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Observability of periodic lines in three-dimensional lid-driven cylindrical cavity flows

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This study employs three-dimensional particle-tracking velocimetry (3D-PTV) for experimental investigation of the existence and properties of periodic lines in 3D lid-driven time-periodic flows inside a cylindrical cavity. These periodic lines, consisting of material points that periodically return to their initial position, play a central role in the transport properties of laminar flows, yet their existence has so far been demonstrated only in numerical simulations. The formation and characteristics of periodic lines are inextricably linked with spatiotemporal symmetries of the flow. 3D-PTV measurements determined that relevant symmetries, identified with previous symmetry analyses, are satisfied within experimental error bounds. These measurements subsequently isolated periodic lines in the designated symmetry planes, thus offering first experimental evidence of their physical existence and their fundamental reliance on symmetries. Experimental periodic lines are topologically equivalent to those in simulated flows with identical symmetries and exhibit the same response to changes in forcing conditions. The laboratory experiments by these observations bridge the gap from theoretical and numerical predictions on periodic lines to real 3D flows.

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I. INTRODUCTION

Transport of scalar quantities (chemical species, pollutants, heat) under laminar flow conditions occurs over a large range of length and time scales in many natural and industrial systems. This includes geophysical phenomena such as magma flow and convective mixing in the Earth’s mantle [1,2], dispersion of nutrients and pollutants in oceanographic flows [3,4],¹ and a plethora of industrial applications, extending from mixing and thermal processing of viscous fluids as, e.g., polymers and foodstuffs [5] via compact equipment for process intensification [6,7] to sophisticated lab-on-a-chip systems for analytical chemistry and bioengineering [8–12]. Compact and microfluidic systems, in particular, given their key role in emerging technologies, are an important class of laminar flows, and a primary motivation for our study. Deeper insight into the three-dimensional (3D) transport properties of such flows is essential to improve their performance. Moreover, this has the potential to greatly enhance the functionality of microfluidic systems [13].

Key to the 3D transport properties of laminar flows are coherent structures formed by the Lagrangian fluid trajectories due to continuity. Such structures are the building blocks of the flow topology and geometrically constrain and determine the motion of passive tracers² advected by the flow [14–17]. This has since the early 1980s been widely studied in the context of chaotic advection of passive tracers, the kinematic equivalent to efficient mixing, in two-dimensional (2D) time-periodic flows. These studies exposed material points that periodically return to their initial position (‘periodic points’) as prime coherent structures determining the transport properties: elliptic and hyperbolic points. Former and latter kinds are, respectively, the centers of nonmixing regions and regions with efficient mixing by chaotic advection [14,15,17,18]. Despite the great progress made on 2D systems, the 3D case has been considerably less investigated. Moreover, experiments particularly at low Reynolds numbers $Re$, notwithstanding their great relevance to many physical processes in nature and industry, remain rare to date. This motivates this study, which seeks to increase understanding of transport in 3D time-periodic laminar flows by way of laboratory experiments. The investigation concentrates on the 3D counterpart to periodic points in 2D systems: periodic lines. Such lines typically consist of elliptic and hyperbolic segments that (locally) determine the 3D transport properties in essentially the same way as periodic points in 2D flows [16,19,20].

The formation of coherent structures usually is intimately related to spatiotemporal symmetries in the Lagrangian equations governing the tracer motion [21–24], and the emergence of periodic lines in fact is a direct consequence of certain discrete symmetries [16,25]. Hence, symmetry analyses are an integral part of the experimental study on periodic lines hereafter.

The examined configuration consists of a lid-driven cylinder flow in which the fluid is set in motion by time-periodic in-plane translation of one of the endwalls via predefined forcing protocols [16,20]. This system has been explored extensively by way of analytical and numerical studies, revealing rich and essentially 3D Lagrangian tracer dynamics, rendering the cylinder flow representative of generic 3D time-periodic flows [16,25–27]. The cylinder flow accommodates periodic lines in the noninertial limit $Re = 0$. This limit, by virtue of
a continuous symmetry, furthermore admits invariant material surfaces that define a family of concentric spheroids. Tracer particles are entrapped within these surfaces and perform an effectively 2D motion. Brouwer’s fixed point theorem implies periodic points within these spheroids that in the 3D flow domain merge into periodic lines [16]. Thus, the periodic lines and invariant spheroids, and their underlying symmetries, are inextricably linked.

Fluid inertia (Re > 0) causes disintegration of the noninertial flow topology and thus facilitates the onset of 3D chaotic advection [25–27]. The periodic lines and invariant spheroids at Re = 0 play a fundamental role in this route to 3D chaos in that weak inertia expands tracer migration to thin shells forming around the latter and connecting via tubes centered on elliptic segments of periodic lines. This response scenario, termed resonance-induced merger (RIM), has been observed in all numerical simulations of the time-periodic cylinder flow involving forcing by one endwall only, suggesting a universal mechanism [25–27]. Demonstrating the physical formation of periodic lines and invariant spheroids in the cylinder flow via laboratory experiments is the principal aim of this study and constitutes a first important step towards experimental validation of RIM.

The experimental analysis has been carried out by 3D particle tracking-velocimetry (3D-PTV). Exploratory measurements in the cylinder flow demonstrated the great potential of this technique for 3D quantitative experimental studies on Lagrangian fluid trajectories and corresponding coherent structures [16]. Experimental characterizations of Lagrangian transport in 3D laminar flows are relatively rare and typically restricted to visualization of cross sections or projections of 3D flow features and concentration patterns by laser-induced fluorescence [28–34]. One of the few quantitative transport studies include quasi-3D-PTV (i.e., 2D-PTV in two projections) using a single tracer particle [35] and planar velocimetry in 2D cross sections via 2D particle-image velocimetry [33,34]. Quantitative 3D-PTV experiments on truly 3D Lagrangian tracer dynamics in a cavity flow comparable to the present cylinder flow are discussed in [36,37]. However, these investigations concern substantially higher Reynolds numbers than those relevant to laminar scalar transport and concentrate on individual fluid trajectories instead of coherent structures. This study expands on these experimental transport studies by quantitatively investigating key features of the flow topology of a prototypical 3D low-Re flow.

