictions, already recorded in 1.8.7 (II) and (III), are incorporated in the definition.

3 DEFINITION A variator network is a coherent system of three-poles and branches, in which all angular velocities are determined by stating one angular velocity, and in which all torques are determined by stating one torque.

The acting as a variator, determined by the two restrictions in the definition, yields two important theorems, one regarding the input and output shafts, the other the number of nodes.
3.1. An example of a variator network, in actual design (a) and schematically (b). See also Figs. 1.1.1, 1.1.2, 1.1.3

4. **THEOREM**: A variator network has two shaft ends not incorporated in a branch.

These two shaft ends are known as input shaft and output shaft. In our theory a distinction between input and output shaft is not relevant.

5. **THEOREM**: In a variator network the number of epicyclic gear trains is equal to the number of nodes.

**Proof of both theorems**: The number of unknowns must be equal to the number of equations. The enumeration in the table below yields the two conditions in the lowest level. It results in

\[ \rho = q \] i.e. the number of epicyclic gear trains \( \rho \) is equal to the number of nodes \( q \).

\[ n = 2 \] i.e. in a variator network there are two shaft ends not directly connected to other shaft ends, in other words, not incorporated in a branch.
### 3.1. Number of Terminology

<table>
<thead>
<tr>
<th>number of shaft ends</th>
<th>terminology</th>
<th>number of equations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>ang. velocities</td>
</tr>
<tr>
<td>(3p)</td>
<td>(p) epicyclic gear trains</td>
<td>(p)</td>
</tr>
<tr>
<td>(3q)</td>
<td>(q) nodes</td>
<td>(2q)</td>
</tr>
<tr>
<td>(2r)</td>
<td>(r) transmissions</td>
<td>(r)</td>
</tr>
<tr>
<td>(2v)</td>
<td>(v) variators</td>
<td>(v)</td>
</tr>
</tbody>
</table>

\(\text{total number of shaft ends, of which } n \text{ shaft ends are not connected to other shaft ends, and } \text{specified because of conditions } 1, 8, 7 (\text{II, III})\)

\[
(3p + 3q + 2r + 2v) = 1 + (\frac{5}{2} p + \frac{7}{2} q + 2r + 2v - \frac{1}{2} n)
\]

\[
(3p + 3q + 2r + 2v) = 1 + (\frac{7}{2} p + \frac{5}{2} q + 2r + 2v - \frac{1}{2} n)
\]

### 3.2. Junction and Addition of Transmissions

In the classification and analysis of variator networks it is often practical to reduce the number of transmissions, without disturbing the relations between angular velocities and between torques of the essential parts of the variator network. The most elementary case is the junction of two or more transmissions in series.

**Assertion** Two transmissions in series connection are equivalent to a transmission with a transmission ratio equal to the product of the original transmission ratios, and with an efficiency equal to the product of the original efficiencies. See Figs. 3.2.1 and 3.2.2.

**Proof** The angular velocities and the torques satisfy

\[
\omega_c = i_{C/E} i_{E/D} \omega_0 \\
\tau_0 = -i_{C/E} i_{E/C} \tau_{E/C} \tau_{E/D}/\epsilon
\]
If one of the transmissions or both are variators, then the equivalent transmission is also variator.

**2 COROLLARY** Each branch may be characterised by only one transmission or variator.

For classification purposes the branches are now distinguished in 'hard branches' and 'soft branches'; see Fig. 3.2.3.

![Diagram](image)

Fig. 3.2.3 The variator network of Fig. 3.1.1 schematically. The two transmissions and the variator are joint to one variator, forming a soft branch. The other branch is a hard branch. The three-poles are divided into an epicyclic gear train (circle) and a node (dot).

**3 DEFINITION** A hard branch is a branch without a variator.

**4 DEFINITION** A soft branch is a branch with a variator.

The other method of reducing the number of transmissions, or of simplifying the variator network in another respect, is the adding, in series with an existing transmission, of such a transmission that the branch concerned is simplified to a direct connection. To maintain the relations for angular velocities and torques it is necessary to add identical transmissions in other places, namely round a 'subnetwork'; see Fig. 3.2.4.

![Diagram](image)

Fig. 3.2.4 Addition of transmissions to a subnetwork σ

**5 DEFINITION** A subnetwork is a coherent part of a variator network with at least one three-pole.

**6 ASSERTION** A variator network maintains its properties if equal transmissions are added to all branches which connect an arbitrary subnetwork with the remaining part of the variator network.

**Proof** All relations between angular velocities or between torques
in the subnetwork are homogeneous linear equations. These equations remain valid if all angular velocities in the subnetwork are multiplied by the same value \( i \) and all torques are divide by the same value \( \eta \).

The assertions 1 and 6 induce

7 **THEOREM** Each variator network can be described as a network in which any three-pole is interconnected by any other three-pole along hard branches.

**Proof** Suppose there are two three-poles in a variator network not satisfying 7. Then, a subnetwork exists containing one of the three-poles mentioned, and connected to the remaining part of the variator network by soft branches exclusively. To all these soft branches and possibly to the input shaft or output shaft, equal variators may be added in such a way that one of the soft branches becomes a hard branch. This process may be continued until all three-poles are interconnected along hard branches.

Theorem 7 is applied to simplify the treatment of variator networks by

8 **CONDITION** A variator network will be described in such a way that any three-pole is interconnected to any other three-pole along hard branches.

3.3. **Rapid determination of torque ratios and efficiencies**

In 1.5.12 the torque ratio was written as a product of two parameters, each with two suffixes, \( \frac{a}{b} \eta_{BA} \). The labour of this complicated writing is rewarded by the theorem below expressing the close relationship between torque ratios and transmission ratios.

As an introductory remark it is mentioned here that a torque ratio \( \eta \) equals the corresponding transmission ratio \( i \) if the efficiency \( \eta = 1 \). Now, assumption 1.8.9 (IV), to the effect that a dissipative power does not influence any relation between angular velocities, involves that transmission ratios \( i \) may be found if torque ratios and efficiencies are given. Conversely, one can surmise that torque ratios may be found by adding factors \( \eta \) to the corresponding transmission ratios \( i \), not only for a single unit, but also for combinations of units and especially for the complete variator network. Indeed, formulae of torque ratios may be identical, term by term, with the corresponding formulae of transmission ratios.

The equations between the angular velocities of a three-pole in a variator network are of the types \( 1, 4, 2 \) or \( 2, 1, 1 \). Those for the torques are of the types \( 1, 4, 3 \) or \( 2, 2, 1 \). These are all homogeneous linear equations.
3.3.

The coefficients are either +1 or -1, or depend on the parameters $\tilde{\eta}_{A/B}$ and $\tilde{\eta}_{BA}$. Such pairs of parameters $\tilde{\eta}_{A/B}$ and $\tilde{\eta}_{BA}$ are contributed to the formulae by each epicyclic gear train, transmission and variator. The parameter $\tilde{\eta}_{BA}$ always appears in the product $\tilde{\eta}_{A/B} \tilde{\eta}_{BA}$. In consequence of the homogeneous linearity, 1.8.7 (II) and (III) now read in formulae

1. $\omega_k = \Omega_{A/B} \omega_{A/B}, \ldots, \omega_{BG}, \ldots \omega_k$

2. $T_k = -\Theta_{A/B} \omega_{A/B} \omega_{A/B} \ldots \omega_{BG} \omega_{BG} \ldots T_k$

in which $K$, $G$, ..., stand for arbitrary shaft ends in the variator network. If the notation of 1.5.7 and 1.5.15 is extended to these arbitrary shaft ends, then

3. $\tilde{r}_{A/G} \tilde{r}_{B/G} \omega_{A/B} = \Omega_{A/B} \omega_{A/B}, \ldots, \omega_{BG}, \ldots$

4. $\tilde{r}_{A/G} \tilde{r}_{B/G} \omega_{A/B} = -\Omega_{A/B} \omega_{A/B} \omega_{A/B} \ldots \omega_{BG} \omega_{BG} \ldots T_k$

5. **THEOREM** The functions $\Omega_{A/B}$ and $\Theta_{A/B}$ are identical if the ternary efficiency $\tilde{\eta}_{A/B}$ is made $\tilde{\eta}_{A/B}$ by the assumption that the dissipative powers are zero in the part of the variator network specified in the formulae.

**Proof** Because of $\tilde{\eta}_{A/B}$ and because of lemma 2.2, 13 both functions have the same value for the same independent parameters $\tilde{\eta}_{A/B}, \tilde{\eta}_{BG}, \ldots$. Hence they must be identical.

The most important application of this theorem concerns the efficiency of a variator network. Let $A$ be the input shaft, $Z$ the output shaft, and $C_A/Z = \omega_A/\omega_Z$ the transmission ratio of the variator network expressed in the binary ratios

6. $C_A/Z = \Omega_{A/Z} \omega_{A/B}, \ldots, \omega_{BG}, \ldots$

then because of $\tilde{\eta}_{A/B} = 1$ for $\rho = 0$, theorem 5 holds, i.e.

7. $\tilde{r}_{A/Z} \tilde{r}_{Z/A} = \Omega_{A/Z} \omega_{A/B} \omega_{A/B} \ldots \omega_{BG} \omega_{BG} \ldots$

and the efficiency of the variator network is

8. $\tilde{\eta}_{Z/A} = \frac{\Omega_{A/Z} \omega_{A/B} \omega_{A/B} \ldots \omega_{BG} \omega_{BG} \ldots}{\tilde{r}_{A/Z} \omega_{A/B} \omega_{A/B} \ldots \omega_{BG} \omega_{BG} \ldots}$

This means that for the calculation of the efficiency it suffices to have the disposal of the conjugated efficiencies and the equation of the transmission ratios. It is not necessary to deduce separately a formula for efficiencies.

9. The conjugation of an efficiency to the corresponding transmission ratio is expressed by the efficiency function.
If the transmission ratio is given by a function like \( \hat{b} \), the efficiency function is defined by

\[
eff \left( \Omega_{\text{A}/\text{Z}} \right) = \frac{\Omega_{\text{A}/\text{Z}} \left( \Omega_{\text{A}/\text{B}}, \ldots, \Omega_{\text{H}/\text{I}}, \ldots \right)}{\Omega_{\text{N}/\text{Z}} \left( \Omega_{\text{A}/\text{B}}, \ldots, \Omega_{\text{H}/\text{I}}, \ldots \right)}
\]

**Example** Suppose the transmission ratio \( y \) of a variator network as a whole depends on the transmission ratio \( x \) of the variator and on the binary ratios \( \hat{c}_1 \) and \( \hat{c}_2 \)

\[
y = \frac{(\hat{c}_1 - 1)x + (\hat{c}_2 - 1)\hat{c}_2}{(\hat{c}_1 - 1)x + (\hat{c}_2 - 1)}
\]

The efficiency of the variator is \( \eta_x \), the binary efficiency \( \hat{\eta}_1 \) is conjugated to \( \hat{c}_1 \), likewise \( \hat{\eta}_2 \) to \( \hat{c}_2 \). The assumption that all dissipative powers are zero makes the efficiency of the variator network \( \eta_y = 1 \), hence theorem 5 holds and the efficiency \( \eta_y \) can directly be written

\[
\eta_y = \text{eff}(y) = \frac{\left( \hat{c}_1 \eta_{y-1} + (\hat{c}_2 - 1) \hat{\eta}_1 \right) \left( \hat{c}_1 \eta_{y-1} + (\hat{c}_2 - 1)x \hat{c}_2 \right)}{\left( \hat{c}_1 - 1 \right) x + (\hat{c}_2 - 1)}
\]

**EXAMPLE** Equation 2.2.7 for the binary efficiency of an epicyclic gear train

\[
\hat{\eta}_{\text{CA}} = \frac{1 - \hat{c}_{\text{A}/\text{B}} \hat{\eta}_{\text{AB}}}{1 - \hat{c}_{\text{AB}}}
\]

results from equation 2.1.6

\[
\hat{c}_{\text{AC}} = 1 - \hat{c}_{\text{AB}}
\]

by application of theorem 5.

### 3.4. Conversion

'Conversion' is a mathematical artifice in the analysis and classification of variator networks. It is based upon the distinct identity of the equations for angular velocities and torques. An interchange between \( \omega \) and \( \tau \) produces a system with properties closely related to the original one. Such an interchange will be called 'conversion'.

**DEFINITION** Conversion is the replacement of a variator network by a variator network of similar structure for which in any part between the angular velocities the same equations are operative as originally between the torques; and likewise between the torques the same equations are operative as originally between the angular velocities.
This definition directly leads to

2 PROPERTY The conversion of a converted variator network yields the original variator network.

We shall investigate what conversion means to each of the elements of a variator network.

3 ASSERTION A transmission converts into a transmission with a new transmission ratio and yet, for the original direction, with the original efficiency. See representation in diagram 3.4.1.

\[
\begin{array}{c|c}
\text{C} & \text{D} \\
\hline
\omega_C = \omega_D & \omega_C = \omega_D \\
J_C = J_D & J_C = J_D
\end{array}
\]

Fig. 3.4.1 Conversion of a transmission

REMARK The direction for which the parameters are defined (sequence of suffixes) is reversed for the sake of a simpler representation. The positive direction of rotation is unchanged.

4 ASSERTION A branch without a transmission (direct connection) converts into a particular transmission, i.e. one which simply inverts the direction of rotation. See representation in diagram 3.4.2.

\[
\begin{array}{c|c}
\text{C} & \text{D} \\
\hline
\omega_C = \omega_D & \omega_C = \omega_D \\
J_C = J_D & J_C = J_D
\end{array}
\]

Fig. 3.4.2 Conversion of a direct connection

5 ASSERTION A node converts into a fictitious element called converse node, that is a hypothetical epicyclic gear train with mutually equal torques. See representation in diagram 3.4.3.

\[
\begin{array}{c|c}
\omega_A = \omega_B = \omega_C & \omega_A = \omega_B = \omega_C \\
J_A = J_B = J_C & J_A = J_B = J_C
\end{array}
\]

Fig. 3.4.3 Conversion of a node
ASSERTION An epicyclic gear train converts into a node with two transmissions. See representation in diagram 3.4.4.

\[
\begin{align*}
\omega_A - \tau_{AB} \omega_B + \tau_{BA}^{-1} \omega_C &= 0 \\
\gamma_B &= -\tau_{AB} \eta_{BA} \gamma_A \\
\gamma_C &= (\tau_{AB} \eta_{BA}^{-1}) \gamma_A
\end{align*}
\]

Fig. 3.4.4 Conversion of an epicyclic gear train.

The symbol for an epicyclic gear train is explained in 2.2.15.

The relationship between the converse node and epicyclic gear trains is applied in

ASSERTION An epicyclic gear train can be represented by a converse node with two transmissions. See representation in Fig. 3.4.5.

\[
\begin{align*}
\omega_A - \tau_{AB} \omega_B + \tau_{BA}^{-1} \omega_C &= 0 \\
\gamma_B &= -\tau_{AB} \eta_{BA} \gamma_A \\
\gamma_C &= (\tau_{AB} \eta_{BA}^{-1}) \gamma_A
\end{align*}
\]

Fig. 3.4.5 An epicyclic gear train represented by a converse node with two transmissions.

Proof The epicyclic gear train is converted; then, the node, the two transmissions, and the third shaft (branch without transmission) are converted successively, after which to each shaft end a transmission with \( i = -1 \) and \( \eta = 1 \) is added.

3.5. Interchange of adjacent nodes

The classification of variator networks is considerably simplified by the device of interchange, because several configurations will prove to be equivalent to each other.

DEFINITION Adjacent nodes are nodes connected by a hard branch, thus forming a subnetwork.
By means of assertion 3.2.6 the subnetwork of the adjacent nodes represented in Fig. 3.5.1 can be replaced by that of Fig. 3.5.2. Obviously the subnetwork of Fig. 3.3.3 is equivalent to that of Figs. 3.5.2 and 3.5.1.

![Adjacent nodes](image1)

Figs. 3.5.1 and 3.5.2

![Interchanged adjacent nodes](image2)

Fig. 3.5.3

2 ASSERTION A subnetwork consisting of two adjacent nodes can be replaced by an equivalent one, in which two shaft ends originally connected to the same node are now connected to different nodes. This replacement is called interchange of adjacent nodes.

Proof It is easy to verify for both subnetworks

\[ w_6 = w_2 = i w_1 = i w_5 \]

\[ i \eta_6 + i \eta_2 + \eta_5 + \eta_5 = 0 \]

3 If nodes are interconnected by a soft branch, interchange will cause an increase of the number of variables, analogous to the increase of transmissions in Fig. 3.5.3. Therefore, such an interchange is not allowed.

3.6 Interchange of adjacent epicyclic gear trains

1 DEFINITION Adjacent epicyclic gear trains are epicyclic gear trains connected by a hard branch, thus forming a subnetwork.

2 ASSERTION A subnetwork consisting of two adjacent epicyclic gear trains can be replaced by an equivalent one, in which two shaft ends originally connected to the same epicyclic gear train are now connected to different ones. This replacement is called interchange of adjacent epicyclic gear trains.

Proof Theorem 3.3.5 can be applied to the (binary) efficiencies, because two of the four shaft ends may be supposed to be blocked. So, there is no objection to starting the proof taking all (binary) efficiencies equal to unity. The equations for torques of the sub-

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networks in Figs. 3.6.1 and 3.6.2 are identical to those for angular velocities of the converted subnetworks in Figs. 3.6.3 and 3.6.4.

