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AdS non-linear curvature-squared and curvature-quartic multidimensional (D=8) gravitational models with stabilized extra dimensions

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Abstract

We investigate $D$-dimensional gravitational model with curvature-quadratic and curvature-quartic correction terms: $R + R^2 + R^4$. It is assumed that the corresponding higher dimensional spacetime manifold undergoes a spontaneous compactification to a manifold with warped product structure. Special attention is paid to the stability of the extra-dimensional factor space for a model with critical dimension $D = 8$. It is shown that for certain parameter regions the model allows for a freezing stabilization of this space. The effective four-dimensional cosmological constant is negative and the external four-dimensional spacetime is asymptotically AdS.

1 Introduction

Multidimensionality of our Universe is one of the most intriguing assumption in modern physics. It follows from theories which unify different fundamental interactions with gravity, such as M or string theory \cite{1}, and which have their most consistent formulation in spacetimes with more than four dimensions. Thus, multidimensional cosmological models have received a great deal of attention over the last years.

Stabilization of additional dimensions near their present day values (dilaton/geometrical moduli stabilization) is one of the main problems for any multidimensional theory because a dynamical behavior of the internal spaces results in a variation of the fundamental physical constants. Observations show that internal spaces should be static or nearly static at least from the time of recombination (in some papers arguments are given in favor of the assumption that variations of the fundamental constants are absent from the time of primordial nucleosynthesis \cite{2}). In other words, from this time the compactification scale of the internal space should either be stabilized and trapped at the minimum of some effective potential, or it should be slowly varying (similar to the slowly varying cosmological constant in the quintessence scenario). In both cases, small fluctuations over stabilized or slowly varying compactification scales (conformal scales/geometrical moduli) are possible.

Stabilization of extra dimensions (moduli stabilization) in models with large extra dimensions (ADD-type models) has been considered in a number of papers (see e.g., Refs. \cite{3}-\cite{10}). In the corresponding approaches, a product topology of the $(4 + D')$-dimensional bulk spacetime was constructed from Einstein spaces with scale (warp) factors depending only on the coordinates of the external 4-dimensional component. As a consequence, the conformal excitations had the form of massive scalar fields living in the external spacetime. Within the framework of multidimensional cosmological models (MCM) such excitations were investigated in \cite{11}-\cite{13} where they were called gravitational excitons. Later, since the ADD compactification approach these geometrical moduli excitations are known as radions \cite{3, 5}.

Most of the aforementioned papers are devoted to the stabilization of large extra dimensions in theories with a linear multidimensional gravitational action. String theory suggests that the usual linear Einstein-Hilbert action should be extended with higher order nonlinear curvature terms. In our papers \cite{17}-\cite{20} we considered a simplified model with multidimensional Lagrangian of the form $L = f(R)$, where $f(R)$ is an arbitrary smooth function of the scalar curvature. Without connection to stabilization of the extra-dimensions, such models (4-dimensional as well as multidimensional ones) were considered e.g. in Refs. \cite{14}-\cite{16}. There, it was shown

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that the nonlinear models are equivalent to models with linear gravitational action plus a minimally coupled scalar field with self-interaction potential. Similar approach was elaborated in Refs. [21] where the main attention was paid to a possibility of the late time acceleration of the Universe due to the nonlinearity of the model. In our papers [17–20], we advanced the equivalence between the nonlinear models and the linear ones with a minimally coupled scalar field towards investigating the stabilization problem for extra dimensions1).

Particular attention was paid to models where usual linear curvature term \( R \) was supplemented with either \( R^2 \) or \( R^3 \) or \( R^{-1} \) nonlinear terms. All of these models can be investigated analytically. It was shown that for certain parameter ranges, the extra dimensions are stabilized. In the present paper we extend this consideration to a model with Lagrangian of the type \( f(R) = R + R^2 + R^4 \). Such simple generalization enriches considerably the qualitative behavior of the model because it results in either one-branch or three-branch models. For each of these branches the stability analysis should be performed separately. Our paper is the first one among two papers devoted to this model. Here, we investigate the one-branch model postponing the three-branch model for our forthcoming paper. This model can be solved analytically. However, this analysis is very cumbersome for an arbitrary number of dimensions \( D \). So, we restrict our consideration to a critical dimension \( D = 8 \). Critical dimension is defined by doubled degree of the scalar curvature polynomial \( f(R) \), i.e. in our case \( D = 2 \times 4 \) (see Appendix A below). We show that for certain parameter regions the model allows for a freezing stabilization of the internal space. Here, the stabilization takes place in negative minimum of the effective potential. Thus the effective four-dimensional cosmological constant is also negative and the homogeneous and isotropic external four-dimensional spacetime is asymptotically AdS.

The paper is structured as follows. In section 2 we present a brief technical outline of the transformation from a non-Einsteinian purely gravitational model with general scalar curvature nonlinearity of the type \( f(R) \) to an equivalent curvature-linear model with additional nonlinearity carrying scalar field. Afterwards, we derive criteria which ensure the existence of at least one minimum for the effective potential of the internal space scale factors (volume moduli). These criteria are then used in Sec. 3 to obtain the regions in parameter space which allow for a freezing stabilization of the scale factors in the model with scalar curvature nonlinearities of the type \( f(R) = R + R^2 + R^4 \). A brief discussion of the obtained results is presented in the concluding Sec. 4. In appendix A it is shown that \( D = 2 \times \text{deg}_g(f) \) is the critical dimension of the considered models. Finally, graphical visualization of the effective potential with a global minimum is given in appendix B.