The paper is organized as follows. The physical problem is introduced in Sec. II and its symmetry properties and topological makeup are addressed in Sec. III. The experimental setup and 3D-PTV tracking procedure are described in Sec. IV. Moreover, a hybrid particle-tracking method for numerical simulation of Lagrangian tracer dynamics using a Eulerian flow field constructed from 3D-PTV data is presented. Section V investigates the symmetries of the steady base flow that underlies the time-periodic forcing protocols and indirectly verifies the existence of the invariant spheroids. The formation of periodic lines in time-periodic flows, including an analysis of their key properties, is demonstrated in Sec. VI for a number of representative cases. Conclusions are summarized in Sec. VII.

II. PROBLEM DEFINITION

This study considers time-periodic flows inside a square cylinder \( D : [r, \theta, z] = [0, R] \times [0, 2\pi] \times [-H/2, H/2] \), with \( R \) and \( H = 2R \) its radius and height, respectively. The fluid is set in motion via time-periodic repetition of a sequence of \( p \) piecewise steady translations of the bottom wall (forcing steps) by prescribed forcing protocols. These forcing steps are of equal duration \( T_{\text{step}} = T/p \), with \( T \) the period time of one cycle, and consist of reorientations of the base flow induced by the steady translation of the bottom wall at velocity \( U_{\text{wall}} \) in the \( x \) direction.

Highly viscous flow conditions are assumed such that unsteady transients between forcing steps are negligible: \( T_{\nu}/T_{\text{step}} \approx 1 \), with \( T_{\nu} = R^2/\nu \) the viscous time scale and \( \nu \) the kinematic fluid viscosity. Under this premise, which is verified in Sec. IV for the present experimental study, the time-periodic flow consists of piecewise steady velocity fields. The corresponding steady base flow is governed by the nondimensional steady momentum and continuity equations

\[
Re u \cdot \nabla u = -\nabla p + \nabla^2 u, \quad \nabla \cdot u = 0, \tag{1}
\]

with \( u \) and \( p \) the nondimensional fluid velocity and pressure, respectively, and the unit cylinder \( D : [r, \theta, z] = [0, 1] \times [0, 2\pi] \times [-1, 1] \) as associated nondimensional flow domain. Dimensional analysis yields two control parameters, viz., the Reynolds number and nondimensional wall displacement, respectively defined as

\[
Re = \frac{U_{\text{wall}} R}{\nu}, \quad D = \frac{D_{\text{wall}}}{R}, \tag{2}
\]

with \( D_{\text{wall}} \) the physical wall displacement during one forcing step.

The motion of passive tracers released in the flow field \( u(x,t) \) is described by the kinematic equation

\[
\frac{dx}{dt} = u(x,t), \quad x(0) = x_0, \tag{3}
\]

with \( x(t) \) and \( x_0 \) the current and initial tracer positions, respectively. The formal solution to Eq. (3) reads as

\[
x(t) = \Phi(x_0), \tag{4}
\]

and defines the continuous Lagrangian flow from the initial to the current tracer position along trajectory \( X(t;x_0) = \{ \Phi(x_0) \circ \xi, 0 \leq \xi \leq t \} \). The corresponding discrete mapping of time-periodic flows is given by

\[
x_{k+1} = \Phi(x_k), \tag{5}
\]

with \( x_k := x(kT) \) the tracer position after \( k \) periods.

The time-periodic forcing protocols are composed of a sequence of reorientations of the base flow. Thus, the generic mapping (5) takes the form

\[
\Phi = \Phi_p \Phi_{p-1} \cdots \Phi_1, \quad \Phi_n = R^{n-1} \Phi_b R^{1-n}, \tag{6}
\]

with \( 1 \leq n \leq p \), \( \Phi_b \) the mapping associated with the base flow and \( R : (r, \theta, z) \rightarrow (r, \theta + \theta_{\text{step}}, z) \) the reorientation operator. The forcing protocols can be distinguished on the basis of whether the bottom wall describes a closed or an open path within one forcing cycle. Both open [16] and closed
Symmetries in the flow topology emanate from symmetries in the conservation laws and boundary conditions governing the Eulerian velocity field. The boundary conditions of the base flow consist of no-slip conditions \( u|_{z=1} = 0 \) on the top wall \((z = 1)\) and \( u|_{r=1} = 0 \) on the cylinder wall \((r = 1)\) and condition \( u|_{z=-1} = U(x,y) \) at the bottom wall \((z = -1)\) in the nondimensional formulation (Sec. II). The latter identifies with \( U(x,y) = (1,0,0) \) in case of a rigid wall and in the noninertial limit \( Re = 0 \) leads to an internal velocity field with symmetries

\[
S_xu_x = u_x, \quad S_yu_y = -u_y, \quad S_zu_z = -u_z, \quad \text{(9)}
\]

and

\[
S_xu_{x,z} = u_{x,z}, \quad S_yu_{y} = -u_{y}, \quad S_zu_{z} = -u_{z}, \quad \text{(10)}
\]

where \( S_x: (x,y,z) \rightarrow (-x,y,z) \) and \( S_y: (x,y,z) \rightarrow (x,-y,z) \), due to the linearity of the momentum equation (1) for vanishing inertial term \([38,39]\).

Essential is that the above symmetries are not specific to forcing by a rigid bottom wall: any boundary condition \( U(x,y) \) with properties (9) and (10) imparts those symmetries onto the base flow. Thus, noninertial base flows with boundary conditions according to Eqs. (9) and (10) in fact constitute a family of flows with identical symmetries. Moreover, given that symmetries in the flows due to the forcing protocols derive from those of the base flow \([16,20]\), this implies an associated family of time-periodic flows with identical symmetries. These symmetries, in turn, organize the flow topologies, meaning that (flows constructed from) base flows with boundary condition \( U \) meeting (9) and (10) are topologically equivalent.

Symmetries (9) and (10) manifest themselves in the Lagrangian flow (4) through

\[
\Phi_t = S_x\Phi_t^{-1}S_x, \quad \Phi_t = S_y\Phi_t S_y, \quad \phi_t = S_z\phi_t S_z, \quad (11)
\]

with \( S_x \) and \( S_y \) as before. This has fundamental ramifications for the topology of the base flow \([38,39]\):

(i) Streamlines \( \mathcal{C} \) crossing symmetry plane \( I_s = \{ x \in \mathbb{D} \}_{x=0} \), with \( \mathbb{D} \) the nondimensional domain (Sec. II), are closed and self-symmetric about this plane: \( \mathcal{C} = S_y \mathcal{C} \).

