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L. Pahlavan a, C. Kassapoglou a, A.S.J. Suiker b & Z. Gürdal a

a Faculty of Aerospace Engineering, Delft University of Technology, The Netherlands
b Department of the Built Environment, Eindhoven University of Technology, The Netherlands


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A wavelet-based spectral finite element method for elastic wave propagation

L. Pahlavan*, C. Kassapoglou, A.S.J. Suiker and Z. Gürdal

Faculty of Aerospace Engineering, Delft University of Technology, The Netherlands; Department of the Built Environment, Eindhoven University of Technology, The Netherlands

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A wavelet-based spectral finite element method (WSFEM) is presented that may be used for an accurate and efficient analysis of elastic wave propagation in two-dimensional (2D) structures. The approach is characterised by a temporal transformation of the governing equations to the wavelet domain using a wavelet-Galerkin approach, and subsequently performing the spatial discretisation in the wavelet domain with the finite element method (FEM). The final solution is obtained by transforming the nodal displacements computed in the wavelet domain back to the time domain. The method straightforwardly eliminates artificial temporal edge effects resulting from the discrete wavelet transform and allows for the modelling of structures with arbitrary geometries and boundary conditions. The accuracy and applicability of the method is demonstrated through (i) the analysis of a benchmark problem on axial and flexural waves (Lamb waves) propagating in an isotropic layer, and (ii) the study of a plate subjected to impact loading. The wave propagation response for the impact problem is compared to the result computed with standard FEM equipped with a direct time-integration scheme. The effect of anisotropy on the response is demonstrated by comparing the numerical result for an isotropic plate to that of an orthotropic plate, and to that of a plate made of two dissimilar materials, with and without a cut-out at one of the plate corners. The decoupling of the time-discretised equations in the wavelet domain makes the method inherently suitable for parallel computation, and thus an appealing candidate for efficiently studying high-frequency wave propagation in engineering structures with a large number of degrees of freedom.

Keywords: discrete wavelet transform; wavelet-Galerkin approach; elastic wave; finite-element modelling; mechanical behaviour; numerical method; parallel computing; anisotropic elasticity

1. Introduction

The online detection and characterisation of small and medium-scale defects and inhomogeneities in engineering structures is commonly referred to as 'structural health monitoring'. A deviation in structural performance as a result of the generation of
damage can be identified by periodically sampling the dynamic response of the structure with a spatial arrangement of sensors. Because a change in the dynamic response is characterised by the scattering of waves at newly damaged areas, the location and size of the damage may be estimated by carefully reconstructing the paths of the scattered waves. For practical applications related to engineering structures with a large number of degrees of freedom, this requires the availability of fast analysis tools that accurately simulate high-frequency wave propagation signals necessary for the identification of small-scale instabilities and inhomogeneities.

In the study of wave propagation problems, direct time integration methods, such as the Newmark-β scheme [1], have been very popular and widely used in commercial finite element method (FEM) codes. However, the continuous increase in size and complexity of engineering problems has initiated a search for more efficient time discretisation methods that make optimal use of the advantages of parallel computing. An appealing tool that may be applied for efficiently and accurately solving wave propagation problems is the discrete wavelet transform (DWT), of which the most commonly-used set was introduced by Daubechies at the end of the 1980s [2,3]. In comparison to the fast Fourier transform (FFT), the DWT of Daubechies wavelet has several advantages, namely (i) its compact support in both the time domain and frequency domain (which allows for removing noise from time series and does not require the application of periodic boundary conditions as in FFT) and (ii) the possibility of decomposing a signal into different frequency components, where, for each frequency component, the time history can be kept with a specific (frequency-dependent) resolution (which forms the basic ingredient for a multi-resolution analysis).

The characteristics of Daubechies wavelets and their usefulness for solving partial differential equations have been discussed in various papers [4–9], which has led to their application in the fields of fluid dynamics [5,7], electromagnetics [10,11], dynamics of discrete systems [12–14] and damage detection [15,16]. Daubechies wavelets have further been applied for representing the spatial derivative operators in one-dimensional [17] and two-dimensional [18,19] elastic wave propagation formulations, leading to an accurate description of the spatial response. In addition, Mitra and Gopalakrishnan have used Daubechies wavelets for a spectral analysis of wave propagation phenomena, where 2D problems were solved by transforming both the time parameter and one of the spatial coordinates with DWT, and subsequently solving the ordinary differential equation in terms of the remaining spatial coordinate with FEM [20,21]. Despite the advantages of this approach in terms of (i) allowing for a decoupling of the governing equations in the transformed domain (which naturally enables parallel computation), and (ii) providing a time discretisation scheme that is unconditionally stable, undesirable artificial edge effects in space and time are observed. The elimination of these edge effects requires the application of elaborate extrapolation schemes at the temporal and spatial boundaries of the problem under consideration [20–23]. With respect to a spatial coordinate such a scheme only works adequately if the geometrical boundary and the corresponding boundary conditions are relatively straightforward, thus limiting the method to engineering problems with basic boundary conditions and geometries.

In the present paper, a wavelet-based spectral finite element method (WSFEM) is proposed that keeps the appealing features of the above-mentioned wavelet-based
spectral method, but overcomes its restrictions. The approach, which is elaborated for a two-dimensional (2D) orthotropic structure, is based on a temporal transformation of the governing equations to the wavelet domain using a wavelet-Galerkin approach. The resulting equations are decoupled by means of an eigenvalue analysis, and subsequently spatially discretised in the wavelet domain with FEM. For the FEM discretisation, a 9-noded element with exponential shape functions is derived, where the complex arguments of the shape functions are determined from the characteristics of compressional and shear waves propagating along the principal axes of orthotropy. The final solution is obtained by transforming the FEM nodal displacements computed in the wavelet domain back to the time domain. Since geometrical edge effects are absent when all spatial coordinates are discretised with FEM, the WSFEM formulation is suitable for studying wave propagation phenomena in structures with arbitrary shape and boundary conditions. Furthermore, the temporal edge effects emerging from the DWT of the time parameter can be eliminated in a relatively straightforward fashion, by assuming zero initial conditions and performing the wavelet-Galerkin formulation over a finite time window instead of an infinitely large time domain. Since the governing equations in the transformed domain are uncoupled for each wavelet point, the WSFEM approach is inherently suitable for parallel computation.

The accuracy and applicability of the WSFEM approach are demonstrated by examining the dispersion relations of a benchmark problem on axial and flexural waves (known as “Lamb waves”) propagating in an isotropic, plane-strain layer with traction-free surfaces, and comparing the numerical result to the well-known Rayleigh–Lamb solution. In addition, the in-plane propagation of compressional and shear waves in a plate subjected to an impact load is simulated, and the result is compared to FEM solutions computed with linear and quadratic elements and a direct numerical time-integration scheme. The effects of anisotropy and irregular material interfaces and boundaries are illustrated by comparing the WSFEM solutions for an isotropic plate to that of an orthotropic plate, and to that of a plate made of two dissimilar materials, with and without a cut-out at one of the plate corners.

