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EXISTENCE OF SOLUTIONS TO THE DIFFUSIVE VSC MODEL*

J. HULSHOF[†], R. NOLET[†], AND G. PROKERT[‡]

Abstract. We prove existence of classical solutions to the so-called diffusive vesicle supply center (VSC) model describing the growth of fungal hyphae. It is supposed in this model that the local expansion of the cell wall is caused by a flux of vesicles into the wall and that the cell wall particles move orthogonally to the cell surface. The vesicles are assumed to emerge from a single point inside the cell (the VSC) and to move by diffusion. For this model, we derive a nonlinear, nonlocal evolution equation and show the existence of solutions relevant to our application context, namely, axially symmetric surfaces of fixed shape, traveling along with the VSC at constant speed. Technically, the proof is based on the Schauder fixed point theorem applied to Hölder spaces of functions. The necessary estimates rely on comparison and regularity arguments from elliptic PDE theory.

Key words. VSC model, fungal hyphae, moving boundary problems, PDE

AMS subject classifications. Primary, 35Q92; Secondary, 35R37, 35C07

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1. Introduction. Describing the growth behavior of living cells is a challenging pursuit, both from the point of view of biological modeling and from the point of view of the mathematical treatment of the resulting models. Since growth of a cell proceeds primarily by incorporating new material into the cell wall and membrane, models for cell growth have to describe how the shape of a cell changes as a result of this process. In geometric models, the cell wall and the membrane are treated as a single surface without thickness. This allows one to mathematically describe the cell wall as an embedded two-dimensional manifold. In this case, the well-known “first variation of area formula” relates the local growth of cell surface area to the velocity of its particles—more precisely, to their normal velocity and the divergence of their tangential velocity.

Extreme growth behavior can be observed in fungal hyphae cells, i.e., very long, hair-shaped cells that form the mycelium of fungi. Accordingly, modeling their growth has attracted particular interest, with an emphasis on solutions given by a fixed, traveling profile. In most models for these cells, it is assumed that cell wall particles move in a direction orthogonal to the cell surface. (An exception to this is the isometric model described by Tindemans [11].) This assumption of orthogonal growth is mostly justified by observations, with turgor pressure given as a possible physical mechanism. It will also be adopted in the present paper. Moreover, conservation of mass dictates that the surface area growth equals the local flux F of material into the cell boundary. In section 2 we show that these assumptions determine the normal velocity as $v_n = -F/H$, where H is the mean curvature of the manifold.

Some models express this flux solely as a function of the local geometry of the cell wall. For example, Goriely, Károlyi, and Tabor [8] define the flux as a function of

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the curvature. The vesicle supply center (VSC) models, first proposed by Bartnicki-Garcia, Hergert, and Gierz [1] (see also Bartnicki-Garcia et al. [2] and Bartnicki-Garcia and Gierz [3]), assume that material is transported toward the wall in so-called vesicles, i.e., small “sacks” bounded by a membrane. These vesicles are created at the Golgi apparatus and transported via the cytoskeleton to the VSC, from which they are released and transported to the cell wall. On arrival at the cell wall the contents of the vesicles are used to make cell wall material while the vesicle membrane merges with the cell membrane. For modeling purposes it is not important whether the VSC acts as a distribution center for vesicles created elsewhere or whether it produces them itself; in both cases it can be treated as a source of vesicles. In models for tip growth, the location of the VSC often coincides with an organelle called the Spitzenkörper.

The VSC models are divided into two classes, depending on how vesicles move from the VSC to the cell wall. In the so-called ballistic model, vesicles travel in straight lines toward the cell wall. The model by Bartnicki-Garcia, Hergert, and Gierz [1] is of this kind, with vesicles sent in every direction isotropically. The advantage of ballistic models lies in their mathematical simplicity: The flux of vesicles arriving at a point on the cell wall can be calculated directly from its distance to the VSC and the slope of the wall. A traveling wave ansatz then yields an ODE for the shape of the hypha. In a previous article [9] we used this to show that this model has unique, stable, traveling solutions. These solutions are tubular elongating cells growing mostly at the tip, as observed in fungal hyphae. Possible variations of the ballistic model involve including a directional preference to the release of vesicles so that more of them are focused on the tip, or having multiple sources.

One criticism of the ballistic model, given, e.g., by Koch [10], is that inside a living cell, it is highly unlikely that a vesicle will travel in a straight line to its destination. Instead it will perform a random walk and will be absorbed when it hits the cell boundary. Accordingly, the concentration of vesicles obeys a Poisson equation with a point source at the VSC and homogeneous Dirichlet boundary conditions.

Numerical calculations on this model were done by Tindemans, Kern, and Mulder [12]. Possible variations of the diffusive model include further physical properties of the cell wall, e.g., elasticity or reduced absorption due to aging [5]. A very good overview of many of the models available is given by de Keijzer, Emons, and Mulder [4].

The aim of all of these models is to find a traveling solution corresponding to a fungal hypha. These are solutions which are stationary in a frame of reference traveling along with the VSC (or at some fixed velocity if no VSC is present in the models). As usual, we introduce the assumption of cylindrical symmetry; i.e., the surface can then be expressed as a curve rotated around the z axis. We seek solutions which asymptotically approximate a cylinder as $z \rightarrow -\infty$.

We want to stress that the diffusive VSC models are essentially nonlocal. In fact, the equations for the tip shape involve an unknown flux function which itself depends on the tip shape. Therefore this shape is determined by a condition which cannot be formulated as an ODE.

So far, most research has focused on numerically approximating the tip shape for these models. In this article we provide a theoretical foundation for the simplest of them by rigorously proving the existence of these traveling solutions using a Schauder fixed point argument. However, the methods described in this article should work as well for certain related models involving orthogonal growth and a flux dependent on the cell shape; on this, see also the conclusions section.

1.1. Notation and conventions. In this article we will often make use of the following notation: Ω is an open subset of $\mathbb{R}^3$, not necessarily compact, rotationally symmetric around the z axis with boundary $\partial\Omega$. Since we are working in this axially symmetric case, we will use a cylindrical coordinate system $(r, z, \theta)^T$ in $\mathbb{R}^3$. The transformation to Euclidean coordinates is given by $(x_1, x_2, x_3)^T = (r \cos \theta, r \sin \theta, z)^T$. Often we will not mention the coordinate θ in our calculations. The rotationally symmetric surface $\partial\Omega$ will be parametrized by two functions $s \mapsto r(s)$ and $s \mapsto z(s)$. The surface implied is the curve parametrized by these two functions at some fixed value of θ , rotated around the z axis. When we refer to a point on the surface at pathlength s or the point $(r(s), z(s))$, we mean a point $(r(s), z(s), \theta) \in \partial\Omega$ at some arbitrary but fixed value of θ .

For the (scalar) mean curvature H we use the conventions as found in [6]. The mean curvature is the sum (and thus not the true mean) of the principal curvatures with respect to an outward pointing normal $\hat{n}$. For example, the sphere of radius R has mean curvature $-\frac{2}{R}$ at every point.

When u is a (harmonic) function defined on Ω , the flux F_u of u will always be the negative normal derivative of u on $\partial\Omega$. Often u will depend on a parameter ξ ; we will denote this as u_ξ . If it is clear the flux mentioned is the flux of u , we will write F_ξ to denote the flux at parameter value ξ .

The proof in this article relies heavily on the use of Hölder spaces. For these spaces and their norms we will use the notation as found in [7], with $\|\cdot\|_{k,\alpha;X}$ denoting the $C^{k,\alpha}$ norm on the domain of definition X . The space of continuous functions from X to Y with bounded Hölder norm is denoted as $C^{k,\alpha}(X;Y)$. The domain X or codomain Y will be omitted if they are clear from the context.

2. The diffusive VSC model.

2.1. The diffusive flux. In the diffusive VSC model, we assume the VSC is a source of vesicles which diffuse outward toward the cell wall, where they are completely absorbed, causing growth. Each vesicle is a membrane sack containing the materials to build into the cell wall. Upon absorption, the membrane of the vesicle merges with the cell membrane. In the VSC model, the membrane and cell wall are treated as a single manifold, and each vesicle contributes a fixed amount of surface area to this manifold. The total amount of surface area produced by the VSC per unit of time is denoted by P . The VSC is moving in the positive z direction at speed c . We assume the motion of the cell wall is slow on the diffusion time scale of the vesicles, and so the density of vesicles is always in equilibrium. As such, it can be found by solving a Poisson equation. The assumption that vesicles are completely absorbed at the boundary yields a homogeneous Dirichlet boundary condition. Furthermore, we assume that the motion of the cell wall is slow on the diffusion time scale of the vesicles.

