Ruin probabilities with compounding assets for discrete time finite horizon problems, independent period claim sizes and general premium structure

Citation for published version (APA):

Document status and date:
Published: 01/01/2003

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
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• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Ruin probabilities with compounding assets for discrete time finite horizon problems, independent period claim sizes and general premium structure
Ton G. de Kok
WP 82
Ruin probabilities with compounding assets for
discrete time finite horizon problems, independent
period claim sizes and general premium structure

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Abstract

In this paper we present fast and accurate approximations for the probability of ruin over a
finite number of periods, assuming inhomogeneous independent claim size distributions and
arbitrary premium income in subsequent periods. We develop exact recursive expressions for
the non-ruin probabilities in subsequent periods. These recursive expressions provide the
basis for a computationally efficient recursive approximation scheme based on two-moment
gamma distribution fits. An extensive simulation experiment showed the accuracy of the
approximations for values of the horizon up to 20. Especially for small values of the ruin-
probability the approximations are very accurate. Having shown the validity of the
approximations, we applied them to gain insight into two non-stationary investment problems.
1 Introduction

In this paper we present fast and accurate approximations for the probability of ruin in finite time for a discrete time model with inhomogeneous independent claim size distributions and arbitrary premium income in subsequent periods. This model provides a versatile instrument to consider questions, such as the impact of different premium income policies on the probability of ruin as well as on profitability, and the impact of interest on the initial surplus.

The approximations are based on a recursive approximation scheme. Starting with the first period, we subsequently compute the (non-)ruin probability up to the current period. The approximation scheme only employs information about the first two moments of the claim size distribution in each period. This can be considered both a strength and a weakness: on the one hand historical data about claim size distributions do not allow for much more than accurate computation of the sample mean and sample variance, while on the other hand the ruin probability may well be sensitive to more than the first two moments of the claim size distribution. In this paper we restrict to the two-moment approximation scheme and give some indications how to extend the scheme, such that higher moments can be taken into account.

The literature on discrete time models with inhomogeneous claim size distributions is rare. We refer to De Vylder and Goovaerts (1988) and De Vylder and Marceau (1996) for numerical methods for continuous time models with homogeneous claim size distributions. In De Vylder and Goovaerts (1988) a recursive scheme is derived to compute subsequent ruin probabilities for a discrete time, finite horizon problem. The recursion is derived by conditioning on the period claim size in the first period. Interestingly, the recursive scheme derived in this paper is based on conditioning on the period claim size in the last period. Most of the numerical methods for continuous time models use the discrete time model as the basis for approximations, typically by deriving difference equations. We refer to Dickson and Waters (1999) for a discussion of some of these methods, that enable to include compounding assets, i.e. interest in included. As much as possible we follow the terminology of Dickson and Waters (1999), though our model is more general, since we allow for inhomogeneity in both claim size distributions and premium incomes in subsequent periods.

The paper is organized as follows. In section 2 we present the model in detail and discuss its versatility in modelling real-world problems. In section 3 we derive exact recursive expressions for the probability of ruin in subsequent periods. Based on this exact recursive
formulation we propose a recursive approximation scheme with negligible computational burden. We validate the approximation scheme extensively in section 4. In section 5 we apply the approximation scheme to study two investment problems that exhibit inhomogeneity. In section 6 we summarize our conclusions.

2 Probability of ruin in finite time

We consider a discrete time ruin problem over $N$ time periods, where the cumulative claim size in period $i$, $i=1,\ldots,N$, is a random variable (rv), which is denoted by $X_i$. The cumulative distribution function (cdf) of $X_i$ is given by $F_i$, i.e. $F_i(x) = P\{X_i \leq x\}, x \geq 0$.

We assume the $X_i, i=1,\ldots,N$, to be mutually independent. Let $U_0 \geq 0$ denote the initial surplus and let $P_i, i=1,\ldots,N$ denote the premium income in period $i$. We assume $P_i$ to be a constant, i.e. the future premium incomes are assumed to be known in advance. The interest rate per period is denoted by $r$.