2 General setup

We consider a \( D = (4 + D') \)-dimensional nonlinear pure gravitational theory with action functional

\[
S = \frac{1}{2\kappa_D^2} \int_M d^Dx \sqrt{|g|} f(\bar{R}) ,
\]

where \( f(\bar{R}) \) is an arbitrary smooth function with mass dimension \( \mathcal{O}(m^2) \) (\( m \) has the unit of mass) of a scalar curvature \( \bar{R} = R[\bar{g}] \) constructed from the \( D \)-dimensional metric \( g_{ab} \) \((a, b = 1, \ldots, D)\). \( D' \) is the number of extra dimensions and \( \kappa_D^2 \) denotes the \( D \)-dimensional gravitational constant which is connected with the fundamental mass scale \( M_{\ast(4+D')} \) and the surface area \( S_{D-1} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2) \) of a unit sphere in \( D-1 \) dimensions by the relation \( 20, 27 \)

\[
\kappa_D^2 = 2S_{D-1}/M_{\ast(4+D')}^2 .
\]

Before we endow the metric of the pure gravity theory \( 2.1 \) with explicit structure, we recall that this \( \bar{R} \)-nonlinear theory is equivalent to a theory which is linear in another scalar curvature \( \bar{R} \) but which contains an additional self-interacting scalar field. According to standard techniques \( 14, 15, 13 \), the corresponding \( R \)-linear theory has the action functional:

\[
S = \frac{1}{2\kappa_D^2} \int_M d^Dx \sqrt{|g|} \left[ R[\bar{g}] - g^{ab}\phi_{,a}\phi_{,b} - 2U(\phi) \right] ,
\]

where

\[
f'(\bar{R}) = \frac{df}{d\bar{R}} := e^{A\phi} > 0 , \quad A := \sqrt{\frac{D-2}{D-1}} ,
\]

and where the self-interaction potential \( U(\phi) \) of the scalar field \( \phi \) is given by

\[
U(\phi) = \frac{1}{2} (f'(\bar{R}))^{-D/(D-2)} \left[ \bar{R} f' - f \right] ,
\]

\[
= \frac{1}{2} e^{-B\phi} \left[ \bar{R}(\phi)e^{A\phi} - f(\bar{R}(\phi)) \right] , \quad B := \frac{D}{\sqrt{(D-2)(D-1)}} .
\]

1) A different approach to the problem of the extra dimension stabilization was proposed in Ref. 22. This method can be applied to Lagrangians containing high-order curvature invariants. In the case of the models with Lagrangians \( L = f(R) \) both of these approaches result in the same conclusions.
This scalar field \( \phi \) carries the nonlinearity degrees of freedom in \( \bar{R} \) of the original theory, and for brevity we call it the nonlinearity field. The metrics \( g_{ab}, \bar{g}_{ab} \) of the two theories (2.4) and (2.8) are conformally connected by the relations

\[
    g_{ab} = \Omega^2 \bar{g}_{ab} = \left[ f'(\bar{R}) \right]^{2/(D-2)} \bar{g}_{ab}.
\]  

(2.7)

As next, we assume that the D-dimensional bulk space-time \( M \) undergoes a spontaneous compactification to a warped product manifold

\[
    M = M_0 \times M_1 \times \ldots \times M_n
\]  

(2.8)

with metric

\[
    \bar{g} = \bar{g}_{ab}(X) dX^a \otimes dX^b = \bar{g}^{(0)} + \sum_{i=1}^n e^{2\beta^i(x)} g^{(i)}.
\]  

(2.9)

The coordinates on the \((D_0 = d_0 + 1)\)-dimensional manifold \( M_0 \) (usually interpreted as our observable \( (D_0 = 4)\)-dimensional Universe) are denoted by \( x \) and the corresponding metric by

\[
    \bar{g}^{(0)} = \bar{g}_{\mu\nu}^{(0)}(x) dx^\mu \otimes dx^\nu.
\]  

(2.10)

For simplicity, we choose the internal factor manifolds \( M_i \) as \( d_i \)-dimensional Einstein spaces with metrics

\[
    g^{(i)} = g^{(i)}_{m_i n_i} (y_i) dy_i^{m_i} \otimes dy_i^{n_i},
\]  

so that the relations

\[
    R^{(i)}_{m_in_i} \left[ g^{(i)} \right] = \lambda^i g_{m_in_i}, \quad m_i, n_i = 1, \ldots, d_i
\]  

(2.11)

and

\[
    R \left[ g^{(i)} \right] = \lambda^i d_i = R_i
\]  

(2.12)

hold. The specific metric ansatz (2.9) leads to a scalar curvature \( \bar{R} \) which depends only on the coordinates \( x \) of the external space: \( \bar{R}[^\bar{g}] = \bar{R}(x) \). Correspondingly, also the nonlinearity field \( \phi \) depends on \( x \) only: \( \phi = \phi(x) \).