If at time $t = 0$ the tip of the cell wall is at the origin, and the VSC is at distance ξ from the tip, then the density u_ξ of vesicles can be found by solving

$$(2.1) \quad \begin{aligned} \Delta u_\xi &= -P\delta(r, z + \xi - ct) && \text{in } \Omega, \\ u_\xi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The flux of material arriving at a point is now given by $F_\xi = -\frac{\partial u_\xi}{\partial \hat{n}}$, where $\hat{n}$ is the outward pointing normal of $\partial\Omega$. This flux gives the rate per unit of area at which the surface area increases.

2.2. Mass balance. If one takes an arbitrary bounded region $\Gamma \subset \partial\Omega$ of the surface of the cell, with surface area $\|\Gamma\|$ and boundary curve $\partial\Gamma$, then the total flux of material absorbed in Γ is given by

$$(2.2) \quad \frac{d\|\Gamma\|}{dt} = \int_{\Gamma} F_{\xi} dS.$$

If one assumes Γ is transported by a velocity field v , then Gauss's formula for the first variation of area states that

$$(2.3) \quad \frac{d\|\Gamma\|}{dt} = - \int_{\Gamma} H(\hat{n} \cdot v) dS + \oint_{\partial\Gamma} (\hat{m} \cdot v) dl,$$

where $\hat{m}$ is the outward pointing normal to $\partial\Gamma$ tangent to $\partial\Omega$ and H is the (scalar) mean curvature. By assumption, the surface of the cell moves orthogonally to the cell surface, so $v = v_n \hat{n}$ and the integral over $\partial\Gamma$ vanishes. As Γ was chosen arbitrarily, we get from (2.2) and (2.3) that

$$(2.4) \quad v_n = -\frac{F_{\xi}}{H}.$$

2.3. Main result. Equation (2.4), with flux defined by (2.1) as explained in section 2.1, describes the evolution of an embedded two-dimensional manifold. The aim of this article is to show there exist solutions to this evolution which evolve similarly to the growth of fungal hyphae.

THEOREM 2.1 (main result). *For each speed c of the VSC and production rate P there exists a manifold and distance ξ such that, when evolving under evolution equation (2.4), this manifold appears stationary in a coordinate frame moving at speed c in the z direction. The manifold is cylindrically symmetric and approximates a cylinder of radius $\frac{P}{2\pi c}$ as $z \rightarrow -\infty$.*

This theorem will be proved in the course of this article.

2.4. Scaling. We define the typical length scale X and time scale T of the model as follows:

$$(2.5) \quad X = \frac{P}{4\pi c}, \quad T = \frac{P}{4\pi c^2}.$$

Rescaling our spatial coordinates by $x = X\tilde{x}$ and $t = T\tilde{t}$, we denote the rescaled domain as $\tilde{\Omega}$. It is now natural to rescale the dependent variables and constants as

$$(2.6) \quad \begin{aligned} u_{\xi} &= \frac{X}{T} \tilde{u}_{\tilde{\xi}}, & F_{\xi} &= \frac{1}{T} \tilde{F}_{\tilde{\xi}}, & H &= \frac{1}{X} \tilde{H}, & v_n &= \frac{X}{T} \tilde{v}_n, \\ \xi &= X\tilde{\xi}, & \tilde{c} &= 1, & \tilde{P} &= 4\pi. \end{aligned}$$

We now see that the rescaled model satisfies

$$(2.7) \quad \tilde{v}_n = -\frac{\tilde{F}_{\tilde{\xi}}}{\tilde{H}},$$

where $\tilde{F}_{\tilde{\xi}} = -\frac{\partial \tilde{u}_{\tilde{\xi}}}{\partial \tilde{n}}$ and $\tilde{u}_{\tilde{\xi}}$ satisfies

$$(2.8) \quad \begin{aligned} \Delta \tilde{u}_{\tilde{\xi}} &= -4\pi\delta(\tilde{r}, \tilde{z} + \tilde{\xi} - t) && \text{in } \tilde{\Omega}, \\ \tilde{u}_{\tilde{\xi}} &= 0 && \text{on } \partial\tilde{\Omega}. \end{aligned}$$

For the remainder of this article we will drop the tildes and work with this rescaled model.

FIG. 1. Traveling wave profile with model definitions. The solid curve, rotated around the z axis, is the cell boundary $\partial\Omega$. The arrows indicate the normal velocity, and the dotted lines are the particle trajectories. The dashed curve indicates the location of the cell boundary at some later time.

2.5. The unbounded traveling wave problem. We wish to find a surface satisfying the evolution equation (2.4) which moves along with the VSC; see, for example, Figure 1. In other words, in a coordinate system moving along with the VSC, $\partial\Omega$ appears stationary. In this coordinate system, Gauss's formula for the variation of area on an arbitrary subsurface Γ states that

$$(2.9) \quad \int_{\Gamma} F_{\xi} dS = - \int_{\Gamma} H(\hat{n} \cdot (v - \hat{e}_z)) dS + \oint_{\partial\Gamma} (\hat{m} \cdot (v - \hat{e}_z)) dl.$$

For a stationary solution, $v - \hat{e}_z$ must lie tangent to $\partial\Omega$ and the integral over Γ vanishes. By assumption, v is perpendicular to $\hat{m}$, and so

$$(2.10) \quad - \oint_{\partial\Gamma} \hat{m} \cdot \hat{e}_z dl = \int_{\Gamma} F_{\xi} dS.$$

We now choose Γ to be the region from the tip up to the plane located at $z = z(s)$; then $\hat{m} \cdot \hat{e}_z$ is constant over $\partial\Gamma$. We choose cylindrical coordinates r and z , and describe $\partial\Omega$ as the curve $(r(s), z(s))$ rotated around the z axis. We parametrize such that s is the pathlength over $\partial\Omega$ from the tip.

In these coordinates $\hat{m} = r'(s)\hat{e}_r + z'(s)\hat{e}_z$, and (2.10) simplifies to

$$(2.11) \quad z'(s) = -\frac{G_{\xi}(s)}{r(s)}, \quad r'(s) = \sqrt{1 - \left(\frac{G_{\xi}(s)}{r(s)}\right)^2},$$

where

$$(2.12) \quad G_{\xi}(s) = \frac{1}{2\pi} \int_{\Gamma} F_{\xi} dS = \int_0^s F_{\xi}(\sigma) r(\sigma) d\sigma,$$

and $F_{\xi}(\sigma)$ is the flux passing through the point on the boundary at pathlength σ .

We wish to find functions $r(s)$, $z(s)$ and a number ξ^* such that when (2.1) is solved on the domain defined by the functions, the corresponding cumulative flux G_{ξ^*} and the

functions r and z satisfy (2.11) with boundary conditions $r(0) = z(0) = 0$, $r'(0) = 1$. We wish $r(s)$ to remain bounded. Since by the divergence theorem $G_\xi(s) \rightarrow 2$, this can only be accomplished if $r(s) \rightarrow 2$ as $s \rightarrow \infty$. In the rest of the article we will refer to this as the unbounded traveling wave problem.

2.6. The bounded traveling wave problem. The fact that the domain of the functions r and z is infinite, and therefore that the domain Ω is unbounded, makes analysis difficult. In order to handle these difficulties, we first restrict ourselves to bounded domains with a no-flux condition at $z = z(s_{max})$ for some sufficiently large s_{max} . We apply the method of reflection and define the following problem: Given functions $r(s)$ and $z(s)$ on $(0, s_{max})$ we define $\partial\Omega$ to be the curve $(r(s), z(s))$ rotated around the z axis and reflected at the plane $z = z(s_{max})$. If the VSC is located at a distance ξ from the tip, the density of vesicles $u_\xi(r, z)$ is found by solving

$$(2.13) \quad \begin{aligned} \Delta u_\xi &= -4\pi\delta(r, z + \xi) - 4\pi\delta(r, z + \eta) && \text{in } \Omega, \\ u_\xi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

where $\eta = -2z(s_{max}) - \xi$ is the distance from the reflected VSC to the tip at $z = 0$. The functions F_ξ and G_ξ are still defined as above. Note that by symmetry and the divergence theorem, $G_\xi(s_{max}) = 2$.