(ii) Streamlines form symmetry pairs \( (\mathcal{C}, S_y \mathcal{C}) \) about symmetry plane \( I_s = \{ x \in \mathbb{D} \}_{y=0} \).

Figure 2 gives this flow topology for the rigid-wall condition \( U = (1,0,0) \) obtained by numerical simulation with the semi-analytical solution to the noninertial limit of the momentum equation according to Ref. \([38]\). The symmetric arrangement of closed streamlines outlines a vortical flow consisting of a large single eddy that occupies the entire flow domain, save localized weak corner eddies near the upper rim of the cylinder wall (not shown).

The topological equivalence of flows subject to other boundary conditions \( U \) satisfying symmetries (9) and (10) with the streamline portrait in Fig. 2 is demonstrated for \( U(x,y) = [(x^2 + y^2 - 1)^2, 0, 0] \), which serves as smooth version of the rigid-wall condition \( U(x,y) = (1,0,0) \) \([26,27,40]\). Figure 3 shows the streamline patterns of both cases in the symmetry plane \( I_s \) and clearly exposes identical topologies comprising closed streamlines symmetrically arranged about plane \( I_s \). (Simulations for the smooth conditions utilize the spectral flow solver by \([40]\) for \( Re = 0 \)). Important in the present scope is that the boundary conditions in the experimental setup differ from the above conditions due to placement of the cylinder.
in a large tank of quiescent fluid and leaving a small gap \( \Delta z \) between its lower rim \( z = -1 \) and the translating bottom wall. This alters the velocity field in the plane \( z \) that in all its components become nonzero and dependent on position (Sec. IV). However, symmetries (9) and (10) are preserved, meaning that the experimental system nonetheless belongs to the before-mentioned family of flows. Retention of the boundary symmetries thus suffices to facilitate experimental investigation of the generic topological flows. Note that the adopted ansatz addresses an aspect of greater physical significance compared to an analysis on one-to-one correspondence between coherent structures in case of identical boundary conditions. The latter concerns in essence the geometry (identity in shape) of coherent structures. This study, on the other hand, concerns their topology (equivalence in shape), which is the more fundamental (and thus generic) property [41].

The following furnishes a concise recapitulation of those portions of the symmetry analyses in Refs. [16,20,25–27] that are relevant for this study. In particular, the consequences for the flow topology are highlighted.

Symmetries (9) and (10) in conjunction with the linearity of the momentum equation for \( \text{Re} = 0 \) impose separation of variables upon the velocity field following

\[
\begin{align*}
 u_{r,z}(r,\theta,z) &= \tilde{u}_{r,z} \cos \theta, \\
 u_\theta(r,\theta,z) &= \tilde{u}_\theta \sin \theta,
\end{align*}
\]

with

\[
\begin{align*}
 \tilde{u}_{r,\theta}(r,z) &= \tilde{u}_{r,\theta}(-r,z), \\
 \tilde{u}_{r}(r,z) &= -\tilde{u}_{r}(-r,z),
\end{align*}
\]

in the cylindrical frame of reference [16,20]. This gives

\[
\frac{dz}{dr} = f(r,z), \quad \sin \theta = g(r,z; r_0, z_0) \sin \theta_0,
\]

as partially integrated equation of motion, with \( f(r,z) = \tilde{u}_r/\tilde{u}_r \) and \( g(r,z; r_0, z_0) = \exp(\int_{r_0}^r \nabla \cdot \mathbf{u}' ds/|\mathbf{u}'|) \). Here, \( \nabla = (\partial/\partial r, \partial/\partial \theta, \partial/\partial z) \) and \( \mathbf{u}' = (\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_z) \), and the integral is taken along a streamline \( \mathcal{C}' \) in the \( r \) plane \( \mathcal{D}' : [r, z] = [-1,1] \times [-1,1] \) [20]. Hence, the 3D motion of a tracer is dictated by its dynamics within \( \mathcal{D}' \). Property \( f(r,z) = -f(-r,z) \) implies an analogous symmetry as (11) for the planar flow \( \mathbf{F}_i \), i.e.,

\[
\Phi_i = S_r \Phi_i^{-1} S_r, \quad \Phi_i = S_a \Phi_i S_a^{-1},
\]

with \( S_r : (r,z) \to (-r,z) \) and \( S_a : (r,\theta, z) \to (r, \theta + \alpha, z) \) and \( 0 \leq \alpha \leq 2\pi \). This has further consequences for the topology of the base flow [26,27]:

(i) Streamlines \( \mathcal{C}' \) crossing symmetry axis \( I_s = I_s I_r = \{ (r,z) \in \mathcal{D}'| r = 0 \} \), are closed and self-symmetric about this axis: \( \mathcal{C}' = S_a \mathcal{C}' \).

(ii) Continuous symmetry \( S_a \) implies axisymmetric invariant surfaces, defined by the surfaces of revolution of streamlines \( \mathcal{C}' \), onto which 3D streamlines \( \mathcal{C} \) are confined. Thus, the projection \( \mathcal{P} : \mathcal{D} \to \mathcal{D}' \) of the 3D streamline pattern \( \mathcal{C} \) into \( \mathcal{D}' \) collapses onto the 2D streamline pattern \( \mathcal{C} : \mathcal{P} \mathcal{C} = \mathcal{C}' \).

Velocity \( \mathbf{u}' \) identifies with \( (u_z, u_r) \) for \( \theta = 0 \), meaning that the 2D streamline pattern \( \mathcal{C}' \) in \( \mathcal{D}' \) coincides with that in symmetry plane \( I_s \) (Fig. 3). The corresponding surfaces of revolution define concentric invariant spheroids; Fig. 4 shows the latter for the smooth boundary condition \( \mathbf{U} \) [Fig. 3(b)].