Fig. 3.6.1 Adjacent epicyclic gear trains

Fig. 3.6.2 Interchanged adjacent epicyclic gear trains

Fig. 3.6.3 Converted adjacent epicyclic gear trains with binary efficiencies put equal to unity

Fig. 3.6.4 Converted interchanged adjacent epicyclic gear trains with binary efficiencies put equal to unity

The original subnetwork and the one with interchanged epicyclic gear trains have to be equivalent. This leads via

\[ \begin{align*}
\dot{\theta}_u^* &= \dot{\theta}_u + \dot{\theta}_u \dot{\theta}_R = \dot{\theta}_u \dot{\theta}_R \\
\dot{\theta}_u \dot{\theta}_R (\dot{\theta}_u + 1) &= (\dot{\theta}_u + 1) \\
(\dot{\theta}_u + 1) &= \dot{\theta}_u (\dot{\theta}_u + 1)
\end{align*} \]

to

\[ \begin{align*}
\dot{\theta}_u^* &= 1 + \dot{\theta}_u \dot{\theta}_R = \dot{\theta}_u \dot{\theta}_R \\
\dot{\theta}_u^* &= \frac{\dot{\theta}_u \dot{\theta}_R}{1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u} \\
\dot{\theta}_R^* &= \frac{1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u}{1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u}
\end{align*} \]

and by virtue of theorem 3.3.5 to

\[ \begin{align*}
\dot{\eta}_u^* &= \frac{(1 + \dot{\theta}_u \dot{\theta}_R) \dot{\eta}_u \dot{\eta}_R - \dot{\theta}_u \dot{\theta}_R \dot{\eta}_u \dot{\eta}_R}{(1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u) \dot{\eta}_u \dot{\eta}_R - \dot{\theta}_u \dot{\theta}_R \dot{\eta}_u \dot{\eta}_R} \\
\dot{\eta}_u^* &= \frac{\dot{\theta}_u \dot{\theta}_R \eta_u \eta_R (1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u)}{(1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u) \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R - \dot{\theta}_u \dot{\theta}_R \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R} \\
\eta_R^* &= \frac{(1 + \dot{\theta}_u \dot{\theta}_R) \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R - \dot{\theta}_u \dot{\theta}_R \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R}{(1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u) \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R - \dot{\theta}_u \dot{\theta}_R \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R} \\
\eta_R^* &= \frac{(1 + \dot{\theta}_u \dot{\theta}_R) \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R - \dot{\theta}_u \dot{\theta}_R \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R}{(1 + \dot{\theta}_u \dot{\theta}_R - \dot{\theta}_u) \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R - \dot{\theta}_u \dot{\theta}_R \dot{\theta}_u \dot{\theta}_R \eta_u \eta_R}
\end{align*} \]
The subnetwork of interchanged epicyclic gear trains is equivalent to the original one if 4 and 5 are satisfied. See Fig. 3.6.5.

Fig. 3.6.5 Example of an interchange of adjacent epicyclic gear trains in actual design (a, b), and schematically (c, d)
4.1. Clos ed netw ork

As appears from definition 1.8.7 (also 3.1.3), a variator network as a whole is equivalent to a variator. The introduction of that equivalent variator, called 'reticulator', presents a convenient mathematical artifice to analyse seemingly different structures at one go.

1 DEFINITION The reticulator of a variator network is a fictitious variator for which the relations between the angular velocities and between the torques are identical with those of the variator network as a whole.

Apart from being a substitute for the variator network, a reticulator may be introduced as a supplement to the variator network; see Fig. 4.1.1.

Fig. 4.1.1 A closed network is a variator network supplemented by a reticulator

2 DEFINITION A closed network is a variator network fictitiously supplemented with the reticulator.

If the reticulator is considered a substitute for the variator network, its dissipative power is equal to the dissipative power of the variator network. If the reticulator is considered a supplement, the power

For the analysis of a variator network we have the disposal of the concepts interchange of epicyclic gear trains (3.6.) or nodes (3.5.), and conversion (3.4.). A few more concepts will be added, viz. closed network, mesh, primitive subnetwork. Especially by the study of meshes and subnetworks certain kinds of networks will prove to be reducible to simpler ones or will prove to have undesired properties. The analysis concerns structures generally with more elements than would be used in practice. Therefore, examples of actual designs will scarcely be given.
difference in the reticularator may be seen as a supplementary power with a value opposite to the dissipative power of the variator network. A reticularator can only in this particular respect be distinguished from a variator in the variator network.

3 COROLLARY The removal of any variator of a closed network leaves a variator network of which the reticularator is identical to the removed variator.

4.2. Meshes

1 DEFINITION A mesh is a closed figure formed by a collection of a number of branches interconnected by the same number of three-poles.

2 DEFINITION A hard mesh is a mesh including no variator, but possibly the reticularator.

3 DEFINITION A soft mesh is a mesh including a variator which is not the reticularator.

Meshes in a variator network can be counted in several ways. To avoid a chaos in which the concept of 'mesh' loses significance, we also define the number of meshes. See Fig. 4.2.1.

Fig. 4.2.1 A five-mesh variator network, with eight three-poles \((p=4)\), two variators \((v=2)\), five meshes \((m=5)\), of which two are hard meshes \((h=2)\). Meshes number 2 and 3 are hard meshes. If A and B indicate the input and output shafts, the lowermost 'variator' is the reticularator.

4 DEFINITION The number of meshes in a variator network, closed by its reticularator, is equal to the largest number of meshes less unity, by which all branches, including the reticularator, are described twice.

5 THEOREM For a variator network with \(p\) epicyclic gear trains the number of meshes \(m\) is \(m = p + 1\).
4.2. **Proof** In a closed network the number of three-poles is \(2p\) and the number of branches including the reticulator is \(3p\). Now the theorem of Euler relating to the numbers of vertices (here three-poles), meshes and branches, reads \(2p + m = 3p + 1\) (Euler's topologic theorem is treated in [39] and [38].)

6 **DEFINITION** The number of hard meshes is the number of meshes in a closed network if all variators, excepting the reticulator, are removed.

7 **THEOREM** For a variator network satisfying 3.3.8, with \(p\) epicyclic gear trains and \(v\) variators, the number of hard meshes is \(h = p - v + 1\).

**Proof** Let the removal of \(v\) variators be coupled with the removal of \(2v\) shaft ends. The removal of each shaft end causes the removal of a three-pole and the junction of two branches into one branch. So \((2p-2v)\) three-poles and \((3p-3v)\) branches remain. If the variator network satisfies 3.2.8, in each hard mesh at least one three-pole does not vanish. Thus, the number of hard meshes remains the same and the theorem of Euler reads \((2p-2v) + h = (3p-3v) + 1\).

**REMARK** If the variator network, contrary to 3.2.8, contains a hard mesh connected to the remaining part of the network by soft branches exclusively, then this hard mesh may vanish in the procedure mentioned above. In that case \(h \geq p - v + 1\).

4.3. **Reiterative networks**

The concept of 'reiterative networks' has importance to the classification of variator networks, and is especially significant for the analysis and later on the synthesis of meshes in variator networks.

1 **DEFINITION** A variator network is a reiterative network if a subnetwork can be indicated that is a variator network in itself. An interchange of adjacent epicyclic gear trains or an interchange of adjacent nodes may precede to the indication of such a subnetwork.

2 Such a subnetwork, considered as a variator network, can be replaced by its reticulator. So, the variator network is reduced to one with fewer three-poles. See examples in Figs. 4.3.1 and 4.3.2. The reduction to a variator network with fewer three-poles justifies the

3 **RESTRICTION** Reiterative networks will be left out of consideration.

The exclusion of reiterative networks has consequences for the constitution of meshes in variator networks with two or more epicyclic gear trains, with respect to the number of three-poles.
Fig. 4.3.1 A reiterative network, in actual design (a), schematically (b), and simplified (c).

Fig. 4.3.2 A reiterative network with interchange of adjacent epicyclic gear trains.

4 ASSERTION In a variator network with two or more epicyclic gear trains the smallest possible mesh, with respect to its number of three-poles, is a soft mesh. This smallest soft mesh or its converse consists of one epicyclic gear train, two nodes, and a variator between the nodes. One of the remaining branches may have a variator as well.

Proof The various configurations conceivable of the smallest soft mesh are easily determined and verified. See Figs. 4.3.3 and 4.3.4. The example in Fig. 4.3.5 shows a mesh which leads to a reiterative network, and thus does not satisfy 4.

Fig. 4.3.3 and 4.3.4 Smallest meshes with one variator.

Fig. 4.3.5 A mesh which leads to a reiterative network.
5. **ASSERTION** The smallest hard mesh consists of two epicyclic gear trains and two nodes in alternating sequence. See Fig. 4.3.6.

**Proof** The possible configurations of the smallest hard mesh are easily found out, and thus the proof can readily be supplied.

![Fig. 4.3.6 Smallest hard mesh](image1)

6. **ASSERTION** In every hard mesh two epicyclic gear trains and two nodes can be traced which constitute a smallest hard mesh if all other epicyclic gear trains and nodes are removed. See Fig. 4.3.7.

![Fig. 4.3.7 Hard mesh. The dotted lines may contain an arbitrary number of epicyclic gear trains and nodes in any sequence](image2)

**Proof** Adjacent nodes in the hard mesh can, according to assertion 3.5.2, be replaced by one node inside the mesh and the other outside; see Fig. 4.3.8. Adjacent epicyclic gear trains can be transposed similarly. Thus, the remaining epicyclic gear trains alternate with nodes, and constitute a mesh with at least two epicyclic gear trains and two nodes satisfying assertion 5.

![Fig. 4.3.8 Transposition of adjacent nodes](image3)

7. **ASSERTION** A soft mesh with one variator has the configuration of a hard mesh in which either one epicyclic gear train or one node is replaced by a variator.

**Proof** The smallest soft mesh may be related to the smallest hard mesh in this way. Further see the proof of assertion 6.

4.4. **Inconsistent networks**

The two restrictions 1.8.7 (II) and (III) in the definition of a variator network, namely all angular velocities to be determined by stating one
angular velocity and all torques to be determined by stating one torque, mean a stringent condition for the degrees of freedom in the constitution of a variable network. The first conclusions were theorem 3.1.4 concerning the input and output shafts, and theorem 3.1.5 concerning the number of epicyclic gear trains and the number of nodes. We shall now investigate the degrees of freedom of subnetworks.

If a subnetwork consists of such units that its set of equations for angular velocities and for torques is inconsistent, then the variator network will be called 'inconsistent'. For instance, the only solution for a number of angular velocities may be \( \omega = 0 \), and the corresponding torques may have arbitrary values, or in another case the torques may be zero and the angular velocities arbitrary.

1. **DEFINITION** A variator network is an inconsistent network if the angular velocities or the torques of any subnetwork do not have one and only one value for each presumed angular velocity or presumed torque in the variator network.

2. In general, the inconsistency is not a property of the subnetwork itself, but a relation between the subnetwork and the remaining part of the variator network.

3. In a subnetwork are found:
   - \( p_{sub} \) epicyclic gear trains,
   - \( q_{sub} \) nodes,
   - \( b \) branches,
   - \( c \) connecting shaft ends to the remaining part of the variator network.

\[ c + 2b = 3p_{sub} + 3q_{sub} \]

4. The remaining part of the variator network determines the angular velocities of \( c_w \) connecting shaft ends and the torques of \( c_t \) connecting shaft ends.

\[ \begin{cases} c_w \geq 1 \\ c_t \geq 1 \end{cases} \]

5. **THEOREM** The angular velocities and the torques of a subnetwork have one and only one value for each presumed angular velocity or presumed torque in the variator network, if the subnetwork satisfies the conditions:

\[ \begin{cases} c_w = \frac{1}{2}(c + p_{sub} - q_{sub}) \geq 1 \\ c_t = \frac{1}{2}(c - p_{sub} + q_{sub}) \geq 1 \end{cases} \]
which are equivalent to the conditions

\[
\begin{align*}
\mathcal{C}_w & \geq 1 \\
\mathcal{C}_r & \geq 1 \\
\mathcal{C}_w + \mathcal{C}_r & = \mathcal{C} \\
\mathcal{C}_w - \mathcal{C}_r & = \mathcal{P}_{\text{sub}} - \mathcal{Q}_{\text{sub}} 
\end{align*}
\]

Proof The total number of shaft ends in the subnetwork is \((3\mathcal{P}_{\text{sub}} + 3\mathcal{Q}_{\text{sub}} + 2\mathcal{b})\). The \((3\mathcal{P}_{\text{sub}} + 3\mathcal{Q}_{\text{sub}} + 2\mathcal{b})\) angular velocities of the shaft ends are determined by \((\mathcal{P}_{\text{sub}} + 2\mathcal{Q}_{\text{sub}} + 3\mathcal{b})\) equations in the subnetwork and by \(\mathcal{C}_w\) angular velocities of the connections. The number of equations must be equal to the number of unknowns, so

\[
\mathcal{P}_{\text{sub}} + 2\mathcal{Q}_{\text{sub}} + 3\mathcal{b} + \mathcal{C}_w = 3\mathcal{P}_{\text{sub}} + 3\mathcal{Q}_{\text{sub}} + 2\mathcal{b}
\]

From 4 and 13 we derive 9 and, by conversion, 10. The sum and the difference of 9 and 10 lead to the equivalent conditions 11 and 12.

The theorem 8 and the definition 1 result in

14 THEOREM A variator network is an inconsistent network if any subnetwork does not satisfy the conditions 9 and 10.

The subnetworks in Figs. 4.4.1 and 4.4.2 do not satisfy 9 and 10.

\begin{align*}
\text{Fig. 4.4.1 A subnetwork by which the shaft ends are blocked} \\
\mathcal{C} = 3 \\
\mathcal{P}_{\text{sub}} = 0 \\
\mathcal{Q}_{\text{sub}} = 3 \\
\frac{1}{2}(\mathcal{C} + \mathcal{P}_{\text{sub}} - \mathcal{Q}_{\text{sub}}) = 0 \\
\frac{1}{2}(\mathcal{C} - \mathcal{P}_{\text{sub}} + \mathcal{Q}_{\text{sub}}) = 3
\end{align*}

\begin{align*}
\text{Fig. 4.4.2 A subnetwork by which the shaft ends are blocked} \\
\mathcal{C} = 3 \\
\mathcal{P}_{\text{sub}} = 3 \\
\mathcal{Q}_{\text{sub}} = 6 \\
\frac{1}{2}(\mathcal{C} + \mathcal{P}_{\text{sub}} - \mathcal{Q}_{\text{sub}}) = 0 \\
\frac{1}{2}(\mathcal{C} - \mathcal{P}_{\text{sub}} + \mathcal{Q}_{\text{sub}}) = 3
\end{align*}

The undesired properties of an inconsistent network justify the

15 RESTRICTION Inconsistent networks will be left out of consideration.
4.5. Reducible networks

1 DEFINITION A reducible network is a variator network in which a subnetwork can be replaced by a subnetwork with fewer three-poles or by a branch, without disturbing the relations for angular velocities and for torques in the remaining part of the variator network.

If a variator network is non-reducible, none of its subnetworks will by definition be equivalent to one with fewer three-poles. One can surmise that there are a limited number of such subnetworks and it may be interesting to examine their configurations. They will be called 'primitive subnetworks'.

2 DEFINITION A subnetwork without variators and without the reticulator is called a primitive subnetwork if the equations of the angular velocities and the torques of its connecting shaft ends cannot be realised by a subnetwork with fewer three-poles.

A primitive subnetwork cannot possibly contain soft branches, since each variator network might then be considered reducible, viz. reducible to its reticulator.

3 A step-by-step investigation of primitive subnetworks, indicated in tables 4.5.1 to 4.5.3 inclusive, has been made by means of the coefficient matrices of the equations for angular velocities. For that purpose, the following two theorems were proved.

4 THEOREM Two subnetworks with identical coefficient matrices for angular velocities will also have identical coefficient matrices for torques, following adaptation of the (binary) efficiencies in one of the subnetworks.

Proof By virtue of restriction 4.4.15 a subnetwork satisfies the conditions 4.4.9 and 4.4.10. The subnetwork determines a number of \( c - c_w = C \) homogeneous linear equations for the angular velocities of the connecting shaft ends and a number of \( c - c_T = C_t \) equations for the torques. The equations for these angular velocities are represented by a coefficient matrix with \( c \) columns and \( C \) rows, those for the torques by a coefficient matrix with \( c \) columns and \( C_t \) rows.

Let the subnetworks \( \alpha \) and \( \alpha^* \) have identical coefficient matrices for angular velocities, then both have the same number of columns \( c \), the same number of rows \( C_t \), and the same difference \( c - C_t = C_w \) of these two numbers. A subnetwork \( \alpha \) with \( (c_w - 2) \) nodes and a shaft end \( A \) is connected to \( C_t \) shaft ends of the subnetwork \( \alpha \). A subnetwork \( \beta \) with \( (c_T - 2) \) epicyclic gear trains and a shaft end \( B \) is connected to \( C_t \) shaft ends of the subnetwork \( \alpha \). See Fig. 4.5.1. Because of 4.4.12

\[
\rho = \rho_{\alpha \beta} + (c_T - 2) = \rho_{\alpha \beta} + (c_w - 2) = q
\]

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In this way, a complete variator network is generated with an input shaft A and an output shaft B.

