Type III Einstein-Yang-Mills solutions

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We consider Kundt’s class of metrics, described by the line element:

$$ds^2 = -2du (dv + W dz + \overline{W} d\bar{z} + H du) + 2P^{-2}dzd\bar{z}, \quad P_{\nu \bar{\nu}} = 0$$  \hspace{1cm} (1)

Here, $P$ and $H$ are real functions while $W$ is complex; $u = (t - z)/\sqrt{2}$, $v = (t + z)/\sqrt{2}$ are light-cone coordinates and $z = (x + iy)/\sqrt{2}$, $\bar{z} = (x - iy)/\sqrt{2}$ complex conjugate coordinates in the transverse plane. We consider solutions of type III. In this case we can take $P = 1$ without loss of generality. As a consequence the Gaussian curvature of the wave surfaces $u =$ constant, $K = 2P^2(\ln P)_{z\bar{z}}$, vanishes. Type III solutions are therefore characterized by plane wave surfaces. Details on specific aspects of the metrics (1) can be found in [1].

It can be seen that two distinct cases arise for vacuum solutions of type III (for references, see [1]). In the first case the function $W$ does not depend on the light-cone coordinate $v$. In the second case the dependence is of $v$. These two cases correspond to the Newman-Penrose quantity $\tau$ being zero or $\neq 0$, respectively.

### A. Case $W,v = 0$

Consider the metric (1) with:

$$P = 1 \quad \hspace{1cm} (2a)$$

$$W = W(u, \bar{z}) \quad \hspace{1cm} (2b)$$

$$H = H^0(u, z, \bar{z}) + \frac{1}{2}(W_{z\bar{z}} + \overline{W_{z\bar{z}}}) v \quad \hspace{1cm} (2c)$$

Here, $W$ is an arbitrary complex function and $H^0$ is a real function. This is a vacuum solution of Einstein equations if the function $H^0$ satisfies:

$$\frac{1}{2}R_{43} = H^0_{z\bar{z}} - \text{Re} \left( W^2_{z\bar{z}} + WW_{z\bar{z}} + W_{z\bar{z}}u \right) = 0$$  \hspace{1cm} (3)

Numerical indices are tetrad indices$^1$. All other components of the Ricci tensor vanish. This class of solutions degenerates to type N when $\Psi_3 = \frac{1}{2}W_{z\bar{z}} = 0$. In this case one can use the remaining coordinate transformation (see [1]) to make $W = 0$. The metric reduces then

---

$^1$ Tetrad vectors as in [1]: $e_1 = \partial_t = \partial_z, \quad e_4 = \partial_u + \overline{W} \partial_\bar{z} + W \partial_z - (H + \overline{W}) \partial_v, \quad e_4 = \partial_v$
to that of familiar pp waves \[18\],

\[ ds^2 = 2 \, dz \, d\bar{z} - 2 \, du \, dv - H(u, z, \bar{z}) \, du^2 \]  

(4)

where \( H \) satisfies \( H_{,z\bar{z}} = 0 \).

The function \( W \) can have an arbitrary \( u \) dependence but must be at least quadratic in \( \bar{z} \) so that \( \Psi_3 \neq 0 \). This holds for pure radiation solutions as well: they only enter the solution through the function \( H^0 \) so \( W \) is as in vacuum.

We can generalize the solution to include a null Yang-Mills field \( (\Phi_0 = \Phi_1 = 0) \) of the form:

\[ A = \alpha^a(u, z, \bar{z}) T_a \, du, \quad \alpha^a = \chi^a(u, z) + \bar{\chi}^a(u, \bar{z}) \]  

(5)

Here, \( \chi^a \) are arbitrary complex functions. This field was considered in curved spacetime for the first time in \[6\].