Mapping \( \mathbf{F}_p \) associated with the base flow (Sec. II) adopts the above symmetries of the underlying continuous flow. This manifests itself in the symmetries

\[
\Phi_n = S_p \Phi_{p+1-n} S_p, \quad 1 \leq n \leq p
\]

between the individual forcing steps \( \Phi_i \), according to Eq. (6) for the present case of closed forcing protocols [25]. Here, \( S_p : (r, \theta, z) \to (r, 2\beta - \theta, z) \), with \( \beta = (\pi - \theta_{\text{step}})/2 \), effectuates reflection about the symmetry plane \( I_p = I_s I_p = \{ x \in \mathcal{D} | z = 0 \} \). Symmetry (16), in turn, gives rise to the global symmetry

\[
\Phi = S_p \Phi_{p+1-n} S_p
\]

in the mapping corresponding with the forcing protocol [25]. This constitutes a so-called time-reversal reflectional symmetry and has the fundamental implication that the mapping always accommodates a period-1 line \( \mathcal{L} \) following (8) in the corresponding symmetry plane \( I_p \) [16,25–27]. The latter coincides with \( \beta = \pi/6 \) and \( \pi/4 \) for protocols \( T \) and \( S \), respectively.
The Lagrangian equations of motion attain the form $dz/dr = f$ and $\sin \theta^* = g \sin \theta_0^*$, with $\theta^* = \theta - (n-1)\theta_{\text{step}}$ and $f, g$ as before, for forcing step 1 $\leq n < p$, meaning that structure (14) and thus symmetries (15) are preserved. This implies that the invariant spheroids, illustrated in Fig. 4, persist for the class of time-periodic forcing protocols under investigation here.

The above advances symmetries in the base flow as primary organizing “mechanisms” for the flow topology of the time-periodic forcing protocols. They cause the formation of two of its key building blocks: invariant spheroids and period-1 lines. The principal aim of this study is experimental investigation of these features by symmetry analysis of the base flow (Sec. V) and direct measurement of the period-1 lines (Sec. VI). The employed experimental methods are discussed below.

IV. EXPERIMENTAL METHODS

A. Laboratory setup and tracking method

The laboratory setup consists of a square perspex cylinder with radius $R = 35$ mm, submerged sideways in a bath of highly viscous silicone oil (AK10000, Wacker GmbH, Germany), with kinematic viscosity $\nu = 10^{-2}$ m$^2$/s$^{-1}$ and density $\rho = 970$ kg m$^{-3}$, contained in a large perspex container (Fig. 5). One endwall is fixed and the other is mounted on support structures that admit flow forcing through 2D in-plane translations accomplished by an automated motion-control system [16].

Smooth-wall motion is ensured by leaving a small z-wise gap $Z \approx 1–2$ mm with the fixed cylinder so as to avoid vibrations induced by physical contact. This results in a localized inflow/outflow between the interior of the cylinder and the ambient fluid in the tank. The gap itself is inconsequential for this study since the fundamental symmetry properties underlying the flow topology are retained (Sec. III). However, limited positioning accuracy admits adjustment of the gap only within a tolerance $\Delta Z \approx 0.5$ mm, which introduces a geometric asymmetry $\epsilon_Z = \Delta Z/(2R) = 0.0071 \sim O(1\%)$.

Maximum wall displacement is $D_{\text{wall}} = 150$ mm, with corresponding dimensionless displacement $D = 4.3$, and a fixed wall velocity $U_{\text{wall}} = 1$ mm$^{-1}$ is employed. This yields $Re = 3.5 \times 10^{-3}$, meaning that symmetry breaking of $S_3$ (and its derivative symmetries $S_{\alpha, \beta}$) due to fluid inertia is negligible for experiments encompassing only single forcing periods [27]. (Note that inertia leaves symmetry $S_3$ intact.) Long-term effects, on the other hand, may become significant [25–27]. This is beyond the present scope, however. Moreover, forcing steps have durations $T_{\text{step}} = 50–150$ s, rendering transients negligible: $T_s/T_{\text{step}} \sim O(10^{-3}) \ll 1$ (Sec. II). Time-periodic experimental flows thus indeed correspond with piecewise steady noninertial flows.

3D Lagrangian fluid trajectories and velocities are obtained by means of standard 3D-PTV measurements. To this end, the fluid is seeded with minute polystyrene particles of diameter $d_p = 250$ $\mu$m and density $\rho_p = 1050$ kg m$^{-3}$. Particle inertia and particle buoyancy must be negligible so as to ensure adequate representation of the fluid motion [42]. The particle Stokes number equals $St = \tau_p/\tau_f \sim O(10^{-3})$, with $\tau_p = \rho_p d_p^2/18 \nu$ the inertial response time of particles and $\tau_f = R/U_{\text{wall}}$ the characteristic time scale of the flow. The ratio particle drift velocity $U_p = \rho_p - \rho|g d_p^2/18 \nu$ to characteristic flow velocity amounts to $U_p/U_{\text{wall}} \sim O(10^{-4})$. These rule out both inertia and buoyancy, meaning particles behave as passive tracers and thus are well-suited for 3D-PTV experiments.

Particle motion is recorded by four 8-bit 1600 $\times$ 1200 pixels CCD cameras (MegaPlus II ES2020, Kodak, United Kingdom) positioned in a square arrangement facing the moving wall at an angle $\alpha_a \approx 9^\circ$ with the cylinder axis. This is a compromise between a large field of view (“small” $\alpha_a$) and reliable particle tracking (“large” $\alpha_a$). Particles are illuminated by an array of 238 LEDs (Luxeon K2, Philips, The Netherlands) [43]. Their particular spectrum (intensity peaks around 440–460 nm) and pulsed operation synchronous with the cameras (20-ms pulses at $f = 0.5$ Hz) prevents undesired buoyancy effects due to heating. Density namely varies as $\Delta \rho/\rho = \kappa DT$, with $\kappa = 9.2 \times 10^{-4}$ K$^{-1}$ the thermal expansivity, meaning a fluctuation $\Delta \rho/\rho \sim O(0.5\%)$, which
is acceptable, requires $\Delta T \lesssim \mathcal{O}(1)$. Temperature monitoring by thermocouples confirmed compliance with this condition.

Data processing of the particle imagery is performed with the 3D-PTV algorithm developed at ETH, Zürich, Switzerland [44–48]. This algorithm leans on an optical model assuming separation of cameras and flow domain by a single solid wall. However, the present configuration includes two solid walls between cameras and flow domain, viz., the container wall and the cylinder endwall, introducing additional refraction unaccounted for by the model. Estimation yields an additional refractive shift of $\mathcal{O}(125 \mu m)$, which is well below the particle diameter and thus acceptable (see Appendix A). This admits utilization of the above-mentioned 3D-PTV algorithm.