Let the same subnetworks α and β now be connected to corresponding shaft ends of the subnetwork σ*. Let both variator networks have the same \( \omega_A \) and the same \( \sigma_B \). Because of theorem 3.3.5

5. \[ \omega_B = \Omega_{BA}(\varepsilon \ldots \ldots \cdot) \omega_A \]
6. \[ \tau = -\Omega_{BA}(\varepsilon \eta \ldots \ldots \cdot) \sigma_B \]
7. \[ \omega^* = \Omega_{\alpha A}(\varepsilon \ldots \ldots \cdot) \omega_A \]
8. \[ \tau^* = -\Omega_{\alpha A}(\varepsilon \eta^* \ldots \ldots \cdot) \sigma_B \]

The identical matrices of σ and σ* and the equally complemented subnetworks α and β cause \( \omega_B = \omega^*_B \); whence

9. \[ \Omega_{BA}(\varepsilon \ldots \ldots \cdot) = \Omega_{BA}^*(\varepsilon \ldots \ldots \cdot) \]

The adaptation of the (binary) efficiencies \( \eta^* \ldots \ldots \cdot \) in the subnetwork σ* is defined by

10. \[ \Omega_{\alpha A}(\varepsilon \eta \ldots \ldots \cdot) = \Omega_{\alpha A}^*(\varepsilon \eta^* \ldots \ldots \cdot) \]

through which

11. \[ \tau = \tau^* \]

The combined subnetworks α and β must now have the same coefficient matrix for torques as the combined subnetworks α and σ*. The subnetworks σ and σ* must then have identical coefficient matrices for torques.

12. **THEOREM** Two subnetworks have identical coefficient matrices for angular velocities if the coefficient matrices for torques can be made identical by adaptation of the (binary) efficiencies in both subnetworks.

**Proof** Similar to that of theorem 4 with an adaptation defined by \( \eta = 1 \) and \( \eta^* = 1 \).

**REMARK** An economical adaptation of parameters in only one variator network would suffice, viz. an adaptation of \( \varepsilon^* \) and \( \eta^* \) under the condition...
4.5.

that \( \eta' \) remains unchanged. However, the adaptation \( \eta_1', \eta''_1 \) is the most convenient one.

13 A step-by-step and (up to five connecting shaft ends) exhaustive investigation of primitive subnetworks has resulted in tables 4.5.1 to 4.5.3 inclusive. Examples of reducible subnetworks are given in Figs. 4.5.2 to 4.5.5 inclusive.

Table 4.5.1 Complete survey of primitive subnetworks with three connecting shaft ends

<table>
<thead>
<tr>
<th>C0 = 2</th>
<th>C0 = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>A B C</td>
</tr>
<tr>
<td>([1 \ 1 \ 1] ) ([ \frac{a_{03}}{a_{03}} ] ) = 0</td>
<td>([1 \ 1 \ 0] ) ([ \frac{a_{03}}{a_{03}} ] ) = 0</td>
</tr>
</tbody>
</table>

Table 4.5.2 Complete survey of primitive subnetworks with four connecting shaft ends

<table>
<thead>
<tr>
<th>C0 = 3</th>
<th>C0 = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>A B C</td>
</tr>
<tr>
<td>([1 \ 1 \ 0] ) ([ \frac{a_{03}}{a_{03}} ] ) = 0</td>
<td>([1 \ 0 \ 0] ) ([ \frac{a_{03}}{a_{03}} ] ) = 0</td>
</tr>
<tr>
<td>([0 \ 0 \ 1] ) ([ \frac{a_{03}}{a_{03}} ] ) = 0</td>
<td>([0 \ 1 \ 0] ) ([ \frac{a_{03}}{a_{03}} ] ) = 0</td>
</tr>
</tbody>
</table>

Fig. 4.5.2 A reducible subnetwork and its equivalent primitive subnetwork
Table 4.5.3 Complete survey of primitive subnetworks with five connecting shaft ends

<table>
<thead>
<tr>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
<th>C₆</th>
<th>C₇</th>
<th>C₈</th>
<th>C₉</th>
<th>C₁₀</th>
<th>C₁₁</th>
<th>C₁₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 4.5.3 A reducible subnetwork and its equivalent primitive subnetwork
The avoidance of unnecessarily complicated variator networks justifies the

14 RESTRICTION Reducible networks will be left out of consideration.
A subnetwork with two connecting shaft ends is equivalent to a branch, in other words

15 COROLLARY Reiterative networks are reducible.
Restriction 4.3.3 is included in restriction 14. Finally, an important
limitation of the number of epicyclic gear trains will be discussed.

16 THEOREM Variator networks in which the number, \( p \), of epicyclic
gear trains is more than twice the number, \( v \), of variators, are
reducible.
Proof Consider the input shaft, the output shaft, and the \( 2v \) shaft ends
of the \( v \) variators in an arbitrary variator network with \( p \) epicyclic gear
trains. The equations between the angular velocities of these \( (2v+2) \)
shaft ends consist of linear equations which can be imitated by those of
a new variator network. At first, four shaft ends may be connected to
a smallest hard mesh. Then, each pair of remaining shaft ends may
be connected with the aid of two additional epicyclic gear trains and
two additional nodes. Hence, \( p = 2 + (2v-2) = 2v \) suffices to constitute a
variator network with \( v \) variators. See Fig. 4.5.6.

17 LIMITATION \( p \leq 2v \)
4.6. Parallel branches

1 The motive to define parallel branches is the presupposition that they may be characteristic of a variator network, especially for the distribution of the power flow. The distribution of angular velocities, torques or powers over \( t \) parallel branches is specified by \((t-1)\) mutual ratios. If it is desired to give each of these ratios its own value, \((t-1)\) independent quantities have to be chosen, which implies that at least \((t-1)\) variators are required.

This consideration is disputable, for it is not sure that an inter-independency of the parallel branches is of importance. Consequently, the parallel branches are mentioned here for the sake of completeness only. They are insignificant within the scope of the present study.

2 DEFINITION Complementary subnetworks are two subnetworks of a variator network, one containing the input shaft, the other the output shaft, together including all three-poles only once.

The branches between the two complementary subnetworks are called parallel branches and are distinguished in hard and soft parallel branches, more or less similar to the previous concepts of hard and soft branches. See Fig. 4.6.1.

![Fig. 4.6.1 Complementary subnetworks and parallel branches](image)

3 DEFINITION Hard parallel branches are the branches interconnecting two complementary subnetworks for which, via each of these branches, the connections between input shaft and output shaft are maintained if all variators are removed.

4 DEFINITION Soft parallel branches are the branches interconnecting two complementary subnetworks for which, via each of these branches, the connections between input shaft and output shaft are interrupted if one or more variators are removed.

5 THEOREM If \( p \) is the number of epicyclic gear trains, \( v \) the number of variators, and \( t \) the number of hard parallel branches, then \( t \leq p-v+1 \).

Proof Let one of the complementary subnetworks have \( k_A \) three-poles and \( v_A \) variators, the other \( k_B \) three-poles and \( v_B \) variators. There are \( v \) soft parallel branches with a variator in each of the branches themselves, and \( s \) remaining soft parallel branches.

\[ k_A + k_B = 2p \]
If in the complementary subnetwork $A$ all variators are removed and all parallel branches are disconnected, then there are $(1 + 2v_A + v_v + t + s)$ unconnected shaft ends, including the input shaft. In the simplest case, the complementary subnetwork consists of one three-pole with three shaft ends, so that

$$1 + 2v_A + v_v + t + s = 3$$

for $k_A = 1$

If $k_A$ increases to $(k_A + 1)$ the number of unconnected shaft ends either decreases by 1, or increases by 1, or increases by 3. The latter case can always be avoided by applying theorem 3.2.7. Hence

$$1 + 2v_A + v_v + t + s \leq k_A + 2$$

and also

$$1 + 2v_B + v_v + t + s \leq k_B + 2$$

From 6, 7, 8, 9 follows $t + s \leq p - v + 1$, and, after omission of $s$, the theorem.

The number $p - v + 1$ is the number of hard meshes, $h$, according theorem 4.2.7. The number of hard parallel branches, $t$, depends on the place of the dissection; see Fig. 4.6.2. If, instead of the problematic number $t$, the maximum number of hard parallel branches, $t_{max}$, is considered, theorem 5 may be written $t_{max} \leq h$.

![Fig. 4.6.2 Different dissections of a variator network](image)

**Assertion** A network for which

contains at least one hard mesh without the reticulator, and at least three variators.

**Proof** The inequality $t_{max} \geq 0$ and supposition 11 lead to $h \geq 1$. This means that at least one hard mesh exists. If this hard mesh contains the reticulator, a hard parallel branch exists, $t_{max} \geq 1$, and by 11 $h \geq 2$. Finally, a hard mesh without the reticulator will be found. Between the hard mesh with the reticulator and the hard mesh without it, there is one hard branch to satisfy condition 3.2.8, and no more than this.
one hard branch to satisfy \( 11 \). In addition to the hard branch, at least two soft branches must interconnect the two hard meshes to avoid a reiterative network. Then, one interconnection must be added to avoid an inconsistent network. Hence, in addition to the hard branch, three soft branches interconnect the two hard meshes. An example is given in Fig. 4.6.3.

![Variator network with \( t < p - v + 1 \). The dissection presents two soft parallel branches and one hard branch](image-url)

**Fig. 4.6.3** Variator network with \( t < p - v + 1 \). The dissection presents two soft parallel branches and one hard branch.
5.1. Responsivity

The 'responsivity' is a measure for the influence of a slight virtual variation of the transmission ratio $x$ of an individual variator on the transmission ratio $y$ of the variator network. It must have a multiplicative character because of the multiplicative character of the transmission ratios. The derivative $\frac{dy}{dx}$ would be inadequate because of its additive character. The concept of 'responsivity' is identical to the concept of sensitivity in older control system theories. Another name is introduced here to avoid confusion with modern interpretations of sensitivity, and to emphasise the significance of the concept in the present variator network theory, especially for the distribution of power.

**DEFINITION** In a closed network the shaft ends A and B are interconnected by some variator or by the reticulator. The same is true of the shaft ends C and D. See Fig. 5.1.1. When the other variators that are possibly built in are supposed to have fixed values of their transmission ratios, a responsivity is

![Diagram of a variator network](image)

**Fig. 5.1.1** A closed network. The reticulator between input shaft A and output shaft B is indicated by its transmission ratio $y$, and a variator between shaft ends C and D is indicated by $x$

$$\lambda_{AC} = \frac{d \log(u_C)}{d \log(u_B)}$$

**Table 1**

<table>
<thead>
<tr>
<th>Permutation</th>
<th>Responsivity</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A,B,C,D</td>
<td></td>
<td>A,B,C,D permutable in pairs</td>
</tr>
</tbody>
</table>

The 8 permutations of A, B, C, D produce eight responsivities of a variator network with one variator. If one responsivity is given, the others are determined by the relations

$$\lambda_{CA} = \frac{1}{\lambda_{AC}}$$

A,B,C,D permutable

$$\lambda_{AD} = -\lambda_{AC}$$

A,B,C,D permutable
It is convenient to use abbreviations for the transmission ratios

\[ x = \frac{w_c}{w_b} \quad \text{(variator)} \]

\[ y = \frac{w_A}{w_B} \quad \text{(reticulator)} \]

and to confine ourselves to one of the responsivities, say

\[ \lambda_{AC} = \frac{y}{y} = \frac{x}{y} \frac{dy}{dx} \]

8. The number of mutually independent responsivities is equal to the number of variators. For instance, the 24 responsivities of a variator network with two variators have mutual relations like 3 and 4, which reduces the responsivities to be considered to

\[ \begin{align*}
\lambda_{AC} &= \frac{x}{y} \frac{\partial y}{\partial x} \\
\lambda_{CE} &= \frac{z}{x} \frac{\partial x}{\partial z} \\
\lambda_{EA} &= \frac{y}{z} \frac{\partial z}{\partial y}
\end{align*} \]

The product of these responsivities is

\[ \lambda_{AC} \lambda_{CE} \lambda_{EA} = -1 \]

and therefore only two are mutually independent.

5.2. Distribution of power

For practical applications it is important to know the power transmitted by an individual variator compared with the power supplied to the input shaft. In a variator network with shaft ends denoted according to 5.1.1 the distribution of power is represented by the ternary efficiency

\[ \hat{\eta}_{CA} = -\frac{P_c}{P_A} \quad \text{A.B permutable} \]

\[ \hat{\eta}_{CA} = -\frac{P_c}{P_A} \quad \text{c.c. permutable} \]

\[ \hat{\eta}_{CA} = -\frac{P_c}{P_A} \quad \text{A.B.C.D permutable in pairs} \]

There are eight ternary efficiencies to represent the distribution of power between a variator and the reticulator. Their mutual relations are

\[ \hat{\eta}_{AC} = \frac{1}{\hat{\eta}_{CA}} \quad \text{A.A.C.D permutable} \]

\[ \hat{\eta}_{DA} = \hat{\eta}_{CA} \hat{\eta}_{DC} \quad \text{A.A.C.D permutable} \]

It is convenient to use abbreviations for the efficiencies
and to confine ourselves to one of the above ternary efficiencies, say $\eta_{CA}$. If $\eta_{CA}, \eta_A$ and $\eta_B$ are given, all ternary efficiencies are determined.

The distribution of power has a very important relation to the responsivity, expressed in the following three theorems.

**THEOREM** If the dissipative powers in a variator network are zero, then the responsivity is equal to a ternary efficiency which represents the distribution of power

\[ \hat{\eta}_{CA} = \lambda_{AC} \quad \text{for} \quad \eta_A = \eta_B = 1 \]

**Proof** For the subnetwork with shaft ends A, B, C, D, see Fig. 5.2.1.

![Fig. 5.2.1](image_url) A closed network. The reticulator between input shaft A and output shaft B is indicated by its transmission ratio $y$, and a variator between shaft ends C and D is indicated by $x$

the sum of powers is zero. Hence

\[ dp_A + dp_B + dp_C + dp_D + dp_v = 0 \]

and

\[ \tau_A \frac{dw_A}{w_B} - \eta_A \frac{w_A}{w_B} \frac{dw_B}{w_B} + \tau_C \frac{w_C}{w_D} \frac{dw_C}{w_D} + \frac{dp_v}{w_v} = 0 \]

Following substitution of $\eta_A = 1, \eta_B = 1, dp_v = 0$,

\[ \tau_A \frac{dw_B}{w_B} - \frac{w_A}{w_B} \frac{dw_B}{w_B} + \tau_C \frac{w_C}{w_D} \frac{dw_C}{w_D} = 0 \]

\[ \tau_A \frac{d(w_A)}{w_B} + \tau_C \frac{w_C}{w_D} \frac{dw_C}{w_D} = 0 \]

\[ \frac{P_A}{y} + \frac{P_C}{x} = 0 \]

\[ \hat{\eta}_{CA} = -\frac{P_A}{P_A} = \frac{dy}{y} + \frac{dx}{x} = \lambda_{AC} \]

Hence the theorem.

**THEOREM** If a function $\theta$ expresses the relation between the torques...
on shaft ends A and C in the form

\[ \frac{T}{\lambda} = \Theta_{\text{eff}}(\Theta) \]

where \( \Theta_{\text{eff}}(\Theta) \) represents the efficiency function defined in 3.3.10, the angular velocities satisfy

\[ \frac{\omega_A}{\omega_C} = \frac{\Theta}{\lambda_{AC}} \]

and the distribution of power satisfies

\[ \Phi_{CA} = \lambda_{AC} \Theta_{\text{eff}}(\Theta) \]

Proof The ratio of angular velocities

\[ \frac{\omega_A}{\omega_C} = \frac{1}{\lambda_{AC} \Theta_{\text{eff}}(\Theta)} \]

does not change if the dissipative powers are supposed to be zero. Theorem 6 yields 10, and then, 11.

**THEOREM** If a function \( \Omega \) expresses the relation between angular velocities on shaft ends A and C in the form

\[ \frac{\omega_A}{\omega_C} = \Omega \]

then the torques satisfy

\[ \frac{T}{\lambda} = -\Omega \lambda_{AC} \Theta_{\text{eff}}(\Omega \lambda_{AC}) \]

and the distribution of power satisfies

\[ \Phi_{CA} = \lambda_{AC} \Theta_{\text{eff}}(\Omega \lambda_{AC}) \]

Proof An indirect proof, using \( \Theta = \Omega \lambda_{AC} \)

**EXAMPLE** Consider a variator network with one variator and no other transmissions (nota bene), schematically represented by Fig. 5.2.1. The torques mentioned in assumption 1.8.5 (III) are torques of the variator and the reticulator.

\[ (1-x\eta_x)\frac{T}{\lambda} + (1-y\eta_y) = 0 \]

which leads to

\[ \frac{T}{\lambda} = \frac{(1-y\eta_y)}{(1-x\eta_x)} \]

\[ \frac{\omega_A}{\omega_C} = \frac{1}{\lambda_{AC}} \frac{(1-y)}{(1-x)} \]

\[ \Phi_{CA} = \lambda_{AC} \frac{(1-x)(1-y\eta_y)}{(1-y)(1-x\eta_x)} \]
5.3. **Backbone chain**

The concept 'backbone chain' will be introduced as a mathematical artifice to have a convenient deduction method for the formulae of a variator network.

1 **DEFINITION** A backbone chain is a collection of \( n \) nodes and \( n \) converse nodes connected to each other in one line, including neither a transmission nor a mesh.

2 **COROLLARY** There are \((2n-1)\) interconnections between the three-poles and \((2n+2)\) shaft ends. See example in Fig. 5.3.1.

![Fig. 5.3.1 A backbone chain](image)

3 **ASSERTION** A collection of \( n \) epicyclic gear trains and \( n \) nodes interconnected to each other in one line can be represented by a backbone chain with transmissions added to each of the shaft ends of the backbone chain.