Passing from the \( \bar{R} \)-nonlinear theory (2.4) to the equivalent \( R \)-linear theory (2.8), the metric (2.9) undergoes the conformal transformation \( \bar{g} \mapsto g \) [see relation (2.7)]

\[
    g^{(i)} = \Omega^2 \bar{g}^{(i)} = (e^{A\phi})^{2/(D-2)} \bar{g}^{(0)} + \sum_{i=1}^n e^{2\beta^i(x)} g^{(i)}
\]  

(2.13)

with

\[
    g^{(0)}_{\mu\nu} := (e^{A\phi})^{2/(D-2)} \bar{g}^{(0)}_{\mu\nu}, \quad \beta^i := \beta^i + \frac{A}{D-2}\phi.
\]  

(2.14)

The main subject of our subsequent considerations will be the stabilization of the internal space components. A strong argument in favor of stabilized or almost stabilized internal space scale factors \( \beta^i(x) \), at the present evolution stage of the Universe, is given by the intimate relation between variations of these scale factors and those of the fine-structure constant \( \alpha \) [28]. The strong restrictions on \( \alpha \)-variations in the currently observable part of the Universe [29] imply a correspondingly strong restriction on these scale factor variations [28]. For this reason, we will concentrate below on the derivation of criteria which will ensure a freezing stabilization of the scale factors. Extending earlier discussions of models with \( \bar{R}^2, \bar{R}^4 \) and \( \bar{R}^{-1} \) scalar curvature nonlinearities [17]-[20] we will investigate here models of the nonlinearity type \( \bar{R}^2 + \bar{R}^4 \).

In Ref. [13] it was shown that for models with a warped product structure (2.4) of the bulk spacetime \( M \) and a minimally coupled scalar field living on this spacetime, the stabilization of the internal space components requires a simultaneous freezing of the scalar field. Here we expect a similar situation with simultaneous freezing stabilization of the scale factors \( \beta^i(x) \) and the nonlinearity field \( \phi(x) \). According to (2.14), this will also imply a stabilization of the scale factors \( \beta^i(x) \) of the original nonlinear model.

In general, the model will allow for several stable scale factor configurations (minima in the landscape over the space of volume moduli). We choose one of them \( ^2 \), denote the corresponding scale factors as \( \beta^i_0 \), and work further on with the deviations

\[
    \beta^i(x) = \beta^i(x) - \beta^i_0
\]  

(2.15)

as the dynamical fields. After dimensional reduction of the action functional (2.13) we pass from the intermediate Brans-Dicke frame to the Einstein frame via a conformal transformation

\[
    g^{(0)}_{\mu\nu} = \hat{\Omega}^2 \tilde{g}^{(0)}_{\mu\nu} = \left( \prod_{i=1}^n e^{A\beta^i} \right)^{-2/(D_0-2)} g^{(0)}_{\mu\nu}
\]  

(2.16)

\(^2 \) Although the toy model ansatz (2.14) is highly oversimplified and far from a realistic model, we can roughly think of the chosen minimum, e.g., as that one which we expect to correspond to current evolution stage of our observable Universe.
with respect to the scale factor deviations \( \dot{\beta}(x) \). As result we arrive at the following action

\[
S = \frac{1}{2\kappa^2_D} \int d^{D_0} x \sqrt{|g^{(0)}|} \left\{ \hat{R} \left[ g^{(0)} \right] - \hat{G}_{ij} \ddot{g}^{(0)}_{\mu\nu} \partial^i \dot{\beta}^j \partial^\nu \dot{\beta} - \ddot{g}^{(0)}_{\mu\nu} \partial^i \dot{\beta}^j \partial^\nu \dot{\beta} - 2\kappa^2_D \partial^i \dot{\beta}^i \right\},
\]  

(2.17)

which contains the scale factor offsets \( \beta^0 \) through the total internal space volume

\[
V_{D^*} = V_I \times v_0 = \prod_{i=1}^n \int_M d^d y \sqrt{|g^{(i)}|} \times \prod_{i=1}^n e^{d_i \beta^0_i}
\]

(2.18)
in the definition of the effective gravitational constant \( \kappa^2_{D_0} \) of the dimensionally reduced theory

\[
\kappa^2_{(D_0=4)} = \kappa^2_D/V_{D^*} = 8\pi/M^4 \implies M^4 = \frac{4\pi}{S_{D-1}} V_{D^*} M^{2+D'}.
\]

(2.19)

Obviously, at the present evolution stage of the Universe, the internal space components should have a total volume which would yield a four-dimensional mass scale of order of the Planck mass. The tensor components of the midisuperspace metric (target space metric on \( \mathbb{R}^n_T \)) reads: \( \hat{G}_{ij} = d_i \delta_{ij} + d_i d_j / (D_0 - 2) \), where \( i, j = (1, \ldots, n) \), see \[30\, [31]. The effective potential has the explicit form

\[
U_{eff}(\dot{\beta}, \phi) = \left( \prod_{i=1}^n e^{d_i \beta^0_i} \right)^{-\frac{n^2-1}{2}} \left[ -\frac{1}{2} \sum_{i=1}^n \hat{R}_i e^{-2\hat{\beta}^i} + U(\phi) \right],
\]

(2.20)

where we abbreviated

\[
\hat{R}_i := R_i \exp (-2\beta^0_i).
\]

(2.21)