2. Review of the discrete wavelet transform

Before presenting the wavelet-Galerkin formulation, for clarity reasons first the main characteristics of the discrete wavelet transform are summarised in relation to Daubechies wavelet. The discrete wavelet transform (DWT) uses (i) a scaling function \( \varphi_{k,n}(\tau) \) and (ii) a wavelet function \( \psi_{k,n}(\tau) \) for decomposing a time-dependent function \( f(\tau) \) into a specific number of frequency components in the transformed (or wavelet) domain. For reasons of convenience, the time parameter \( \tau \) is taken here as dimensionless, but will be connected to the actual time \( t \) when discussing time discretisation aspects in Section 3. Basic forms of the scaling function and the wavelet function are given by [3,24]

\[
\varphi_{k,n}(\tau) = 2^{k/2} \varphi(2^k \tau - n), \quad k, n \in \mathbb{Z},
\]  

(1)
where the integers $k$ and $n$ are known as the dilation and translation indices, respectively. In engineering problems, the function $f(\tau)$ that needs to be transformed typically falls within the space of square-integrable functions $L^2(\mathbb{R})$. The functions $\psi_{k,n}(\tau)$ and $\psi_{k,n}(\tau)$ presented in Equations (1) and (2) can be used to construct approximations of the $L^2$-function $f(\tau)$ as \[ P_k f(\tau) = \sum_{n=-\infty}^{\infty} c_{k,n} \psi_{k,n}(\tau), \] and \[ Q_k f(\tau) = \sum_{k=-\infty}^{\infty} d_{k,n} \psi_{k,n}(\tau), \] in which $c_{k,n}$ and $d_{k,n}$ are the approximation coefficients and detail coefficients, respectively. Essentially, $P_k f(\tau)$ and $Q_k f(\tau)$ represent projections of the time-dependent function $f(\tau)$ on, respectively, the subspaces $V_k$ and $W_k$ of the $L^2$-space, with $\psi_{k,n}(\tau)$ and $\psi_{k,n}(\tau)$ representing the basis functions spanning these subspaces. It is noted that the subspace characterised by the scaling function $\psi_{k,n}(\tau)$ reduces from $V_{k+1}$ to $V_k$ when the resolution of the sample points used for the projections of $f(\tau)$ reduces in accordance with a decrease of the dilation index from $k + 1$ to $k$. Here, an important property of subspaces characterised by $\psi_{k,n}(\tau)$ is that $V_k \subseteq V_{k+1}$, $\forall k \in \mathbb{Z}$. In order to quantify the difference between the subspaces $V_{k+1}$ and $V_k$, it is postulated that (i) the subspaces $V_k$ and $W_k$ are orthogonal, and (ii) construct the subspace at a higher sampling resolution, $V_{k+1}$, in accordance with \[ V_{k+1} = V_k \oplus W_k, \] where the symbol $\oplus$ thus denotes the orthogonal (or direct) sum. In line with this expression, the projection $P_{k+1} f(\tau)$ can be decomposed into projections $P_k f(\tau)$ and $Q_k f(\tau)$ at a lower sampling resolution as \[ P_{k+1} f(\tau) = P_k f(\tau) + Q_k f(\tau). \] From Equation (5), it may be concluded that $W_{k+1} \perp (V_k \oplus W_k)$, from which it follows that $W_{k+1} \perp W_k$. Correspondingly, a wavelet subspace for a specific sampling resolution is orthogonal to a wavelet subspace for any other sampling resolution, which implies that wavelet functions $\psi_{k,n}(\tau)$ spanning subspaces of different sampling resolutions are orthogonal. Equation (6) is the basic ingredient for a multi-resolution (or multiple scales) analysis using wavelets.

In accordance with Equation (5), the subspace $V_0$ (corresponding to a dilation index $k = 0$) relates to the subspace $V_1$ as $V_0 \subseteq V_1$, due to which a basis function of $V_0$ may be expressed as a linear combination of the basis functions spanning $V_1$ \[ \varphi(\tau) = \sum_{n=-\infty}^{\infty} a_n \varphi(2\tau - n), \] i.e.

\[ \varphi(\tau) = \sum_{n=-\infty}^{\infty} a_n \varphi(2\tau - n), \]
where $a_n$ are the so-called filter coefficients. Note that the expression for the scaling function in the right-hand side follows from Equation (1) using $k = 1$. A similar two-scale relationship can be established for a wavelet function $\psi(\tau)$ spanning the subspace $W_0$. From Equation (5) it may be concluded that $W_0 \subseteq V_1$, as a result of which the wavelet function can be expressed as a linear combination of the scaling functions spanning $V_1$ [2,3], i.e.

$$\psi(t) = \sum_{n=-\infty}^{\infty} (-1)^n a_{1-n} \varphi(2t - n).$$  \hspace{1cm} (8)

For Daubechies orthogonal, compactly supported wavelet, only a finite number of the filter coefficients $a_n$ in Equations (7) and (8) are non-zero. The values of these filter coefficients can be derived uniquely by imposing the following three conditions on the scaling and wavelet functions [3,24]:

$$\int_{-\infty}^{\infty} \varphi(\tau) d\tau = 1,$$

$$\int_{-\infty}^{\infty} \varphi(\tau + i) \varphi(\tau + j) d\tau = \delta_{ij},$$  \hspace{1cm} (9)

$$\int_{-\infty}^{\infty} \psi(\tau) \tau^p d\tau = 0, \quad p = 0, 1, 2, \ldots, M - 1.$$

The first relation above is a normalisation condition that warrants uniqueness of the scaling function, the second relation represents orthonormality of the scaling function under translation (as indicated by the Kronecker delta symbol $\delta_{ij}$), and the third relation reflects that $\psi(\tau)$ has vanishing moments $M$, which guarantees that the wavelets provide an exact representation of polynomials up to the order $M = N/2$, with $N$ being the (even) order of Daubechies wavelet $\psi(\tau)$. Note that this last condition for $p = 0$ specifies the requirement of a zero mean value for the wavelet function, $\int_{-\infty}^{\infty} \psi(\tau) d\tau = 0$. The values of the filter coefficients can be computed recursively by combining the two-scale relations, Equations (7) and (8), with the three conditions provided by Equation (9). More details on the algorithm for the computation of the filter coefficients can be found in [7].

In the present study, the scale is assumed as fixed, corresponding to a specific choice for the dilation index, which is $k = 0$. In accordance with Equations (3) and (1), the function $f(\tau)$ can then be approximated by projecting it onto the subspace $V_0$, i.e.

$$f(\tau) \approx P_0 f(\tau) = \sum_{n=-\infty}^{\infty} c_{0,n} \varphi(\tau - n),$$  \hspace{1cm} (10)

in which the scaling function $\varphi(\tau)$ for Daubechies wavelet is determined from Equation (7), with the filter coefficients $a_n$ computed from the recursive algorithm explained above.

3. Single-scale wavelet-Galerkin approach

Assume a differential equation of the form $F(f^{(0)}, \frac{\partial f^{(1)}}{\partial x}, \frac{\partial^2 f^{(2)}}{\partial x^2}, \ldots) = 0$, which is characterised by functions $f^{(i)}: (x, y, \tau) \rightarrow \mathbb{R}; i = 0, 1, 2, \ldots$, where $\tau$ reflects the
(dimensionless) time and \( x \) and \( y \) are spatial coordinates. Note that different functions \( f^{(i)} \) (indicated by a different superindex \( i \)) relate to a time derivative of a different order. In a partial differential equation this difference in functions results from the fact that one function typically is obtained as the spatial derivative of another function. The specific forms of the spatial derivatives are not included in the above (generic) formulation of the differential equation, since these are irrelevant for explaining the time discretisation procedure. However, spatial derivatives will be addressed in detail when discussing the spatial discretisation procedure in Section 4.