In the sequel we distinguish between case (a), where interest is added to the surplus at the end of a period just after cumulative claims must be paid and the case (b) where interest is added to the surplus at the end of a period, just before cumulative claims must be paid. We assume that interest payments are based on the actual surplus immediately before the payments are made. Where needed we will use the superscripts (a) and (b) to distinguish between the two cases. The difference between the two cases is best explained by considering explicit expressions for the surpluses at the end of a period, after both claims and interests have been included. We define the random variables $U_i^{(a)}$ and $U_i^{(b)}$, $i=1,\ldots,N$, as follows:

| $U_i^{(a)}$ ($U_i^{(b)}$) | surplus at the end of period $i$ after claims have been paid and interest is added after (before) claims have been paid. |

Then it easy to see that the following equations hold,
\[ U_i^{(a)} = U_0 (1 + r)^i + \sum_{j=1}^{i} P_j (1 + r)^{i-j} - \sum_{j=1}^{i} X_j (1 + r)^{i-j} \]
\[ U_i^{(b)} = U_0 (1 + r)^i + \sum_{j=1}^{i} P_j (1 + r)^{i-j+1} - \sum_{j=1}^{i} X_j (1 + r)^{i-j} \]

Equations (1) suggest to define cumulative claim size rv's \( \hat{X}_i, i=1,...,N \), and premium incomes \( \hat{P}_i, i=1,...,N \), as follows

\[ \hat{X}_i = \frac{X_j}{(1 + r)^i} \]
\[ \hat{P}_i = \begin{cases} \frac{P_j}{(1 + r)^i}, & \text{case (a)} \\ \frac{P_j}{(1 + r)^{i-1}}, & \text{case (b)} \end{cases} \]

With these definitions we find that both \( \hat{U}_i^{(a)} \) and \( \hat{U}_i^{(b)}, i=1,...,N \), can be written in the standard form

\[ \hat{U}_i = U_0 + \sum_{j=1}^{i} \hat{P}_j - \sum_{j=1}^{i} \hat{X}_j , \]
which is identical to the expression for the surplus without interest. Furthermore note that

\[ U^{(a)}_i > 0 \iff \hat{U}^{(a)}_i > 0 \]
\[ U^{(b)}_i > 0 \iff \hat{U}^{(b)}_i > 0 \]

This implies that for further analysis of (non-)ruin probabilities in discrete time we can restrict to the situation without interest, i.e. \( r=0 \). Thus we can drop the superscripts \( (a) \) and \( (b) \) until section (5), where we study the impact of the interest rate on the probability of ruin.

Let us define the ruin probabilities \( \psi(U_0,i), i=1,...,N \), as follows,

| \( \psi(U_0,i) \) | probability that the insurer's surplus, starting from \( U_0 \) at time 0, is negative at the end of one or more periods \( 1,2,...,i \) |

For convenience we also define the non-ruin probabilities \( \overline{\psi}(U_0,i) \) by

\[ \overline{\psi}(U_0,i) = 1 - \psi(U_0,i), U_0 \geq 0, i = 1,...,N . \]

In the sequel we will concentrate on expressions for \( \overline{\psi}(U,i) \). From the above definitions we derive that

\[ \overline{\psi}(U_0,i) = P\left\{ \sum_{k=1}^{i} X_k \leq U_0 + \sum_{k=1}^{i} P_k, 1 \leq j \leq i \right\} \]

Introducing the variables \( \xi_i, i=1,...,N \), defined as

\[ \xi_i = U_0 + \sum_{j=1}^{i} P_j , \]

we obtain the following canonical form of the non-ruin probability \( \overline{\psi}(U,i) \),

\[ \overline{\psi}(U_0,i) = P\left\{ \sum_{k=1}^{i} X_k \leq \xi_j, 1 \leq j \leq i \right\} . \]  (4)
In section 3 we derive an exact recursive characterization of this canonical non-ruin probability, that enables us to derive a fast and accurate approximation scheme to determine the (non-)ruin probability in discrete time over a finite horizon with independent claim sizes and arbitrary premium incomes in subsequent periods.

3 Analysis and approximations

For the analysis of the rhs of equation (4) it is convenient to introduce the function $G_i(\xi_1, \ldots, \xi_i)$ defined as

$$G_i(\xi_1, \ldots, \xi_i) = P \left\{ \sum_{k=1}^i X_k \leq \xi_j, j = 1, \ldots, i \right\},$$

Now we introduce artificial random variables $Y_i$, $i = 1, \ldots, N$, that are associated with the above function as follows,

$$P \{ Y_i \leq x \} = \frac{G_i(\xi_1, \ldots, \xi_{i-1}, x)}{G_{i-1}(\xi_1, \ldots, \xi_{i-1})}, x \geq 0.$$

It is easy to see that the rv’s $Y_i$, $i = 1, \ldots, N$, are positive random variables. Note that we suppressed the dependence of $Y_i$ on the rv’s $X_1, \ldots, X_i$ and the constants $\xi_1, \ldots, \xi_i$. With this notation Theorem 1 recursively characterizes the rv’s $Y_i$, $i = 1, \ldots, N$.