The Yang-Mills equation in flat space is just:

\[ \chi = 0 \]  

(6)

This equation remains unchanged in the spacetime \[11\]-\[20\] because of two reasons. First, the absence of components of the YM field strength of the form \( F^{\mu \nu} \) in this geometry. In the second place, the only non-vanishing Christoffel symbols of the form \( \Gamma_{\rho \nu \rho} \) are precisely \( \Gamma_{\rho u} \). As a result only the ordinary derivative survives in the (curved) Yang-Mills equation:

\[ \partial_\lambda F^{\lambda \mu} + \Gamma_{\mu u} \rho F^{u \mu \rho} - [A_u, F^{u \rho}] = 2\alpha_{,z\bar{z}} = 0 \]  

(7)

Of course Eq. \[3\] has to be modified to account for the energy-momentum tensor of the Yang-Mills field:

\[ T_{uu} = \frac{1}{2\pi} \gamma_{ab} \alpha^a_{,z} \alpha^b_{,\bar{z}} \]  

(8)

Here, \( \gamma_{ab} \) is the invariant metric of the Lie group. In tetrad indices we have \( T_{33} = T_{uu} \). Therefore, Eq. \[4\] transforms to:

\[ H_{,z\bar{z}}^0 - \text{Re} \left( W^2_{,z} + WW_{,z\bar{z}} + WW_{,u} \right) = 2\gamma_{ab} \alpha^a_{,z} \alpha^b_{,\bar{z}} \]  

It is straightforward to solve for \( H^0 \):

\[ H^0(u, z, \bar{z}) = f(u, z) + \bar{f}(u, \bar{z}) + 2\gamma_{ab} \alpha^a_{,z} \alpha^b_{,\bar{z}} + \text{Re} \left\{ (W_{,u} + WW_{,\bar{z}}) \, z \right\} \]  

(9)

Here \( f \) is an arbitrary complex function. Eqs. \[4\] and \[8\] with an arbitrary (at least quadratic) \( W \) and the given \( H^0 \) describe an exact type III solution of the EYM system. It is a generalized Goldberg-Kerr solution. The type N reduction of this solution has been known for a long time \[6\].

\[ \text{B. Case } W_{,u} = -2/(z + \bar{z}) \]

We consider now the metric \[11\] with:

\[ P = 1 \]  

(10a)

\[ W = W^0(u, z) - \frac{2u}{z + \bar{z}} \]  

(10b)

\[ H = H^0(u, z, \bar{z}) + \frac{W^0 + \bar{W}^0}{z + \bar{z}} - \frac{u^2}{(z + \bar{z})^2} \]  

(10c)

Here, \( W^0 \) is an arbitrary complex function and \( H^0 \) is a real function. This is a vacuum solution of Einstein equations if the function \( H^0 \) satisfies the differential equation:

\[ \frac{1}{2} R_{33}^0 = (z + \bar{z}) \left( \frac{H^0 + \bar{W}^0}{z + \bar{z}} \right) - W^0_{,z} \bar{W}^0_{,\bar{z}} = 0 \]  

(11)

This is the only non-vanishing component of the Ricci tensor\(^2\). This class of solutions degenerate to type N as well when \( \Psi_3 = \bar{W}^0_{,\bar{z}}/(z + \bar{z}) = 0 \). In this case one can transform \( W^0 \) to zero \[11\] and the resulting metric is known as Kundt waves.

We consider the field \[3\] in this background. Features in this geometry are the same ones as in the previous case, except for an extra term in the Yang-Mills equation:

\[ (\partial_z + \Gamma_{u u} + \Gamma_{v v}) F^{u v} + (z \leftrightarrow \bar{z}) = 0 \]  

(12)

After further inspection this extra term is seen to vanish as \( \Gamma_{u u} = -\Gamma_{v v} \), \( \Gamma_{u \bar{z}} = -\Gamma_{v \bar{z}} \). In this way, the Yang-Mills equation in the spacetime \[11\], \[10a\], \[11\] is still the same as in flat space, Eq. \[6\]. Finally, Eq. \[11\] has to be modified to include the YM energy-momentum tensor \( T_{33} = (\gamma_{ab}/2\pi) \alpha^a_{,z} \alpha^b_{,\bar{z}} \).

We refine our ansatz in order to solve explicitly for \( H^0 \). We choose the specific function \( W^0 = \chi^a(u, z) \), \( g(u, z) \) being an arbitrary complex function. This is the simplest form of \( W^0 \) such that \( \Psi_3 \) is non-vanishing. On the other hand we limit the Yang-Mills field \[3\] to:

\[ \chi^a = \lambda^a(u) \, z \]  

(13)