A freezing stabilization of the internal spaces will be achieved if the effective potential has at least one minimum with respect to the fields \( \dot{\beta}(x) \). Assuming, without loss of generality, that one of the minima is located at \( \beta^i = \beta^0 \Rightarrow \dot{\beta}^i = 0 \), we get the extremum condition:

\[
\frac{\partial U_{eff}}{\partial \beta^i} \bigg|_{\beta=0} = 0 \implies \hat{R}_i = \frac{d_i}{D_0 - 2} \left( -\sum_{j=1}^n \hat{R}_j + 2U(\phi) \right).
\]

(2.22)

From its structure (a constant on the l.h.s. and a dynamical function of \( \phi(x) \) on the r.h.s.) it follows that a stabilization of the internal space scale factors can only occur when the nonlinearity field \( \phi(x) \) is stabilized as well. In our freezing scenario this will require a minimum with respect to \( \phi \):

\[
\left. \frac{\partial U(\phi)}{\partial \phi} \right|_{\phi_0} = 0 \iff \left. \frac{\partial U_{eff}}{\partial \phi} \right|_{\phi_0} = 0.
\]

(2.23)

We arrived at a stabilization problem, some of whose general aspects have been analyzed already in Refs. \[11\,\, [13\] and \[17\,\, [20\]. For brevity we only summarize the corresponding essentials as they will be needed for more detailed discussions in the next sections.

1. Eq. (2.22) implies that the scalar curvatures \( \hat{R}_i \) and with them the compactification scales \( e^{d_i \beta^0} \) [see relation (2.21)] of the internal space components are finely tuned

\[
\frac{\hat{R}_i}{d_i} = \frac{\hat{R}_j}{d_j}, \quad i, j = 1, \ldots, n.
\]

(2.24)

2. The masses of the normal mode excitations of the internal space scale factors (gravitational excitons/radions) and of the nonlinearity field \( \phi \) near the minimum position are given as \[13\]:

\[
m^2_1 = \ldots = m^2_n = -\frac{4}{D-2} U(\phi_0) = -\frac{2\hat{R}_i}{d_i} > 0,
\]

(2.25)

\[
m^2_\phi := \left. \frac{d^2 U(\phi)}{d\phi^2} \right|_{\phi_0} > 0.
\]

(2.26)
3. The value of the effective potential at the minimum plays the role of an effective 4D cosmological constant of the external (our) spacetime $M_0$:

$$
\Lambda_{\text{eff}} := U_{\text{eff}} \bigg|_{\beta^i=0, \phi=\phi_0} = \frac{D_0 - 2}{D - 2} U(\phi_0) = \frac{D_0 - 2}{2} \frac{\partial_i}{d_i} .
$$

(2.27)

4. Relation (2.24) implies

$$
\text{sign } \Lambda_{\text{eff}} = \text{sign } U(\phi_0) = \text{sign } R_i .
$$

(2.28)

Together with condition (2.25) this shows that in a pure geometrical model stable configurations can only exist for internal spaces with negative curvature$^3$:

$$
R_i < 0 \quad (i = 1, \ldots, n) .
$$

(2.29)

Additionally, the effective cosmological constant $\Lambda_{\text{eff}}$ as well as the minimum of the potential $U(\phi)$ should be negative too:

$$
\Lambda_{\text{eff}} < 0, \quad U(\phi_0) < 0 .
$$

(2.30)

Plugging the potential $U(\phi)$ from Eq. (2.6) into the minimum conditions (2.23), (2.26) yields with the help of $\partial_\phi \tilde{R} = A f'/f''$ the conditions

$$
\frac{dU}{d\phi} \bigg|_{\phi=\phi_0} = \frac{A}{2(D-2)}(f''-D/(D-2)h) \bigg|_{\phi_0} = 0, \quad h := D f - 2 \tilde{R} f', \implies h(\phi_0) = 0 ,
$$

(2.31)

$$
\frac{d^2 U}{d\phi^2} \bigg|_{\phi=\phi_0} = \frac{1}{2} A e^{(A-B)\phi_0} [\partial_\phi \tilde{R} + (A-B)\tilde{R}] \bigg|_{\phi_0} = \frac{1}{2(D-1)}(f'')^{-2/(D-2)} \frac{f''}{f''} \partial_\phi h \bigg|_{\phi_0} > 0 ,
$$

(2.32)

where the last inequality can be reshaped into the suitable form

$$
f'' \partial_\phi h \big|_{\phi_0} = f'' \left[(D-2) f' - 2 \tilde{R} f'' \right] \big|_{\phi_0} > 0 .
$$

(2.33)

Furthermore, we find from Eq. (2.31)

$$
U(\phi_0) = \frac{D - 2}{2D} (f'')^{-\frac{D}{D-2}} \tilde{R}(\phi_0) .
$$

(2.34)

so that (2.31) leads to the additional restriction

$$
\tilde{R}(\phi_0) < 0
$$

(2.35)

at the extremum.

In the next section we will analyze the internal space stabilization conditions (2.24) - (2.30) and (2.31) - (2.35) on their compatibility with particular scalar curvature nonlinearity $f(\tilde{R})$. According to our definition (2.4) we shall consider the positive branch

$$
f'(\tilde{R}) > 0 .
$$

(2.36)

Although the negative $f' < 0$ branch can be considered as well (see e.g. Refs. [15, 16, 20]), we postpone this case for our future investigations.