The 3D-PTV algorithm requires determination of the coordinate transformation between 2D pixel and 3D physical reference frames. This is accomplished by camera-wise calibration with a planar reference grid translated axially through the flow domain. The seeding density was around 500 particles, which corresponds to a typical particle separation of about 8 mm, and is a factor of 8 lower compared to previous experiments using this 3D-PTV methodology [43,49]. The primary reason is that this study concentrates on fluid trajectories, whereas those in Refs. [43,49] focus on instantaneous velocity fields. The former benefits from a “low” seeding density in that this enhances particle matching and thus promotes isolation of long trajectories; the latter relies on the fine spatial resolution afforded by a “high” seeding density. The adopted strategy in the present experiments enabled the 3D-PTV algorithm to typically track 100–200 particles for the full duration of a forcing period.

B. Hybrid particle-tracking method

The range of the laboratory setup is restricted to displacements $D \leq 4.3$. However, relevant phenomena often take place substantially beyond that operating limit [25]. The reach of the laboratory experiments is augmented by a hybrid numerical-experimental method: numerical tracking of artificial particles released in a velocity field determined by 3D-PTV measurements. This method, based on a similar ansatz adopted for 2D flows [50], is complementary to the direct 3D-PTV measurements in that it enables investigation of forcing protocols inaccessible by the experimental setup.

The hybrid method involves a preprocessing step consisting of evaluation of the base-flow velocity field in the 3D-PTV data points by segment-wise polynomial approximations of the trajectories following [49] and its subsequent interpolation onto a regular grid. The steady nature of the base flow admits combination of the velocity fields of individual time steps into one overall field with a very high data-point density. This enables accurate interpolation onto a very fine regular grid and thus a close approximation of the true experimental velocity field. The experiments in Sec. V span 250 time steps and yield a grid of $N \approx 8 \times 10^6$ velocity vectors, signifying a typical spacing $\Delta x = \sqrt{\pi HR^2/N} \sim \mathcal{O}(d_p)$.

The interpolated velocity field serves as input for a numerical integration algorithm for the kinematic equation (3) by the standard explicit Euler scheme $x_{n+1} = x_n + u(x_n, t_n)\Delta t$, with time levels $t_n = n\Delta t$ and discrete time step $\Delta t$. The latter is determined via the Courant-Friedrichs-Lewy (CFL) condition $\Delta t \ll \Delta x/\max(|u|) [51]$ and $u(x_n, t_n)$ is evaluated by interpolation of the velocity field in the regular grid onto the arbitrary positions $x_n$. Time-periodic forcing is accomplished by stepwise reorientation of the base flow.

The hybrid method is implemented in the high-level programming language MATLAB and employs built-in routines for the polynomial approximations and spatial interpolation during preprocessing and tracking.

V. ANALYSIS OF THE BASE FLOW

A. Symmetries in space

The symmetry analysis in Sec. III advanced symmetries in the Eulerian and Lagrangian representations of the base flow as primary organizing “mechanisms” for the flow topology. Presence of these symmetries in real flows is examined hereafter by way of 3D-PTV experiments and thus expands on first exploratory studies on 3D trajectories and velocity fields in the present configuration [16].

Figure 6 gives selected 3D streamlines from a data set obtained with 3D-PTV measurements for steady translation of the bottom wall over its maximum nondimensional displacement $D = 4.3$ (Sec. IV). The streamline pattern clearly exposes the generic vortical structure predicted by numerical simulations (Fig. 2). This provides a first visual agreement between numerical and experimental studies. Note that reliable determination of a particle location by the 3D-PTV algorithm requires monitoring by at least three different cameras. However, due to the camera arrangement, this condition is not satisfied near the top rim $(r/R, z/R) = (1, 1)$. This deteriorates the tracking yield and, in consequence, substantially lowers the number of particles tracked in that region, as is clearly visible in shown streamline portrait.

Symmetries $S_x$ and $S_y$ according to Eq. (11) are the principal symmetries of the base flow. They dictate that (sections of) the streamline pattern $C$ identify via reflections

$S_xC = C$, \quad $C_{|y<0} = S_xC_{|y>0}$.

\begin{equation}
S_xC = C, \quad C_{|y<0} = S_xC_{|y>0}.
\end{equation}

FIG. 6. 3D streamlines of the base flow obtained by 3D-PTV experiments for displacement $D = 4.3$. 

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Presence of former and latter symmetries is investigated in Figs. 7 and 8, respectively, by overlaying the actual streamline patterns $C$ and $C_{|y=0}$ (black) with their corresponding reflections $S_x C$ and $S_y C_{|y=0}$ (cyan/gray). This reveals a good qualitative agreement in both cases and thus verifies existence of $S_x$ and $S_y$ in the experimental flow. Note that instances of crossing between actual and reflected streamlines in the overlays for the full spanwise extent (top panels) emanate predominantly from different streamline curvatures at different spanwise positions $y$ instead of symmetry breaking. Such crossing is namely absent in shown slices; here, original and reflected streamlines closely shadow one another.

The continuous symmetry $S_\alpha$ according to Eq. (15) causes the projection $P$ of the 3D streamline pattern $C$ into the rz plane to collapse onto the 2D streamline pattern in the symmetry plane $y = 0$ (Sec. III):

\[ P C = C_{|y=0}. \]  

This property is examined in Fig. 9 by overlaying projection $P C$ (black) and $C_{|y=0}$ approximated by streamlines within the slice $|y| \leq 0.025$ (cyan/gray). This again verifies the presence of the symmetry in question. First, the projected streamlines run parallel and, save localized erratic “jumps” due to measurement inaccuracies and data-processing artifacts, are devoid of crossings. This reflects the restriction of 3D streamlines to invariant spheroids topologically equivalent to those shown in Fig. 4 and thus indirectly demonstrates existence of these entities. Second, the projected streamlines coincide well with the approximated streamline pattern $C_{|y=0}$.

Symmetries $S_\alpha$ and $S_\alpha$, and thus indirectly also their Cartesian counterparts $S_x$ and $S_y$, are investigated quantitatively via the velocity field. To this end, the 3D velocity field is interpolated onto a regular grid in planes $\theta = \alpha$ by the same procedure as employed in the hybrid method following Sec. IV B. Departures from symmetry $S_\alpha$ are evaluated by testing compliance of the measured velocity field with the particular structures (12) and (13) through metrics

\[ \epsilon_r(r,z;\alpha) = |u_r(r,\alpha,z) - u_r(-r,\alpha,z)|, \]

\[ \epsilon_\theta(r,z;\alpha) = |u_\theta(r,\alpha,z) - u_\theta(-r,\alpha,z)|, \]  

\[ \epsilon_z(r,z;\alpha) = |u_z(r,\alpha,z) + u_z(-r,\alpha,z)|, \]  

which vanish identically in case of perfect symmetry. Performing this analysis for several $\alpha$ evaluates the degree of satisfaction of symmetry $S_\alpha$.  