**Proof** The epicyclic gear trains are replaced by converse nodes and transmissions in accordance with assertion 3.4.7. The transmissions between the three-poles are eliminated by applying assertions 3.2.1 and 3.2.6. See example in Fig. 5.3.2.

![Fig. 5.3.2 Illustration of assertion 5.3.3](image)

4 A backbone chain has no parameters, neither a transmission ratio nor an efficiency.

5.4. **Classification of variator networks**

**Exclusions**

Several variator networks have already been excluded, viz.
- inconsistent networks (restriction 4.4.15)
- reducible networks (restriction 4.5.14 and limitation 4.5.17), among which
- reiterative networks (corollary 4.5.15);
while condition 3.2.8 protects the configuration of a variator network against an unnecessary increase of the number of variators.

Moreover, it is important to note that

1. a variator is a variator network.

2. Consequently, variator networks with at least one epicyclic gear train, which contain a variator on the input shaft or on the output shaft, are reiterative networks.

3. By virtue of theorem 3.2.7 a variator network without hard parallel branches is equivalent to one with a variator on the input shaft or on the output shaft. Such a variator network is a reiterative network. See Fig. 5.4.1.

![Image](image_url)

**Fig. 5.4.1** A double variable transmission and its equivalent reiterative network

### Parameters

4. The parameters applied to the classification of variator networks are

- $\rho$ number of epicyclic gear trains,
- $\nu$ number of variators.

A third parameter may be used, viz.

$t_{\text{max}}$ maximum number of hard parallel branches

for which in accordance with theorem 4.6.5

$$t_{\text{max}} \leq \rho - \nu + 1$$

5. A fourth parameter

- $h$ number of hard meshes

depends on $\rho$ and $\nu$ according to theorem 4.2.7.

6. $h = \rho - \nu + 1$

From 5 and 6 it follows that

7. $t_{\text{max}} \leq h$
Classification of variator networks

Table 5.4.1 gives a classification scheme for variator networks, arranged in order of the parameters \( v \) and \( h \). The only 'variator network' to be mentioned for \( h = 0 \) is a variator itself, according to 1 and 2. Variator networks with \( p > 2v \) are reducible, according to 4.5.16. Variator networks with one or two variators satisfy \( t_{\text{max}} = \frac{h}{v} \), as may be concluded from assertion 4.8.10. The variator networks in this group, \( v = 1, p \leq 2 \) and \( v = 2, p \leq 4 \) will be dealt with briefly in this chapter. In section 5.7, a recapitulation will be given, based on this classification scheme.

Table 5.4.1 Classification scheme

<table>
<thead>
<tr>
<th>( h = p - v + 1 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v = 1 )</td>
<td>( p = 0 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>( t_{\text{max}} = h )</td>
<td>( p &gt; 2v )</td>
</tr>
<tr>
<td>( v = 2 )</td>
<td>( p = 0 )</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>( p &gt; 2v )</td>
<td></td>
</tr>
<tr>
<td>( v = 3 )</td>
<td>( p = 0 )</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>( p &gt; 2v )</td>
</tr>
<tr>
<td>( v = 4 )</td>
<td>( p = 0 )</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

5.5. Variator networks with one variator

1 The removal of the variator in a variator network with one variator leaves a subnetwork with four connecting shaft ends. This subnetwork has to be a primitive one, in accordance with table 4.5.2, and must have equal numbers of epicyclic gear trains and nodes. From the primitive subnetworks via the closed networks the operational schemes of the variator networks are made up. Variator networks with three or more epicyclic gear trains are reducible. But for the variator network mentioned in 5.4.1 (variator by itself) there are two types of variator networks with one variator, called 'variable shunt' and 'variable bridge', defined below.

The variable shunt

2 DEFINITION A variable shunt is a variator network with 1 epicyclic gear train, 1 node, and 1 variator, of which the closed network is given in Fig. 5.5.1 and the operational scheme in Fig. 5.5.2.

3 It is of no interest which of the shaft ends A and B is input shaft or output shaft. Therefore, it suffices to study only one variable shunt.
Fig. 5.5.1 Closed network of a variable shunt. The reticulator and the variator are represented by dotted lines.

Especially for comparison with other variator networks it is profitable to represent the variable shunt by a backbone chain and two transmissions. The quantities in the representation with a backbone chain are marked by an asterisk; see Fig. 5.5.3. The equations for the transmission ratios and efficiencies can be derived by one of the following methods.

The first method is to write the equations in matrix notation, via

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & x^* \\
1 & 0 & y^*
\end{pmatrix}
\begin{pmatrix}
x^*
\end{pmatrix}
= 0,
\quad
\omega^*_0 = \omega^*_g
\]

and elimination of \(\omega^*_a, \omega^*_b, \omega^*_c, \omega^*_d,\)

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & x^*
\end{pmatrix}
\begin{pmatrix}
x^*+y^*+1 = 0
\end{pmatrix}
\]

resulting in

\[
x^*+y^*+1 = 0
\]

and, by virtue of theorem 3.2.5,

\[
x^*\eta^*_a+y^*\eta^*_c+1 = 0
\]
Another method is the direct reading of Fig. 5.5.4. The sum of angular velocities of the three shaft ends of the converse node is zero, resulting in equation 6. The sum of torques of the three shaft ends of the node is zero, resulting in equation 7.

\[ \lambda_{AC} = \frac{x^n}{y^n} \]

For a more extensive analysis of the variable shunt the reader is referred to chapter 6.

The variable bridge

DEFINITION A variable bridge is a variator network with 2 epicyclic gear trains, 2 nodes, and 1 variator, of which the closed network is given in Fig. 5.5.5 and the operational scheme in either Fig. 5.5.6 or 5.5.7.

The configurations of Figs. 5.5.6 and 5.5.7 are deduced from each other by a transposition of letters or by conversion. Therefore, it suffices to study only one variable bridge, say in the configuration of Fig. 5.5.6. The representation of a variable bridge by means of a backbone chain is given in Fig. 5.5.8.
The transmission with a transmission ratio \( \rho^* \) and an efficiency \( \eta_0^* \) is a non-variable transmission resulting from the reduction of a hard mesh to a composing chain.

The equations for the transmission ratios and efficiencies can be derived by one of the following methods.

The first method is to write the equations in matrix notation, via

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & \rho^*
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5
\end{pmatrix} = 0
\]

and elimination of \( w_1 \) to \( w_5 \)

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & \rho^*
\end{pmatrix}
\begin{pmatrix}
w_2 \\
w_3 \\
w_4 \\
w_5 \\
w_5
\end{pmatrix} = 0
\]

resulting in

\[ x^*y^* + x^* - y^* - \rho^* = 0 \]

and by virtue of theorem 3.3.5

\[ x^*\eta_0^*y^*n_0^* + x^*n_0^* - y^*\eta_0^* - \rho^*\eta_0^* = 0 \]

Another method is the direct reading of Fig. 5.5.9, where the angular velocities are expressed in \( \omega_0^* \) and the torques in \( T_C^* \).

\[ \lambda_{AC} = \frac{x^*(1+y^*)}{y^*(1-x^*)} = \frac{y^*+\rho^*}{x^*-\rho^*} \]

For a more extensive analysis of the variable bridge the reader is referred to chapter 7.
The removal of the variators leaves a subnetwork with six connecting shaft ends. Instead of an investigation into primitive subnetworks with six connecting shaft ends, even if restricted to those with three epicyclic gear trains and three nodes, it is preferable to make a deduction of the variator networks directly.

The double variable shunt

DEFINITION A double variable shunt is a variator network with two epicyclic gear trains, two nodes, and two variators.

The four three-poles are interconnected by hard branches either in chain, see Fig. 5.6.1, or in star, see Fig. 5.6.2. All arrangements in star constitute reiterative networks. So, the arrangements of a double variable shunt are derived from three-poles in chain. To avoid other reiterative networks, the epicyclic gear trains and nodes must alternate in this line. The closed network is given in Fig. 5.6.3 and the operational schemes in Figs. 5.6.4 to 5.6.6 inclusive.

![Diagram](image)

Figs. 5.6.1 and 5.6.2 Four three-poles in chain and in star

![Diagram](image)

Fig. 5.6.3 Closed network of a double variable shunt

Figs. 5.6.4 ... 5.6.6 Double variable shunts

The configurations of Figs. 5.6.4 to 5.6.6 inclusive are deduced from each other by transposition of letters. Therefore, it suffices to study only one variable bridge, say in the configuration of Fig. 5.6.4.

The representation of a double variable shunt by a backbone chain and three transmissions is given in Fig. 5.6.7. The equations for the
transmission ratios and the efficiencies, resulting from

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & x^0 \\
0 & 0 & 1 & y^0 \\
1 & 0 & 0 & z^0 \\
\end{bmatrix} = 0
\]

and theorem 3.3.5, are

\[x^n y^x + x^n - y^n - z^n = 0\]

\[x^n y^x y^n y^x + x^n y^n y^n - z^n y^n = 0\]

Fig. 5.6.7 The backbone chain in the representation of the double variable shunt

Let the considered responsivities be \(\lambda_{AC}, \lambda_{CE}, \lambda_{EA}\), and the considered ternary efficiencies \(\eta_{CA}, \eta_{EC}, \eta_{AE}\). These quantities are represented in table 5.6.1.

Table 5.6.1 Double variable shunt; closed network, backbone chain, responsivities, ternary efficiencies

<table>
<thead>
<tr>
<th>Network</th>
<th>Responsivities</th>
<th>Ternary Efficiencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(\lambda_{AC} = \frac{y+z}{x-z})</td>
<td>(\eta_{CA} = \lambda_{AC} \text{eff}(\frac{1}{x-z}))</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{CE} = \frac{y+z}{x-z})</td>
<td>(\eta_{EC} = \lambda_{CE})</td>
</tr>
<tr>
<td></td>
<td>(\lambda_{EA} = \frac{x-z}{z})</td>
<td>(\eta_{AE} = \lambda_{EA} \text{eff}(1-z))</td>
</tr>
</tbody>
</table>

The double variable bridge

**Definition** A double variable bridge is a variator network with 3 epicyclic gear trains, 3 nodes, and 2 variators.

It has one hard mesh without the reticulator. With the help of assertion 4.3.6 all double variable bridges can be generated by adding one node and one epicyclic gear train to the smallest hard mesh. The working-out in 10 closed networks or 24 variator networks is listed in table 5.6.2.

The extended double variable bridge

**Definition** An extended double variable bridge is a variator network with 4 epicyclic gear trains, 4 nodes, and 2 variators.
It has two hard meshes without the reticulator. These hard meshes have two, one, or no common hard branches. Variator networks based on two hard meshes without common hard branch, shown in Figs. 5.6.8 and 5.6.9, constitute group 4 in table 5.6.3. A non-reducible network based on two hard meshes with one common hard branch contains a subnetwork, an example of which is shown in Fig. 5.6.10.

![Diagram of two hard meshes without common hard branch](image)

Figs. 5.6.8 and 5.6.9 Two hard meshes without common hard branch

![Diagram of two hard meshes with one common hard branch](image)

Fig. 5.6.10 Two hard meshes with one common hard branch

The smallest hard mesh in this subnetwork may be connected to the remaining chain in several ways, see group 2 in table 5.6.3. A variator network, or its converse, based on two hard meshes with two common hard branches can be described by the subnetwork in Fig. 5.6.11, to which a node has to be added. Addition of the node to the places indicated in Fig. 5.6.11 results in variator networks of group 1 or 2. Addition in all other places results in variator networks of group 3 in table 5.6.3. In the variator networks of each group primitive subnetworks are recognisable with a structure more or less characteristic of that group.

![Diagram of two hard meshes with two common hard branches](image)

Fig. 5.6.11 Two hard meshes with two common hard branches in equivalent configurations. To generate an extended variable bridge, a node has to be added to either of the indicated branches.
Table 5.6.2 Double variable bridges; closed networks, number of variable networks represented by those closed networks, backbone chains, responsivities, ternary efficiencies

<table>
<thead>
<tr>
<th>1 network</th>
<th>$xy-xz-yz+(p+1)\cdot p = 0$</th>
<th>$\lambda_{AC} = \frac{x(y-z)}{y(x-z)}$</th>
<th>$\eta_{CA} = \lambda_{AC} \cdot \text{eff}(z+y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 networks</td>
<td>$xy+yz-xz=x+(p+1)\cdot p = 0$</td>
<td>$\lambda_{AC} = \frac{x(yz+1)}{y(xz+1)}$</td>
<td>$\eta_{CA} = \lambda_{AC} \cdot \text{eff}(x+y+1)$</td>
</tr>
<tr>
<td>3 networks</td>
<td>$xz+y+z+x+y+(p+1)\cdot p = 0$</td>
<td>$\lambda_{AC} = \frac{x(yz+1)}{y(xz+1)}$</td>
<td>$\eta_{CA} = \lambda_{AC} \cdot \text{eff}(x+y+1)$</td>
</tr>
<tr>
<td>3 networks</td>
<td>$xz+y+z+x+y+(p+1)\cdot p = 0$</td>
<td>$\lambda_{AC} = \frac{x(yz+1)}{y(xz+1)}$</td>
<td>$\eta_{CA} = \lambda_{AC} \cdot \text{eff}(x+y+1)$</td>
</tr>
</tbody>
</table>
Table 5.6.3 Extended double variable bridges; closed networks, number of variator networks and converse networks represented by those closed networks. Winding lines indicate a split for a backbone chain.

**Group 1**

- \(3+0\)
- \(1+0\)

**Group 2**

- \(3+3\) \(3+3\) \(3+0\) \(3+0\) \(3+3\) \(3+3\)

**Group 3**

- \(3+3\) \(3+3\) \(2+2\) \(3+3\)
- \(2+2\) \(2+2\) \(3+3\)
- \(3+3\) \(3+3\) \(3+3\)
- \(2+2\)

**Group 4**

- \(2+2\) \(2+2\) \(2+0\) \(3+0\)

networks with non-alternating nodes in the containing chain
5.7. Recapitulation of variator networks with one or two variators

An outline of variator networks with one or two variators, based upon the classification scheme in table 5.4.1 is given in table 5.7.1. The name given to the variator networks in a certain class has been borrowed from one of these variator networks which is reproduced as a stylised character.

Table 5.7.1 Outline of variator networks with \( v \) variators, \( v = 1 \) or 2, \( p \) epicyclic gear trains, and \( h \) hard meshes

<table>
<thead>
<tr>
<th>( p )</th>
<th>( v )</th>
<th>( h )</th>
<th>Transmission</th>
<th>Character</th>
<th>Operational Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>variable</td>
<td>character</td>
<td>backbone chain</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>transmission</td>
<td></td>
<td>closed network</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>operational scheme</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( p )</th>
<th>( v )</th>
<th>( h )</th>
<th>Bridge</th>
<th>Character</th>
<th>Operational Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>shunt</td>
<td>character</td>
<td>backbone chain</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>closed network</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>operational scheme</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( p )</th>
<th>( v )</th>
<th>( h )</th>
<th>Bridge</th>
<th>Character</th>
<th>Operational Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>double</td>
<td>character</td>
<td>backbone chain</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>shunt</td>
<td></td>
<td>closed network</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>operational scheme</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>double</td>
<td>character</td>
<td>backbone chain</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>shunt</td>
<td></td>
<td>closed network</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>operational scheme</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( p )</th>
<th>( v )</th>
<th>( h )</th>
<th>Bridge</th>
<th>Character</th>
<th>Operational Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>extended</td>
<td>character</td>
<td>backbone chains</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>double</td>
<td></td>
<td>closed networks</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>variable</td>
<td></td>
<td>networks represented in table 5.5.2</td>
</tr>
</tbody>
</table>

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CHAPTER 6
VARIATOR NETWORK OF THE VARIABLE SHUNT TYPE

6.1. Description of a variable shunt

The definition of a variable shunt was given in 5.5.2. The most condensed representation was given in Fig. 5.5.3 and in formula 5.5.6. Now, step by step, information about the epicyclic gear train will be introduced, first the parameters $i$ and $\bar{e}$ and later the position of the planet carrier. The relation between Figs. 5.5.3 and 6.1.1 is given by the equations

1. $x^* = \frac{\bar{e}}{\bar{e}-1} x$
2. $y^* = \frac{1}{\bar{e}-1} y$

in which the parameters $i$ and $\bar{e}$ are abbreviations

3. $\bar{e} = \bar{e}_{AC}$
4. $\bar{e} = \bar{e}_{CA}$

The equation for the transmission ratios is (see Fig. 6.1.2)

5. $2\bar{e} - y - 1 + 1 = 0$

and can be written explicitly

6. $x = \frac{y + \bar{e} - 1}{\bar{e}}$
7. $y = \bar{e} x - \bar{e} + 1$
8. $\bar{e} = \frac{(y-1)}{(x-1)}$

The responsivity is $\lambda_{AC}$, abbreviated to $\lambda$ (see Fig. 6.1.3)
Fig. 6.1.2 The transmission ratios of a variable shunt

\[ \lambda = \frac{\omega_1}{\omega_2} = \frac{x(y-1)}{y(x-1)} \]

The expressions for the powers are read from Fig. 6.1.4

Fig. 6.1.4 Angular velocities and torques in a variable shunt expressed in \( \omega_0 \) and \( \frac{1}{k} \)
The dissipative power of the epicyclic gear train is (see the arrow marked \( -P_e \) in Fig. 6.1.4)

\[
P_e = \eta_x (1-\eta_f) \omega_{BA}^T = (1-y)(1-\eta) \omega_{BA}^T
\]

The dissipative power of the variator is

\[
P_v = \gamma \eta_x (1-\eta_f) \omega_{BA}^T
\]

The efficiency of the variable shunt follows from 5 and 3.3.8

\[
\eta_f = \frac{\eta_0 - \eta + 1}{\eta_x - \eta + 1}
\]

The distribution of power is characterised by the ternary efficiency

\[
\eta_{CA} = -\frac{P_e}{P_A} = \frac{\gamma \eta_x}{\gamma}
\]

\[
\eta_{CA} = \lambda_{AC} \eta_0
\]

The efficiency \( \eta \) of the epicyclic gear train depends on the value of the sun efficiency \( \eta_s \) and on the position of the sun gears between which the sun efficiency is defined. In other words, \( \eta \) depends on \( \eta_s \) and on the

Table 6.1.1  The binary efficiency \( \eta \)
position of the planet carrier with respect to the shaft ends between which \( l \) and \( r \) are defined. In table 6.1.1 for three positions of the planet carrier the binary efficiency \( \eta \) is expressed in the binary ratio \( l \) and the sun efficiency \( \eta_s \).