3 The $R^2 + R^4$-model

In this section we analyze a model with curvature-quadratic and curvature-quartic correction terms of the type

$$
f(\tilde{R}) = \tilde{R} + \alpha \tilde{R}^2 + \gamma \tilde{R}^4 - 2\Lambda_D .
$$

(3.1)

We start our investigation for an arbitrary number of dimensions $D$. First of all, we should define the relation between the scalar curvature $\tilde{R}$ and the nonlinearity field $\phi$. According to eq. (2.4) we have:

$$
f' = e^{A\phi} = 1 + 2\alpha \tilde{R} + 4\gamma \tilde{R}^3 .
$$

(3.2)

$^3$) Negative constant curvature spaces $M_i$ are compact if they have a quotient structure: $M_i = H^{d_i}/\Gamma_i$, where $H^{d_i}$ and $\Gamma_i$ are hyperbolic spaces and their discrete isometry group, respectively.
This equation can be rewritten in the form
\[ \bar{R}^3 + \frac{\alpha}{2\gamma} \bar{R} - \frac{X(\phi)}{4\gamma} = 0, \]  
(3.3)

where
\[ X \equiv e^{A\phi} - 1, \quad -\infty < \phi < +\infty \iff -1 < X < +\infty. \]  
(3.4)

Eq. (3.3) has three solutions \( \bar{R}_{1,2,3} \), where one or three of them are real valued. Let
\[ q := \frac{\alpha}{6\gamma}, \quad r := \frac{1}{8\gamma} X. \]  
(3.5)

The sign of the discriminant
\[ Q := r^2 + q^3 \]  
(3.6)
defines the number of real solutions (see, e.g., Ref. [32]):
\[ Q > 0 \quad \Rightarrow \quad \exists R_1 = 0, \quad \exists R_{2,3} \neq 0 \]
\[ Q = 0 \quad \Rightarrow \quad \exists R_i = 0, \quad i = 1, 2, 3, \quad \bar{R}_1 = \bar{R}_2 \]
\[ Q < 0 \quad \Rightarrow \quad \exists \bar{R}_i = 0, \quad i = 1, 2, 3. \]  
(3.7)

It is most convenient to consider \( \bar{R}_i = \bar{R}_i(X) \) as solution family depending on the two additional parameters \( (\alpha, \gamma) \). Physical scalar curvatures correspond to real solutions \( \bar{R}_i(X) \). For \( Q > 0 \) the single real solution \( \bar{R}_1 \) is given as
\[ \bar{R}_1 = \left[ r + Q^{1/2} \right]^{1/3} + \left[ r - Q^{1/2} \right]^{1/3}. \]  
(3.8)
The three real solutions \( \bar{R}_{1,2,3}(X) \) for \( Q < 0 \) read
\[ \bar{R}_1 = s_1 + s_2, \]
\[ \bar{R}_2 = \frac{1}{2} (-1 + i\sqrt{3}) s_1 + \frac{1}{2} (-1 - i\sqrt{3}) s_2 = e^{i\frac{2\pi}{3}} s_1 + e^{-i\frac{2\pi}{3}} s_2, \]
\[ \bar{R}_3 = \frac{1}{2} (-1 - i\sqrt{3}) s_1 + \frac{1}{2} (-1 + i\sqrt{3}) s_2 = e^{-i\frac{2\pi}{3}} s_1 + e^{i\frac{2\pi}{3}} s_2, \]  
(3.9)

where we can fix the Riemann sheet of \( Q^{1/2} \) by setting in the definitions of \( s_{1,2} \)
\[ s_{1,2} := \left[ r \pm i|Q|^{1/2} \right]^{1/3}. \]  
(3.10)

In this paper we investigate the case of positive \( Q(\phi) \) that is equivalent to the condition
\[ Q(\phi) > 0 \quad \Rightarrow \quad \text{sign } \alpha = \text{sign } \gamma. \]  
(3.11)
The case \( \text{sign } \alpha \neq \text{sign } \gamma \) that corresponds to different signatures of the discriminant \( Q \) will be considered in our forthcoming paper.

To define the conditions for minima of the effective potential \( U_{eff} \), first we obtain the extremum positions of the potential \( U(\phi) \). The extremum condition (2.31) for our particular model (3.1) reads:
\[ \bar{R}_{(0)1}^4 \left( \frac{D}{2} - 4 \right) + \bar{R}_{(0)1}^2 \left( \frac{D}{2} - 2 \right) + \bar{R}_{(0)1} \left( \frac{D}{2} - 1 \right) - D\Lambda_D = 0, \]  
(3.12)

where subscript 1 indicates that we seek the extremum positions for the solution (3.3). Eq. (3.12) clearly shows that \( D = 8 \) is the critical dimension for the model (3.1) in full agreement with the result of the Appendix (see (A.3)). In what follows, we investigate this critical case. For \( D = 8 \) eq. (3.12) is reduced to a quadratic one
\[ \bar{R}_{(0)1}^2 + \frac{3}{2\alpha} \bar{R}_{(0)1} - \frac{4\Lambda_8}{\alpha} = 0; \quad \Lambda_8 \equiv \Lambda_{D=8} \]  
(3.13)

with the following two roots:
\[ \bar{R}_{(0)1}^{(\pm)} = -\frac{3}{4\alpha} \pm \sqrt{\left( \frac{3}{4\alpha} \right)^2 + \frac{4}{\alpha} \Lambda_8}. \]  
(3.14)
These roots are real if parameters $\alpha$ and $\Lambda_8$ satisfy the following condition:

$$
\left(\frac{3}{4\alpha}\right)^2 + \frac{4}{\alpha}\Lambda_8 \geq 0.
$$

(3.15)