066320-7
FIG. 9. (Color online) Symmetry $S_\alpha$ in the 3D experimental streamline pattern: projection of actual streamlines $C$ (black) into the $rz$ plane versus streamlines in plane $y = 0$ (cyan/gray). The latter are approximated by streamlines in the slice $|y/R| \leq 0.05$.

Metrics (20) are computed in terms of their mean

$$\mu_i(\alpha) = \frac{\sum_{k=1}^{N} \epsilon_i(r_k,z_k;\alpha)}{N}$$  \hspace{1cm} (21)

and corresponding standard deviation

$$\sigma_i(\alpha) = \sqrt{\frac{\sum_{k=1}^{N} (\epsilon_k(r_k,z_k;\alpha) - \mu_i(\alpha))^2}{N}},$$  \hspace{1cm} (22)

with $i \in \{r,\theta,z\}$, over the $N$ data points. For $\alpha = \pi/4$, this yields $(\mu_r,\mu_\theta,\mu_z) = (0.0060,0.0016,0.0073)$ and $(\sigma_r,\sigma_\theta,\sigma_z) = (0.0058,0.0014,0.0070)$, signifying an asymmetry well below 1%. Figure 10(a) gives the planar velocity field $(u_r,u_z)_{|r,z|}$ (black) and its corresponding reflection $(u_r,-u_z)_{|-r,z|}$ about the axis $r = 0$ (cyan/gray) for $\alpha = \pi/4$. [Note that the latter relates to the 3D velocity field in the symmetry plane $I_\xi$ via $u = (u_r,0,u_z) = (u_r,0,u_z)/\cos \alpha$, reflecting the intimate relation between symmetries $S_\alpha$ and $S_{\alpha/2}$.] Original and reflected velocity fields, consistent with measures (20), exhibit a close resemblance and deviations are restricted primarily to the direct proximity of the bottom rim $(r/R,z/R) = (1,-1)$ and the cylinder wall $r/R = 1$. Compromised performance of the interpolation scheme due to the substantially lower density of data points in this region, e.g., visualized in Fig. 9 by the spatial extent of the 3D streamline pattern, is believed the most probable cause of this effect. The associated normal component $u_\theta(r,z)$ (black) and reflection $u_\theta(-r,z)$ (cyan/gray) are shown for $r = 0.5$ in Fig. 10(b) and display a comparable correlation as the planar components.

FIG. 10. (Color online) Symmetry $S_\alpha$ in the velocity field demonstrated for the plane $\theta = \pi/4$: actual velocity (black) versus reflected velocity (cyan/gray): (a) planar components $u_r,u_z$ evaluated on a $16 \times 16$ equidistant grid; (b) normal component $u_\theta$ evaluated on lines $r = \pm 0.5$.

Figure 11 shows the mean departures $(\mu_r,\mu_\theta,\mu_z)$ and standard deviations $(\sigma_r,\sigma_\theta,\sigma_z)$ for $0 \leq \alpha \leq \pi$. This reveals that violations of symmetries $S_r$ and $S_\alpha$, and, in consequence, of $S_x$ and $S_y$, typically remain below the 1% mark (dashed line). This is within the geometric asymmetry $\epsilon_Z$ due to positioning inaccuracies of the bottom wall (Sec. IV A), and puts forth the latter as the most likely cause for these violations. Other effects such as particle and fluid inertia, refractive shift, buoyancy, etc., can namely be ruled out on grounds of their negligible contributions.

B. Symmetry in time: Reversibility

The linearity of the momentum equation (1) in the non-inertial limit $Re = 0$ leads to the time-reversal symmetry (11). This, in turn, implies reversibility of the Lagrangian motion upon reversing the translation direction of the driving wall, and forward and backward displacement are equal. This forward-backward forcing namely corresponds with a two-step protocol

$$\Phi = \Phi_2 \Phi_1,$$  \hspace{1cm} (23)
The reversibility offers a further way to investigate the symmetry properties of the experimental base flow. Introduce to this end the displacement $\delta x = |\Delta x|/2$, with $|\Delta x| = |\Phi(x_0) - x_0|$ the separation between final and the initial particle positions after a single period of forcing protocol (23), as a measure for asymmetry (ideally $\delta x = 0$).

Forcing protocol (23) was investigated experimentally for nondimensional displacement $D = 4.3$. Figure 12 displays the normalized distribution of $\delta x$, yielding a mean displacement $\mu_{\delta x} = 0.18$ mm, which is well below the particle diameter $d_p = 0.25$ mm. The distribution reveals that $\delta x \leq d_p$ and $\delta x \leq 2d_p$ for 75% and 98%, respectively, of the particles. This amounts to a typical asymmetry $\delta x / R \sim \mathcal{O}(0.005)$, which is consistent with the findings in Sec. V A.

The asymmetry $\delta x$ proves uncorrelated with the initial particle location $x_0$. This is demonstrated by the scatter plots in Fig. 13, exposing a random spatial distribution of displacements $\delta x$ versus initial radial and axial position. Important to note here is that the accumulation of particles at intermediate $r/R$, instead of signifying a spatial dependence, emanates from the interplay of two effects: (i) the volume

![FIG. 11. Symmetry $S_\alpha$ in the velocity field in terms of the mean $(\mu_r,\theta,z)$ and standard deviation $(\sigma_r,\theta,z)$ of metrics (20) evaluated on $16 \times 16$ equidistant grids in plane $\theta = \alpha$ for $0 \leq \alpha \leq \pi$.](image)

![FIG. 12. Distribution of asymmetry $\delta x$ over the experimental streamlines for forcing-reversal protocol (23) at $D = 4.3$. The dotted line indicates $d_p$; the arrow indicates the mean asymmetry of 180 $\mu$m.](image)

![FIG. 13. Asymmetry $\delta x$ for forcing-reversal protocol (23) at $D = 4.3$ versus initial particle position $(r/R,z/R)$ in the rz plane. Dashed lines indicate $d_p/2$.](image)
of annular segments \([r, r + \Delta r]\) (and, inherently, the number of contained particles) increases linearly as \(V \sim r\); (ii) the seeding density rapidly decreases near the cylinder wall. Particle accumulation in the lower part of the cylinder stems from the limited optical access to the upper part due to the particular camera arrangement (Sec. IV A).