Theorem 1

$$P \{ Y_i \leq x \} = P \{ X_i \leq x \}$$

$$P \{ Y_i \leq x \} = \frac{P \{ X_i + Y_{i-1} \leq x, Y_{i-1} \leq \xi_{i-1} \}}{P \{ Y_{i-1} \leq \xi_{i-1} \}}, i = 2, \ldots, N. \quad (5)$$

Proof of Theorem 1.

The equation characterizing $Y_i$ follows directly from the definition of the cdf of $Y_i$ and the definition of $G_i(.)$. 
Let us assume that equations (5) hold for all \( i \leq n - 1 \). Let us consider the rv \( Y_n \). Substituting the expression for \( G_n(\cdot) \) we find

\[
P \left\{ Y_n \leq x \right\} = \frac{P \left\{ \sum_{k=1}^{\ell} X_k \leq \xi_j, j = 1, \ldots, n - 1, \sum_{k=1}^{n} X_k \leq x \right\}}{P \left\{ \sum_{k=1}^{\ell} X_k \leq \xi_j, j = 1, \ldots, n - 1 \right\}}
\]

By conditioning on \( X_n \) we can write the numerator in equation (6) as

\[
P \left\{ \sum_{k=1}^{\ell} X_k \leq \xi_j, j = 1, \ldots, n - 1, \sum_{k=1}^{n} X_k \leq x \right\} = \int_{0}^{x} \int_{0}^{\max(0, x - \xi_{n-1})} \ldots \int_{0}^{\max(0, x - \xi_{n-1})} P \left\{ \sum_{k=1}^{\ell} X_k \leq \xi_j, j = 1, \ldots, n - 1, \sum_{k=1}^{n-1} X_k \leq x - y \right\} dF_{X_n}(y)
\]

Now we apply the induction assumption on the arguments of the integral yielding

\[
P \left\{ \sum_{k=1}^{\ell} X_k \leq \xi_j, j = 1, \ldots, n - 1, \sum_{k=1}^{n} X_k \leq x \right\} = \int_{0}^{x} \int_{0}^{\max(0, x - \xi_{n-1})} \ldots \int_{0}^{\max(0, x - \xi_{n-1})} P \left\{ \sum_{k=1}^{\ell} X_k \leq \xi_j, j = 1, \ldots, n - 2, \sum_{k=1}^{n-2} X_k \leq x - y \right\} dF_{X_n}(y)
\]
Reversing the arguments used above to derive the equation before applying the induction assumption we find

\[
P \left\{ \sum_{k=1}^{j} X_k \leq \xi_j, j = 1, \ldots, n-1, \sum_{k=1}^{n} X_k \leq x \right\} = P \left\{ \sum_{k=1}^{j} X_k \leq \xi_j, j = 1, \ldots, n-2 \right\} \left( \int_{0}^{\max(0,x-\xi_{n-1})} P\{Y_{n-1} \leq \xi_{n-1}\} \, dF_{X_n}(y) \right.
\]
\[
+ \int_{\max(0,x-\xi_{n-1})}^{x} P\{Y_{n-1} \leq x-y\} \, dF_{X_n}(y) \right). \]
\]

Substituting this expression for the numerator of equation (6) yields

\[
P\{Y_n \leq x\} = \frac{P\{Y_{n-1} \leq \xi_{n-1}, X_n + Y_{n-1} \leq x\}}{P\left\{ \sum_{k=1}^{j} X_k \leq \xi_j, j = 1, \ldots, n-1 \right\}} \cdot \frac{P\left\{ \sum_{k=1}^{j} X_k \leq \xi_j, j = 1, \ldots, n-2 \right\}}{P\{Y_{n-1} \leq \xi_{n-1}\}}.
\]

Using the definition of the function $G_{n-1}()$, we obtain

\[
P\{Y_n \leq x\} = \frac{P\{X_n + Y_{n-1} \leq x, Y_{n-1} \leq \xi_{n-1}\}}{P\{Y_{n-1} \leq \xi_{n-1}\}}.
\]

This completes our proof.