If $\text{sign}(\alpha) = \text{sign}(\Lambda_8)$, then condition (3.15) is automatically executed, else

$$
|\Lambda_8| \leq \frac{9}{64|\alpha|}, \quad \text{sign}(\alpha) \neq \text{sign}(\Lambda_8).
$$

(3.16)

To insure that roots correspond to a minimum value of $U(\phi)$, they should satisfy the condition (2.33):

$$
f'' \left[ (D-2)f' - 2\bar{R}f'' \right]_{\phi_0} > 0 \iff f'' \left[ (3 + 4\alpha \bar{R})_{\phi_0} > 0, \right.
$$

(3.17)

where

$$
f'' = 2\alpha + 12\gamma \bar{R}^2.
$$

(3.18)

Because for $Q > 0$ eq. (3.18) is the single real solution of the cubic eq. (3.13), then $\bar{R} = \bar{R}_1(\phi)$ is a monotonic function of $\phi$. Thus, the derivative $\partial_\phi \bar{R}_1 = A \bar{f}^2 / f''$ does not change its sign. Keeping in mind that we consider the $f' > 0$ branch, the function $\bar{R}_1(\phi)$ is a monotone increasing one for $f'' > 0$. As apparent form eq. (3.8), for increasing $\bar{R}_1$ we should take $\gamma > 0$. In a similar manner, the function $\bar{R}_1(\phi)$ is a monotone decreasing one for $f'' < 0$. Thus, for the minimum position $\bar{R}_{(0)1}$, inequality (3.17) leads to the following conditions (we remind that according to eq. (3.11), the minimum position $\bar{R}_{(0)1}$ should be negative and according to eq. (3.14) sign $\alpha = \text{sign} \gamma$):

**I.** $f'', \gamma, \alpha > 0$ :

$$
3 + 4\alpha \bar{R}_{(0)1}^+ > 0 \iff |\bar{R}_{(0)1}^+| < \frac{3}{4\alpha}.
$$

(3.19)

**II.** $f'', \gamma, \alpha < 0$ :

$$
3 + 4\alpha \bar{R}_{(0)1}^- < 0 \iff -|\bar{R}_{(0)1}^-| > \frac{3}{4|\alpha|}.
$$

(3.20)

Obviously, inequality (3.21) is impossible and we arrive to the conclusion that the minimum of the effective potential $U_{\text{eff}}$ is absent if $\text{sign} \alpha = \text{sign} \gamma = -1$.

Additionally, it can be easily seen that in the case

$$
\text{sign} \alpha = \text{sign} \gamma = \text{sign} \Lambda_D = +1
$$

(3.21)

the effective potential $U_{\text{eff}}$ has no minima also. This statement follows from the form of the potential $U(\phi)$ for the model (3.14). According to eq. (3.8), $U(\phi)$ reads:

$$
U(\phi) = (1/2)e^{-B\phi} \left( \alpha \bar{R}^2 + 3\gamma \bar{R}^4 + 2\Lambda_D \right).
$$

(3.22)

Thus, this potential is always positive for parameters satisfying (3.21) and we arrive to the contradiction with the minimum condition (2.30). Therefore, the investigation carried above indicates that the internal space stable compactification is possible only if the parameters satisfy the following sign relation:

$$
\alpha > 0, \gamma > 0, \Lambda_8 < 0.
$$

(3.23)

Let us investigate this case in more detail. For this choice of signs of the parameters, it can be easily seen that both extremum values $\bar{R}_{(0)1}^{(\pm)}$ from eq. (3.14) satisfy the condition (2.33): $\bar{R}_{(0)1}^{(\pm)} < 0$. However, the expression

$$
f' \left( \bar{R}_{(0)1}^{(\pm)} \right) = 1 + 2\alpha \bar{R}_{(0)1}^{(\pm)} + 4\gamma \bar{R}_{(0)1}^{(\pm)3} = -\frac{1}{2} + \sqrt{\frac{9}{4} - 16\alpha |\Lambda_8| - 4\gamma |\bar{R}_{(0)1}^{(\pm)}}^3\right]
$$

(3.24)

shows that only $\bar{R}_{(0)1}^{(+)1}$ can belong to $f' > 0$ branch. To make $f' \left( \bar{R}_{(0)1}^{(+)1} \right)$ positive, parameter $\gamma$ should satisfy the condition

$$
\gamma < \frac{-\frac{1}{2} + \sqrt{\frac{9}{4} - 16\alpha |\Lambda_8|}}{4 |\bar{R}_{(0)1}^{(+)1}|^3}.
$$

(3.25)