VI. ANALYSIS OF PERIOD-1 LINES

A. Isolation of period-1 lines using symmetries

Period-1 lines are, aside from the invariant spheroids, important coherent structures that organize the flow topology and thereby determine the transport properties of the present flows (Sec. I). Key features of period-1 lines, exposed by theoretical analysis and direct numerical simulations, are in the following compared to experimental results obtained by 3D-PTV measurements.

The time-reversal symmetry (16) causes the present forcing protocols to always accommodate a period-1 line \(\mathcal{L}\) following (8) in the corresponding symmetry plane \(I_\beta = \{x \in \mathcal{D} | \phi = \beta\}\). These planes coincide with \(\beta = \pi/6\) and \(\pi/4\) for protocols \(T\) and \(S\), respectively. The underlying symmetries of the base flow furthermore cause the period-1 line during its progression within one forcing cycle to stepwise wander through the sequence of planes

\[
I_n = \{x \in \mathcal{D} | y = \phi_1 + n \phi_{\text{step}}\}, \quad 1 \leq n \leq p \tag{25}
\]

following

\[
\mathcal{L}^{(n)} = \Phi^{(n)}(\mathcal{L}), \quad \mathcal{L}^{(n)} \in I_n, \tag{26}
\]

with \(\Phi^{(n)} = \Phi \ldots \Phi_1\) the mapping associated with the first \(n\) forcing steps [25]. (Note that \(I_p = I_\beta\).) The intermediate positions \(\mathcal{L}^{(n)}\) in fact constitute period-1 lines of the reoriented forcing protocol \(\Phi = R^n \Phi R^{-n}\), with \(R\) according to Eq. (6), and relate with the sought-after period-1 line via \(\mathcal{L}^{(n)} = R^n \mathcal{L}\). This implies that particles switching from plane \(I_{n-1}\) towards its consecutive plane \(I_n\), with \(1 \leq n \leq p\), during any of the forcing steps belong to \(\mathcal{L}^{(n)}\) and, in consequence, to \(\mathcal{L}\).

The above properties form the basis for the isolation of (segments of) the period-1 line from the experimental data. Particles switching from planes \(I_{n-1}\) to \(I_n\) undergo a rotation exactly over an angle \(\theta_{\text{step}}\) between two forcing steps. We assume that only particles associated with the period-1 line exhibit this systematic reorientation. Thus, isolation of (candidate) period-1 points involves identifying the particles for which \(\theta_n - \theta_{n-1} - \theta_{\text{step}} \leq \epsilon\), with \(n\) indicating the forcing step and \(\epsilon\) some tolerance. This isolation procedure can be performed for any of the \(p\) forcing steps, rendering this approach considerably more robust and efficient than an ansatz based on monitoring particles for the full forcing cycle.

B. Period-1 lines

The random distribution of tracer particles and the relatively low seeding density (Sec. IV) necessitate combination of data sets of multiple forcing periods so as to obtain a sufficient number of candidate period-1 points for demarcating the period-1 line. Moreover, the above tolerance \(\epsilon\) for determining the stepwise azimuthal displacement \(\theta_n - \theta_{n-1}\) is set by defining a spherical search volume of radius \(r_\epsilon = 1.75 \text{mm}\) around each particle. (This tolerance is determined empirically so as to capture a maximum number of candidate period-1 points with the lowest possible error).

Figures 14(a) and 15(a) give the candidate period-1 points in the associated planes \(I_n\) following (25) for protocols \(T\) and \(S\), respectively, by employing the above procedure to a 3D-PTV measurement over 500 periods and at a displacement of \(D = 4.3\). Note that the finite tolerance \(\epsilon\) yields candidate period-1 points that are slightly off the planes \(I_n\), resulting in a certain “thickness” of the data sets. The fact that the identified particles nonetheless sit in the direct proximity of planes \(I_n\) confirms the previous hypothesis that only material points associated with the period-1 line in symmetry plane \(I_\beta\) undergo rotations exactly over an angle \(\theta_{\text{step}}\). Unifying the data sets \(S_n\) of each plane \(I_n\) into one set \(S\) in the symmetry plane \(I_\beta\) via

\[
S = R^{-1} S_1 \cup \ldots \cup R^{-1} S_{n-1} \cup R^{-n} S_n \tag{27}
\]

yields the experimental characterization of the period-1 line \(\mathcal{L}\). The data sets \(S\) thus obtained are given in Figs. 14(b), 15(b), and 15(c).
and 15(b) and in both cases to good approximation indeed delineate one coherent curve. Departures from a perfectly smooth curve must, since data set $S$ results directly from inaccuracies in data sets $S_n$, be attributed primarily to the finite tolerance $\epsilon$.

The reoriented data sets $S_n$ collapsing on one curve in the symmetry plane $I_\beta = \pi/4$ through the union (27) further substantiates the presence of the predicted symmetries in the experimental field. Moreover, the shape of the period-1 line and its variation with changing displacement $D$, consistent with the topological equivalence of base flows subject to boundary conditions (9) and (10), qualitatively correlates with numerical simulations using other boundary conditions. This is investigated in the following.

C. Period-1 lines versus wall displacement

Numerical studies on the cylinder flow revealed that the shape and properties of the period-1 lines $L$ depend strongly on the wall displacement [20,25]. Both rigid-wall and smooth conditions on the bottom wall, though resulting in quantitative differences, yield qualitatively similar behavior on grounds of their fundamental topological equivalence (Sec. III):

(i) Progressive convolution of the period-1 line with increasing $D$.

(ii) A common attachment point $(r/R, z/R) = (0, 1)$ at the top wall.

(iii) A common interior stagnation point $(r/R, z/R) = (0, z_0)$ on the cylinder axis, with $-1 < z_0 < 0$.