6.2. Conditions for the power flow

Only those power flows are possible the dissipative powers of which are negative. Thus, \( P_{\text{in}} < 0 \) and \( P_{\text{out}} < 0 \). From \( \frac{P_{\text{in}} - P_{\text{out}}}{P_{\text{in}}} > 0 \) with 6.1.14 and 6.1.15 follows

1 CONDITION The power flow of a variable shunt satisfies

\[
(1 - \eta)(1 - \frac{1}{l})(1 - \frac{1}{r}) < 0
\]

This condition decides which direction of the power flow in the epicyclic gear train (represented by \( \eta \)) belongs to a given direction of the power flow in the variator (represented by \( \eta_v \)). In a similar way follows a condition for \( \eta_v \) from 6.1.10 and 6.1.14.

2 CONDITION The power flow of a variable shunt satisfies

\[
(1 - \eta)(1 - \frac{1}{l})(1 - \frac{1}{r}) > 0
\]

The inequalities 2 and 4 are related to each other by transposition of letters and reversal of the inequality sign. The evaluation of the two conditions is given in table 6.2.1 as a mathematical complete.

Table 6.2.1 Inequalities for the power flow of a variable shunt
relationship. In most practical applications the values of \( \eta_s \) and \( \bar{\eta}_s \) are fairly close to 1. For such cases a conveniently arranged extract of table 6.2.1 is presented in table 6.2.2. If, for instance, the transmission ratios \( x \) and \( y \) and the direction of the power flow of the variable shunt (represented by \( \eta_s \)) are given, one can read from table 6.2.2 whether \( \eta_s \) and \( \bar{\eta}_s \) are below 1 (shaded areas) or above 1 (blank areas).

Table 6.2.2 Extract of table 6.2.1 for values of \( \eta_s \) and \( \bar{\eta}_s \) close to 1
§ 3. **Self-locking variable shunt**

The value of $\eta$ determines the direction of the power flow.

1. $\eta > 1$ means positive power on shaft end B and negative power on shaft end A.
2. $0 < \eta < 1$ means positive power on shaft end A and negative power on shaft end B.
3. $\eta < 0$ means positive power on shaft end A as well as positive power on shaft end B.

In the third case all input power transforms into dissipative power.

4. In accordance with 1, 7, 5 a variable shunt with negative efficiency is called **self-locking**. For closer investigation the efficiency of a variable shunt given in § 1. 16 may be rewritten in various ways.

\[ \eta_\eta = 1 - \lambda(1 - \bar{\eta}_B) \]

\[ \eta_\eta = \frac{\bar{\eta}(x - iC_B\bar{\eta}_B)}{\bar{\eta}_B} \]

The passing of the value $\eta = 1$ has already been dealt with in table 6. 2. 2. The limits $\eta = 0$ and $\eta = \infty$ will now be investigated.

The limit $\eta = 0$ separates the self-locking area from the area with a power flow from input shaft A to output shaft B. This limit consists of the line

\[ x\bar{\eta}_B - iC_B\bar{\eta}_B = 0 \]

The other limit $\eta = \infty$ separates the self-locking area from the area with a power flow from input shaft B to output shaft A. This limit consists of the lines

\[ \begin{cases} 
\bar{\eta} = \infty \\
\bar{\eta}_B = \infty \\
\eta_\eta = \infty 
\end{cases} \]

in which 10 stands for the uninteresting case of a freely rotating shaft end C and a blocked shaft end D. The self-locking areas are shown in table 6. 3. 1.
6.4. Restrictions imposed by practical requirements

Running through zero is usually allowed for the transmission ratio $x$ of the variator, and sometimes for the transmission ratio $y$ of the variator network. This does not impose restrictions here. An unfavourable relation between $x$ and $y$ may be caused by extremely small or extremely high values of the binary ratio $\tilde{f}$. Therefore, the only restriction for the transmission ratios is
6.4.

1 RESTRICTION \[ \xi_{\text{min}} \leq |\xi| \leq \xi_{\text{max}} \]

A variable shunt must not be self-locking for the principal direction of power flow and commonly not for the reversed power flow either.

2 RESTRICTION \[ \eta_{r} > 0 \]

Another plausible restriction limits the maximum power through the variator, to a certain fraction \( e \) of the power through the input shaft.

3 RESTRICTION

4 \[ \eta_{\text{CA}} \leq e \quad \text{for} \quad \eta_{y} < 1 \quad \text{and} \quad \eta_{r} < 1 \]

5 \[ \eta_{\text{DA}} \leq e \quad \text{for} \quad \eta_{y} > 1 \quad \text{and} \quad \eta_{r} < 1 \]

6 \[ \eta_{\text{CA}} \leq e \quad \text{for} \quad \eta_{y} < 1 \quad \text{and} \quad \eta_{r} > 1 \]

7 \[ \eta_{\text{DA}} \leq e \quad \text{for} \quad \eta_{y} > 1 \quad \text{and} \quad \eta_{r} > 1 \]

while the fraction \( e \) satisfies

8 \[ 0 < e \leq 1 \]

The inequalities 4 to 7 inclusive are transformed into

9 \[ \lambda \leq +\frac{e}{\eta} \quad \text{for} \quad \eta_{y} < 1 \quad \text{and} \quad \eta_{r} < 1 \]

10 \[ \lambda \leq -\frac{e}{\eta_{y}} \quad \text{for} \quad \eta_{y} > 1 \quad \text{and} \quad \eta_{r} < 1 \]

11 \[ \lambda \leq -\frac{e_{y}}{\eta} \quad \text{for} \quad \eta_{y} < 1 \quad \text{and} \quad \eta_{r} > 1 \]

12 \[ \lambda \leq +\frac{e_{y}}{\eta_{y}} \quad \text{for} \quad \eta_{y} > 1 \quad \text{and} \quad \eta_{r} > 1 \]

The restrictions are shown in table 6.4.1. The diagrams would be more complete if the curves of Fig. 6.1.2 were entered in the unrestricted areas. These curves for transmission ratios, drawn as far as there are no restrictions, would show values of the parameter \( \lambda \), the range of \( x \) and the range of \( y \) that are adequate from the point of view of distribution of power and restrictions 1 and 2.
Table 6.4.1 Areas for practical applications, calculated with
\( \eta_1 = 0.96 \) or \( 1/0.96 \), \( \eta_2 = 0.80 \) or \( 1.25 \), \( e = 0.4 \).
The transmission ratio \( x \) is defined for the direction
represented by the oval. The transmission ratio \( y \) is
defined \( y = \frac{\omega_A}{\omega_B} \). The diagrams in the right-hand column
are repeated with reciprocal \( x \) and \( y \), to facilitate a
convenient comparison with those in the left-hand column.
7.1. Description of a variable bridge

The definition of a variable bridge was given in 5.5.9. The most condensed representation was given in Fig. 5.5.8 and in formula 5.8.14. Information about the epicyclic gear trains will now be introduced, first the parameters \( \bar{t}_1, \bar{t}_2, \bar{n}_1, \bar{n}_2 \) and later the position of the planet carrier in each of the epicyclic gear trains. The relation between Figs. 5.5.8 and 7.1.1 is given by the equations

\[
\begin{align*}
x^* &= \frac{(\bar{t}_1-1)}{(\bar{t}_2-1)} t_2 x \\
y^* &= \bar{t}_2 y \\
p^* &= \bar{t}_1 \bar{t}_2
\end{align*}
\]

in which the parameters \( \bar{t}_1, \bar{t}_2, \bar{n}_1, \bar{n}_2 \) are abbreviations

\[
\begin{align*}
\bar{t}_1 &= \bar{t}_{AC}/\bar{t}_{BC} \\
\bar{t}_2 &= \bar{t}_{BODA}\bar{t}_{AD} \\
\bar{n}_1 &= \bar{n}_{GLAC} \\
\bar{n}_2 &= \bar{n}_{AD}/\bar{t}_{BD}
\end{align*}
\]

The equation for the transmission ratios is

\[
(\bar{t}_1-\bar{n}_2 x y - (\bar{t}_1-1)x + (\bar{t}_2-1)y - (\bar{t}_2-1)\bar{t}_1 = 0
\]
and can be written explicitly

\[ x = \frac{(i_2-1)(i_2-\gamma)}{(i_1-1)(i_2y-1)} \]

\[ y = \frac{(i_1-1)x + (i_2-1)\gamma_i}{(i_1-1)i_2x + (i_2-1)} \]

The responsivity is \( \lambda_{AC} \), abbreviated to \( \lambda \), and written in various ways

\[ \lambda = \frac{(i_1-1)(i_2-1)(1-\xi_2)}{(i_1-1)x + (i_2-1)i_1} \{ (i_1-1)\xi_2x + (i_2-1) \} \]

\[ \lambda = \frac{(y-\xi_1)(i_2y)}{y(i_2-1)} \]

\[ \lambda = \frac{(y-\xi_1)(y-1)}{(i_1-1)(y-1)y} \]

\[ \lambda = -\frac{(i_2-1)x(y-1)}{(i_2-1)y(x-1)} \]

The expressions for the powers are read from Fig. 7.1.2

\[ P_A = y \left( 1 + \eta_n \xi_2^\frac{\xi_i - 1}{\xi_i - 1} \right) \omega_B \]

\[ P_B = -\left( \xi_i \eta_i + \eta_n \xi_2^\frac{\xi_i - 1}{\xi_i - 1} \right) \omega_B \]

\[ P_C = \frac{(\xi_i \eta_i - 1)}{(i_1-1)} \omega_B \]

\[ P_0 = -\eta_n \left( \frac{\xi_i - 1}{\xi_i - 1} \right) \omega_A \]

\[ P_{AC} = \eta_n \omega_B \]

\[ P_{AD} = x \eta_n \xi_2^\frac{\xi_i - 1}{\xi_i - 1} \omega_B \]

\[ P_{BC} = -\xi_i \eta_i \omega_B \]

\[ P_{BD} = -y \eta_n \xi_2^\frac{\xi_i - 1}{\xi_i - 1} \omega_B \]

The dissipative powers of the epicyclic gear trains are

\[ P_1 = \frac{\xi_i(1-\eta_i)}{(i_1-1)} \omega_B \]

\[ P_2 = -\frac{\xi_i(1-\eta_i)}{(i_2-1)} \eta_n (y-1) \omega_B \]
The dissipative power of the variator between the epicyclic gear trains is

\[
\rho_x = \frac{(I_1^{\alpha} - 1)(1 - \eta_x)(I_1 - y)}{(I_1 - 1)} \cdot \omega_B \tau_{AC}
\]

or rewritten

\[
\rho_x = \frac{(I_2^{\alpha} - 1)}{(I_2 - 1)} (1 - \eta_x) x(I_2 y - 1) \omega_B \tau_{AC}
\]
The dissipative powers are restituted by the supply power \( P_y = P_x + P_o \)

\[
P_y = \frac{y(T_{n_1}^2T_{n_2}^2 - 1)}{(T_{n_2}^2 - 1)(1 - \eta_y)\omega_bT_c}
\]

For a variable bridge with the variator between the nodes, \( P_y \) is the dissipative power of that variator, and \( P_x \) is the supply power.

The distribution of power is characterised by the ternary efficiency

\[
\eta_{CM} = -\frac{P_y}{P_x} = \frac{(T_{n_1}^2 - 1)(T_{n_2}^2 - 1)(1 - \eta_y)x}{(T_{n_2}^2 - 1) + x\eta_yT_2(T_{n_1}^2 - 1)}
\]

The efficiencies \( \eta_x \) and \( \eta_y \) of the variator and the variator network may be deduced from 8 and 10.

\[
\eta_y = \frac{1}{y} \frac{(T_{n_1}^2 - 1)x + (T_{n_2}^2 - 1)T_{n_1}}{(T_{n_1}^2 - 1)T_{n_2}x + (T_{n_2}^2 - 1)}
\]

\[
\eta_x = \frac{x}{(T_{n_1}^2 - 1)(T_{n_2}^2 - 1)y}\frac{(T_{n_1}^2 - 1)\eta_y}{(T_{n_2}^2 - 1)\eta_x}
\]

The efficiencies \( \eta_1 \) and \( \eta_2 \) of the epicyclic gear trains depend on which shaft end in each of the epicyclic gear trains is planet carrier, and on the values of the sum efficiencies \( \eta_h \) and \( \eta_g \).

### 7.2. Conditions for the power flow

1. Only those power flows are possible the dissipative powers of which are negative. The sum of the input power and output power is positive.

   For a variable bridge with the variator between the epicyclic gear trains follows from \( P_1 < 0 \), \( P_x < 0 \), via \( \frac{P_y}{P_x} > 0 \) with \( \eta_x < 0 \) and \( \eta_1, \eta_2 \).

   2. CONDITION 

   \[
   \left\{ \frac{y-\xi}{y-1} \right\} \left\{ \frac{\eta_y - \eta_h}{\eta_h - 1} \right\} > 0
   \]

   if variator between epicyclic gear trains

   For a variator between the nodes follows from \( P_1 < 0 \),

   \( P_x = -P_1 - P_2 - P_y > 0 \) via \( \frac{P_y}{P_x} > 0 \)

   3. CONDITION 

   \[
   \left\{ \frac{y-\xi}{y-1} \right\} \left\{ \frac{\eta_y - \eta_h}{\eta_h - 1} \right\} < 0
   \]

   if variator between nodes

In a similar way the comparison of \( P_2 \) with \( P_x \) results in
4 CONDITION \( \left\{ \frac{y_{\gamma}-1}{y-1} \right\} \left\{ \frac{\eta_{y}-1}{\eta_{x}-1} \right\} \left\{ \frac{\eta_{1}-1}{\eta_{2}-1} \right\} > 0 \) if variator between epicyclic gear trains

5 CONDITION \( \left\{ \frac{y_{\gamma}-1}{y-1} \right\} \left\{ \frac{\eta_{y}-1}{\eta_{x}-1} \right\} \left\{ \frac{\eta_{1}-1}{\eta_{2}-1} \right\} < 0 \) if variator between nodes.

From 2 and 4 as well as from 3 and 5, with 7.1.30 follows

6 CONDITION \( \left\{ \frac{y_{\gamma}-1}{y-1} \right\} \left\{ \frac{\eta_{y}-1}{\eta_{x}-1} \right\} \left\{ \frac{\eta_{1}-1}{\eta_{2}-1} \right\} > 0 \)

A variable bridge satisfies \( \frac{P_{\gamma}}{P_{y}} < 0 \) which via \( \frac{P_{\gamma}}{P_{y}} = \frac{1-\eta_{y}}{1-\eta_{x}} \frac{\eta_{1}}{\eta_{2}} \) and \( \frac{\omega_{c}}{\omega_{\gamma}} = \frac{(\bar{r}_{1}-\bar{r})}{(\bar{r}_{1}-\bar{r}_{2})} \) because of 5.2.16 (no transmissions in the branches), results in

7 CONDITION \( \left\{ \frac{\bar{r}_{1}-\bar{r}}{\bar{r}_{2}} \right\} \left\{ \frac{1-\eta_{x}}{1-\eta_{y}} \right\} \left\{ \frac{1-\eta_{x}}{1-\eta_{y}} \right\} < 0 \)

8 Suppose the epicyclic gear trains are not of a self-locking type, viz. neither \( \frac{\bar{r}_{1}}{\bar{r}_{2}} \) lies between 1 and \( \bar{h}_{1} \), nor \( \frac{\bar{r}_{1}}{\bar{r}_{2}} \) between 1 and \( \bar{h}_{2} \), and that the variator is not self-locking. The efficiencies of the epicyclic gear trains and the variator may each have two values in accordance with 2.8.7.

9 Then, because of 2, 4 and 7.1.29, a variable bridge with the variator between the epicyclic gear trains has two equilibrium.

10 Then, because of 3, 5 and 7.1.29, a variable bridge with the variator between the nodes has two equilibrium if the term \( \left\{ \frac{y_{\gamma}-1}{y-1} \right\} \left\{ \frac{\eta_{y}-1}{\eta_{x}-1} \right\} \left\{ \frac{\eta_{1}-1}{\eta_{2}-1} \right\} \) does not change its sign.

7.3. Power flow through the branches

1 There are three meshes in a variable bridge. With regard to these meshes four types of power flow can be distinguished. Power flows are left out of consideration if an epicyclic gear train dissipates all the power received from three branches, and/or if the variator dissipates all the power received. Power flows which become identical following transposition of suffixes belong to the same type. A type is characterized by relations between the directions of the branch powers \( P_{\gamma}, P_{x}, P_{y}, P_{z} \). Formulae 7.1.19 to 7.1.22 inclusive are used to find the decisive inequalities given in table 7.3.1.