The approximate solution of the above differential equation is calculated applying the wavelet-Galerkin approach, where the inner product of the differential equation and the translated form of the scaling function \( \phi(\tau - m) \) is taken, i.e.

\[
\int_{-\infty}^{\infty} F\left(f^{(0)}, \frac{\partial f^{(1)}}{\partial \tau}, \frac{\partial^2 f^{(2)}}{\partial \tau^2}, \ldots \right) \phi(\tau - m) d\tau = 0, \quad \forall m \in \mathbb{Z}.
\] (11)

In order to further develop Equation (11), in correspondence with Equation (10) a function \( f^{(0)} \) is approximated by its discrete wavelet transform as

\[
f^{(0)}(x, y, \tau) \approx P_0 f^{(0)} = \sum_{n=-\infty}^{\infty} f_n^{(0)}(x, y) \phi(\tau - n), \quad n \in \mathbb{Z}, i = 0, 1, 2, \ldots
\] (12)

in which \( f_n^{(0)}(x, y) \) are the approximation coefficients, which here depend on the spatial coordinates \( x \) and \( y \). From this expression, the \( j \)th-order time-derivative of the function \( f_n^{(0)}(x, y) \) may be elaborated as

\[
\frac{\partial^j f^{(0)}(x, y, \tau)}{\partial \tau^j} \approx \sum_{n=-\infty}^{\infty} f_n^{(0)}(x, y) \frac{d^i \phi(\tau - n)}{d\tau^i}, \quad i, j = 1, 2, \ldots
\] (13)

Applying Equations (12) and (13) to Equation (11) results in inner products of a time-derivative of the scaling function and the scaling function itself. Such a product is formally known as a ‘connection coefficient’ \( \Gamma_{m-n}^{j} \) [6], and has the form

\[
\Gamma_{m-n}^{j} = \int_{-\infty}^{\infty} \frac{d^j \phi(\tau - n)}{d\tau^j} \phi(\tau - m) d\tau, \quad j = 0, 1, 2, \ldots
\] (14)

For the zeroth order time derivative \( (j = 0) \), the connection coefficient turns into the Kronecker delta symbol, which immediately follows from the orthonormality condition Equation (9). Since Daubechies wavelets are compactly supported, the connection coefficients \( \Gamma_{m-n}^{j} \) are only non-zero for a specific, finite range of indices \( m, n \in [-N + 2, L + N - 3] \) [9, 25], with \( L \) being the length of the discretised time window and \( N \) being the order of Daubechies wavelet. For this reason, in the subsequent expressions the summation from \( n = -\infty \) to \( n = \infty \) is replaced by a summation over a finite number \( n \), where the actual value corresponds to the non-zero values of the connection coefficients.

The approximation coefficients corresponding to index values \( n \) located outside the domain of interest \( [0, L - 1] \) may furnish the wavelet approximation with artificial discrepancies at the bounds of the domain of interest, which are known as artificial edge effects [22]. In order to alleviate these edge effects, instead of performing the time integration for the determination of the connection coefficients...
over an unbounded time domain $\tau \in (-\infty, \infty)$, see Equation (14), in the present study, the time integration is carried out over a finite time window $\tau \in [0, L - 1]$. This has the advantage that temporal edge effects related to response contributions from after the time period of interest become automatically zero, since the connection coefficients become zero for $\tau > L - 1$ [7]. Additionally, temporal edge effects related to response contributions from before the time period of interest (i.e. $\tau < 0$) become zero only when the initial conditions are taken as zero. Since a wide range of practical wave propagation problems is characterised by zero initial conditions, this assumption will be made in the present study. It is emphasised that the current strategy of alleviating edge effects is straightforward, and differs from other, more elaborative approaches where wavelet extrapolation schemes are established; see [20,22,25]. Performing the computation of the connection coefficients over a finite time window of length $L$ turns Equation (14) into

$$
\Gamma_{m-n}^j = \int_0^{L-1} d\tau \frac{d\varphi(\tau - n)}{d\tau} \varphi(\tau - m) d\tau, \quad j = 1, 2, \ldots.
$$

Because the integration is done over a finite time interval, in contrast to Equation (14) for the zeroth time derivative ($j = 0$) the connection coefficient will not be equal to the Kronecker delta. Moreover, in order to warrant the accuracy of $\Gamma_{m-n}^j$, the width $L$ of the time window should be sufficiently large to avoid aliasing. This requirement is indeed satisfied for the case studies considered in this communication, where the specific choice of $L$ is set by the highest frequency taken into account for reaching satisfactory response accuracy.

The integration of Equation (15) is carried out using the method proposed in [7,8] for solving Burgers and population balance equations. Incorporating the computed connection coefficients in the present wavelet-Galerkin formulation leads to a coupled system of $D$ $(\text{spatial})$ differential equations (with $D$ representing the spatial dimension of the problem), where the time dependency is accounted for in a discretised sense.

In order to explain the time discretisation of the system of coupled equations in more detail, the above wavelet-Galerkin approach is applied to the case of the in-plane dynamic response of an orthotropic layer. Assuming a layer of finite thickness in the $z$-direction and adapting the plane-stress assumption, the equations of motion in the $x$- and $y$-directions of the layer (which are assumed to coincide with the principal axes of orthotropy) can be deduced from the general three-dimensional case presented in [26] as

$$
\begin{align*}
Q_{11} \frac{\partial^2 u}{\partial x^2} + (Q_{12} + Q_{66}) \frac{\partial^2 v}{\partial x \partial y} + Q_{66} \frac{\partial^2 u}{\partial y^2} &= \rho \frac{\partial^2 u}{\partial t^2}, \\
Q_{22} \frac{\partial^2 v}{\partial y^2} + (Q_{21} + Q_{66}) \frac{\partial^2 u}{\partial x \partial y} + Q_{66} \frac{\partial^2 v}{\partial x^2} &= \rho \frac{\partial^2 v}{\partial t^2},
\end{align*}
$$

(16)

where $\rho$ is the material density, $u$ and $v$ are the displacements in $x$- and $y$-directions, respectively, and $t$ is the (real) time. Furthermore, $Q_{ij}$ are the in-plane stiffness components, with the indices $i$ and $j$ denoting, respectively, the row and column of the general $6 \times 6$ stiffness matrix for a 3D-configuration. Note that, in accordance with Voigt’s notation, the stress and strain tensor components ‘11, 22, 33, 23, 13, 12’
correspond to the rows (and columns) 1–6 of the stiffness matrix, respectively. Under plane-stress conditions, the (non-zero) stiffness components are

\[
Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \\
Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad \left(= Q_{21} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \right), \\
Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \\
Q_{66} = \mu_{12}.
\]

(17)

From Equation (17) it follows that symmetry of the stiffness matrix requires that \(\nu_{12}E_2 = \nu_{21}E_1\), which essentially leaves the two-dimensional orthotropic model with four independent stiffness parameters, i.e. the axial stiffnesses \(E_1, E_2\), the Poisson’s ratio \(\nu_{12}\), and the shear modulus \(\mu_{12}\). Furthermore, since the layer thickness in the \(z\)-direction does not influence wave propagation in the \(x-y\) plane of the layer, this parameter does not appear in Equations (16) and (17). The time discretisation of the problem is performed by connecting the real time \(t\) to the dimensionless time \(\tau\) via [25]

\[t = \tau \Delta t \quad \text{with} \quad \tau \in [0, L - 1],\]