As apparent from this equation, parameter $\gamma$ remains positive if $\Lambda_8$ belongs to the interval

$$
\Lambda_8 \in \left( -\frac{1}{8\alpha}, 0 \right).
$$

(3.26)
For this values of $\Lambda_8$, the condition (3.16) is automatically satisfied. We also note, that for positive $\alpha$ and negative $\bar{R}^{(+)}_{(0)1}$ the condition (3.19) is also satisfied. Taking into account the interval (3.26), the corresponding allowed interval for $\gamma$ reads:

$$\gamma \in \left(0, \frac{1}{4 \left| \bar{R}^{(+)}_{(0)1} \right|} \right).$$

(3.27)

Thus, for any positive value of $\alpha$, Eqs. (3.26) and (3.27) define allowed intervals for parameters $\Lambda_8$ and $\gamma$ which ensure the existence of a global minimum of the effective potential $U_{eff}$. Here, we arrive to the required stable compactification of the internal space. The position of the minimum $(\beta_0^1, \phi_0)$ and its value can be easily found (via the root $\bar{R}^{(+)}_{(0)1}$) with the help of Eq. (3.2) and corresponding Eqs. from section 3. The Fig.1 - Fig.2 (see Appendix B) demonstrate such minimum for a particular choice of the parameters: $\alpha = 1, \gamma = 1, \Lambda_8 = -0.1$. To conclude this section, we want to note that limit $\Lambda_8 \to 0$ corresponds to $\bar{R}^{(+)}_{(0)1} \to 0$ which results in the decompactification of the internal space $\beta_0^1 \to \infty$.

4 Conclusions

In our paper we analyze the model with curvature-quadratic and curvature-quartic correction terms of the type (3.1) and show that the stable compactification of the internal space takes place for the sign relation (3.23). Moreover, the parameters of the model should belong to the allowed intervals (regions of stability) (3.26) and (3.27). The former one can be rewritten in the form

$$\Lambda_8 = \frac{\xi}{8\alpha}, \quad \xi \in (-1, 0).$$

(4.1)

Thus, for the root $\bar{R}^{(+)}_{(0)1}$ and parameter $\gamma$ we obtain respectively

$$\bar{R}^{(+)}_{(0)1} = \frac{\eta}{\alpha}, \quad \eta \equiv \frac{1}{4} \left(-3 + \sqrt{9 + 8\xi} \right) < 0$$

(4.2)

and

$$\gamma = \frac{\zeta\alpha^3}{4|\eta|^3}, \quad \zeta \in (0, 1).$$

(4.3)

Eq. (4.2) shows that $\bar{R}^{(+)}_{(0)1} \in (-\frac{1}{2\alpha}, 0)$.

It is of interest to estimate the masses of the gravitational excitons (2.25) and of the nonlinearity field $\phi$ (2.26) as well as the effective cosmological constant (2.27). From Eqs. (3.1) - (3.3) follows that $\bar{R}^{(+)}_{(0)1} \sim \Lambda_8 \sim -\alpha^{-1}, \gamma \sim \alpha^3 \implies f'(\phi_0) \sim \mathcal{O}(1), f''(\phi_0) \sim \alpha, U(\phi_0) \sim -\alpha^{-1}$. Then, the corresponding estimates read:

$$-\Lambda_{eff} \sim m_1^2 \sim m_\phi^2 \sim \alpha^{-1}.$$

(4.4)

From other hand (see Eqs. (2.21) and (2.25))

$$U(\phi_0) \sim \exp(-2\beta_0^1) = b_{(0)1}^{-2}.$$

(4.5)

So, if the scale factor of the stabilized internal space is of the order of the Fermi length: $b_{(0)1} \sim L_F \sim 10^{-17}$ cm, then $\alpha \sim L_F^2$ and for the effective cosmological constant and masses we obtain: $-\Lambda_{eff} \sim m_1^2 \sim m_\phi^2 \sim 1$ TeV$^2$.

In the present paper the analysis of the internal space stable compactification was performed in the case $Q(\phi) > 0 \implies \text{sign } \alpha = \text{sign } \gamma$. In our forthcoming paper we extend this investigation to the case of negative $Q(\phi)$ where the function $\bar{R}(\phi)$ has three real-valued branches.

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4) Similar interval for the allowed values of $\gamma$ was also found in [20] for the curvature-quartic model.
A  On critical dimensions in nonlinear models

The existence of a critical dimension (in our case $D = 8$) is a rather general feature of gravitational theories with polynomial scalar curvature terms (see, e.g., Refs. [23, 24, 25]). Following our paper [20], it can be easily demonstrated for a model with curvature nonlinearity of the type

$$f(\bar{R}) = \sum_{k=0}^{N} a_k \bar{R}^k$$

(A.1)

for which the ansatz

$$e^{A\phi} = f' = \sum_{k=0}^{N} k a_k \bar{R}^{k-1}$$

(A.2)

leads, similar like [24], to a potential

$$U(\phi) = \frac{1}{2} (f')^{-D/(D-2)} \sum_{k=0}^{N} (k-1)a_k \bar{R}^k.$$  

(A.3)

The condition of extremum [24] for this potential reads:

$$D f' - 2 \bar{R} f'' = \sum_{k=0}^{N} (D - 2k) a_k \bar{R}^k = 0.$$  

(A.4)

Thus, at the critical dimension $D = 2N$ the degree of this equation is reduced from $N$ to $N - 1$. In this case the search of extrema is considerably simplified.