Given s_{max} we wish to find functions $r(s)$, $z(s)$ and a number ξ^* , such that when (2.13) is solved on the domain defined by these functions, the corresponding cumulative flux G_{ξ^*} and the functions r and z satisfy (5.2) with boundary conditions $r(0) = z(0) = 0$, $r'(0) = 1$, and $r(s_{max}) = 2$. We will refer to this problem as the bounded traveling wave problem. In section 7 we take the limit as $s_{max} \rightarrow \infty$ to show the existence of a solution for the unbounded traveling wave problem described in the previous section.

3. The Schauder map. Our approach to solving this problem relies on a Schauder fixed point argument on a subset of the product space $C^{1,\alpha} \times C^{0,\alpha}$ containing Hölder continuous functions $s \mapsto r(s)$ and $s \mapsto z'(s)$ for some $\alpha \leq \frac{1}{2}$. We equip this space with the $C^{1,\beta} \times C^{0,\beta}$ topology for some $\beta < \alpha$. Defining $z(s) = \int_0^s z'(\sigma) d\sigma$, the functions $r(s)$ and $z(s)$ describe a boundary $\partial\Omega$ with certain properties. (Note that since $z \rightarrow -\infty$ if $s_{max} \rightarrow \infty$, we cannot claim that $z \in C^{1,\alpha}$ is bounded uniformly in s_{max} ; instead we demand this only of its derivative z' .) Since the Schauder fixed point theorem requires that we work on a closed convex subset of this product space, we cannot require that $\partial\Omega$ is parametrized by pathlength ($r'^2 + z'^2 = 1$ is not a convex requirement). Instead we require that $r'^2 + z'^2 \leq 1$ and $r' - z' \geq 1$; see Figure 2. Note, however, that the image of the Schauder map does satisfy the pathlength requirement $r'^2 + z'^2 = 1$.

3.1. The domain of the Schauder map. Given a sufficiently large s_{max} we define the domain of the Schauder map $\Xi(M, A, C; s_{max})$ as the subset of $C^{1,\alpha}([0, s_{max}]; \mathbb{R}) \times C^{0,\alpha}([0, s_{max}]; \mathbb{R})$ containing functions r and z' satisfying the following convex properties:

$$(3.1) \quad \|r\|_{1,\alpha} \leq M, \quad \|z'\|_{0,\alpha} \leq M,$$

$$(3.2) \quad r'(s)^2 + z'(s)^2 \leq 1, \quad r'(s) - z'(s) \geq 1,$$

$$(3.3) \quad r(s_{max}) = 2, \quad r'(s_{max}) = 0,$$

FIG. 2. Convex bounds on r' and z' .

$$(3.4) \quad \frac{r'(s_2) - r'(s_1)}{s_2 - s_1} \leq A, \quad \frac{z'(s_2) - z'(s_1)}{s_2 - s_1} \leq A \quad \text{for } s_1 < s_2,$$

$$(3.5) \quad s - \frac{1}{9}C^2s^3 \leq r(s) \leq 2 \quad \text{for } 0 \leq s \leq C^{-1}.$$

In section 4 we will show that one can solve the Dirichlet problem (2.13) for every ξ , $\xi_{\min} \leq \xi \leq \xi_{\max}$, with $\xi_{\min}$ and $\xi_{\max}$ to be determined later, and obtain a family G_ξ of cumulative fluxes, parametrized by ξ , with certain properties. This defines a map $\Psi_1 : \Xi(M, A, C; s_{\max}) \rightarrow C^1([\xi_{\min}, \xi_{\max}]; C^{1,\alpha}([0, s_{\max}]; \mathbb{R}))$. Given $G_\xi \in \text{Im}(\Psi_1)$ and a value of the parameter ξ , (2.11) can be seen as an ordinary differential equation, which can be solved to obtain functions $r_\xi(s)$ and $z_\xi(s)$. In section 5 we will show that one can find a unique value ξ^* such that $r_{\xi^*}(s_{\max}) = 2$. This defines a map $\Psi_2 : \text{Im}(\Psi_1) \rightarrow \Xi(\tilde{M}, \tilde{A}, \tilde{C})$. In section 6 we will choose M , A , and C such that the composition $\Psi = \Psi_2 \circ \Psi_1$ maps from $\Xi(M, A, C; s_{\max})$ to itself. We then use Schauder's fixed point theorem to show that the map Ψ , which we will refer to as the Schauder map, has a fixed point. Since solutions to (2.11) satisfy $r'^2 + z'^2 = 1$, this fixed point describes a surface parametrized by pathlength and solves the bounded traveling wave problem defined in section 2.6.

LEMMA 3.1. *The set $\Xi(M, A, C; s_{\max})$ is closed in the $C^{1,\beta} \times C^{0,\beta}$ topology.*

Proof. Let (r_n, z'_n) be a sequence in $\Xi(M, A, C; s_{\max})$ which converges to (r, z') in $C^{1,\beta} \times C^{0,\beta}$. We need to prove that (r, z') satisfies (3.1) to (3.5). We can clearly take the limit to see that (r, z) satisfies (3.2) to (3.5), so we need only concern ourselves with the Hölder norm established in (3.1). Now since r_n is bounded in the $C^{1,\alpha}$ norm it has a convergent subsequence in the $C^{1,\beta}$ norm; clearly the limit of this subsequence is r and thus $\|r\|_{1,\alpha} \leq M$. Similarly $\|z'\|_{0,\alpha} \leq M$. $\square$

3.2. Estimates on $r(s)$, $z(s)$ and distances. The definition of the set $\Xi(M, A, C; s_{\max})$ yields several estimates on $r(s)$, $z(s)$ and distances between points on the boundary that will be used throughout this article.

First of all, (3.2) gives $0 \leq r' \leq 1$, $-1 \leq z' \leq 0$, $r'^2 + z'^2 \geq \frac{1}{2}$, $r(0) = 0$, $r'(0) = 1$, and $z'(0) = 0$. The estimate given by (3.5) gives an asymptotic approximation for $r(s)$ in the tip at small s . The monotonicity of r then gives a lower bound away from the tip:

$$(3.6) \quad \frac{8}{9}C^{-1} \leq r(s) \leq 2 \quad \text{for } C^{-1} \leq s \leq s_{\max}.$$

Using this we can establish an asymptotic estimate for $z(s)$ at small s ,

$$(3.7) \quad z(s) \geq -\sqrt{s^2 - r(s)^2} \geq -\frac{1}{2}Cs^2 \quad \text{for } 0 \leq s \leq C^{-1},$$

while the requirement that $r' - z' \geq 1$ implies that

$$(3.8) \quad -s \leq z(s) \leq 2 - s$$

for all s .

The choice of parametrization of the curve $s \mapsto (r(s), z(s))$ yields various useful bounds on the distances between points on the curve.

LEMMA 3.2. *The distance between points $(r(s_2), z(s_2))$ and $(r(s_1), z(s_1))$ is bounded from above and below by the difference in parameter values $s_2 - s_1$:*

$$(3.9) \quad \frac{1}{2}(s_2 - s_1)^2 \leq (r(s_2) - r(s_1))^2 + (z(s_2) - z(s_1))^2 \leq (s_2 - s_1)^2.$$

Proof. Let $d(s)$ be the distance from $(r(s), z(s))$ to $(r(s_1), z(s_1))$:

$$(3.10) \quad d(s) = \sqrt{(r(s) - r(s_1))^2 + (z(s) - z(s_1))^2}.$$

By the Cauchy–Schwarz inequality,

$$(3.11) \quad \left(\frac{d}{ds} d(s) \right)^2 = \left(\frac{r(s) - r(s_1)}{d(s)} r'(s) + \frac{z(s) - z(s_1)}{d(s)} z'(s) \right)^2 \leq r'(s)^2 + z'(s)^2 \leq 1,$$

so $d(s_2) \leq (s_2 - s_1)$. For the lower bound we see that

$$(3.12) \quad \begin{aligned} d(s_2)^2 &= \frac{1}{2} ((r(s_2) - r(s_1)) - (z(s_2) - z(s_1)))^2 + \frac{1}{2} ((r(s_2) - r(s_1)) + (z(s_2) - z(s_1)))^2 \\ &\geq \frac{1}{2}(s_2 - s_1)^2, \end{aligned}$$

since $r'(s) - z'(s) \geq 1$.

Geometrically, the upper bound is achieved when the path between the points at pathlengths s_1 and s_2 consists of a straight line. The lower bound is achieved when the path consists solely of vertical and horizontal segments. $\square$

Furthermore, the estimates for $r(s)$ and $z(s)$ established in the previous section allow us to calculate a lower bound for the distance between the VSC and the boundary of the cell, an important ingredient for bounding the flux.