This behavior is demonstrated in Fig. 16 for forcing protocol $T$ by numerical simulations using the rigid-wall and smooth boundary conditions on the bottom wall. Both progressions clearly reveal the interior stagnation point that acts as a “pivot” around which period-1 lines curl up with increasing $D$. This stagnation point coincides with the center of the vortex in the symmetry plane $I_y$ of the base flow and sits at $z_0 \approx -0.56$ (Fig. 3). It exists for any forcing involving only the bottom wall and, irrespective of particular protocol, always dictates a similar progression with increasing $D$ as shown in Fig. 16. The period-1 lines for rigid-wall and smooth conditions approximately coincide for
Hence, period-1 lines of both cases, aside from the qualitative implications formation of a period-1 line in the symmetry plane must exhibit topologically equivalent behavior (Sec. III). This to the same family as the flows simulated above and thus to the particular boundary conditions. Moreover, this further save the particular displacement $D$, is relatively insensitive quantitative agreement, signifying a formation process that, period-1 lines using this rule of thumb. This exposes a close agreement and equal quantitative resemblance.

The experimental flow due to forcing protocol $T$ belongs to the same family as the flows simulated above and thus must exhibit topologically equivalent behavior (Sec. III). This implies formation of a period-1 line in the symmetry plane $I_{β}$ according to the above scenario. This is investigated below.

Figure 17 shows the period-1 lines $L$ (symbols) within the symmetry plane $I_{β}$ for displacements $D = [1.4, 2.9, 4.3]$ obtained from direct 3D-PTV experiments via the procedure outlined in Sec. VI.B. Shape and dependence upon displacement $D$ are entirely consistent with the generic progression shown in Fig. 16. Note in particular the intersection of the period-1 lines at the common interior stagnation point on the cylinder axis, which again sits at $z_{0} ≈ −0.56$. Comparison with the simulations for the rigid-wall conditions reveals that the employed displacement $D_{sim}$ roughly correlates with $D$ as $D_{sim}/D ≈ 2/3$. The curves in Fig. 17 indicate the simulated period-1 lines using this rule of thumb. This exposes a close quantitative agreement, signifying a formation process that, save the particular displacement $D$, is relatively insensitive to the particular boundary conditions. Moreover, this further demonstrates the essential topological equivalence of period-1 lines in the family of flows under investigation here. Recall in this respect that a similar scaling rule connects the period-1 lines of the rigid-wall and smooth boundary conditions. Hence, comparison of experimental period-1 lines with those simulated for the smooth conditions yields identical qualitative agreement and equal quantitative resemblance.

Period-1 lines beyond the operating limit $D = 4.3$ of the experimental setup have been determined by means of the hybrid particle-tracking method proposed in Sec. IV.B. Figure 18 gives the period-1 lines (symbols) for displacements $D = [4.3, 8.6, 17.1]$ computed with the hybrid method. The line for $D = 4.3$ coincides with that found by direct processing of 3D-PTV data (Fig. 17) within an acceptable error margin, advancing the hybrid method as a reliable data-processing tool that greatly extends the reach of the laboratory experiments. Curves in Fig. 18 correspond with simulated period-1 lines, again using $D_{sim}/D ≈ 2/3$. The semiexperimental period-1 lines, similar to their simulated counterparts in Fig. 16, undergo progressive convolution upon augmenting $D$ while remaining fixated at the two designated positions on the cylinder axis. Moreover, the close agreement with the simulated period-1 lines is retained, notwithstanding the considerably greater geometric complexity. Shown results thus substantiate the previous findings in that the experimental period-1 lines behave entirely in accordance with the generic scenario following Fig. 16.

VII. CONCLUSIONS

The work discussed here concerns an experimental investigation on the formation of coherent structures in the Lagrangian fluid trajectories of three-dimensional (3D) time-periodic laminar flows. These structures geometrically determine the advection of passive tracers and thus are essential to transport and mixing properties in a wide range of physical processes in nature and industry. Key coherent structures are so-called periodic lines, that is, material curves consisting of material points that periodically return to their initial position. These entities have to date been observed only in numerical simulations. The aim of this study is to demonstrate their existence and investigate their properties in laboratory experiments.

The 3D time-periodic flow inside a finite cylinder, driven by piecewise steady translations of the bottom endwall via a given forcing protocol, serves as flow configuration. This flow admits periodic lines in the noninertial limit $Re = 0,$
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APPENDIX: REFRACTIVE SHIFT

The employed 3D-PTV algorithm hinges on an optical model assuming separation of cameras and flow domain by a single solid wall [44–48]. The present configuration, on the other hand, includes two solid walls of identical material (i.e., perspex) between cameras and flow domain, where the interior wall is surrounded by the same liquid (i.e., silicone oil). This introduces additional refraction that is unaccounted for in the algorithm. The total refractive shift is estimated below so as to determine the significance of this effect.

The optical configuration comprises of two fluids, air (“a”) and silicone oil (“o”), separated by two perspex walls (“p”) of thickness $d_{1,3}$ and separated by a gap $d_2$ following schematic Fig. 19. Snell’s law of refraction [54] applied to each fluid-solid interface yields

$$
\delta = \delta_1 + \delta_2 + \delta_3
= (d_1 + d_2 + d_3) \frac{\sin \gamma_{ap} \cos \alpha_p}{\cos \alpha_o} + d_2 \left( \frac{\sin \gamma_{ao}}{\cos \alpha_o} - \frac{\sin \gamma_{ap}}{\cos \alpha_p} \right)
$$

as total shift $\delta$, where $\gamma_{ap} = \alpha_a - \alpha_p$, $\gamma_{ao} = \alpha_a - \alpha_o$, which can be decomposed as $\delta = \delta_{\text{singlewall}} + \delta\Delta$. Here, $\delta_{\text{singlewall}}$ represents the shift of a virtual wall of effective thickness

![Diagram](https://via.placeholder.com/150)

FIG. 19. Refractive shift $\delta$ in the experimental setup due to solid-fluid interfaces between air (“a”), perspex (“p”), and silicon oil (“o”).
$d_1 + d_2 + d_3$ separating air and oil, which can be accounted for by the optical model of the 3D-PTV algorithm. 

The small viewing angle by the optical model of the 3D-PTV algorithm. Contribution corresponds to the additional refraction that is beyond the model. However, the small viewing angle $\alpha_s \simeq 9^\circ$ of the cameras (Sec. IV A) and the small difference between refractive indices of perspex and silicone oil, i.e., $n_p = 1.47$ and $n_o = 1.40$, yields an upper bound of $125 \mu m$ for $\Delta \delta$. This is well below the particle diameter and thus acceptable.