(18)

by dividing the time window of interest into \(L - 1\) time intervals \(\Delta t\). Applying the wavelet-Galerkin approximation given by Equation (11) to Equations (14) and (15) to the equations of motion, Equation (16), and invoking Equation (18), results in

\[
\sum_{n=0}^{L-1} \Gamma_{m-n}^{0} \left( Q_{11} \frac{\partial^2 u_n}{\partial x^2} + (Q_{12} + Q_{66}) \frac{\partial^2 v_n}{\partial x \partial y} + Q_{66} \frac{\partial^2 u_n}{\partial y^2} \right) = \sum_{n=0}^{L-1} \left[ \Delta t^2 \rho \Gamma_{m-n}^{0} \right] u_n \\
\sum_{n=0}^{L-1} \Gamma_{m-n}^{0} \left( Q_{22} \frac{\partial^2 v_n}{\partial y^2} + (Q_{21} + Q_{66}) \frac{\partial^2 u_n}{\partial x \partial y} + Q_{66} \frac{\partial^2 v_n}{\partial x^2} \right) = \sum_{n=0}^{L-1} \left[ \Delta t^2 \rho \Gamma_{m-n}^{0} \right] v_n
\]

(19)

with \(m = 0, 1, \ldots, L - 1\), where \(u_n = u_n(x, y)\) and \(v_n = v_n(x, y)\) are the approximation coefficients following from the application of Equation (12). Expressing these \(2 \times L\) equations in matrix-vector form, after some minor rearrangement, leads to

\[
Q_{11} \frac{\partial^2 \mathbf{u}}{\partial x^2} + (Q_{12} + Q_{66}) \frac{\partial^2 \mathbf{v}}{\partial x \partial y} + Q_{66} \frac{\partial^2 \mathbf{u}}{\partial y^2} = \Delta t^2 \rho \Gamma_{0}^{-1} \Gamma_{2} \mathbf{u} = \mathbf{0}, \\
Q_{22} \frac{\partial^2 \mathbf{v}}{\partial y^2} + (Q_{21} + Q_{66}) \frac{\partial^2 \mathbf{u}}{\partial x \partial y} + Q_{66} \frac{\partial^2 \mathbf{v}}{\partial x^2} = \Delta t^2 \rho \Gamma_{0}^{-1} \Gamma_{2} \mathbf{v} = \mathbf{0}
\]

(20)

where the matrices with the connection coefficients are given by

\[
\Gamma_{0}[m, n] = \Gamma_{m-n}^{0}, \quad m, n \in [0, 1, \ldots, L - 1], \\
\Gamma_{2}[m, n] = \Gamma_{m-n}^{2}, \quad m, n \in [0, 1, \ldots, L - 1],
\]

(21)
and the displacement vectors are $\mathbf{u}[n] = u_n$ and $\mathbf{v}[n] = v_n$. Computing the orthonormal eigenvectors of the matrix (product) $\Gamma_0^{-1}\Gamma_2$ appearing in Equation (20) and assembling these column-wise in the matrix $\Phi$, the matrix $\Gamma_0^{-1}\Gamma_2$ can be diagonalised into the matrix $D$ as

$$D = \Phi^{-1}(\Gamma_0^{-1}\Gamma_2)\Phi,$$

(22)

with each diagonal element of $D$ reflecting an eigenvalue of the matrix $\Gamma_0^{-1}\Gamma_2$. From here on, these (dimensionless) eigenvalues are scaled with the time interval $\Delta t$ as

$$\kappa_n^2 = \frac{D_{nn}}{\Delta t^2} \quad \text{with} \quad n \in [0, 1, \ldots, L - 1],$$

(23)

where it is emphasised that the ‘frequency’ $\kappa_n$ is analogous to the eigenfrequency appearing in a modal analysis of vibration problems. Further, as can be noted from Equation (23), $n$ formally runs from $n = 0$ to $n = L - 1$, where the lowest eigenvalue relates to $n = 0$ and the highest eigenvalue to $n = L - 1$. Note from Equations (21) and (22) that the connection coefficients $\Gamma_0, \Gamma_2$, the matrix $\Phi$, and the matrix inverses $\Gamma_0^{-1}, \Phi^{-1}$, need to be recalculated each time the number of sampling points $L$ changes during a numerical simulation. These are computationally demanding operations if the number of sampling points is large, which may be perceived as a disadvantage of the method. However, for a fixed number of sampling points these operations can be minimised by computing the above matrices and their inverses only once, and storing these for further use in the computational algorithm.

Now, employing the projections

$$\mathbf{\hat{u}} = \Phi^{-1}\mathbf{u},$$

$$\mathbf{\hat{v}} = \Phi^{-1}\mathbf{v},$$

(24)

in Equation (20), and multiplying the result by the inverted matrix $\Phi^{-1}$, together with Equations (22) and (23) leads to a set of $2 \times L$ uncoupled, spatial differential equations

$$\begin{align*}
Q_{11} \frac{\partial^2 \mathbf{\hat{u}}_n}{\partial x^2} + (Q_{12} + Q_{66}) \frac{\partial^2 \mathbf{\hat{v}}_n}{\partial x \partial y} + Q_{66} \frac{\partial^2 \mathbf{\hat{v}}_n}{\partial y^2} - \rho \kappa_n^2 \mathbf{\hat{u}}_n &= 0 \\
Q_{22} \frac{\partial^2 \mathbf{\hat{v}}_n}{\partial y^2} + (Q_{21} + Q_{66}) \frac{\partial^2 \mathbf{\hat{u}}_n}{\partial x \partial y} + Q_{66} \frac{\partial^2 \mathbf{\hat{u}}_n}{\partial x^2} - \rho \kappa_n^2 \mathbf{\hat{v}}_n &= 0
\end{align*}$$

(25)

with $n = 0, 1, \ldots, L - 1$.

The decoupling of the differential equations allows these to be solved independently at each wavelet point $n$, permitting a parallel implementation of the time discretisation procedure. The spatial discretisation of Equations (25) is performed fully with the finite element method (FEM). This makes the present approach different from earlier wavelet-based solution procedures for 2D problems [20,21], in which, in addition to a wavelet transformation used for the discretisation of the time coordinate, a second wavelet transformation was applied for discretising one spatial coordinate, and subsequently FEM was employed for discretising the resulting ordinary differential equations in terms of the other spatial coordinate. As mentioned in the introduction, the advantage of discretising Equation (25) fully...
with FEM (in this case a 2D FEM) is that the necessity for specifying requirements on the type of boundary conditions and geometry is circumvented.