LEMMA 3.3. *If $\frac{1}{2}C^{-1} < \xi_{\min} \leq \xi$, then the distance $d_\xi(s)$ from the point $(r(s), z(s))$ to the VSC is bounded from below by a nonzero constant $d_{\min}$ depending only on C and $\xi_{\min}$.*

Proof. Using the asymptotics for r and z at small s , (3.5) and (3.7),

$$d_\xi(s) = \sqrt{r(s)^2 + (\xi + z(s))^2} \geq \xi + z(s) \geq \xi_{\min} - \frac{1}{2}C^{-1} \quad \text{for } 0 \leq s \leq C^{-1},$$

while for large s ,

$$d_\xi(s) \geq r(s) \geq \frac{8}{9}C^{-1} \quad \text{for } C^{-1} \leq s \leq s_{\max}$$

by (3.6). The minimum of these two estimates gives a lower bound for the distance. $\square$

3.3. Exterior spheres. For bounds on the flux in the next section we require it be possible to touch a sphere of fixed radius to every point of the boundary in such a way that the interior of the sphere does not intersect Ω . If the second derivatives of r and z were bounded from above, this would be a relatively straightforward task involving the calculation of the first principle curvature. The upper bound on the difference quotient given by (3.4) is in fact sufficient for this task.

LEMMA 3.4. *Let B_R be a ball of radius $R \leq \frac{1}{4A}$ touching $\partial\Omega$ at the point $(r(s_1), z(s_1))$. Then the distance from any point $(r(s_2), z(s_2)) \in \partial\Omega$ to the center (r_c, z_c) of B_R is always greater than R .*

Proof. Let $\frac{1}{2}\pi \leq \theta \leq \pi$ be such that

$$(3.13) \quad \cos \theta = \frac{z'(s_1)}{\sqrt{r'(s_1)^2 + z'(s_1)^2}}, \quad \sin \theta = \frac{r'(s_1)}{\sqrt{r'(s_1)^2 + z'(s_1)^2}}.$$

The center of the ball B_R is then given by

$$(3.14) \quad r_c = r(s_1) - R \cos \theta, \quad z_c = z(s_1) + R \sin \theta.$$

The distance d_c between the point $(r(s_2), z(s_2))$ and the center of this sphere is given by

$$(3.15) \quad d_c^2 = (r(s_2) - r(s_1))^2 + (z(s_2) - z(s_1))^2 + R^2 + 2R((r(s_2) - r(s_1)) \cos \theta - (z(s_2) - z(s_1)) \sin \theta).$$

Integrating the difference quotients (3.4) we can estimate

$$(3.16) \quad \begin{aligned} r(s_2) - r(s_1) &\leq r'(s_1)(s_2 - s_1) + \frac{1}{2}A(s_2 - s_1)^2, \\ z(s_2) - z(s_1) &\leq z'(s_1)(s_2 - s_1) + \frac{1}{2}A(s_2 - s_1)^2. \end{aligned}$$

Substituting this, the linear terms in $(s_2 - s_1)$ drop out and

$$(3.17) \quad d_c^2 \geq (r(s_2) - r(s_1))^2 + (z(s_2) - z(s_1))^2 + R^2 + AR(s_2 - s_1)^2(\cos \theta - \sin \theta).$$

The distance between points at parameter values s_2 and s_1 can be estimated using Lemma 3.2, and thus

$$(3.18) \quad d_c^2 \geq \frac{1}{2}(s_2 - s_1)^2 + R^2 - 2AR(s_2 - s_1)^2,$$

and so if we choose $R \leq \frac{1}{4A}$, this distance will always be greater than R . $\square$

4. The Dirichlet problem. Given a domain Ω given by the functions $r(s)$ and $z(s)$ as described in the previous section, we wish to find a solution u_ξ to (2.13). We then wish to find various estimates for the flux $F_\xi(s)$ passing through the point $(r(s), z(s))$ and the cumulative flux $G_\xi(s)$, defined as

$$(4.1) \quad G_\xi(s) = \int_0^s F_\xi(\sigma) r(\sigma) \sqrt{r'(\sigma)^2 + z'(\sigma)^2} d\sigma.$$

Note that if $\partial\Omega$ is parametrized by pathlength, which, for example, is the case in the fixed point of the Schauder map, then this definition is equivalent to (2.12).

4.1. The domain Ω .

LEMMA 4.1. *If the boundary $\partial\Omega$ is given by $C^{1,\alpha}$ Hölder continuous functions $r(s)$ and $z(s)$ as described previously, then the enclosed domain Ω is of class $C^{1,\alpha}$.*

Proof. For the purposes of this proof, we extend the functions r and z to the interval $[0, 2s_{max}]$ by reflection, so for $s > s_{max}$, $r(s) = r(2s_{max} - s)$, and $z(s) = 2z(s_{max}) - z(2s_{max} - s)$. Now $r'(s_{max}) = 0$ and $z'(s_{max}) = -1$, so the derivatives are continuous. For $s_1 < s_{max} < s_2$,

$$\begin{aligned} |r'(s_2) - r'(s_1)| &\leq |r'(s_2) - r'(s_{max})| + |r'(s_{max}) - r'(s_1)| \\ (4.2) \quad &\leq \|r\|_{1,\alpha} |s_2 - s_{max}|^\alpha + \|r\|_{1,\alpha} |s_{max} - s_1|^\alpha \\ &\leq 2 \|r\|_{1,\alpha} |s_2 - s_1|^\alpha, \end{aligned}$$

and similarly for the Hölder quotient of z' ; therefore these functions are Hölder continuous. We now need to prove that each point of $\partial\Omega$ has a neighborhood which can be described as the graph of a $C^{1,\alpha}$ function. We examine the point x^* at pathlength s^* and angle θ^* . Without loss of generality we can assume that $\theta^* = 0$ due to the rotational symmetry. A point x given by the parameters s and θ in the neighborhood of x^* has Euclidean coordinates $(x_1, x_2, x_3)^T = (r(s) \cos \theta, r(s) \sin \theta, z(s))^T$. We now introduce new coordinates ξ such that the origin lies on x^* , rotated such that the direction ξ_1 lies tangent to the curve $r(s), z(s)$ and the direction ξ_2 lies in the direction of rotation by θ . Then

$$(4.3) \quad \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{bmatrix} -z'(s^*) & 0 & r'(s^*) \\ 0 & 1 & 0 \\ -r'(s^*) & 0 & -z'(s^*) \end{bmatrix} \begin{pmatrix} r(s) \cos \theta - r(s^*) \\ r(s) \sin \theta \\ z(s) \end{pmatrix}.$$

Now at $(s, \theta) = (s^*, 0)$ we have $\frac{\partial \xi}{\partial s} = (0, 0, -1)^T$ and $\frac{\partial \xi}{\partial \theta} = (0, r(s^*), 0)^T$. Therefore, if $s^* \neq 0$, then $r(s^*) \neq 0$, and by the implicit function theorem there is a neighborhood around x^* where we can write ξ_3 (and s and θ) as a $C^{1,\alpha}$ function of ξ_1 and ξ_2 .

If, on the other hand, $s^* = 0$, then since $r'(0) = 1$, $r(s)$ is invertible in a neighborhood of zero, and its inverse is $C^{1,\alpha}$. The tip can now be described as the graph $x_3 = z(r^{-1}(\sqrt{x_1^2 + x_2^2}))$. $\square$

This lemma implies that the domain Ω is of class $C^{1,\alpha}$, which ensures (see, for example, [7, Theorem 8.34]) that there is a unique solution to the Dirichlet problem (2.13) which is $C^{1,\alpha}$. Since $\partial\Omega$ is not C^2 , it does not satisfy an interior sphere condition everywhere, and we cannot use the boundary point lemma to conclude that $F_\xi > 0$ everywhere. This motivates the following lemma for less smooth domains.

LEMMA 4.2. *Let Ω be a (not necessarily rotationally symmetric) domain sufficiently smooth that the maximum principle and divergence theorem hold and a normal direction $\hat{n}$ to the boundary can be defined almost everywhere. Let $u > 0$ be a weak solution of*

$$(4.4) \quad \begin{aligned} \Delta u &= f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where f has compact support away from the boundary. Then there is no open subset $\Gamma \subset \partial\Omega$ such that $F_u = -\frac{\partial u}{\partial \hat{n}} = 0$ on Γ .