2 If a power circulates in a mesh in unchanged direction then this mesh has what is known as a blind power.

3 It is emphasised that a blind power in a mesh containing the variator does not imply that the variator is highly loaded.
7.4. **Reverse-symmetric variable bridge**

The representation of the variable bridge in Fig. 5.5.8 with a composing chain and a minimum number of parameters has the disadvantage that it is only a symbolic representation. In a real design, the minimum number of parameters, equal to the number in the representation with the composing chain, can be achieved by putting \( i_1 = i_2 \) and \( \tilde{n}_{12} = \tilde{n}_{21} \).

1. **DEFINITION** A reverse-symmetric variable bridge is a variable bridge with equal binary ratios of the epicyclic gear trains, cyclically directed in the hard mesh, and with equal sun efficiencies.

2. For comparison with a variable bridge with ratios \( x \) and \( y \) the corresponding ratios of the reverse-symmetric variable bridge are...
written \( ax \) and \( by \), see Fig. 7.4.1 in comparison with Fig. 7.1.1. The equations 7.1.1 to 7.1.3 inclusive become

\[
\begin{align*}
x^* &= -i\alpha x \\
y^* &= -i\beta y \\
p^* &= \tau^2
\end{align*}
\]

**Fig. 7.4.1 Reverse-symmetric variable bridge**

From 5 and 7.1.3 via \( i^2 = \imath_1 \imath_2 \) instead of concluding to \( \imath = \sqrt{\imath_1 \imath_2} \) we have the freedom to write

\[
i = \imath_2 \sqrt{\frac{\imath_1}{\imath_2}}
\]

by which

\[
a = \frac{(\imath_1 - 1) \sqrt{\imath_1}}{(\imath_2 - 1) \imath_2}
\]

\[
b = \sqrt{\frac{\imath_1}{\imath_2}}
\]

The particular form of 6 associates the values of \( \imath \) with the signs of \( \imath_1 \) and \( \imath_2 \). The value \( \imath \) is imaginary if \( \imath_1 \) and \( \imath_2 \) have different signs.

The ratios \( a \) and \( b \) are independent of \( x \) and \( y \). They are imaginary if \( \imath_1 \) and \( \imath_2 \) have different signs. The areas of these ratios are specified in Figs. 7.4.2 and 7.4.3. The symbol \( j \) means \( j = \sqrt{-1} \). The transmission ratios satisfy

**Figs. 7.4.2 and 7.4.3 Areas of \( a \) and \( b \)**
7.4.

\[ axby - ax + by - \bar{t} = 0 \]

The responsibility is invariable.

\[ \lambda_{AC} = \frac{(y-\bar{t})(iy-\bar{t})}{y(i^2+1)} = \frac{by-b\bar{t}}{by(i^2+1)} = \frac{ax(by+1)(by-\bar{t})}{by(ax+1)(ax-\bar{t})} \]

The transmission ratios and the responsibility are illustrated in Figs. 7.4.4 to 7.4.7, where \( j \) means \( j=\sqrt{-1} \).

\[ \lambda_{AC} = \frac{(y-\bar{t})(iy-\bar{t})}{y(i^2+1)} = \frac{by-b\bar{t}}{by(i^2+1)} = \frac{ax(by+1)(by-\bar{t})}{by(ax+1)(ax-\bar{t})} \]

Figs. 7.4.4 and 7.4.5 Transmission ratios of reverse-symmetric variable bridges.

Figs. 7.4.6 and 7.4.7 Responsivity of reverse-symmetric variable bridges. The relation between \( \lambda_{CA} \) and \( ax \) is derived by the permutation 
- \( by \) becomes \( ax \), \( \bar{t} \) becomes \( \bar{t} \)
- \( \lambda_{AC} \) becomes \( \lambda_{CA} \)

A variable bridge with \( \bar{t}^2 < 0 \) has no advantages in comparison with a
variable shunt, its responsivity does not progress more favourably while its design is more complicated. See Figs. 7.4.6 and 6.1.3.

15 A variable bridge with $t^2 > 0$, on the contrary, may have a remarkably favourable responsivity, see Fig. 7.4.7. The gradual change of the responsivity $\lambda_{AC}$ in a wide interval of the ratio $by$ is important because the ternary efficiency $\eta_{CA}$ tends to approximate the value of $\lambda_{AC}$.

16 A more detailed analysis of a variable bridge exceeds the scope of this study. The capricious lines found in table 6.4.1 for the practical application of a variable shunt are a warning against the complications inherent in a variable bridge. Suffice it to observe a coincidence of a branched power flow through the branches (table 7.3.1) with a favourable responsivity. A rough indication is made in Fig. 7.4.8.

Fig. 7.4.8 Rough indication of areas favourable to a variable bridge with the variator between the epicyclic gear trains and with rather low dissipative powers. The dotted lines indicate minor important parts of the favourable areas. The same diagram holds for a variable bridge with the variator between the nodes following the permutation

$ax$ becomes $by$, $i$ becomes $-i$
$by$ becomes $ax$, $\lambda_{AC}$ becomes $\lambda_{CA}$
7.5. **Double epicyclic gear train**

The two epicyclic gear trains of a variable bridge constitute a noteworthy design called 'double epicyclic gear train'. See Fig. 7.5.1.

![Double epicyclic gear train diagram](image)

**Fig. 7.5.1 Double epicyclic gear train**

1. **DEFINITION** A double epicyclic gear train consists of two single epicyclic gear trains and two nodes, configured as a smallest hard mesh.

   The angular velocities of the four shaft ends satisfy the proportion deduced from 2.1.1:

   $$\frac{(\omega_A - \omega_B)}{(\omega_C - \omega_B)} : \frac{(\omega_A - \omega_C)}{(\omega_B - \omega_C)} : \frac{(\omega_B - \omega_C)}{(\omega_D - \omega_C)} = 
   \frac{(\ell_2 - 1)}{(\ell_1 - 1)} : \frac{(\ell_2 - 1)}{(\ell_1 - 1)} : \frac{(\ell_2 - 1)}{(\ell_1 - 1)}$$

2. The inequalities of the angular velocities are summed up in Fig. 7.5.2.

![Inequalities for angular velocities](image)

**Fig. 7.5.2 Inequalities for angular velocities. The abbreviation \(\{ABCD\}\) means either \(\omega_A < \omega_B < \omega_C < \omega_D\) or \(\omega_D < \omega_C < \omega_B < \omega_A\)**
With respect to the positions of the planet carriers four types of double epicyclic gear trains may be distinguished, as indicated in Table 7.5.1.

Table 7.5.1 Four types of double epicyclic gear trains, not symmetrically related to each other with respect to the positions of the planet carriers

A double epicyclic gear train may be treated as a reiteration of a single epicyclic gear train. A special case, however, will be considered here, viz. the double epicyclic gear train with common sun gears and common planet carriers.

If the two epicyclic gear trains have a common sun gear, then their planet carriers must be coupled. Coupled planet carriers are equivalent to a common planet carrier. A transformation of the design after the example of Fig. 7.5.3 demonstrates the equivalence of a double epicyclic gear train with coupled planet carriers to one with a common planet carrier in which planet gears in a planet group have a common shaft.

**DEFINITION** A Wolf from transmission is a double epicyclic gear train with a common planet carrier in which planet gears in a planet group have a common shaft.
A Wolfrom-transmission may be split into two epicyclic gear trains in three ways, each having a different symbolic representation. This leads to a classification into two groups, each group with two principal designs and three equivalent symbolic representations. See Fig. 7.5.4 and Table 7.5.2. In most existing designs the planet carrier of a Wolfrom-transmission is not connected to any element outside the transmission. Thus, the transmission is equivalent to a single epicyclic gear train between the shaft ends of the three sun gears.
Fig. 7.5.4 A Wolfrum-transmission split in three different ways

The efficiency of a Wolfrum-transmission may be poor for the main power flow and generally negative for the reversed power flow (self-locking).
CHAPTER 8
DYNAMIC RESPONSE OF A VARIATOR NETWORK

8.1. Schematic dynamic representation of an epicyclic gear train

An epicyclic gear train was represented in 3.1.7 as a converse node having no transmission parameter at all. The transmission parameters were transferred to the shafts. If in addition the dynamic properties of planet gears could be transferred to the shafts, the mathematical model of an epicyclic gear train would be as empty as the model of a node. Then, the study of dynamic properties of a variator network would be reduced to the study of the dynamics of shafts, combined into a system by parameter-free relations of three-poles. This approach will be answered by the deductions below.

Glossary of symbols

suffixes A, B sun gear
suffix C planet carrier
suffixes a, b planet gear

\( \hat{e}_a \) unit vector, perpendicular to axis a (planet gear), coplanar with axis a and axis C (planet carrier)

\( \hat{e}_c \) unit vector, perpendicular to axis a and axis C

\( \hat{e}_L \) unit vector, along axis a

\( \hat{f}_{A}, \hat{f}_{B}, \hat{f}_{C} \) binary ratio between sun gear and planet gear

\( J_1, J_b, J_c \) transferred moment of inertia

\( J_1, J_b, J_c \) moment of inertia of planet gear about its own axis

\( J_1, J_b, J_c \) moment of inertia of planet gear about a co-ordinate axis

\( m_s, m_b, m_c \) mass of planet gear

\( M_s, M_b, M_c \) transferred torque

\( M_s, M_b \) torque in elastic connection between planet gears

\( r_1, r_b, r_c \) radius from central axis to centre of gravity of planet gear

\( S_A, S_b, S_c \) transferred torsional stiffness

\( S_b \) torsional stiffness of connection between planet gears

\( T_{A', B', C'} \) transferred torque

\( T_{A, B, C} \) torque transmitted by planet gear

\( T_{A', B', C'} \) connecting torque

\( T_{A', B', C'} \) component of the torque due to the acceleration of the planet gear

\( \delta_{A, B, C} \) binary efficiency between sun gear and planet gear

\( \phi_a, \phi_b, \phi_c \) angle between planet axis and central axis

\( \phi_{A, B, C} \) connecting rotating angle

\( \phi_{A, B, C} \) connecting rotating angle
\( \theta \) rotating angle of planet gear about its own axis
\( \omega_0, \omega_c, \omega_b \) connecting angular velocity
\( \omega, \omega_0 \) angular velocity of planet gear about its own axis
\( \omega, \omega_0, \omega_b \) component of angular velocity

Acceleration of a planet gear

With respect to its dynamic performance, a planet gear should not be restricted to a design with parallel axes, as was done in 2.6.2. The most general case will be treated here.

The torque due to the acceleration of planet gear \( a \) (see Fig. 8.1.1) is

\[
T_x \hat{e}_x + T_y \hat{e}_y + T_z \hat{e}_z = -J_x \ddot{\omega}_x - J_y \ddot{\omega}_y - J_z \ddot{\omega}_z
\]

where

\[
J_x = \left( \frac{1}{2} J_a + \frac{m_b r_b^2}{\sin^2 \theta_b} \right)
\]

\[
J_y = \left( \frac{1}{2} J_a + \frac{m_b r_b^2}{\sin^2 \theta_b} \right)
\]

\[
J_z = J_a
\]

\[
\omega_x = -\omega_c \sin \theta_a
\]

\[
\omega_y = 0
\]

\[
\omega_z = +\omega_c \cos \theta_a + \omega_b
\]

\[
\hat{e}_x = \ddot{\omega}_c \times \hat{e}_x + \omega_c \cos \theta_a \hat{e}_y
\]

\[
\hat{e}_y = \ddot{\omega}_c \times \hat{e}_y - \omega_c \cos \theta_a \hat{e}_x - \omega_c \sin \theta_a \hat{e}_z
\]

\[
\hat{e}_z = \ddot{\omega}_c \times \hat{e}_z + \omega_c \sin \theta_a \hat{e}_y
\]
which results in the components

\[
\begin{align*}
T_x &= \left( \frac{1}{2} J_a + \frac{m_b r_b^2}{\sin^2 \beta_a} \right) \omega_c^2 \sin \beta_a \\
T_y &= \left( \frac{1}{2} J_a + \frac{m_b r_b^2}{\sin^2 \beta_a} \right) \omega_c^2 \cos \beta_a - J_a (\omega_c \cos \beta_a + \dot{\omega}_b) \omega_c^2 \sin \beta_a \\
T_z &= -J_b (\omega_c \cos \beta_a + \dot{\omega}_b)
\end{align*}
\]

9 The torques \( T_x \) and \( T_y \cos \beta_a \) act on the bearings. They are important for the calculation of the bearing load, but here they can be left out of consideration if the friction in the bearings is sufficiently low.

10 RESTRICTION The friction torques due to \( T_x \) and \( T_y \cos \beta_a \) are ignored.

11 The torque \( T_y \sin \beta_a \) acts on the planet carrier

\[
T_y \sin \beta_a = \left( \frac{1}{2} J_a \sin^2 \beta_a + m_b r_b^2 \right) \omega_c^2
\]

This torque is equivalent to the torque caused by an additional moment of inertia \( \frac{1}{2} J_a \sin^2 \beta_a + m_b r_b^2 \) on the planet carrier, hence

\[
J_c = \left( \frac{1}{2} J_a \sin^2 \beta_a + m_b r_b^2 \right) + \left( \frac{1}{2} J_a \sin^2 \beta_a \right)
\]

13 The torque \( T_z \) partly acts on the sun gear \( A \) via the planet transmission \( a-A \), partly on the sun gear \( B \) via the elastic connection between the planet gears and the planet transmission \( b-B \).

\[
M_{ab} = \tau_a = J_a \cos \beta_a \omega_c^2 + J_a \dot{\omega}_a = J_a \cos \beta_a \omega_c^2 + J_a \dot{\omega}_a
\]

With

\[
M_{ab} = S_{ab} (\phi_b - \phi_a)
\]

\[
\tau_a = -\tau_{aA} \tau_{aA}^T \quad \text{a-A, b-B permutable in pairs}
\]

\[
\phi_a = \tau_{aA} (\phi_b^* - \phi_a^*) \quad \text{a-A, b-B permutable in pairs}
\]

\[
\tau_{AB} = \tau_{aA} \tau_{aB} \quad \text{for } \tau_{aB} = +1
\]

Equation 15 yields

\[
\tau_{AB} \tau_{aA}^T - S_{ab} \left( \phi_b^* - \tau_{AB} \phi_a^* + (\tau_{aA} - \tau_{aB}) \phi_c^* \right) - \tau_{aA} J_a \cos \beta_a \phi_c^* = J_a (\phi_b^* - \phi_a^*) = 0
\]

\( a-A, b-B \) permutable in pairs

Elimination of \( S_{ab} \) from the two equations 20 results in

112
When the configuration of Fig. 8.1.1 is replaced by that of Fig. 8.1.2 the torques \( \tau_a^*, \tau_b^* \) and the angles \( \varphi_a^*, \varphi_b^*, \varphi_c^* \) must remain unaffected.

Fig. 8.1.2 The moments of inertia of the planet gears and the torsional stiffness of their connection transferred to the shafts. The omitted shaft end B is similar to A.

Hence, the two equations 20, and consequently equation 21, have to be satisfied. The configuration of Fig. 8.1.2 satisfies

\[ I_{AA} \ddot{\varphi}_a^* + I_{BB} \ddot{\varphi}_b^* = (J_a \cos \varphi_a - J_b \cos \varphi_b) \ddot{\varphi}_a^* + \dot{\varphi}_a^* (\dot{\varphi}_a^* - \ddot{\varphi}_a^*) + \dot{\varphi}_b^* (\dot{\varphi}_b^* - \ddot{\varphi}_b^*) \]

Equations 23 and 24 yield

\[ \ddot{\varphi}_a^* = \ddot{\varphi}_a + J_a \dddot{\varphi}_a^* - J_a \dddot{\varphi}_a \]

Substitution of 23, 26, 27 in 21 yields

\[ (I_{AA} \ddot{\varphi}_a^* - \ddot{\varphi}_a^*) \ddot{\varphi}_a^* + \{I_{AA} \ddot{\varphi}_a^* - \ddot{\varphi}_a^* (J_a \cos \varphi_a - J_b \cos \varphi_b) \} \ddot{\varphi}_a^* + \{I_{BB} \ddot{\varphi}_b^* - \ddot{\varphi}_b^* (J_a \cos \varphi_a - J_b \cos \varphi_b) \} \ddot{\varphi}_b^* = 0 \]

which proves to be an identity for the following values of \( J_a, J_b, J_c \), to which the previous equation 13 is supplemented.

\[ \begin{align*}
J_a &= \frac{I_{AA}^2 \ddot{\varphi}_a^*}{I_{AA} - \ddot{\varphi}_a^*} \\
J_b &= \frac{I_{BB}^2 \ddot{\varphi}_b^*}{I_{BB} - \ddot{\varphi}_b^*}
\end{align*} \]
\[ \begin{align*}
\lambda_a &= \frac{\tau_{ca} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{ca} \cos \beta_a} (\cos \gamma_a - \frac{\tau_{ca}}{\tau_{ab}}) J_a + (\cos \beta_a - \frac{\tau_{ca}}{\tau_{ab}}) J_b \\
\lambda_b &= \frac{\tau_{cb} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} (\cos \gamma_a - \frac{\tau_{cb}}{\tau_{ab}}) J_a + (\cos \beta_a - \frac{\tau_{cb}}{\tau_{ab}}) J_b \\
J_c &= \left( \frac{1}{2} J_a \sin^2 \beta_a + m_a \gamma_a^2 \right) + \left( \frac{1}{2} J_b \sin^2 \beta_b + m_b \gamma_b^2 \right)
\end{align*} \]

Equations 23, 25 and 30 result in

\[ \kappa_c^* = s_a (\phi_c^* - \phi_a) + \frac{\tau_{ca} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} \left( \phi_c^* - \phi_b \right) \]

Substitution in 20 of 32 and 26 and elimination of \( \phi_c^* \) yields

\[ \begin{align*}
\frac{\tau_{ca} \tau_{ab} \bar{r}_{ab} - \tau_{ba} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} (\phi_c^* - \phi_a) - \left( \frac{\tau_{cb} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} \right) (\phi_c^* - \phi_b) = 0
\end{align*} \]

which proves to be an identity when both

\[ \begin{align*}
s_a &= \frac{\tau_{ca} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} J_a + (\cos \beta_a - \frac{\tau_{ca}}{\tau_{ab}}) J_a \bar{r}_{ab} \\
s_b &= \frac{\tau_{cb} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} J_b + (\cos \beta_a - \frac{\tau_{cb}}{\tau_{ab}}) J_b \bar{r}_{ab}
\end{align*} \]

The equations 30, 31, 33 and 34 determine the complete transfer of dynamic properties to the shafts. A special case occurs when between the planet gears no elastic element exists, \( s_{ab} = \infty \), \( \phi_a = \phi_b \).

\[ \begin{align*}
\lambda_a + \lambda_b &= \frac{\tau_{ca} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} (\tau_{ca} \cos \beta_a + \tau_{cb} \cos \beta_a) (J_a + J_b) \\
J_a + J_b &= \frac{\tau_{ca} \tau_{ab} \bar{r}_{ab}}{\tau_{ba} \cos \beta_a + \tau_{cb} \cos \beta_a} (\tau_{ca} \cos \beta_a + \tau_{cb} \cos \beta_a) (J_a + J_b) \\
J_c &= \frac{1}{2} (J_a + J_b) \sin^2 \beta_a + m_a \gamma_a^2 + m_b \gamma_b^2 \\
s_a &= s_{ba} = \infty
\end{align*} \]

For \( s_{ab} = \infty \), \( \phi_a = \phi_b \).