4. Spatial discretisation with the finite element method

For solving wave propagation problems in the wavelet domain, it is a natural choice to perform the spatial discretisation of Equation (25) by choosing the basis of the displacement fields in accordance with a Fourier series, i.e.

\[
\hat{u}_n(x, y) = \sum_p \sum_q C_{pq}^n \exp\left(i (k_{x,p} x + k_{y,q} y)\right)
\]

\[
\hat{v}_n(x, y) = \sum_p \sum_q D_{pq}^n \exp\left(i (k_{x,p} x + k_{y,q} y)\right)
\]

with \( n = 0, 1, \ldots, L - 1 \),

where \( C_{pq}^n \) and \( D_{pq}^n \) are the Fourier coefficients and \( p \) and \( q \) are integers. The constants \( k_{x,p}^n \) and \( k_{y,q}^n \) can be determined in accordance with the characteristics of specific waves propagating through the structure. In the present study, these constants, which are related to wavenumbers (as will be shown below), are derived from the characteristics of the compressional (P) and vertically polarised shear (SV) waves propagating along the principal axes of orthotropy, i.e. the \( x \)- and \( y \)-directions. For plane P and SV waves propagating in the \( x \)-direction, the wavelength in \( y \)-direction is infinitely large, which relates to a wavenumber equal to zero. This condition is in agreement with

\[
k_{y,0}^n = 0.
\]

The corresponding wavenumbers in the \( x \)-direction can be computed by substituting Equation (27) into Equation (26), followed by inserting the result into Equation (25) and calculating the non-trivial solutions. The expression in terms of the displacement \( \hat{u}_n \) (which is associated to the P wave in \( x \)-direction) then provides

\[
k_{x,1}^n = i k_{x,P}^n = i \sqrt{\frac{\rho C_{11}^2}{Q_{11}}}, \quad k_{x,2}^n = -i k_{x,P}^n,
\]

while for the expression in terms of the displacement \( \hat{v}_n \) (which is associated to the SV wave in \( x \)-direction) it is found that

\[
k_{x,3}^n = i k_{x,SV}^n = i \sqrt{\frac{\rho C_{66}^2}{Q_{66}}}, \quad k_{x,4}^n = -i k_{x,SV}^n.
\]

In Equations (28) and (29), the parameters \( k_{x,P}^n \) and \( k_{x,SV}^n \) have the dimension length\(^{-1}\), and may be interpreted as ‘wavenumbers’ of the P wave and SV wave in \( x \)-direction, respectively. These wavenumbers are generally complex-valued, due to the fact that the eigenvalues \( \lambda_n \) used for the orthonormal decomposition of the governing equations in the wavelet domain are complex-valued.
The propagation of P and SV waves in the y-direction corresponds to a wavenumber in the x-direction equal to zero, which is in agreement with the condition

$$k_{x,0}^n = 0. \tag{30}$$

Combining Equation (30) with Equation (26), followed by inserting the result into Equation (25), leads to the following non-trivial solution for the expression in terms of the displacement $\hat{v}_n$ (which is associated to the P-wave in the y-direction):

$$k_{y,1}^n = ik_{y,P}^n = i\sqrt{\frac{\rho k_{n}^2}{Q_{22}}} k_{y,2}^n = -ik_{y,P}^n. \tag{31}$$

In addition, from the expression in terms of the displacement $\hat{u}_n$ (which is associated to the SV wave in y-direction) it can be derived that

$$k_{y,3}^n = ik_{y,SV}^n = i\sqrt{\frac{\rho k_{n}^2}{Q_{66}}} k_{y,4}^n = -ik_{y,SV}^n. \tag{32}$$

Note from Equations (29) and (32) that the wavenumbers $k_{x,SV}^n$ and $k_{y,SV}^n$, characterising the propagation of the SV wave in, respectively, the x- and y-directions, are equal. However, from Equations (28) and (31) it may be observed that the wavenumbers $k_{x,P}^n$ and $k_{y,P}^n$, characterising the propagation of the P wave in x- and y-directions, are different. This difference indeed can be ascribed to the orthotropic material characteristics of the layer. With the result of Equations (27)–(32), the summations over the integers $p$ and $q$ in Equation (26) specify into

$$\hat{u}_n(x,y) = \sum_{p=0,1,2} \sum_{q=0,3,4} C_{pq}^n \exp\left(i\left(k_{x,P}^n x + k_{y,SV}^n y\right)\right)$$

$$\hat{v}_n(x,y) = \sum_{p=0,3,4} \sum_{q=0,1,2} D_{pq}^n \exp\left(i\left(k_{x,P}^n x + k_{y,SV}^n y\right)\right) \tag{33}$$

with $n = 0, 1, \ldots, L - 1$,

which can be elaborated as

$$\begin{bmatrix} \hat{u}_n(x,y) \\ \hat{v}_n(x,y) \end{bmatrix} = \begin{bmatrix} C_1^n + C_2^n e^{-k_{x,SV}^n x} + C_3^n e^{-k_{y,SV}^n y} + C_4^n e^{-k_{y,SV}^n y} + C_5^n e^{-k_{y,SV}^n y} + C_6^n e^{-k_{x,SV}^n x} e^{-k_{y,SV}^n y} + C_7^n e^{-k_{y,SV}^n y} e^{-k_{y,SV}^n y} + C_8^n e^{-k_{y,SV}^n y} e^{-k_{y,SV}^n y} + C_9^n e^{-k_{y,SV}^n y} e^{-k_{y,SV}^n y} \\ D_1^n + D_2^n e^{-k_{x,SV}^n x} + D_3^n e^{-k_{y,SV}^n y} + D_4^n e^{-k_{y,SV}^n y} + D_5^n e^{-k_{y,SV}^n y} + D_6^n e^{-k_{x,SV}^n x} e^{-k_{y,SV}^n y} + D_7^n e^{-k_{x,SV}^n x} e^{-k_{y,SV}^n y} + D_8^n e^{-k_{x,SV}^n x} e^{-k_{y,SV}^n y} + D_9^n e^{-k_{x,SV}^n x} e^{-k_{y,SV}^n y} \end{bmatrix}^T. \tag{34}$$

Note that, in the resulting expression Equation (34), the wavenumbers of the P and SV waves have been substituted, in accordance with Equations (28)–(32).
Furthermore, for notational convenience the Fourier coefficients $C_{pq}^n$ and $D_{pq}^n$ are designated here as $C_{pq}^k$ and $D_{pq}^k$, respectively, with $k \in \{1, 2, \ldots, 9\}$. When storing these $9 \times 2 = 18$ coefficients in a vector $\mathbf{c}_n = [C_{11}^n, C_{12}^n, \ldots, C_{93}^n; D_{11}^n, D_{12}^n, \ldots, D_{93}^n]^T$, Equation (34) can be expressed in matrix-vector form as

$$\mathbf{\hat{u}}_n = \mathbf{P}_n \mathbf{c}_n \quad \text{with} \quad n = 0, 1, \ldots, L - 1,$$

where the vector $\mathbf{\hat{u}}_n$ contains the two displacement functions $\hat{u}_n(x, y)$ and $\hat{v}_n(x, y)$, and the $2 \times 18$ matrix $\mathbf{P}_n$ contains the exponential functions, their products, and two times unity, in correspondence with Equation (34). In order to solve for the coefficients $\mathbf{c}_n$ within a FEM-setting, a $9$-noded, $C_0$-continuous, rectangular finite element is constructed that has two-degrees of freedom per node. The nodes are located at the four corners of the element, in the middle of each of the four element edges and at the element centre. Subsequently, a standard FEM-approach is followed, where the nodal coordinates $x_j, y_j$ of each element node $j \in \{1, 2, \ldots, 9\}$ are substituted into Equation (35), providing the displacements at the nodes. The nodal displacements of the element may be stored in a vector $\mathbf{\hat{a}}^e_n$, in accordance with

$$\mathbf{\hat{a}}^e_n = \begin{bmatrix} \mathbf{P}_n |_{x=x_1, y=y_1} \\ \mathbf{P}_n |_{x=x_2, y=y_2} \\ \vdots \\ \mathbf{P}_n |_{x=x_9, y=y_9} \end{bmatrix} \mathbf{c}_n = \mathbf{T}_n \mathbf{c}_n.$$  