Proof. Assume such a $\Gamma \subset \partial\Omega$ exists. Let B_R be a ball of radius R centered on a point in the interior of Γ . Then R can be chosen sufficiently small such that

$(\partial\Omega \cap B_R) \subset \Gamma$ and $B_R \cap \text{supp } f = \emptyset$. Let v solve

$$(4.5) \quad \begin{aligned} \Delta v &= 0 && \text{in } B_R, \\ v &= u && \text{on } \partial B_R \cap \Omega, \\ v &= 0 && \text{on } \partial B_R \setminus \Omega. \end{aligned}$$

Then by the maximum principle $v > 0$ on B_R and $u \leq v$ on $B_R \cap \Omega$. Since $u = v$ on $\partial B_R \cap \Omega$, $F_u \leq F_v$ on $\partial B_R \cap \Omega$. The ball B_R satisfies an interior sphere condition, so by the boundary point lemma, $F_v > 0$ on $\partial B_R \setminus \Omega$. By the divergence theorem, the total flux of v over ∂B_R must be zero, so $\int_{\partial B_R \cap \Omega} F_v dS < 0$. This implies that $\int_{\partial B_R \cap \Omega} F_u dS < 0$. By the divergence theorem, the total flux of u over $\partial(B_R \cap \Omega)$ must be zero, so $\int_{\partial\Omega \cap B_R} F_u dS > 0$. This is in contradiction with our assumption that $F_u = 0$ on Γ . $\square$

This lemma implies that $G_\xi(s)$ is strictly monotone in s , even though its derivative might occasionally be zero.

4.2. Bounds on $F_\xi(s)$. The uniform upper bound on the curvature allows us to touch a sphere of radius $R = \frac{1}{4A}$ to any point on $\partial\Omega$ such that this sphere lies outside of Ω . This together with the bounds for the distances to the VSC allows us to establish a uniform upper bound for the flux.

LEMMA 4.3. *For sufficiently large $s_{\max}$, the flux $F_\xi(s) = -\frac{\partial u}{\partial n}$ passing through the point $(r(s), z(s))$ on the boundary is bounded:*

$$(4.6) \quad F_\xi(s) \leq F_{\max}(\xi_{\min}, \xi_{\max}, A, C).$$

This implies that we can estimate $G'_\xi(s) \leq F_{\max}s$ and $G'_\xi(s) \leq 2F_{\max}$.

Proof. Let B_R be a sphere of radius $R = \frac{1}{4A}$ touching $\partial\Omega$ at the point $(r(s), z(s))$. We now solve

$$(4.7) \quad \begin{aligned} \Delta v &= -4\pi\delta(r, z + \xi) - 4\pi\delta(r, z + \eta) && \text{outside of } B_R, \\ v &= 0 && \text{on } \partial B_R. \end{aligned}$$

Since B_R lies outside of Ω , $u \leq v$ and the boundaries touch at $(r(s), z(s))$, the flux of v at this point gives us an upper bound for $F_\xi(s)$. We can determine v using reflection techniques. For simplicity we consider the sources at $z = -\xi$ and $z = -\eta$ separately and write $v = v_1 + v_2$, with v_1 and v_2 the individual contributions from these two sources. For $\frac{1}{2}\pi \leq \theta \leq \pi$ let

$$(4.8) \quad \cos \theta = \frac{z'(s)}{\sqrt{r'(s)^2 + z'(s)^2}}, \quad \sin \theta = \frac{r'(s)}{\sqrt{r'(s)^2 + z'(s)^2}}.$$

Let ρ be the distance from the VSC to the center of B_R ,

$$(4.9) \quad \rho^2 = d_\xi(s)^2 + R^2 + 2R((z(s) + \xi) \sin \theta - r(s) \cos \theta),$$

where $d_\xi(s)$ is the distance from the VSC to the point $(r(s), z(s))$. Let $(\tilde{r}, \tilde{z})$ be the point, on the line from the VSC to the center of B_R , at a distance $\tilde{\rho} = \frac{R^2}{\rho}$ from the center of the sphere. Then

$$(4.10) \quad \begin{aligned} r(s) - \tilde{r} &= \frac{R^2}{\rho^2} r(s) + \left(1 - \frac{R^2}{\rho^2}\right) R \cos \theta, \\ z(s) - \tilde{z} &= \frac{R^2}{\rho^2} (z(s) + \xi) - \left(1 - \frac{R^2}{\rho^2}\right) R \sin \theta. \end{aligned}$$

This point acts as a reflected source of strength $-4\pi\frac{R}{\rho}$. The contribution of the source at the VSC to v is given by

$$(4.11) \quad v_1(r, z) = \frac{1}{d_1(r, z)} - \frac{R}{\rho} \frac{1}{\tilde{d}_1(r, z)},$$

where $d_1(r, z)$ is the distance from the point (r, z) to the VSC, and $\tilde{d}_1(r, z)$ is the distance from (r, z) to the reflected point $(\tilde{r}, \tilde{z})$ inside B_R . Note that $d_1(r(s), z(s)) = d_\xi(s)$. If we denote the VSC as the point O , $(\tilde{r}, \tilde{z})$ as P , $(r(s), z(s))$ as X , and the center of the sphere B_R as C , then the triangles OXC and XPC are similar. Thus $\frac{d_\xi(s)}{d_1(s)} = \frac{R}{\rho} = \frac{\rho}{R}$. The contribution to the flux at $(r(s), z(s))$ is then given by

$$(4.12) \quad \begin{aligned} F_1(s) &= \frac{(z(s) + \xi) \sin \theta - r(s) \cos \theta}{d_\xi(s)^3} - \frac{R}{\rho} \frac{(z(s) - \tilde{z}) \sin \theta - (r(s) - \tilde{r}) \cos \theta}{\tilde{d}_1(r(s), z(s))^3} \\ &= \frac{\rho^2 - R^2}{R d_\xi(s)^3} = 2 \frac{(z(s) + \xi) \sin \theta - r(s) \cos \theta}{d_\xi(s)^3} + \frac{1}{R d_\xi(s)} \\ &\leq 2 \frac{\xi_{max} + 2}{d_\xi^3} + \frac{1}{R d_\xi}. \end{aligned}$$

By Lemma 3.3 we can estimate d_ξ in terms of C and ξ_{min} , while R can be expressed in terms of A . We treat the source at $z = -\eta$ similarly to obtain

$$(4.13) \quad v_2(r, z) = \frac{1}{d_2(r, z)} - \frac{R}{\rho} \frac{1}{\tilde{d}_2(r, z)}$$

and

$$(4.14) \quad \begin{aligned} F_2(s) &= 2 \frac{(z(s) + \eta) \sin \theta - r(s) \cos \theta}{d_\eta(s)^3} + \frac{1}{R d_\eta(s)} \\ &\leq 6 \frac{s_{max}}{d_\eta(s)} + \frac{1}{R d_\eta(s)}, \end{aligned}$$

where $d_2(r, z)$ and $\tilde{d}_2(r, z)$ are the distances from the point (r, z) to the source at $z = -\eta$, respectively, the reflection of this source inside B_R and $d_\eta(s) = d_2(r(s), z(s))$. Note that $d_\eta(s) \geq s_{max} - 2 - \xi_{max}$. Combining these contributions yields that $F_\xi(s) \leq F_1(s) + F_2(s) \leq F_{max}$. Since $\frac{s_{max}}{d_\eta(s)} \rightarrow 1$ and $\frac{1}{d_\eta(s)} \rightarrow 0$ as $s_{max} \rightarrow \infty$, this term can be bounded independently of s_{max} , assuming s_{max} is sufficiently large. Thus F_{max} depends on A , C , ξ_{min} , and ξ_{max} . $\square$

4.3. Bounds on $G_\xi(s)$. By using the upper bound on the flux derived in the previous section and then integrating, we can obtain estimates for the cumulative flux G_ξ . However, it will be important to have estimates on G_ξ which do not depend on the parameter A . In order to do this we will use the following comparison principle.

THEOREM 4.4 (comparison principle). *Let Ω and $\tilde{\Omega}$ be domains with $0 \in \Omega \cap \tilde{\Omega}$, with boundaries sufficiently smooth that the divergence theorem and the strong maximum principle hold. Let u and $\tilde{u}$ solve*

$$\begin{aligned} \Delta u &= -\delta(x) & \text{in } \Omega, & & u &= 0 & \text{on } \partial\Omega, \\ \Delta \tilde{u} &= -\delta(x) & \text{in } \tilde{\Omega}, & & \tilde{u} &= 0 & \text{on } \partial\tilde{\Omega}. \end{aligned}$$

Then

$$\int_{\partial\Omega \setminus \tilde{\Omega}} F_u dS \leq \int_{\partial\tilde{\Omega} \cap \Omega} F_{\tilde{u}} dS, \quad \int_{\partial\tilde{\Omega} \setminus \Omega} F_{\tilde{u}} dS \leq \int_{\partial\Omega \cap \tilde{\Omega}} F_u dS.$$

If $\partial\Omega$ and $\partial\tilde{\Omega}$ satisfy an interior sphere condition, then the inequalities are strict.