Corollary 1 of Theorem 1 recursively characterizes the non-ruin probabilities $G_i(\xi_1, \ldots, \xi_i)$.

**Corollary 1**

\[
G_1(\xi_1) = P\{Y_1 \leq \xi_1\}
\]
\[
G_i(\xi_1, \ldots, \xi_i) = P\{Y_i \leq \xi_i\}G_{i-1}(\xi_1, \ldots, \xi_{i-1}), i = 2, \ldots, N
\]

(7)
The essence of the above characterization is that we have reduced an expression involving $i$ mutually dependent events to expressions involving only two mutually dependent events.

Since the characterization is exact, it could be exploited to derive exact numerical schemes, e.g. applying fast Fourier transforms (cf. Abate and Whitt (1992)). In fact the above theorem implies a recursive characterization of the Laplace-Stieltjes transform of $Y_i$, $i=1,...,N$.

Instead of pursuing, possibly numerically intensive, exact methods, we develop a fast recursive scheme that yields approximations for $G_i(\xi_1,\ldots,\xi_i)$ and thereby for the non-ruin probabilities $\overline{\psi}(U_0,i)$. First of all note that Corollary 1 implies that we need an expression for $P\{Y_i \leq x\}$, $i=1,...,N$. Theorem 1 implies that

$$E[Y_i] = E[X_i] + E[Y_{i-1} | Y_{i-1} \leq \xi_{i-1}], i=1,...,N-1, \tag{8}$$

$$\sigma^2(Y_i) = \sigma^2(X_i) + \sigma^2(Y_{i-1} | Y_{i-1} \leq \xi_{i-1}), i=1,...,N-1 \tag{9}$$

For many well-known probability distributions it is straightforward to compute $E[Y_{i-1} | Y_{i-1} \leq \xi_{i-1}]$ and $\sigma^2(Y_{i-1} | Y_{i-1} \leq \xi_{i-1})$. Hence we can recursively determine approximations for the first two moments of $Y_i$, $i=1,...,N$. Besides the first two moments of $Y_i$, Corollary 1 implies that it suffices to compute $P\{Y_i \leq \xi_i\}$. One possible approach is to fit a probability distribution to the first two moments of $Y_i$, from which $P\{Y_i \leq \xi_i\}$ immediately follows. However, numerical investigations suggest that a more accurate approximation is obtained using the following identity,

$$P\{X_i + Y_{i-1} \leq \xi_i, Y_{i-1} \leq \xi_{i-1}\} = P\{X_i + Y_{i-1} \leq \xi_i\} - P\{X_i + (Y_{i-1} | Y_{i-1} \geq \xi_{i-1}) \leq \xi_i\} P\{Y_{i-1} \geq \xi_{i-1}\} \tag{10}$$

In this case we fit convenient cdfs to the first two moments of $X_i$, $Y_{i-1}$ and $(Y_{i-1} | Y_{i-1} \geq \xi_{i-1})$ as well as to $X_i + Y_{i-1}$ and $X_i + (Y_{i-1} | Y_{i-1} \geq \xi_{i-1})$. The better performance of the approximations derived through equation (10) can be explained from the fact that the impact of the conditional random variable $(Y_{i-1} | Y_{i-1} \geq \xi_{i-1})$ is explicitly taken into account.
In section 4 we validate the approximations derived along the lines given above. Clearly, the choice of the probability distribution functions, that are used for fitting the first two moments of the various random variables, plays a major role. In our case we have chosen to fit mixtures of Erlang-distributions (cf. Tijms (1994)). A similar performance of the approximations results from the application of the gamma distribution to fit to the first two moments. Since routines for the incomplete gamma function are widely available, e.g. as built into Excel™, the approximation scheme can be implemented in standard software with a limited programming burden.