EXAMPLE An epicyclic gear train with the design of Fig. 2. 6. 2 and one planet gear a yields

\[ \begin{align*}
\phi_a &= 0 \\
\phi_b &= \frac{1}{2} (Z_a + Z_b) \\
\tau_{ab} &= \frac{Z_a}{Z_b} = \frac{2}{1 + \frac{Z_a}{Z_b}} \\
(\tau_{ca} \tau_{ab} + \tau_{cb}) &= \frac{Z_a + Z_b}{Z_b - Z_a} = \frac{1}{2}
\end{align*} \]
hence

\[
\begin{align*}
J_a &= \frac{2\pi A_1}{(1 + z_B z_A)^2} \cdot J_1 \\
J_b &= \frac{2\pi A_0}{(1 + z_B z_A)^2} \cdot J_2 \\
J_c &= m_r^2
\end{align*}
\]

8.2. Dynamic stability conditions of an epicyclic gear train

The equilibrium conditions result from the dynamic properties of an epicyclic gear train with three rotating shafts and particularly from the consequences of a slight disturbance in the rotation.

1 DEFINITION An epicyclic gear train has an unstable equilibrium if after a slight disturbance the angular velocities do not regain their original values.

2 DEFINITION An epicyclic gear train has a stable equilibrium if after a slight disturbance the angular velocities tend to regain their original values.

The equilibrium will be examined for an epicyclic gear train of which the moments of inertia and the stiffnesses are transferred to the shafts. See Fig. 8.2.1 for the model and for the notation.

![Diagram of an epicyclic gear train](image-url)

Fig. 8.2.1 Epicyclic gear train of which the moments of inertia and the stiffnesses are transferred to the shafts. The shafts are represented by an elastic element between two inertial elements.
8.2.

The only supposition to be made for the torques \( M_a, M_b, M_c \), is that they are constant. Their mutual ratios need not be those of \( \frac{T_a}{T_b}, \frac{T_b}{T_c} \). Even their sum need not be zero.

The equilibrium is stable if and only if in the model a slight disturbance causes a sinusoidal vibration with a non-increasing amplitude superposed on the angular velocities. In a real design such a vibration will be damped. Assumptions 1.8.1 (VI) and (VIII) give the formulae

\[
M_a - M_a = I_a \ddot{\theta}_ax \quad \text{A-b-c-y permutable in pairs}
\]

\[
\frac{M_a}{M_a} = \frac{I_a}{I_a} \quad \text{A-b-c-y permutable}
\]

\[
M_a = S_a(p_a - \dot{\theta}_a) \quad \text{A-b-c-y permutable in pairs}
\]

Completed with equation 2.5.10

\[
\ddot{\varphi}_a - \ddot{\varphi}_a \frac{I_a}{I_a} \ddot{\theta}_a + (\ddot{\varphi}_a - 1) \ddot{\theta}_c = 0
\]

and the abbreviations

\[
\delta_a = \ddot{\varphi}_a - \ddot{\varphi}_a \quad \text{A-b-c-y permutable in pairs}
\]

\[
\kappa_a = (1 + \xi^2 I_a I_a + I_a I_a I_a + I_a I_a I_a I_a) \frac{I_a}{I_a} \quad \text{A-b-c-y permutable}
\]

\[
\eta_a = (1 + \xi^2 I_a I_a + I_a I_a I_a I_a I_a I_a I_a I_a I_a I_a) \frac{I_a}{I_a} \quad \text{A-b-c-y permutable in pairs}
\]

they lead via

\[
\frac{I_a}{I_a} \cdot \frac{M_a}{I_a} = \ddot{\delta}_a + \frac{(S_a + S_a) \delta_a}{I_a}
\]

\[
\frac{I_a}{I_a} - \ddot{\delta}_a + (\ddot{\varphi}_a - 1) \frac{I_a}{I_a} = \frac{S_a \delta_a - I_a I_a S_a I_a}{I_a} \quad \text{A-b-c-y permutable}
\]

to the three equations of motion

\[
H_a \ddot{\delta}_a + (\kappa_a - 1) \delta_a + I_a I_a \frac{S_a}{I_a} \frac{I_a}{I_a} I_a + I_a I_a \frac{S_a}{I_a} \frac{I_a}{I_a} I_a - H_a \frac{M_a}{I_a} = 0
\]

\text{A-b-c-y permutable in pairs}

These simultaneous differential equations need not be solved here. It suffices to consider the quantity \( \rho \) in the general form of the solution

\[
\begin{align*}
\delta_a &= \Sigma a e^{\rho t} \\
\delta_b &= \Sigma b e^{\rho t} \\
\delta_c &= \Sigma c e^{\rho t}
\end{align*}
\]
This quantity \( p \) satisfies the equation

\[
\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 - K_b - H_b p^2 & 1 \\ 1 & 1 & 1 - K_c - H_c p^2 \end{vmatrix} = 0
\]

or in another form

\[
\begin{align*}
\rho^6 + Q_2 \rho^4 + Q_3 \rho^2 + Q_5 &= 0 \\
Q_2 &= \frac{K_a - 1}{H_a} + \frac{K_b - 1}{H_b} + \frac{K_c - 1}{H_c} \\
Q_3 &= \frac{K_a K_b - K_a K_c}{H_a H_b} + \frac{K_b K_c - K_b K_a}{H_b H_c} + \frac{K_c K_a - K_c K_b}{H_c H_a} \\
Q_5 &= \frac{K_a K_b K_c - K_a K_b K_c}{H_a H_b H_c}
\end{align*}
\]

Odd powers of \( p \) are absent, so the roots of 17 are either imaginary or real. Because of 14, each root \( p \) has to be imaginary or zero, thus \( p^2 = 0 \). According to the theorem of Descartes, 17, considered as an equation of the third degree in \( p^2 \) has either one or three negative roots if all coefficients have the same sign.

\[
\begin{align*}
Q_2 > 0 \\
Q_3 > 0 \\
Q_5 > 0
\end{align*}
\]

For a distinction between one and three negative roots the left-hand side of 17 is considered a function in \( p^2 \), see Fig. 8.2.2.

\[
\begin{align*}
f &= \rho^6 + Q_2 \rho^4 + Q_3 \rho^2 + Q_5 \\
f' &= 3 \rho^4 + 2 Q_3 \rho^2 + Q_5
\end{align*}
\]

Fig. 8.2.2 The left-hand side of equation 8.2.17
8.2.

There are three negative roots \( \rho^2 \) if \( f \) is positive for one and negative for another root of \( f' = 0 \) by which both roots of \( f'' = 0 \) are negative. The roots of \( f'' = 0 \) are negative if in addition to 10

\[ Q_2^2 - 3Q_1 \geq 0 \]

Substitution of \( f' = 0 \) in \((27f - 9Qf' - 3Qf'')\) yields

\[ 27f = (2Q_1^2 - 9Q_0Q_2 + 27Q_0) \pm (Q_2^2 - 3Q_1)^{3/2} \]

for \( f' = 0 \)

There are either different signs for \( f \) in 26 by virtue of 25 or three coinciding roots \( \rho^2 \) if

\[ 4(Q_2^2 - 3Q_1)^3 \geq (2Q_1^2 - 9Q_0Q_2 + 27Q_0)^2 \]

Condition 25 is included in 27. The complete condition for a stable equilibrium, formed by 18, 19, 20 and 27, and after admission of roots \( \rho^2 = 0 \), is

\[
\begin{align*}
Q_2 > 0 & \quad Q_1 = 0 \text{ admitted if } Q_1 = Q_0 = 0 \\
Q_1 > 0 & \quad Q_2 = 0 \text{ admitted if } Q_0 = 0 \\
Q_2 > 0 & \quad Q_0 = 0 \\
-\frac{LQ_2^2Q_0 + Q_2^2Q_1^2 + 18Q_2Q_1Q_0 - 4Q_1^3 - 27Q_0^2}{Q_2^3} & \geq 0
\end{align*}
\]

This condition may be written \( Q_{\text{inf}} \leq Q_0 \leq Q_{\text{sup}} \), in which \( Q_{\text{inf}} \) and \( Q_{\text{sup}} \) are functions of \( Q_1 \) and \( Q_2 \), as illustrated in Fig. 8.2.3.

---

**Fig. 8.2.3** Condition for stable equilibrium when \( M_0, M_0, M_V \) are constant
The exceptional limit cases in which a binary efficiency changes its value (condition 2.5.21) during the disturbance, belong to unfavourable situations with ill-defined internal power flow. They are not considered here.

If the torques $M_\alpha$, $M_\beta$, $M_\gamma$ are time-dependent, supposition 3 is not true and, consequently, functions for $M_\alpha$, $M_\beta$, $M_\gamma$ have to be written in the equations of motion. Apart from being more laborious this procedure would not yield different viewpoints.

### 8.3. Electrical analogue of a variator network

As mentioned in 1.3.4 epicyclic gear trains and nodes do not satisfy Kirchhoff's laws in the common interpretation. A special type of electrical networks has to be considered to make an analogue, viz. electrical networks with transformers.

An epicyclic gear train can be represented in the analogue by a transformer with three coils in series, while a node is represented by a transformer with three coils in parallel.

Evidently, conversion leads to a converse analogue. In table 8.3.1 the correlations for the analogue and the converse analogue are summed up.

<table>
<thead>
<tr>
<th>Table 8.3.1 Correlations for the electrical analogue</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>analogue</strong></td>
</tr>
<tr>
<td>--------------</td>
</tr>
<tr>
<td>angular velocity</td>
</tr>
<tr>
<td>torque</td>
</tr>
<tr>
<td>power</td>
</tr>
<tr>
<td>moment of inertia</td>
</tr>
<tr>
<td>reciprocal torsional stiffness</td>
</tr>
<tr>
<td>epicyclic gear train</td>
</tr>
<tr>
<td>converse node</td>
</tr>
<tr>
<td>node</td>
</tr>
<tr>
<td>transmission ratio</td>
</tr>
<tr>
<td>dissipative power in transmission</td>
</tr>
</tbody>
</table>
2 All parameters of an epicyclic gear train, binary ratio, binary efficiency, moments of inertia, stiffnesses, are transferred to the shafts. The remaining converse node is realised in the analogue by a transformer with mutually equal coils in series.

3 In principle, a node is realised in the analogue by a transformer with three mutually equal coils in parallel, which in practice are replaced by a direct connection of the branches. See Fig. 8.3.1.

![Electrical analogue of a composing chain.](image)

**Fig. 8.3.1** Electrical analogue of a composing chain.
To build an analogue of a variator network ideal transformers, ideal inductors and ideal capacitors have to be added.

4 A transmission is represented by a transformer with two coils, a variator by a variable transformer with two coils.

5 The analogue described above has ideal capacitors, ideal inductors, and ideal transformers. A circuit with non-ideal elements needs corrections, especially to approximate assumption 1.8.9 (IV), i.e. a dissipative power should not influence any relation between angular velocities.

6 The conceptions 'in parallel' and 'in series' are not configurations in the electrical analogue only. They are also recognisable in the mechanisms of epicyclic gear trains and nodes. This may be
elucidated by the more simple analogue of forces of rectilinear motions. See Figs. 8.3.2 and 8.3.3, in which the analogue of a planet gear is a bogie.

Fig. 8.3.2 Forces in series  

Fig. 8.3.3 Forces in parallel
TERMINOLOGY

Principal quantities

1.8.1 angular velocity $\omega$ rad/s $T^{-1}$
   hoekssnelheid Winkelgeschwindigkeit

1.2.5 convention for the sign of an angular velocity

1.8.1 torque $T$ N.m $ML^2T^{-2}$
   moment van een koppel Drehmoment

1.2.6 convention for the sign of a torque

1.8.1 moment of inertia $J$ kg.m$^2$ $ML^2$
   massatraagheidsmoment Trägheitsmoment

1.2.9 torsional stiffness $S$ N.m/rad $ML^2T^{-2}$
   torsiestijheid Drehsteifigkeit

1.8.1 torsional stiffness $S$ N.m/rad $ML^2T^{-2}$
   Drehsteifigkeit

1.2.7 shaft power $P$ W $ML^2T^{-3}$
   asvermogen Wellenleistung

The product of the angular velocity of a shaft end and the torque exerted on that shaft end

1.5.18 dissipative power (lost power) $P$ W $ML^2T^{-3}$
   dissipatievermogen (verliesvermogen)
   Dissipationsleistung (Verlustleistung)
   A power that cannot be described as a shaft power

Parameters

1.6.6 transmission ratio $i_{A/B} = \frac{\omega_B}{\omega_A}$
   overbrengverhouding Übersetzungsverhältnis

1.6.7 efficiency $\eta_{BA} = -\frac{\omega_B}{\omega_A}$
   rendement Wirkungsgrad
1.5.5  binary ratio
2.1.2  binaire overbrengverhouding
Standübersetzungsverhältnis

1.5.14  binary efficiency
2.2.4  binair rendement
Standwirkungsgrad

The stationary shaft end is not necessarily the planet carrier

Other ratios

1.5.8  ternary ratio
2.1.3  ternaire overbrengverhouding
Laufübersetzungsverhältnis

1.5.17  ternary efficiency
2.2.1  ternair rendement
Laufwirkungsgrad, ....

The shaft ends are arbitrary shaft ends

5.1.1  responsivity
5.1.2  responsitviteit
Responsität
Shaft ends A, B, respectively C, D, are shaft ends of the reticulator and a variator

3.3.10  efficiency function
3.3.11  rendementsfunctie
Wirkungsgradfunktion
The conjugation of a ternary efficiency to the corresponding ternary ratio

Energy flow

1.7.1  power flow
vermogensstroom
Leistungsfluss
Diagrammatic representation of the input and output of energy

1.7.3  blind power
blind vermogen
Blindleistung
Smallest power along a closed path of cyclically equally directed powers

1.7.4  branched power flow
verzweigter Leistungsfluss
A power flow without a blind power

1.7.5  self-locking
zelfremming
Selbsthemmung
Concerns a situation in which no shaft power can be withdrawn
Elements

1. 2. 1  shaft end
Achsenende  Terminal part of an element, only characterised by an angular velocity and a torque

1. 8. 1  inertial element
roterend lichaam
Drehungskörper  Element characterised by moment of inertia

1. 2. 0  elastic element
elastisch deel
elastischer Teil  Element characterised by torsional stiffness

1. 8. 1  rotating shaft
draaiende as
drehende Achse  A unit with two shaft ends, considered as a sequence of inertial and elastic elements

1. 6. 2  (fixed) transmission
(vaste) overbrugging
(föstes) Getriebe  A unit with two shaft ends having a fixed ratio for angular velocities and a fixed ratio for torques

1. 8. 5  variator
stufenloses Getriebe  A transmission of which the ratio between angular velocities can be changed continuously by arbitrary intervention independent of the situation

3. 1. 1  branch
Zweig  A rotating shaft, a transmission, or a variator

3. 2. 3  hard branch
harde tak
harter Zweig  A branch without variator

3. 2. 4  soft branch
zachte tak
weicher Zweig  A variator

1. 3. 1  three-pole
driepool
Dreipol  A unit with three shaft ends

1. 1. 2  node
knoop
Knoten  A three-pole for which the angular velocities are equal and the sum of torques is zero
An imaginary element with three shaft ends for which the sum of angular velocities is zero and the torques are mutually equal.

A three-pole for which the angular velocities satisfy a linear equation, permitting three equal arbitrary angular velocities, and for which the torques have fixed mutual ratios.

Design of an epicyclic gear train

An epicyclic gear train with three and no more than three co-axial shafts.

A gear on one of the co-axial shafts.