Proof. Let v solve

$$\Delta v = -\delta(x) \quad \text{in } \Omega \cap \tilde{\Omega}, \quad v = 0 \quad \text{on } \partial(\Omega \cap \tilde{\Omega}).$$

Then by the maximum principle $0 \leq v \leq u$ and $0 \leq v \leq \tilde{u}$ on $\Omega \cap \tilde{\Omega}$. Moreover, since $v = u$ on $\partial\Omega \cap \tilde{\Omega}$, $F_v \leq F_u$ on $\partial\Omega \cap \tilde{\Omega}$. Similarly, $F_v \leq F_{\tilde{u}}$ on $\partial\tilde{\Omega} \cap \Omega$. By the divergence theorem,

$$\int_{\partial\Omega \cap \tilde{\Omega}} F_u dS + \int_{\partial\Omega \setminus \tilde{\Omega}} F_u dS = \int_{\partial\Omega \cap \tilde{\Omega}} F_v dS + \int_{\partial\tilde{\Omega} \cap \Omega} F_v dS.$$

Substituting the inequalities for F_v yields the first inequality. The second inequality follows by symmetry. If the boundaries satisfy an interior sphere condition, then by the boundary point lemma $F_v < F_u$ on $\partial\Omega \cap \tilde{\Omega}$ and $F_v < F_{\tilde{u}}$ on $\partial\tilde{\Omega} \cap \Omega$, yielding strict inequalities. $\square$

COROLLARY 4.5. *The result also holds if the source $\delta(x)$ is replaced by several sources $\sum_i w_i \delta(x - x_i)$ of different positive weights w_i for $x_i \in \Omega \cap \tilde{\Omega}$. Similarly we can replace the point source $\delta(x)$ by a positive function $f(x)$ with compact support within $\Omega \cap \tilde{\Omega}$.*

This lemma allows us to compare the integrated flux on $\partial\Omega$ to that of domains for which the solution of the Dirichlet problem is exactly known, for example, spheres and half planes. In this way we establish the following bounds on $G_\xi(s)$.

LEMMA 4.6. *If $\xi \leq \frac{3}{40}$, then there exists a s^* such that $G(s^*) \geq s^*$.*

Proof. The reflected source at $z = -\eta$ contributes positively to the flux. Since we are interested in a lower bound, we can safely ignore it. For some R , $\xi < R < 2$, let B_R be a ball of radius R centered around the VSC. The surface of the ball may intersect $\partial\Omega$ in multiple points; let s^* identify the coordinate of the point farthest from the tip where $\partial\Omega$ intersects the surface of this ball:

$$(4.15) \quad s^* = \max\{s \mid (r(s), z(s)) \in \partial\Omega \cap \partial B_R\}.$$

Let ∂B_R^+ be that part of ∂B_R which lies in the positive z half space. Note that since $z(s) \leq 0$ and $R > \xi$, ∂B_R^+ is nonempty and lies outside of Ω . The comparison principle, Theorem 4.4, now states that

$$(4.16) \quad \begin{aligned} 2\pi G_\xi(s^*) &\geq \int_{\partial\Omega \cap B_R} F_u dS \geq \int_{\partial B_R \setminus \Omega} \frac{1}{R^2} dS \geq \int_{\partial B_R^+} \frac{1}{R^2} dS \\ &\geq \pi \frac{R^2 - (R - \xi)^2}{R^2}. \end{aligned}$$

We now set $R = 2\xi$ to obtain that $G(s^*) \geq \frac{3}{8}$. The point $(r(s^*), z(s^*))$ lies on ∂B_R , so $r(s^*) \leq 2\xi$ and $z(s^*) \geq -3\xi$. By integrating (3.2), $s^* \leq r(s^*) - z(s^*) \leq 5\xi$, and so if $\xi \leq \frac{3}{40}$, then $s^* \leq \frac{3}{8} \leq G_\xi(s^*)$. $\square$

LEMMA 4.7. *If s_{max} is chosen large enough that $s_{max} \geq \xi_{max} + 2$, then*

$$(4.17) \quad G_\xi(s) \leq 8 \left(\frac{s}{\xi} \right)^2 \quad \text{for } 0 \leq s \leq \min \left(C^{-1}, \sqrt{\xi_{min} C^{-1}} \right).$$

Proof. We wish to use the estimates at the tip derived in section 3.2, so we must require that $s \leq C^{-1}$. Furthermore if $s \leq \sqrt{\xi_{\min} C^{-1}}$, then by (3.7), $z(s) \geq -\frac{1}{2}\xi$. Also, if $s_{\max} \geq \xi_{\max} + 2$, then by (3.8), $z(s_{\max}) \leq -\xi_{\max}$, so $\eta = -2z(s_{\max}) - \xi \geq 2\xi_{\max} - \xi$. This enables us to estimate the difference in the z coordinate between the point at pathlength s , the source at the VSC, and the reflected source:

$$(4.18) \quad z(s) + \xi \geq \frac{1}{2}\xi \quad \text{and} \quad z(s) + \eta \geq \frac{1}{2}\xi.$$

We define $\tilde{\Omega}$ to be the half space $\{(r, z) | z \leq z(s)\}$. Let

$$(4.19) \quad \begin{aligned} \tilde{u}(r, z) = & (r^2 + (z + \xi)^2)^{-\frac{1}{2}} - (r^2 + (z - 2z(s) - \xi)^2)^{-\frac{1}{2}} \\ & + (r^2 + (z + \eta)^2)^{-\frac{1}{2}} - (r^2 + (z - 2z(s) - \eta)^2)^{-\frac{1}{2}}. \end{aligned}$$

Then $\tilde{u}$ solves the Dirichlet problem on $\tilde{\Omega}$ with sources at $z = -\xi$ and $z = -\eta$. The comparison principle for integrated fluxes now states that

$$(4.20) \quad 2\pi G_\xi(s) = \int_{\partial\Omega \setminus \tilde{\Omega}} F_u dS \leq \int_{\partial\tilde{\Omega} \cap \Omega} F_{\tilde{u}} dS$$

$$(4.21) \quad = 4\pi \left[1 - \left(1 + \left(\frac{r(s)}{\xi + z(s)} \right)^2 \right)^{-\frac{1}{2}} + 1 - \left(1 + \left(\frac{r(s)}{\eta + z(s)} \right)^2 \right)^{-\frac{1}{2}} \right].$$

Now, using the inequality $1 - 1/\sqrt{1+x} \leq \frac{1}{2}x$, the upper bound $r(s) \leq s$, and the estimates (4.18), we obtain

$$(4.22) \quad G_\xi(s) \leq \left(\frac{r(s)}{\xi + z(s)} \right)^2 + \left(\frac{r(s)}{\eta + z(s)} \right)^2 \leq 8 \left(\frac{s}{\xi} \right)^2. \quad \square$$

4.4. Monotonicity of G_ξ in ξ . We wish to know how the cumulative flux G_ξ changes as the distance ξ between the tip and the VSC is varied while the domain Ω remains fixed. We can write the Dirichlet problem as follows: Let $\tilde{u}_\xi(r, z)$ solve

$$(4.23) \quad \begin{aligned} \Delta \tilde{u}_\xi &= 0 && \text{in } \Omega, \\ \tilde{u}_\xi &= -\frac{1}{(r^2 + (z + \xi)^2)^{\frac{1}{2}}} - \frac{1}{(r^2 + (z + \eta)^2)^{\frac{1}{2}}} && \text{on } \partial\Omega. \end{aligned}$$

Then

$$(4.24) \quad u_\xi(r, z) = \tilde{u}_\xi(r, z) + \frac{1}{(r^2 + (z + \xi)^2)^{\frac{1}{2}}} + \frac{1}{(r^2 + (z + \eta)^2)^{\frac{1}{2}}}$$

solves (2.13). We now differentiate with respect to ξ ; note that $\frac{d\eta}{d\xi} = -1$. Let $\tilde{v}_\xi(r, z)$ solve

$$(4.25) \quad \begin{aligned} \Delta \tilde{v}_\xi &= 0 && \text{in } \Omega, \\ \tilde{v}_\xi &= \frac{z + \xi}{(r^2 + (z + \xi)^2)^{\frac{3}{2}}} - \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}} && \text{on } \partial\Omega. \end{aligned}$$

By Lemma 3.3, $\tilde{v}_\xi$ is bounded on $\partial\Omega$, and so by the maximum principle it is bounded in Ω . We define $v_\xi(r, z)$ as

$$(4.26) \quad v_\xi(r, z) = \tilde{v}_\xi(r, z) - \frac{z + \xi}{(r^2 + (z + \xi)^2)^{\frac{3}{2}}} + \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}}.$$

Then $v_\xi = \frac{\partial}{\partial \xi} u_\xi$, and essentially v_ξ solves a Dirichlet problem with zero boundary condition and two dipoles of opposite orientation at $z = -\xi$ and $z = -\eta$ as sources. We will first show that Ω can be divided into two connected subsets, where v_ξ is positive or negative. We will then show that this also divides the boundary $\partial\Omega$ into two connected subsets, bordering the respective subsets of Ω . Lastly we will examine the cumulative flux of v_ξ and prove a monotonicity result for $G_\xi(s)$.