Here \( \lambda^a(u) \) are bounded complex functions. This field was considered in Minkowski space for the first time in \[11\] and is referred to as non-Abelian plane waves\(^3\). The field \[11\] is the only form of \[3\] which gives rise to a bounded energy-momentum tensor. Under these assumptions the equation to solve reads:

\[ (z + \bar{z}) \left( \frac{H^0 + \bar{g}^0 \bar{z}}{z + \bar{z}} \right)_{,z} - g(u) \bar{g}(u) = 2\gamma_{ab} \lambda^a(u) \bar{\lambda}^b(u) \]  

\[ \text{\footnotesize 2} \] The superscript \( 0 \) in \( R_{33} \) denotes the \( \nu \)-independent part of \( R_{33} \).

\[ \text{\footnotesize 3} \] The original field in \[13\] was \( \chi^a = \lambda^a(u) \, z + \text{some function of } u \) but the later term can be gauged away without affecting gauge-invariant quantities, as pointed out by Coleman.
It is not difficult to see that:
\[
H^0 = (f(u, z) + \bar{f}(u, \bar{z})) (z + \bar{z}) - gg z \bar{z} + \sigma(u)(z + \bar{z})^2 \ln(z + \bar{z}) - 1
\]  
(14)

Here \(f\) is again an arbitrary complex function and \(\sigma\) is the real function \(\sigma(u) = 2\gamma_{ab} \lambda^a(u) \lambda^b(u) + g(u) \bar{g}(u)\). Eqs. (5), (13) and (14) with the given \(W^0\), \(H^0\) describe another exact type III solution of the EYM system. The corresponding vacuum solution \((\lambda^a \equiv 0)\) has to our knowledge not been considered before. The case \(g(u) \equiv 0\) yields the type N reduction and can be regarded as generalized Kundt waves. A related type N solution was very recently given in [16], for a Yang-Mills field of arbitrary waves in the (vacuum) type N reduction. A (type N) solution is obtained by translating the line element into the Kundt canonical form and considering \(N = 0, D = 4\).

C. Geodesics

For completeness, we present the geodesic equations corresponding to the two classes of solutions found, without attempting to provide detailed solutions here.

• Case \(W_{uv} = 0\)

In order to get a feeling for the behavior of geodesics we choose the specific \(W = \bar{z}^2\). This is the simplest \(W\) such that the solution presented in subsection A is of type III. The geodesics are described in terms of real spatial coordinates \(x, y\) by:
\[
\ddot{u} - \sqrt{2} \dot{x} \dot{u}^2 = 0
\]
\[
\ddot{x} + \dot{u}^2 (H_x - x(x^2 - y^2)) + 2 \sqrt{2} y \dot{u} \dot{y} = 0
\]
\[
\ddot{y} + \dot{u}^2 (H_y - 2y^2x) - 2 \sqrt{2} \dot{y} \dot{x} = 0
\]
\[
\ddot{x}^2 + \dot{y}^2 - 2 \dot{x} \dot{v} - 2 H \dot{u}^2 - \sqrt{2} (x^2 - y^2) \dot{x} \dot{u} - 2 \sqrt{2} y x \dot{y} \dot{u} = \epsilon
\]
\(\dot{}\) (15)  
\(\dot{}\) (16)  
\(\dot{}\) (17)  
\(\dot{}\) (18)

Here, the overdot denotes a proper time derivative and \(H\) is:
\[
H = f + \bar{f} + 2(x^4 - y^4) + 2 \gamma_{ab} \lambda^a \lambda^b + \sqrt{2} x y
\]
\(\dot{}\) (19)

We have considered the normalization condition for the 4-velocity, Eq. (18), instead of the (complicated) equation for \(v\). There, \(\epsilon = -1, \) for timelike and null geodesics, respectively.

It can be advantageous to use \(u\) as affine parameter instead of the proper time. If we assume \(\dot{u} > 0\), from Eq. (15):
\[
x = \frac{1}{\sqrt{2}} \frac{d(\ln u)}{du}
\]
\(\dot{}\) (20)

Now, in the hyperplane \(y = 0\), by inserting Eqs. (20) in (19), and recalling \(\dot{x} = \dot{u} \frac{d(x/du)}{du}\), a coupled system for \(x(u), v(u)\) can be obtained.