Let us summarize the steps of the approximation scheme:

1. \( i \rightarrow 1 \)
2. Fit a mixed-Erlang (gamma) distribution to the first two moments of \( X_i \) and \( Y_i \).
3. Compute \( E\left[Y_{i-1} \mid Y_{i-1} \leq \xi_{i-1}\right] \) and \( \sigma^2 \left(Y_{i-1} \mid Y_{i-1} \leq \xi_{i-1}\right) \) (for the case of a gamma distribution fit by using some routine for incomplete gamma functions).
4. Compute \( E\left[Y_i\right] \) and \( \sigma^2 \left(Y_i\right) \) from equations (8) and (9).
5. Compute \( E\left[Y_{i-1} \mid Y_{i-1} \geq \xi_{i-1}\right] \) and \( \sigma^2 \left(Y_{i-1} \mid Y_{i-1} \geq \xi_{i-1}\right) \) (for the case of a gamma distribution fit by using some routine for incomplete gamma functions).
6. Compute \( P\{Y_i \leq \xi_i\} \) using equation (10).
7. \( i \rightarrow i + 1 \). If \( i \leq N \) then goto step 2 else Stop.

4 Validation of approximations

In order to validate the approximations that result from the scheme above, we have set up an extensive simulation experiment, where we varied each of the model parameters. In Table 1 below we describe the experimental design. In all experiments we assumed the cumulative claim sizes to be mixed-Erlang distributed, implying a two-moment fit with a mixture of an Erlang-k and Erlang-(k-1) with the same scale parameter for coefficient of variation below 1, and a two-moment fit with a hyperexponential distribution with the first three moments equal to the first three moments of a gamma distribution with the same first two moments for coefficient of variation above 1 (cf. Tijms (1994)). We normalize the average cumulative claim size by setting \( E[X_i] \) equal to 100. We vary \( N \), the increase or decrease of the average
cumulative claim size per period $E[X_i]$ through parameter $\alpha$, the variability of the cumulative claim size through its coefficient of variation $c_{X_i}$ and the ratio of premium incomes $P_i$ and the average cumulative claim size $E[X_j]$ through parameter $\theta$. We introduce a parameter $\varepsilon$ with values 0 and 1 to distinguish between the case, where the coefficient of variation of the cumulative claim size remains constant ($\varepsilon=0$), and the case, where the coefficient of variation is proportional to the square root of the increase or decrease in the mean of the cumulative claim size ($\varepsilon=1$). Finally we vary the target ruin probability. Given all parameter values we compute $U_0$, such that the target ruin probability is achieved, i.e.

$$\psi(U_0, N) = \psi^*.$$

The total number of experiments equals 675. We have simulated each situation 100,000 times, which guaranteed a sufficiently accurate point estimate for all cases.

<table>
<thead>
<tr>
<th>$N$</th>
<th>2, 5, 10, 15, 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[X_i]$</td>
<td>100</td>
</tr>
<tr>
<td>$c_{X_i}^2$</td>
<td>0.5, 1, 1.5</td>
</tr>
<tr>
<td>$E[X_i]$</td>
<td>$(1 + \alpha)^{i-1} E[X_1], i=2, \ldots, N$</td>
</tr>
</tbody>
</table>
| $c_{X_i}^2$ | $c_{X_i}^2$, if $\varepsilon = 0$  
| $c_{X_i}^2$ | $\frac{c_{X_i}^2}{(\sqrt{1 + \alpha})^{i-1}}$, if $\varepsilon = 1, i=2, \ldots, N$ |
| $\alpha$ | -0.05, -0.02, 0, 0.02, 0.05 |
| $P_i$ | $(1+\theta)E[X]$ |
| $\theta$ | 0.05, 0.1, 0.15 |
| $\psi^*$ | 0.005, 0.01, 0.05 |

Table 1. Experimental design

In the tables below we provide insight into the impact of the various parameters on the accuracy of the approximations. For a given value of the parameter under consideration, we present both results for the absolute error averaged over all cases and the maximum absolute error among all the cases considered. As can be seen from the results below, the parameter $N$ has the greatest impact: the performance of the approximations deteriorates as $N$ increases.
For $N > 20$ the two-moment approximations should be treated with care. This suggests to develop higher-moment approximations, which is left for further research.