Near the dipole source, Ω is divided into two regions where v_ξ is positive or negative with a surface separating these two. The following lemma states this division can be extended to the whole domain.

LEMMA 4.8. *Let v_ξ be defined as in (4.26) on a domain Ω sufficiently smooth that the maximum principle holds and such that the points $(0, -\xi)$ and $(0, -\eta)$ lie in the interior of Ω . Let*

$$(4.27) \quad \begin{aligned} \Omega^+ &= \{(r, z, \theta) \in \Omega \mid v(r, z) > 0 \text{ and } z > z(s_{max})\}, \\ \Omega^- &= \{(r, z, \theta) \in \Omega \mid v(r, z) < 0 \text{ and } z > z(s_{max})\}. \end{aligned}$$

Then for sufficiently large s_{max} , Ω^- and Ω^+ are connected sets.

Proof. Let B be a ball centered around the point $(0, -\xi)$, such that $d = \text{dist}(\partial B, \partial\Omega) > 0$. The boundary condition imposed on $\tilde{v}_\xi$ in (4.25) is bounded and continuous, and thus by the maximum principle, $\tilde{v}_\xi$ is bounded in Ω . By [7, Theorem 2.10] the derivatives of $\tilde{v}_\xi$ are bounded in B , $\sup_B \left| \frac{\partial \tilde{v}_\xi}{\partial z} \right| \leq \frac{3}{d} \sup_\Omega |\tilde{v}_\xi|$. We now examine $v_\xi(r, z)$ on the cylinder defined by $r \leq R$ and $|z + \xi| \leq Z$ for some sufficiently small R and $Z = \frac{1}{\sqrt{3}}R$. Since $\eta = O(s_{max})$, the contribution of the reflected source to v_ξ in this cylinder is $O(s_{max}^{-2})$ and its contribution to $\frac{\partial v_\xi}{\partial z} = O(s_{max}^{-3})$, so we can choose s_{max} sufficiently large that

$$(4.28) \quad \left| \frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}} \right| \leq 1 \quad \text{and} \quad \left| \frac{\partial}{\partial z} \left(\frac{z + \eta}{(r^2 + (z + \eta)^2)^{\frac{3}{2}}} \right) \right| \leq 1.$$

We now estimate v_ξ on the caps of the cylinder,

$$(4.29) \quad \begin{aligned} v_\xi(r, Z - \xi) &\leq \sup_\Omega |\tilde{v}_\xi| + 1 - \frac{Z}{(R^2 + Z^2)^{\frac{3}{2}}} \\ &\leq \sup_\Omega |\tilde{v}_\xi| + 1 - \frac{9}{8} \frac{1}{R^2}, \\ v_\xi(r, -Z - \xi) &\geq -(\sup_\Omega |\tilde{v}_\xi| + 1) + \frac{Z}{(R^2 + Z^2)^{\frac{3}{2}}} \\ &\geq -(\sup_\Omega |\tilde{v}_\xi| + 1) + \frac{9}{8} \frac{1}{R^2}, \end{aligned}$$

while on the cylinder,

$$(4.30) \quad \frac{\partial v_\xi}{\partial z}(R, z) \leq \sup_B \left| \frac{\partial \tilde{v}_\xi}{\partial z} \right| + 1 - \frac{R^2 - 2Z^2}{(R^2 + Z^2)^{\frac{5}{2}}} \leq \sup_B \left| \frac{\partial \tilde{v}_\xi}{\partial z} \right| + 1 - \frac{3\sqrt{3}}{32} \frac{1}{R^3}.$$

Therefore we can choose an R_{max} such that for all $R < R_{max}$, $v_\xi(R, Z - \xi) < 0$, $v_\xi(R, -Z - \xi) > 0$, and $\frac{\partial v_\xi}{\partial z}(R, z) < 0$. Thus $z \rightarrow v_\xi(R, z)$ has a unique zero z_0 . In other words, there exists a function $z_0(r)$ defined on the interval $[0, R_{max}]$ such that $|z_0(r) + \xi| \leq \frac{1}{\sqrt{3}}r$ and $v_\xi(r, z_0(r)) = 0$. By the implicit function theorem, z_0

is continuous and so this function defines a surface Σ which is part of the interface between Ω^+ and Ω^- . By continuity, $\partial\Omega^+ \cap \Sigma$ and $\partial\Omega^- \cap \Sigma$ are both nonempty and both boundaries contain the dipole at $z = -\xi$. Let Ω_1 and Ω_2 be two connected components of Ω^+ or Ω^- . Assume $(0, -\xi) \notin \partial\Omega_{1,2}$; then v_ξ is harmonic in $\Omega_{1,2}$ (note that we excluded the singularity in $(0, -\eta)$ by demanding that $z > z(s_{max})$), and $v_\xi = 0$ on $\partial\Omega_{1,2}$. The maximum principle then implies that $v_\xi = 0$ on $\Omega_{1,2}$, which is a contradiction. However, since all points in the neighborhood of $(0, -\xi)$ for which $v_\xi = 0$ are contained in a subset $\tilde{\Sigma}$ of Σ , $\tilde{\Sigma} \subset \partial\Omega_{1,2}$ so Ω_1 and Ω_2 are connected and must be equal to one another. $\square$

The division of Ω into two regions of positive and negative v_ξ similarly divides the boundary into two parts.

LEMMA 4.9. *Let Ω be a domain sufficiently smooth such that the maximum principle holds, and let Ω^+ , Ω^- , and v_ξ be defined as in the previous lemma. Then $\partial\Omega^+ \cap \partial\Omega$ and $\partial\Omega^- \cap \partial\Omega$ are closed connected sets.*

Proof. This is essentially a two-dimensional argument; since we assume radial symmetry we restrict ourselves to some plane at $\theta = 0$. We examine the region on the boundary with positive or negative flux. Let $I^\pm = \{s \in [0, s_{max}] | (r(s), z(s)) \in \partial\Omega^\pm \cap \partial\Omega\}$ be the set of parameter values whose respective points on the boundary border Ω^+ , respectively, Ω^- . Clearly these are closed sets. Let R_{max} be as in the proof of Lemma 4.8 and let $z^\pm = -\xi \pm \frac{1}{\sqrt{3}}R_{max} - \xi$. By the arguments of the previous lemma, $(0, z^+) \in \tilde{\Omega}^-$ and $(0, z^-) \in \tilde{\Omega}^+$.

Letting $s^- \in I^-$ and $s^+ \in I^+$, we will first show that $s^- \leq s^+$. Assume $s^- > s^+$; since Ω^+ and Ω^- are connected, there exist paths connecting $(r(s^-), z(s^-))$ to $(0, z^+)$ and $(r(s^+), z(s^+))$ to $(0, z^-)$, such that these paths lie completely inside Ω^- , respectively, Ω^+ . Since $z(s^+) > z(s^-)$ by the monotonicity of z , clearly these paths must intersect, which is a contradiction. Thus for $s^- \in I^-$, all points $s < s^-$ are also in I^- ; similarly for $s^+ \in I^+$ all points $s > s^+$ are in I^+ . Thus I^+ and I^- are closed intervals. $\square$

We now have enough information on v_ξ near the boundary to prove the following monotonicity result.

LEMMA 4.10. *For sufficiently large s_{max} the cumulative flux G_ξ is strictly monotone and differentiable in ξ :*

$$(4.31) \quad \frac{\partial G_\xi}{\partial \xi}(s) < 0.$$

Furthermore, this derivative is $C^{1,\alpha}$ Hölder continuous.