(36)

As a next step, the $18 \times 18$ matrix $\mathbf{T}_n$ is inverted to solve for the 18 unknown coefficients, i.e. $\mathbf{c}_n = \mathbf{T}_n^{-1} \mathbf{\hat{a}}^e_n$. Substituting this solution back into Equation (35) results in

$$\mathbf{\hat{u}}_n = \mathbf{P}_n \mathbf{T}_n^{-1} \mathbf{\hat{a}}^e_n = \mathbf{N}_n \mathbf{\hat{a}}^e_n,$$

(37)

where the $2 \times 18$ matrix $\mathbf{N}_n = \mathbf{P}_n \mathbf{T}_n^{-1}$ characterises the (spectral) shape functions of the element, which need to be computed for each wavelet point $n$ separately. Some characteristics of the shape functions are illustrated in Figure 1, using a discretised interval of $L = 150$ wavelet points, where the real and imaginary parts of shape function $N_n$ for a corner node (with local coordinates $x = 0$, $y = 0$) of the 9-noded rectangular element are sketched over the element surface ($= 0.03 \times 0.03 \text{ m}^2$). The wavelet points selected are the initial ($n = 0$, see Figure 1a) and final ($n = 149$, see Figure 1b) wavelet points of the interval $\tau \in [0, L - 1]$ considered.

Figure 1 confirms that (i) the interpolation condition (unity value for the real part $\Re(N_n)$ at the corner node, zero value at all other nodes), and (ii) the local support condition (the real and imaginary parts of the shape function vanish across the element boundaries that do not include the corner node with the unit value) are satisfied. Since the shape functions are $C_0$-continuous, the inter-element compatibility condition is also met. The presence of the imaginary part of a shape function is inherent to a FEM formulation in the wavelet domain (which indeed is characterised by complex-valued parameters, such as the frequency $\kappa_n$ and the wavenumbers presented in Equations (28)–(32)). Figure 1 indicates that the shape function related to the lowest wavelet point $n = 0$ represents relatively long waves (of relatively low frequency), and the shape function related to the highest wavelet point, $n = 149$, ...
represents relatively short waves (of relatively high frequency). Essentially, the convention is such that the wavelength corresponding to the shape function $N_n$ decreases with increasing value of the wavelet point $n$.

The matrix $N_n$ with the element shape functions can be used in the usual way to construct the element stiffness matrix $K_e^n$ and the element mass matrix $M_e^n$ as

$$K_e^n = \int_V (LN_n)^T D (LN_n) dV, \quad M_e^n = \int_V N_n^T R N_n dV,$$

in which the $3 \times 3$ matrix $D$ contains the stiffness moduli of the orthotropic material (as presented in Equation (25)), the $3 \times 2$ matrix $L$ includes the differential operators required for the computation of the strains and the $2 \times 2$ matrix $R$ contains the material density $\rho$, i.e.

$$D = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}, \quad L = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \quad R = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}.$$

![Figure 1. Real part $\Re(N_n)$ (left) and imaginary part $\Im(N_n)$ (right) of a FEM shape function $N_n$ for two different wavelet points, plotted over the surface ($=0.03 \times 0.03 \text{ m}^2$) of a single (9-noded) rectangular element. (a) Shape function $N_0$ of an element corner node (located at $x=0$, $y=0$), corresponding to the initial wavelet point $n=0$. (b) Shape function $N_{149}$ of an element corner node (located at $x=0$, $y=0$), corresponding to the final wavelet point $n=149$ ($=L-1$).](image-url)
In Equation (38), the integration over the element volume required for computing $K_n^e$ and $M_n^e$ can be performed analytically, due to the exponential nature of the shape functions $N_n$.

Following a weak formulation of the governing equations of the structural problem modelled, the system of equations that needs to be solved within a FEM-setting is derived in the usual way, leading to

$$\hat{K}_n \hat{a}_n = \hat{f}_n,$$

where the dynamic stiffness matrix of the structure is given by

$$\hat{K}_n = K_n + \kappa_n^2 M_n.$$  \hfill (41)

The mass matrix $M_n$ and the stiffness matrix $K_n$ in the above expressions are constructed by assembling the element mass and stiffness matrices presented in Equation (38), while the ‘frequency’ $\kappa_n$ follows from Equation (23). Further, the nodal displacement vector $\hat{a}_n$ is constructed from the element displacements presented in Equation (36), and the nodal force vector $\hat{f}_n$ accounts for the traction boundary conditions applied along the surface $S$ of the system via

$$\hat{f}_n = \int_S N_n^T t_n \, dS,$$  \hfill (42)

where, for simplicity, the body forces are omitted. In Equation (42), the tractions in the wavelet domain, $\hat{t}_n$, are computed from the corresponding approximation coefficients, $t_n$, as

$$\begin{bmatrix} \hat{t}_{n=0} \\ \hat{t}_{n=1} \\ \vdots \\ \hat{t}_{n=L-1} \end{bmatrix} = \Phi^{-1} \begin{bmatrix} t_{n=0} \\ t_{n=1} \\ \vdots \\ t_{n=L-1} \end{bmatrix}.$$  \hfill (43)

This expression essentially represents a similar operation as for the displacement fields in Equation (24). The approximation coefficients $t_n = t_n(x, y)$ appearing in Equation (43) are obtained component-wise from the actual, time-dependent traction $t = t(x, y, t)$ using the inverse relation of Equation (12), i.e. through performing an inverse discrete wavelet transform. Note that in Equation (43) the traction vector for an individual wavelet point $n$ is stored row-wise in the corresponding matrices.

After inverting Equation (40) to solve for the nodal displacements $\hat{a}_n$ in the wavelet domain, the approximation coefficients of the nodal displacements $a_n$ can be obtained via

$$\begin{bmatrix} a_{n=0} \\ a_{n=1} \\ \vdots \\ a_{n=L-1} \end{bmatrix} = \Phi \begin{bmatrix} \hat{a}_{n=0} \\ \hat{a}_{n=1} \\ \vdots \\ \hat{a}_{n=L-1} \end{bmatrix}.$$  \hfill (44)
Finally, from the approximation coefficients $a_n$ the time-dependent nodal displacement vector $\mathbf{a}$ is computed with the use of Equation (12).

It should be mentioned that dissipation effects (which, for simplicity reasons, have been omitted in the present analysis) can be included in a straightforward manner in the formulation by extending Equation (41) with a term $\kappa_n C_n$. Here, the matrix $C_n$ contains the characteristics of the linear damping model used. This matrix is constructed in the same fashion as the stiffness matrix $M_n$; see Equations (38) and (39). The implemented WSFEM is presented schematically in Figure 2. The sketch in the right side of the figure illustrates the decoupled FEM solution procedure in the wavelet domain.

5. Numerical results

In order to evaluate the performance of the 9-noded element presented in Section 4, first a benchmark problem with a well-known solution is studied. The benchmark problem relates to the propagation of waves in an infinitely long (in $x$-direction), homogeneous, isotropic layer of thickness $H$ (in $y$-direction) that is subjected to plane-strain conditions in the $z$-direction. The layer is subjected to traction-free boundary conditions at the top ($y = H/2$) and bottom ($y = -H/2$) surfaces. The waves propagating under these conditions are commonly referred to as ‘Lamb waves’, and can be distinguished in axial waves (i.e. the symmetric modes) and flexural waves (i.e., the antisymmetric modes), see, e.g., [27,28].