Errors increase with increasing $c_{X_i}$. The parameters $\varepsilon$, $\theta$ and $\alpha$ do not have an impact on the quality of the approximations, which shows that the heterogeneity of the cumulative claim size distributions and premium income in subsequent periods does not effect the quality of the approximations.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\varepsilon$</th>
<th>$c_{X_i}^2$</th>
<th>$l$</th>
<th>$1.5$</th>
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<td>0.0002</td>
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Table 2a. Average error impact $c_{X_i}$

<table>
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<tr>
<th>$N$</th>
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<th>$c_{X_i}^2$</th>
<th>$l$</th>
<th>$1.5$</th>
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</table>

Table 2b Maximum error impact $c_{X_i}$

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<tr>
<th>$N$</th>
<th>$\varepsilon$</th>
<th>$\alpha$</th>
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<th>$-0.02$</th>
<th>$0$</th>
<th>$0.02$</th>
<th>$0.05$</th>
</tr>
</thead>
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Table 3a. Average error impact $\alpha$

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Table 2b Maximum error impact $\alpha$
Another important observation is that the absolute error is smallest for $\psi(U_0, N)$ equal to 0.005. Hence the approximations are more accurate for small values of the ruin probability. This is further supported by the results in Table 6. In Table 6 we present the relative error $\Delta_r$ in $U_0$, defined by

$$\Delta_r = \frac{|\psi(U_0, N) - \psi_{exact}|}{\psi_{exact}}$$
\[ \Delta_U = \frac{U(\psi^{(sim)}) - U(\psi^*)}{U(\psi^{(sim)})}, \]

where \( U(\psi^*) \) and \( U(\psi^{(sim)}) \) are defined by:

- \( U(\psi^*) \): the value of \( U \) obtained from the approximation scheme for ruin probability \( \psi^* \)
- \( U(\psi^{(sim)}) \): the value of \( U \) obtained from the approximation scheme for ruin probability \( \psi^{(sim)} \), the actual ruin probability found by discrete event simulation

The value of \( \Delta_U \) gives an indication of the relative error caused by the approximation scheme in the decision variable \( U_0 \). The values of \( U(\psi^*) \) and \( U(\psi^{(sim)}) \) have been computed for the cases with \( \varepsilon=0 \).

<table>
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Table 6a. Average value of \( \Delta_U \)

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Table 6a. Maximum value of \( \Delta_U \)
From the results from Table 6 we can conclude that the approximation scheme yields practically useful results for all situations with $\psi^* \leq 0.01$, which is the practically relevant range of target ruin probabilities. It is interesting to see that $\Delta_U$ is largest for small coefficients of variation $c^2_{X_i}$. Apparently the quality of the approximations deteriorates, when there is too much “predictability” of the future claim sizes.

We conclude that the efficient recursive scheme proposed in section 3 provides accurate approximations for both the ruin probability given the value of $U_0$ and vice versa. Our conclusions apply to the situation studied, where we assumed mixed-Erlang claim size distributions. Further research is required to test the validity of the approximation scheme for other claim size distributions with exponential tails, like the Weibull distribution. Based on earlier research on two-moment approximations for various queueing systems, we expect that for claim size distributions with exponential tails the two-moment fit will yield accurate approximations (De Kok and Tijms (1985a,b)). Yet the same research indicates that the approximations obtained cannot be used for models with heavy-tailed claim size distributions, like Pareto-distributed claim sizes.

5 Application to investment problems

Having shown the accuracy of the approximations, we are in a position to discuss applications of the versatile model under consideration. We focus on the impact of the heterogeneity of the claim size distribution on the initial surplus and the premium income policy to be chosen. Furthermore we discuss the impact of interest on the initial surplus requirements.

Market growth and initial surplus

We assume that an insurance company offers a new insurance product to the market. The insurance company has marketing instruments that can influence the market growth until a target market share for the new product is realized. Because of immediate cash-flow problems the company wants to understand the impact of market growth on the initial surplus $U_0$ required to guarantee a target non-ruin probability $\bar{\psi}^*$ over $T$ periods. We consider the following model. Using its marketing instruments the company can reach its target market share after $T_0$ periods (e.g. quarters) following a linear growth path. This linear growth path is modelled as a linearly increasing number of customers, each generating claims according to a Poisson process. The claim sizes are exponentially distributed. The cumulative arrival rate per
period after reaching the target market share equals $\lambda$. Hence the cumulative arrival rate in period $i$ equals $\frac{\lambda}{T_0}i, i=1,2,...,T_0$. The arrival rate $\lambda$ is varied as 1 and 4. The mean claim size is chosen such that the mean period claim size after reaching the market share equals 100, i.e. if $\lambda=1$, then the mean claim size equals 100 and if $\lambda=4$, then the mean claim size equals 25. We assume that the period premium incomes equal 1.1 times the average period claim size. The interest rate $r$ is varied as 0 and 0.03. The target ruin probability $\psi^*$ is varied as 0.01 and 0.005, i.e. $\psi^*$ is varied as 0.99 and 0.995. In figures 1 and 2 we present the results of our experiments for $\lambda=1$ and $\lambda=4$, respectively.