A gear meshing a sun gear. The suffix of a planet gear is a small letter corresponding with the capital letter suffix of the sun gear.

An element on one of the co-axial shafts in which the planet gears are incorporated. In the symbolic representation a planet carrier may be indicated by a line extended in the circle.

Two planet gears, meshing different sun gears, with their mutual transmission in the planet carrier.

The replacement of a sun gear by any coupling between a planet gear and a co-axial shaft.

The gear ratio of the sun gears if the planet carrier does not rotate.

The efficiency of the power flow between the sun gears if the planet carrier does not rotate.
| 2.3.2 | internal power  | The product of a torque exerted on a shaft end and the difference in angular velocities of that shaft end with another |
| 2.7.2 | planet power  | The product of a torque exerted on a sun gear and the difference in angular velocities of that sun gear with that of the planet carrier |
| 2.7.3 | carrier power  | The product of the torque exerted on a sun gear and the angular velocity of the planet carrier |
| 7.5.1 | double epicyclic gear train  | Two single epicyclic gear trains interconnected by two nodes |
| 7.5.6 | Wolfrom-transmission | Double epicyclic gear train with a common planet carrier in which planet gears in a planet group have a common shaft |

**Variator network**

| 1.1.3 | variator network  | A coherent system of three-poles of which the shaft ends, except two of them, are interconnected by branches |
| 1.8.7 | variatornetwerk |  |
| 3.1.3 | Variatorennetz |  |

| 1.1.3 | input shaft, output shaft  | The shaft ends of a variator |
| 3.1.4 | ingaande as, uitgaande as network which does not connect Eintrittswelle, Austrittswelle two three-poles |

| 3.2.5 | subnetwork  | A coherent part of a variator network with at least one three-pole |
| 4.6.2 | complementary subnetworks  | Subnetworks, one with the input shaft, the other with the output shaft, together including all three-poles once |

| 4.6.2 | parallel branches  | The branches which interconnect complementary subnetworks |
| 4.6.3 | parallele takken  |  |
| 4.6.4 | Parallelzweige |  |

<p>| 4.1.1 | reticulator  | An imaginary transmission between the input shaft and the output shaft |
| 4.1.1 | retikulator |  |
| 4.1.1 | Retikulator |  |</p>
<table>
<thead>
<tr>
<th>Section</th>
<th>Term</th>
<th>Description</th>
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<td>closed network</td>
<td>A variator network complemented with the reticulator</td>
</tr>
<tr>
<td>4.2.1</td>
<td>mesh</td>
<td>A collection of branches, interconnected by the same number of three-poles, forming a closed figure</td>
</tr>
<tr>
<td>4.2.4</td>
<td>maas</td>
<td>A mesh including no variator, but possibly the reticulator</td>
</tr>
<tr>
<td>4.2.6</td>
<td>hard mesh</td>
<td>A mesh with one or more variators</td>
</tr>
<tr>
<td>5.3.1</td>
<td>backbone chain</td>
<td>An equal number of nodes and converse nodes, connected to each other in one line, including neither a transmission nor a mesh</td>
</tr>
<tr>
<td>3.4.1</td>
<td>conversion</td>
<td>The replacement of a variator network by a similar one for which between the angular velocities the same equations are operative as originally between the torques, and, likewise, between the torques the same equations are operative as originally between the angular velocities</td>
</tr>
<tr>
<td>4.3.1</td>
<td>reiterative network</td>
<td>A variator network in which a subnetwork is a variator network in itself, whether or not following interchange of adjacent epicyclic gear trains or interchange of adjacent nodes</td>
</tr>
<tr>
<td>4.4.1</td>
<td>inconsistent network</td>
<td>A variator network in which the angular velocities or the torques of any subnetwork do not have one and only one value for each presumed angular velocity or presumed torque in the variator network</td>
</tr>
<tr>
<td>4.5.1</td>
<td>reducible network</td>
<td>A network of which a subnetwork can be replaced by one with fewer three-poles</td>
</tr>
<tr>
<td>4.5.2</td>
<td>primitive subnetwork</td>
<td>A subnetwork without soft branches and without the reticulator, which is not equivalent to a subnetwork with fewer three-poles</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.5.2</td>
<td>variable shunt shunddrijfwerk Überlagerungsgetriebe</td>
<td>A variator network with one variator and one epicyclic gear train</td>
</tr>
<tr>
<td>5.5.9</td>
<td>variable bridge brugdrijfwerk Brückengetriebe</td>
<td>A variator network with one variator and two epicyclic gear trains</td>
</tr>
<tr>
<td>5.6.2</td>
<td>double variable shunt dubbeldshunddrijfwerk</td>
<td>A variator network with two variators and two epicyclic gear trains</td>
</tr>
<tr>
<td>5.6.6</td>
<td>double variable bridge dubbeldbrugdrijfwerk Doppelbrückengetriebe</td>
<td>A variator network with two variators and three epicyclic gear trains</td>
</tr>
<tr>
<td>5.6.7</td>
<td>extended double variable bridge uitgebreid dubbeldbrugdrijfwerk erweitertes Doppelbrückengetriebe</td>
<td>A variator network with two variators and four epicyclic gear trains</td>
</tr>
<tr>
<td>7.4.1</td>
<td>reverse-symmetric variable bridge keersymmetrisch brugdrijfwerk zyklisch-symmetrisches Brückengetriebe</td>
<td>A variable bridge with equal parameters of the epicyclic gear trains cyclically directed in the hard mesh</td>
</tr>
</tbody>
</table>
SUMMARY

This study deals with the transmission of power by means of some combination of epicyclic gear trains and variators, which as a whole acts as a variator. In such combinations, called 'variator networks', sets of three rotating shaft ends are interconnected by epicyclic gear trains and by fixed inter-connections, called 'nodes' (1.1.). The establishment of the network theory started with the admission that nodes are as significant as epicyclic gear trains.

The properties of epicyclic gear trains as well as of nodes are governed by equations between angular velocities and equations between torques. While in other network theories those relations are determined by Kirchhoff's laws, in the variator network theory the equations for epicyclic gear trains have to be distinguished from those for nodes (1.3.). Therefore, no correspondences could be found to other specialised network theories, and a complete new theory had to be established. For analysing the structure and the power transmitting properties of variator networks a mathematical model has been developed, based upon linear equations between angular velocities and between torques. To specify these equations, a few surprisingly simple assumptions have to be made (1.8.). By virtue of consistent conventions for the signs of angular velocities, torques, and other quantities, the establishment and the evaluation of the mathematical model proves to be very lucid. In some respect, the equations mentioned above are a substitute as well as an extension of the common Kirchhoff's laws.

It proves possible to interchange epicyclic gear trains and nodes in a variator network. Such an interchange, called 'conversion', is defined as the interchange of angular velocities and torques. A converted variator network is as significant in both theory and design as the original one (3.4.).

The converse of a node is represented by a 'converse node', being a fictitious epicyclic gear train because of the special form of the relative equations. An actual epicyclic gear train may be represented with the aid of a converse node (3.4.), which is a useful tool in the theory, for instance as a constituent part of what is here called the 'backbone chain' (5.3.). The latter is used for the convenient deduction of formulae for a given variator network.

Special attention is given to meshes in a variator network (4.2. and 4.3.), and to subnetworks (4.4. and 4.5.). In doing so, certain kinds of networks prove to be inconsistent or reducible and have, therefore, been excluded. For instance, a variator network in which the number of epicyclic gear trains exceeds twice the number of variators, is reducible to one with less epicyclic gear trains (4.5.).
The present outline of variator networks is confined to those with one or two variators (5, 7). Those with one variator, called 'variable shunt' (6), and 'variable bridge' (7), are discussed extensively. Those with two variators are discussed only concisely (5, 6).

Dissipative powers are consistently considered in all of the case studies performed. To each unit with a transmission ratio an efficiency is conjugated. When in a variator network a ratio between two angular velocities is given as a function of transmission ratios, then the ratio between the corresponding torques can be written as a similar function of transmission ratios and efficiencies (3, 4). Consequently, the efficiency of a variator network is determined as soon as the relationship between angular velocities is known.

A measure of the influence of a slight virtual variation of the transmission ratio of any individual variator on the transmission ratio of the variator network is its 'responsivity' (5, 1). The responsivity stands for more than this measure only, since it has proved to be decisive for the distribution of power in a variator network (5, 2).

An epicyclic gear train in a variator network may be represented by a converse node, having no transmission ratio nor efficiency. The parameters are transferred to units of the connecting shafts. It is proved that also inertial moments and stiffnesses may be replaced by lumped parameters assigned to the shafts (8, 1). This facilitates the deduction of relationships determining the dynamic performance of epicyclic gear trains (8, 2).

Electric analogues of variator networks are discussed concisely (8, 3).
The history of epicyclic gear trains was described by Förster [1], and earlier by Wilson [2]. Wolfram made the first publication about a double epicyclic gear train with a common planet carrier, later called 'Wolfram-transmission' (7, 5, 6). He referred to the design by the term "Planetenrädergetriebe mit Walzradantrieb" (epicyclic gear train with impelled planet gear), and in an example he calculated a total efficiency of 0.55 for such a transmission with an overall ratio of 100 and a sun efficiency of 0.975. Terplán [4] made an interesting summary of the calculation methods of Kutzbach, Willis, Swamp, Balogh and Szőke, Poppinga, and Koševnikov.

Merritt [5] developed a useful theory independent of the actual design, while Poppinga [6], on the contrary, made tables with 50 cases (of which 8 were double-counted) transformed to unworkable designs. Wolf [7] restored the independence of the actual design, although his theory was not consistent. The symbol in Fig. 1.5.1 and several formulae in 2.1 and 2.2 are similar to those of Wolf. Section 8.1 is an extension of a part of the work of Strömblad [8], while section 2.5 is an extension of Örnhagens theory [9]. Kudrjavcev [10] collected Russian knowledge on the subject.

Literature about testing and about design is left out of consideration, with the exception of four references. Neussel [11] tested a double epicyclic gear train of a special design. Meyer zur Capellen [12] proved the Wankel-engine to be a special case of an epicyclic gear train (Fig. 2.6.8). Tuplin [13] and Hill [14] calculated the number of planet groups (2.6.).

The acceleration of epicyclic gear trains was studied by Tank [15] with formulae reminiscent of the more complete equations 2.5.11 to 2.5.13 inclusive. The deduction of equilibrium conditions for epicyclic gear trains (5.2.) was inspired by a colloquium held by de Beer [16] concerning a worm gear transmission. Denton [17] reported on an unfortunate application of an epicyclic gear train in combination with worm gear drives, but his conclusions were premature.

Rivin [18] proposed simplifications in computing schemes of the dynamics of rotating shafts, while Zinov'iev and Umnov [19] warned against intolerable simplifications in the equation of motion for a mechanical system with a variator.

The oldest design of a variable transmission with split power paths was patented in 1896 [20]. It concerns a hydraulic transmission of which the common casing of pump and motor rotates and acts as output shaft. Modern designs were published by Sadler [22], Himmler [23], and others. Since a single epicyclic gear train and a variable shunt have many
principles in common it is difficult to say that such designs are exclusively equivalent to either epicyclic gear trains or variable shunts. However, a main characteristic in modern designs is an extension of the shaft of the pump to the motor. The extended shaft has the angular velocity of the input shaft, the pump transforms the torque ('node plus transmission'). The extended shaft has the torque of the output shaft, the motor transforms the angular velocity ('epicyclic gear train'). Therefore, such variable hydraulic transmissions may be called 'shunt variators'.

The question when and where variable shunts and variable bridges were invented, will not be answered here while references of the motor-car industry have not been given. Actual designs of variable shunts were reported by Wood [21], Hicks [27], Wahl [28], Jarchow [29], and Olderamaan [34]. One may surmise, that when Macmillan [26] mentioned the discovery by Davies of a variable bridge with the variator between the nodes, Himmler [23] also knew such a design, although his diagram was incorrect and inconsistent with his text. Actual designs of variable bridges were described by Westbury, Smith and Glaze [25] and by Wilson [30]. Since about 1961 publications on variable shunts and variable bridges have appeared regularly, [24] to [37] inclusive, commonly with elementary theoretical considerations. The 'Carnot-theorem' of French [36] may be superfluous by the introduction of 'responsibility' (5.1.).

A mathematical basis for network theories may be found in several books, for instance Ore [38], [39], Seshu and Reed [40]. From the many books dealing with electrical networks Cauer [41] may be referred here. An interesting book covering many other applications is Blackwell [42].

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**Persoonlijke gegevens**

Na het doorlopen van een lagere school te Voorburg bezocht ik de Gemeentelijke HBS te Delft, waar ik in 1943 het diploma HBS-B behaalde. Gedurende de laatste oorlogsjaren volgde ik een wiskunde-opleiding bij prof. dr. F. Schuh. Van 1945 tot 1950 studeerde ik werktuigbouwkunde aan de TH Delft. In 1948 behaalde ik de MO-akte K1 wiskunde, aangevuld met akte Q. Gedurende anderhalf jaar was ik assistent bij de afdeling Wiskunde aan de TH Delft.

STEELINGEN

1. Bij de bewerking van een inwendig evolvent tandwiel ontstaan geen valse ingrijingsverschijnselen, indien zijn topkromtestraal groter is dan de topkromtestraal van het steekwiel. Is daarentegen de topkromtestraal van het steekwiel de grootste, dan ontstaat altijd 'oversnijding', nog voordat er sprake is van echte afwikkeling. Door voldoend diep insnijden kan deze oversnijding worden overwonnen. Polder, J. W.: Overcut, a new theory for tip interference in internal gears (binnenkort te publiceren)

2. Voor een genormaliseerd vertandingssysteem zijn een betrekkelijk klein aantal onderling onafhankelijke geometrische grootheden aan te wijzen, die gezamenlijk een tandwieloverbrenging bepalen en die in een belastbaarheidsberekening elk in slechts één factor optreden. De berekeningsmethoden kunnen daarom zo worden ingericht, dat zonder moeite een optimale vormgeving wordt bereikt.

   Voorlichtingsbladen voor de Metaalindustrie, verzorgd door Werkgroep Tandwielen van de FME:
   VM 09-03 Geometrie van de uitwendige overbrenging,
   VM 09-04 Belastbaarheid van metalen tandwielen.
   Voorstellen van commissie 5 van het NNI in werkgroep 6 van ISO/TC 60.

3. Voor een eenvoudig roterende beweging dient Hz (hertz) te worden gekozen als eenheid van omwentelingsfrequentie ('toerental').

4. Er is geen reden om het begrip 'specifieke glijding' g te handhaven. Waar dat gewenst is kan beter het complement (1-g) worden gebruikt, dat 'meeleeslependement' of 'snelheidsrendement' zou kunnen worden genoemd.

5. De uitdrukking \( i^2 \) voor een gereduced massatraagheidsmoment is onjuist (\( i \) overbrengverhouding, \( J \) massatraagheidsmoment).

6. Het ontwerpen van een overbrengingssysteem met een vertakte vermogensstroom zonder het bij de berekening in acht nemen van alle dissipatievermogens is onverantwoord.

7. De theoretische onderzoeken van Poppinga aan dubbele planeetdrijfwerken leiden niet tot een bruikbaar criterium voor de keuze van zulke planeetdrijfwerken.

   Poppinga, R.: Stirnrad-Planetengetriebe
   Franckh'sche Verlagsbuchhandlung, Stuttgart, 1949
8. De lichaamsvorm van kniptorren (Elateridae) mag als optimaal worden beschouwd voor het effect van de 'sprong' of de knippende beweging die zij maken indien zij in nood verkeren. De nauw verwante goudgerande waterkevers (Dytiscidae) hebben, ondanks een overeenkomstige borststekel, dit springvermogen niet, hetgeen verband kan houden met het geringere effect van een eventuele knipbeweging in water en met de krachtige ontwikkeling van de zwempoten.

Oudemans, A. C.: Het springen der knip- of springkevers
De levende natuur 25(1920)1, 9-17
Doorman, G.: Die Mechanik des Sprunges der Schnellkäfer (Elateriden)
Biol. Centralbl. 40(1920)2/3, 116-119
Binaghi, G.: Sulla meccanica del salto degli Elateridi
Boll. Soc. Entom. Italiana 74(1942)1, 1-6

9. Voor een groep van in bedrijf zijnde gelijke machines kan een schatting worden gemaakt van de kans op storingen en de 'levensduur' op een belangrijk eenvoudiger wijze dan veelal gebruikelijk.

Polder, J. W.: Levensduur van machines
(niet gepubliceerd)

10. Voor een groep van in bedrijf zijnde gelijke machines kan een grafiek van de verdeling van draaiuren (bedrijfsuren, logboektijd) worden gemaakt door langs de abscis draaiuren uit te zetten en langs de ordinaat de machines, die niet door een blijvende storing zijn uitgevallen, te rangschikken naar hun stand van draaiuren op een bepaald tijdstip (kalenderdertijd). De verdelingslijn zal vrij spoedig een vrijwel vaste gebogen vorm aannemen. De differentiaalvergelijking van de verdelingslijn vertoont verwantschap met die van een trillende snaar, onderworpen aan demping.

Polder, J. W.: Levensduur van machines
(niet gepubliceerd)

11. Een meer op de herkenning van het wiskundig model gerichte studie is niet alleen van belang voor de ontwikkeling van de jonge werktuigkundige ingenieur, maar kan ook de studie aan de TH beter toegankelijk maken voor iemand die op oudere leeftijd met de studie begint.

12. Er is behoefte aan de ontwikkeling en de toepassing van een filosofie voor de werktuigbouwkunde.

6 juni 1969, J. W. Polder