The equations of motion, Equation (16), which hold for an anisotropic layer under plane-stress conditions, can be adapted to the case of an isotropic layer under plane-strain conditions by expressing the stiffness coefficients in Equation (16) as:

$$
Q_{12} = Q_{21} = \lambda, \\
Q_{66} = \mu, \\
Q_{11} = Q_{22} = \lambda + 2\mu,
$$

(45)
where the Lamé constants \( \lambda \) and \( \mu \) may be formulated in terms the Young’s modulus \( E \) and the Poisson’s ratio \( \nu \) in the usual way, \( \lambda = \nu E / ((1 + \nu)(1 - 2\nu)) \) and \( \mu = E / (2(1 + \nu)) \). The benchmark problem is solved by choosing the stiffness parameters and density in correspondence with the values for aluminium, i.e., \( E = 70 \) GPa, \( \nu = 0.3 \) and \( \rho = 2700 \) kg/m\(^3\). The dispersion curves are obtained in the standard way, by substituting the harmonic plane wave solution for the displacements \( u \) and \( v \), given by

\[
\begin{align*}
    u(x, y, t) &= \hat{u} \exp(i((\omega t - k_x x - k_y y)), \\
    v(x, y, t) &= \hat{v} \exp(i((\omega t - k_x x - k_y y)),
\end{align*}
\]

into the equations of motion, Equation (16), and numerically computing the non-trivial solution (known as the Rayleigh–Lamb solution) from the roots of the transcendental equation obtained [27,28]. In Equation (46), \( \hat{u} \) and \( \hat{v} \) are the wave amplitudes, \( k_x \) and \( k_y \) are the wavenumbers in \( x \)- and \( y \)-directions, and \( \omega \) is the angular frequency. The Rayleigh–Lamb solution is compared against the dispersion curves computed numerically with the formulation presented in Section 4. In the numerical simulation, a layer of 2 m length and 0.05 m thickness is modelled, which is discretised by 640 9-noded rectangular elements (i.e. four elements across the thickness of the layer and 160 elements across the length). The layer is subjected at its upper left corner to an harmonic load with unit amplitude. The frequency \( f \) of the load is increased stepwise from 2 to 80 kHz, using increments of 2 kHz. For each frequency, the nodal displacements of the layer are computed by solving a system of equations similar to Equation (40), with the frequency parameter \( k_n \) in Equation (41) being replaced by \( i\omega = 2\pi if \). Subsequently, the nodal displacements along the upper boundary of the layer (i.e., at \( y = H/2 \)) are subjected to a fast Fourier transform (FFT), which provides the wavenumbers \( k_x \) characterising the response at the frequency \( f \) considered.

The dispersion curves for the five lowest modes are presented in Figure 3, by plotting the frequency \( f \) versus the wavenumber \( k_x \) in the \( x \)-direction. It may be concluded that the Rayleigh–Lamb solution and the numerical solution are in excellent agreement. The numerical dispersion curves presented in Figure 3 are directly computed in the frequency domain, i.e., a computation of the layer response in time is not required. Hence, the comparison only reveals that the spatial discretisation of the numerical formulation is adequate and precise.

In order to illustrate time discretisation aspects of the WSFEM formulation, a second example is considered in which a 300 x 300 x 1 mm\(^3\) homogeneous, isotropic aluminium plate with \( \rho = 2700 \) kg/m\(^3\), \( E(= E_1 = E_2) = 70 \) GPa and \( \nu(= \nu_{12} = \nu_{21}) = 0.3 \) (which corresponds to a shear modulus of \( \mu = E / (2(1 + \nu)) = 27 \) GPa) is subjected to an in-plane impact load as shown in Figure 4. The load has a unit amplitude (in kN) and a frequency content of 0–40 kHz (as can be observed from the FFT of the load signal \( F(t) \) depicted in the inset of Figure 4b). The load is applied in the vertical (\( y \)) direction at the middle of the lower plate edge. At the two corners of the upper edge the plate is supported in \( x \)- and \( y \)-directions.

In Figure 5 the displacement \( v \) in the \( y \)-direction, evaluated at the centre of the plate (point C), is depicted as a function of time. The figure shows four sets of results, corresponding to (i) an accurate reference solution computed with FEM, (ii) a FEM
computation (FEM-Q) that uses 9-noded quadrilateral elements with quadratic shape functions, (iii) a FEM computation (FEM-L) that uses 4-noded quadrilateral elements with linear shape functions, and (iv) a computation performed with the present WSFEM approach. The time integration of the FEM reference solution (i) and the other two FEM solutions (ii) and (iii) is performed with a second-order central difference scheme with Newmark time-integration parameters equal to $\beta=0$ and $\gamma=0.5$, see Hughes (1987).

The order of the Daubechies wavelet used in the WSFEM approach is $N=6$. The number of nodes used for computations (ii), (iii) and (iv) are the same, namely 289.
The time increment for the FEM computations (ii) and (iii) equals 0.1 \( \text{ms} \), which warrants that the central difference scheme remains stable. The WSFEM formulation is unconditionally stable, so for this method the choice of the time increment is not critical from the stability point of view. From the accuracy point of view, preliminary WSFEM simulations have shown that for the current boundary value problem a time increment of \( \Delta t = 3.0 \text{\mu s} \) provides a result that virtually does not change in accuracy when a smaller time interval is used. Hence, the time interval \( \Delta t = 3.0 \text{\mu s} \) causes the sampling density \( L \) in the wavelet domain apparently to be sufficiently high to accurately capture the response characteristics generated by the loading specified in Figure 4. Note that, for a given time window \( t \) of interest, the connection between \( L \) and \( \Delta t \) is explicitly set through Equation (18).

For generating a FEM reference solution (constructed with 9-noded quadrilateral elements) with a high spatial accuracy, the total number of nodes is taken considerably larger than for the other three solutions, namely 58,081 nodes (corresponding to \( 120 \times 120 = 14,400 \) elements), while the discrete time step is taken significantly smaller, i.e. \( \Delta t = 0.005 \text{\mu s} \). The time window used for all simulations is \( t \in [0, 150 \text{\mu s}] \). It can be observed from Figure 5 that the calculated responses look qualitatively similar, and that the WSFEM solution provides the best approximation to the reference solution. The accuracy of the FEM-Q solution is slightly less than that of the WSFEM solution, while the FEM-L is the least accurate, especially during the final part of the response. Although not illustrated here, it has been confirmed that all solutions become virtually identical under further mesh refinement.