On the other hand, in the hyperplane \(x = 0\) Eqs. (16) and (17) simplify to
\[
\ddot{u} = 0 \leftrightarrow \dot{u} = \text{constant} = \gamma
\]
\(\dot{}\) (21)
\[
\ddot{y} + \dot{u}^2 H_{yy} = 0
\]
\(\dot{}\) (22)

taking the same form as in the type N reduction (see for example [20]). Analytical differences come from the quartic term of (19) which is absent in the type N. This difference can be physically illustrated when we consider the functions \(f = \kappa(u) z^2\), with \(\kappa(u)\) an arbitrary real function\(^4\), and \(\chi^a = \lambda^a(u) z\) in Eq. (13). Then, the motion in the \(y\) direction is governed by:
\[
d^2 y/du^2 - 8y^3 + 2(\gamma_{ab} \lambda^a(u) \lambda^b(u) - \kappa(u)) y = 0
\]
\(\dot{}\) (23)

This equation describes a (parametric) non-linear oscillator in a potential \(V(y) = y^3(\gamma_{ab} \lambda^a \lambda^b - \kappa - 2y^2)\). In the type N reduction there is no cubic term in Eq. (23) and the motion reduces to that of an harmonic oscillator.

Non-linear oscillators can have chaotic motion. Geodesic chaotic motion has been found in the related case of non-homogeneous vacuum \(pp\) waves \((f \sim z^n, n > 2)\), see [21]. However, chaotic behavior is not always present in non-linear systems and therefore possible chaotic motion arising from Eq. (23) would need further investigation.

• Case \(W_{uv} = -2/(z + \bar{z})\)

We take the \(u\) dependence out of \(W^0\) respect to the function considered in subsection B, so now \(W^0 = z\). Geodesics in this second case have the more complicated form:
\[
\ddot{u} - (1 - v/x^2) \dot{u}^2 - 2/x \dot{u} \dot{x} = 0
\]
\(\dot{}\) (24)
\[
\ddot{x} + \dot{u}^2 (H_x + (x - 2v/x)(1 - v/x^2)) + 2 (1 - 2v/x^2) \dot{x} \dot{u} - 2/x \dot{v} \dot{u} = 0
\]
\(\dot{}\) (25)
\[
\ddot{y} + \dot{u}^2 (H_y + y(1 - v/x^2)) + 2y/x \dot{u} \dot{x} = 0
\]
\(\dot{}\) (26)
\[
\ddot{x}^2 + \dot{y}^2 - 2 \dot{x} \dot{v} - 2H \dot{u}^2 - 2(-2v/x + x) \dot{x} \dot{u} + 2y \dot{y} \dot{u} = \epsilon
\]
\(\dot{}\) (27)

Again, the normalization condition of the 4-velocity takes the place of the equation for \(v\). The function \(H\) is:
\[
H = H^0 + v - \frac{v^2}{2x^2}
\]
\(\dot{}\) (28)

and \(H^0\) is given by (14) but with \(g(u) \equiv 1\).

\(\dot{}\)

\(4\) This choice is well motivated as it corresponds to homogeneous \(pp\) waves in the (vacuum) type N reduction.
The geodesic equations can again be rewritten as differential equations in terms of an affine variable \( u \) instead of the proper time. In particular, for the first equation:

\[
\frac{d}{du} \ln \left( \frac{\dot{u}}{x^2} \right) = 1 - \frac{v}{x^2}
\] (29)

Then the remaining geodesics become equations for \((x, y, v)\) in terms of \( u \).

The existence of a singularity at \( x = 0 \) is clear from the geodesic equations, but not its character. Geodesics for a general vacuum type N or conformally flat pure radiation solutions have been analyzed in [22], where it was concluded that \( x = 0 \) is a physical singularity. Whether this is also the case for our type III solutions remains an open question.

III. CONCLUSIONS

In [23] Tafel claimed that null twistfree solutions of the Yang-Mills equations were exhausted by the type N (class of) solutions given in [6]. A new type N solution was recently given in [16]. We have shown that more general solutions exist, as the two type III (classes of) EYM solutions presented here are also null and twistfree.

It has been proved [24] that all spacetimes with vanishing curvature invariants of all orders: 1) are Kundt metrics of Petrov type III, N or O, and 2) are of Plebański-Petrov (PP) type N or O (null radiation or vacuum). These results have been generalized to higher dimensions in [27]. Such spacetimes have been called VSI spacetimes. Our solutions clearly meet requirements 1) and 2) and are therefore new examples of \( D = 4 \) VSI spacetimes. As such, they might have interesting physical applications in supergravity and string theory, in the fashion of pp waves.

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