Figure 1. Initial surplus as a function of $T_0$ for $\lambda=1$.

Figure 2. Initial surplus as a function of $T_0$ for $\lambda=4$. 
From the above results we conclude that a slower market growth indeed reduces the initial surplus requirements, yet the reduction is only moderate: 10% for the situation with $\lambda=1$ and 15% for the situation with $\lambda=4$. Comparing figures 1 and 2, we clearly identify the impact of $\lambda$. Given the mean period claim size a higher value of $\lambda$ leads to a smaller variance and thus requires a lower initial surplus to guarantee the target ruin probability. Furthermore we conclude from the results above that ignoring compounding assets due to interest on capital has a substantial impact of initial surplus requirements: Taking into account the interest rate $r=0.03$ yields a 20% reduction in initial surplus requirements.

**Minimizing Net Present Value of Investments**

We consider the following problem. An insurance company sells an insurance product for which a target non-ruin probability $\overline{\psi}^{*}$ over $N$ periods is set. The company wants to select the optimal investment policy among the class of policies that are defined as follows. At time 0 an initial surplus $U_0$ is invested and each subsequent period a fixed amount $u$ is invested. Assuming an interest rate of $r$ per period, this implies that the Net Present Value $NPV(U_0,u,r,N)$ of the total investment equals

$$NPV(U_0,u,r,N) = U_0 + \frac{u}{r} \left( 1 - \left( \frac{1}{1+r} \right)^N \right)$$

In order to determine the optimal investment policy we consider all combinations $(U_0,u)$ that satisfy

$$NPV(U_0,u,r,N) = NPV(\hat{U}_0,0,r,N) = \hat{U}_0,$$

where $\hat{U}_0$ is determined by

$$\overline{\psi}(\hat{U}_0,0,r,T) = \overline{\psi}^{*}.$$ 

The optimal combination $(U_0,u)$ should yield the lowest ruin probability.

We explore the characteristics of the optimal $(U_0,u)$ combination by considering a numerical example. We assume that the cumulative claim size per period distribution is derived from a compound Poisson claim process with rate 5 per period and average claim size 100. We consider the situation with exponential claim sizes (coefficient of variation $c$ equal to 1) and
Erlang-2 claim sizes (coefficient of variation \( c \) equal to 0.5). We fit a mixture of Erlang distributions (see e.g. Tijms (1994)) to the first two-moments of the resulting cumulative claim size per period distribution. We assume a period premium income equal to 550 (i.e. \( \theta = 0.1 \), with \( \theta \) defined as in table 1 above). The interest rate \( r \) is varied as 0 and 0.03. The results for the four different situation in our experiment are presented in figure 3.

![Figure 3. Ruinprobability \( \psi \) as a function of periodic investment \( u \)](image)

Figure 3 shows that the Net Present Value of investments is minimized by investing in an initial surplus that yields the required non-ruin probability without further investments, i.e. \( u=0 \).

Clearly, we have not formally proven this statement, yet it shows the potential of the model discussed in this paper. An interesting topic for further research would be to combine the two models discussed above by considering a non-stationary situation with respect to the period claim distribution and determining the optimal investment policy.

## 6 Conclusions

In this paper we considered a finite horizon discrete time model with inhomogeneous independent claim size distributions and arbitrary premium income in subsequent periods. We developed exact recursive expressions for the non-ruin probabilities in subsequent periods. These recursive expressions provided the basis for a computationally efficient recursive approximation scheme based on two-moment gamma distribution fits. An extensive simulation experiment showed the accuracy of the approximations for values of the horizon.
up to 20. Especially for small values of the ruin-probability the approximations are very accurate. Having shown the validity of the approximations, we applied them to gain insight into two non-stationary problems. Both problems motivate further research in optimal investment strategies under non-stationary conditions.

**Acknowledgements**

The author would like to thank Marc Goovaerts for useful comments and his encouragement to work out a method originally developed for the numerical optimization of multi-echelon inventory models towards its true origin: ruin probabilities. Much of the work on this paper was completed while the author spent a sabbatical at the Catholic University Leuven in the Winter of 2002-2003.

**References**