In order to quantitatively compare the convergence behaviour of the WSFEM, FEM-L, and FEM-Q solutions under mesh refinement, the ‘root mean square

\[
\begin{align*}
\text{displacement } v_\tau (\text{\mu m}) & \\
\text{time } t (\mu s) & \\
\text{Reference Solution} & \\
\text{WSFEM} & \\
\text{FEM−Q} & \\
\text{FEM−L} & \\
\end{align*}
\]

Figure 5. Vertical displacement \( v \) at the center of the plate (point C in Figure 4). The WSFEM and FEM-Q and FEM-L solutions constructed with meshes of 289 nodes are compared against an accurate FEM reference solution.
The RMS error of the vertical displacement $v_C = v_C(t)$ at the centre of the plate (point C) is computed as

$$\text{RMS}(v_C) = \left( \frac{\int_0^{t_{\text{end}}} [v_C(t) - v_{C,\text{ref}}(t)]^2 \, dt}{\int_0^{t_{\text{end}}} [v_{C,\text{ref}}(t)]^2 \, dt} \right)^{\frac{1}{2}}$$  \hspace{1cm} (47)$$

with $v_{C,\text{ref}}$ the vertical displacement related to the FEM reference solution and $t_{\text{end}} = 150 \, \mu s$. The integral in Equation (47) is computed by means of direct numerical integration. The RMS error is depicted in Figure 6, illustrating that the mesh convergence rate is similar for the three solutions, but that the WSFEM solution has the smallest error for all spatial discretisations considered, followed by the FEM-Q solution and the FEM-L solution. This result is consistent with the result plotted in Figure 5.

In addition to the RMS error, the convergence of the residual error in the strong form of the governing equations may be examined. The residual error is calculated by taking the Euclidian norm of the residuals of the equations of motion, Equations (16)\textsubscript{1,2}, evaluated at the plate centre (i.e., point C, see Figure 4a). The convergence rate of this residual error turns out to be similar for the WSFEM and FEM-Q computations, in a sense that it decreases with a factor 5 to 6 when the number of elements is increased by a factor of 25. Conversely, for the FEM-L simulation the decrease in the residual error is negligible for this increase in mesh refinement.

Finally, the time evolution of the displacement $v$ in $y$-direction is considered over the whole plate using the WSFEM solution for the isotropic plate discretised with 625 nodes (which corresponds to $12 \times 12 = 144$ 9-noded elements); see Figure 7a. The effect of anisotropy is illustrated by comparing the response to that of an orthotropic plate, as depicted in Figure 7b. For the orthotropic plate the stiffness in $x$-direction is taken 25% higher than for the isotropic, aluminium plate, i.e., $E_1 = 87.5 \, \text{GPa}$, while in $y$-direction the stiffness is 25% lower, $E_2 = 52.5 \, \text{GPa}$. 
Figure 7. Contour plots of the displacement $v$ in the $y$-direction at four different time instants for four different cases. (a) Isotropic plate. (b) Orthotropic plate. (c) Composite plate made of isotropic (lower part of plate) and orthotropic (upper part of plate) materials. The material interface is indicated by the black line. (d) Composite plate as in (c), with a cut-out ($50 \times 50 \times 1 \text{mm}^3$) generated at the lower right plate corner. The plate geometry and loading characteristics are presented in Figure 4.
The value of the Poisson’s ratio $\nu_{21} = 0.25$ (which, due to symmetry of the stiffness tensor, corresponds to $\nu_{12} = \nu_{21} E_1/E_2 = 0.42$; see Equation (17)), and the shear modulus is $\mu_{12} = 35$ GPa. In Figure 7c the analysis has been further extended by considering the response of a composite plate consisting of the above-mentioned isotropic material (at the lower part of the plate) and the orthotropic material (at the upper part of the plate). As indicated by the black line, an interface (perfect bonding) with a rectangular tooth geometry (tooth length = tooth width = 0.1 m) connects the two materials in the middle of the plate. The effect of an irregular plate boundary on the response characteristics is examined in Figure 7d by providing the composite plate of Figure 7c with a $50 \times 50 \times 1$ mm$^3$ cut-out at its lower right corner.

The contour plots in Figure 7 illustrate that at $t = 36\mu s$ only part of the load signal has been applied, resulting into a compressional (P) wave propagating along the trajectory $x = 0.15$ m, a shear (S) wave propagating along the trajectory $y = 0$ m, and a combination of these two waves propagating along other trajectories. At $t = 59\mu s$, the front of the P wave has passed the plate centre, where a comparison between Figures 7a and b shows that the wave front for the orthotropic case is more straight than for the isotropic case due to the different stiffness parameters in $x$- and $y$-directions. From the location of the P wave front along $x = 0.15$ m it may be concluded that the P wave velocity in the $y$-direction in the orthotropic plate is lower than in the isotropic plate, which can be ascribed to the lower stiffness of the orthotropic material in $y$-direction. Figures 7a and b further indicate that at $t = 84\mu s$ the fronts of P and S waves have reached the plate boundaries, which has led to interference between incident waves and waves reflected at the boundaries. At $t = 108\mu s$, the tails of incident P and S waves have also reached the plate boundaries, which has further intensified the interference pattern of incident and reflected waves.

From the response of the composite plate in Figure 7c it can be observed that at $t = 84\mu s$ waves propagating from the lower, isotropic part of the plate have reflected (and refracted) at the interface with the orthotropic part, as a result of which the displacement pattern in the isotropic part of the plate has become different from that of the isotropic plate in Figure 7a. Due to additional wave reflections this difference grows with time, as illustrated by the displacement patterns for $t = 108\mu s$ in Figures 7a and c. The effect of the cut-out on the dynamic response is apparent from breakage of the vertical symmetry of the displacement pattern after the P and S waves have reached the left and right plate boundaries, see Figure 7d for $t = 84\mu s$ and $t = 108\mu s$.

The above examples show that the WSFEM method is suitable for simulating elastic wave propagation problems in an accurate and efficient manner. The main strength of the present method is that it combines DWT and FEM in the most optimal way, i.e. DWT is used only for the temporal discretisation of the problem, making the method inherently suitable for parallel computation, and FEM is employed for the spatial discretisation of the problem, as a result of which geometries and boundary conditions of arbitrary complexity can be modelled.

6. Concluding remarks

A WSFEM formulation has been proposed for the analysis of 2D elastic wave propagation problems. The method uses a wavelet-Galerkin approach for
transforming the governing equations from the time domain to the wavelet domain, after which the spatial dependency of the displacement response is solved for in the wavelet domain by means of FEM endowed with exponential shape functions. The final response is obtained by transforming the computed nodal displacements from the wavelet domain back to the time domain. Since the spatial discretisation is performed fully with FEM, there are, in principle, no limitations for extending the present 2D formulation to 3D. This can be done in a similar fashion as for standard FEM, by appropriately including the third spatial ($z$) coordinate in the basis of the displacement field, Equation (26), and deriving the corresponding shape functions of the 3D element in accordance with the procedure in Section 4.

The performance of the present WSFEM formulation is compared to that of a standard FEM approach, showing that the accuracy of WSFEM is similar to that of FEM equipped with quadratic shape functions and a second-order accurate, Newmark-$\beta$ time-integration scheme. In terms of computational efficiency the WSFEM approach offers a significant advantage to FEM equipped with direct time-integration, in a sense that the time-discretised equations can be solved for each wavelet point separately, which naturally enables parallel computation. It is noted that this way of parallelisation differs from approaches based on domain decomposition [29] or an element-by-element formulation [30]. In addition, similar to FEM schemes based on implicit time integration [1], the present WSFEM scheme is unconditionally stable, which (within the required accuracy bounds of the solution) allows for the use of relatively large time intervals in the computation of the response. Another interesting feature of the WSFEM approach is that it allows for straightforwardly combining the spectral basis functions at element level, Equation (37), with h-refinement, such that high-frequency wave propagation signals may be described with high accuracy. The above three aspects make the present WSFEM an appealing candidate for the structural health monitoring of small-scale damage and defects in engineering structures with a large number of degrees of freedom, which is a topic for future research.

References