In the limit $\phi \to +\infty$ the curvature will behave like $\bar{R} \approx ce^{h\phi}$ where $h$ and $c$ can be defined from the dominant term in (A.2):

$$e^{A\phi} \approx N a_N \bar{R}^{N-1} \approx N a_N e^{(N-1)h\phi}.$$  

(A.5)

Here the requirement $f' > 0$ allows for the following sign combinations of the coefficients $a_N$ and the curvature asymptotics $\bar{R}(\phi \to \infty)$:

$$N = 2l : \quad \text{sign}[a_N] = \text{sign}[\bar{R}(\phi \to \infty)]$$

$$N = 2l + 1 : \quad a_N > 0, \quad \text{sign}[\bar{R}(\phi \to \infty)] = \pm 1.$$  

(A.6)

The other combinations, $N = 2l$: $\text{sign}[a_N] = -\text{sign}[\bar{R}(\phi \to \infty)]$, $N = 2l + 1$: $a_N < 0$, $\text{sign}[\bar{R}(\phi \to \infty)] = \pm 1$, would necessarily correspond to the $f' < 0$ sector, so that the complete consideration should be performed in terms of the extended conformal transformation technique of Ref. [15]. Such a consideration is out of the scope of the present paper and we restrict our attention to the cases (A.6). The coefficients $h$ and $c$ are then easily derived as $h = A/(N - 1)$ and $c = \text{sign}(a_N)|N a_N| \rightarrow -\frac{\bar{R}}{e^{h\phi}}$. Plugging this into (A.3) one obtains

$$U(\phi \to +\infty) \approx \text{sign}(a_N) \frac{(N-1)}{2N} |N a_N|^{-\frac{1}{N-1}} e^{-\frac{D}{2} A\phi} e^{\frac{N}{N-1} A\phi}$$  

(A.7)

and that the exponent

$$\frac{D - 2N}{(D - 2)(N - 1)} A$$

changes its sign at the critical dimension $D = 2N$:

$$U(\phi \to +\infty) \rightarrow \text{sign}(a_N) \frac{(N-1)}{2N} |N a_N|^{-\frac{1}{N-1}} \times \left\{\begin{array}{ll}
\infty & \text{for } D > 2N, \\
1 & \text{for } D = 2N, \\
0 & \text{for } D < 2N.
\end{array}\right.$$  

(A.9)

This critical dimension $D = 2N$ is independent of the concrete coefficient $a_N$ and is only defined by the degree $\text{deg}_R(f)$ of the scalar curvature polynomial $f$. From the asymptotics (A.9) we read off that in the high curvature limit $\phi \to +\infty$, within our oversimplified classical framework, the potential $U(\phi)$ of the considered toy-model shows asymptotical freedom for subcritical dimensions $D < 2N$, a stable behavior for $a_N > 0$, $D > 2N$ and a catastrophic instability for $a_N < 0$, $D > 2N$. We note that this general behavior suggests a way how to cure a pathological (catastrophic) behavior of polynomial $\bar{R}^N$—nonlinear theories in a fixed dimension $D > 2N$: By including higher order corrections up to order $N_2 > D/2$ the theory gets shifted into the non-pathological sector with asymptotical freedom. More generally, one is even led to conjecture that the partially pathological behavior of models in supercritical dimensions could be an artifact of a polynomial truncation of an (presently unknown) underlying non-polynomial $f(\bar{R})$ structure at high curvatures — which probably will find its resolution in a strong coupling regime of $M$-theory or in loop quantum gravity.
B Graphical visualizations

Following section 3, we consider the model with one internal space and critical dimension $D = 8$ (as usual, for the external spacetime $D_0 = 4$). Then, the effective potential (2.20) reads:

$$U_{eff}(\hat{\beta}^1, \phi) = e^{-4\hat{\beta}^1} \left[ -\frac{1}{2} \hat{R}_1 e^{-2\hat{\beta}^1} + U(\phi) \right].$$  \hfill (B.1)

To draw this effective potential, we define $\hat{R}_1$ via $U(\phi_0)$ in Eq. (2.25). In its turn, $U(\phi_0)$ is defined in Eq. (2.24) where $\bar{R}(\phi_0) = \bar{R}^{(+)\dagger}$ and $f' \left( \bar{R}^{(+)\dagger} \right)$ can be found from (3.24). In Figs. 1,2 the generic form of the $U_{eff}$ is illustrated by a model with parameters $\alpha = 1, \gamma = 1, \Lambda_8 = -0.1$ from the stability regions (3.26) and (3.27).

![Figure 1](image1.png)

Figure 1: Typical contour plot of the effective potential $U_{eff}(\hat{\beta}^1, \phi)$ given in Eq. (B.1) with parameters $\alpha = 1, \gamma = 1, \Lambda_8 = -0.1$. $U_{eff}$ reaches the global minimum at $(\hat{\beta}^1 = 0, \phi \approx -2.45)$.

![Figure 2](image2.png)

Figure 2: Typical form of the effective potential $U_{eff}(\hat{\beta}^1, \phi)$ given in Eq. (B.1) with parameters $\alpha = 1, \gamma = 1, \Lambda_8 = -0.1$. 
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