Proof. We wish to examine the cumulative flux $H_\xi(s)$ of v_ξ ,

$$(4.32) \quad H_\xi(s) = \frac{\partial}{\partial \xi} G_\xi(s) = \int_0^s F_v(\sigma) r(\sigma) d\sigma,$$

where $F_v(\sigma) = -\frac{\partial v_\xi}{\partial n}(r(\sigma), z(\sigma))$. By Lemmas 4.2 and 4.9 there are two closed intervals I^+ and I^- such that the flux F_v is positive, respectively, negative, everywhere on these intervals and cannot be zero on an open subinterval. If $s^* \in [0, s_{max}] \setminus (I^+ \cup I^-)$, then there would exist an $R > 0$ such that a ball of radius R around $(r(s^*), z(s^*))$ lies neither in Ω^+ nor in Ω^- (where $\Omega^\pm$ is defined as in Lemma 4.8). This is a contradiction since v_ξ cannot be zero on an open subset of Ω . Therefore the union of I^+ and I^- is the whole interval $[0, s_{max}]$. The flux must be zero on the intersection of these two intervals, and so this intersection must be either empty or equal to a

singleton $\{s_0\}$. It cannot be empty since the union of two disjoint closed intervals cannot be an interval. Therefore $I^- = [0, s_0]$ and $I^+ = [s_0, s_{max}]$. The cumulative flux $H_\xi(s)$ is strictly decreasing for $s \in I^-$ and strictly increasing for $s \in I^+$, by the divergence theorem $H_\xi(s_{max}) = 0$ so $H_\xi(s) < 0$ for $s \in (0, s_{max})$. $\square$

5. The traveling wave ODE. In this section we will assume we are given a family of functions $G_\xi(s)$ parametrized by ξ with

$$(5.1) \quad \begin{aligned} 0 < G_\xi(s) < 2 & \quad \text{for } 0 < s < s_{max}, \\ G_\xi(s_{max}) &= 2, \\ G'_\xi(s) &\geq 0, \\ G_\xi(s) &\leq \frac{1}{2}\tilde{C}(\xi)s^2 \quad \text{for } 0 \leq s \leq \tilde{C}(\xi)^{-1}, \end{aligned}$$

where the constant $\tilde{C}(\xi)$ is given in Lemma 4.7. Using this, we will solve the traveling wave ODE for each ξ ,

$$(5.2) \quad r'_\xi(s) = \sqrt{1 - \left(\frac{G_\xi(s)}{r_\xi(s)}\right)^2}, \quad r_\xi(0) = 0, \quad r'_\xi(0) = 1,$$

and show that there exists a ξ^* such that $r_{\xi^*}(s_{max}) = 2$. Since this differential equation is not Lipschitz, we cannot use standard arguments for existence and uniqueness of solutions. In fact, there are many solutions satisfying $r_\xi(0) = 0$; in section 5.1 we will use a contraction argument to show that there is a unique solution $r_{f,\xi}$ which also satisfies $r'_{f,\xi}(0) = 1$. In section 5.3 we then show there is a unique solution $r_{b,\xi}$ satisfying $r_{b,\xi}(s_{max}) = 2$. Finally in section 5.4 we show that for $\xi = \xi^*$ these two solutions match, giving the desired solution.

5.1. The forward solution. In this section we show that for each ξ there exists a solution starting at $s = 0$. We substitute $r_\xi(s) = s - s^3x(s)$. A function $x(s)$ solving the ODE must then be a fixed point of the integral operator Φ :

$$(5.3) \quad \Phi[x](s) = \frac{1}{s^3} \int_0^s 1 - \sqrt{1 - h(x(\sigma), \sigma)} d\sigma,$$

where

$$(5.4) \quad h(x, s) = \left(\frac{G_\xi(s)}{s - s^3x}\right)^2.$$

We examine Φ on the ball B of radius $\frac{1}{9}\tilde{C}(\xi)^2$ in the space of continuous bounded functions on the interval $[0, \tilde{C}(\xi)^{-1}]$ equipped with the supremum norm.

LEMMA 5.1. *The integral operator Φ has a unique fixed point on B .*

Proof. First of all $\Phi : B \rightarrow B$. To see this, assume $x(s) \leq a\tilde{C}(\xi)^2$ for some value of a . If $a \leq \frac{1}{2}$, then for $s \leq \tilde{C}(\xi)^{-1}$, $\frac{G_\xi(s)}{s - s^3x(s)} \leq \frac{1}{2} \frac{\tilde{C}(\xi)}{1-a} s \leq \frac{1}{2(1-a)} \leq 1$. Now for $h \leq 1$, $1 - \sqrt{1-h} \leq h$ and so $\Phi[x](s) \leq \frac{1}{12} \frac{\tilde{C}(\xi)^2}{(1-a)^2}$. If we set $a = \frac{1}{9}$, then $\Phi[x](s) \leq a\tilde{C}(\xi)^2$.

Furthermore, Φ is a contraction on B . Let $\|\cdot\|$ be the supremum norm on B :

$$(5.5) \quad \|\Phi[x_2] - \Phi[x_1]\| \leq \frac{1}{s^3} \int_0^s \sup_{x \in [x_1, x_2]} \left| \frac{\partial}{\partial x} \left(1 - \sqrt{1 - h(x, \sigma)}\right) \right| |x_2(\sigma) - x_1(\sigma)| d\sigma.$$

Now,

$$(5.6) \quad \begin{aligned} \frac{\partial}{\partial x} \left(1 - \sqrt{1 - h(x, s)} \right) &= \frac{G_\xi(s)^2}{(1 - s^2 x)^3 \sqrt{1 - h(x, s)}} \\ &\leq \frac{1}{4(1-a)^3 \sqrt{1 - \frac{1}{4(1-a)^2}}} s^2 \leq \frac{1}{2} s^2 \quad \text{for } a = \frac{1}{9}. \end{aligned}$$

Therefore, $\|\Phi[x_2] - \Phi[x_1]\| \leq \frac{1}{6} \|x_2 - x_1\|$. By the Banach fixed point theorem, Φ must have a fixed point. $\square$

We will denote this fixed point as $r_{f,\xi}$.

5.2. Determination of ξ_{min} and ξ_{max} . We wish to find a value ξ^* such that r_{f,ξ^*} can be continued to the interval $(0, s_{max}]$ with $r_{f,\xi^*}(s_{max}) = 2$. Clearly, since $r_{f,\xi}(s) \leq s$, if at some point s^* , $G_\xi(s^*) \geq s^*$, then we cannot continue the solution at this value of ξ since $r'_{f,\xi}(s^*)$ would not be defined. By Lemma 4.6 such a value of s^* exists, and so $\xi^* \geq \xi_{min} = \frac{3}{40}$.

If there exists an $s^* \leq s_{max}$ such that $r_{f,\xi}(s^*) \geq 2$, the solution can be continued till infinity; however, due to the monotonicity of $r_{f,\xi}$ it will be impossible to meet the requirement that $r_{f,\xi}(s_{max}) = 2$. By Lemma 5.1, $r_{f,\xi}(\tilde{C}(\xi)^{-1}) \geq \frac{8}{9} \tilde{C}(\xi)^{-1}$, so if $\tilde{C}(\xi) \leq \frac{4}{9}$ and $\tilde{C}(\xi)^{-1} < s_{max}$, then the continuation of r_ξ will grow too large. By Lemma 4.7, $\tilde{C}(\xi) = \frac{16}{\xi^2}$ and so $\xi^* \leq \xi_{max} = 6$.

5.3. The backward solution. In this section we will show that there exists a unique solution from $s = s_{max}$ with $r_\xi(s_{max}) = 2$ extending backward till $s = 0$.

LEMMA 5.2. *Given $\xi \in (\xi_{min}, \xi_{max})$ and $s^* \in (0, s_{max})$, then any two functions r_1 and r_2 solving (5.2) on an interval $(s_0, s^*]$ having equal endpoints, $r_1(s^*) = r_2(s^*)$, must be equal over the entire interval.*

Proof. We write the ODE as $r' = f(r(s), s)$ and examine the derivative to r :

$$(5.7) \quad \frac{\partial f}{\partial r} = \frac{1}{f(r, s)} \frac{G_\xi(s)^2}{r^3} > 0.$$

Now assume r_1 and r_2 are two different solutions to (5.2) on an interval $I = (s^* - \delta, s^*)$ such that $r_1(s^*) = r_2(s^*)$. We will show that there is a contradiction. If r_1 and r_2 are different, there exists an $s_0 \in I$ such that $r_2(s_0) \neq r_1(s_0)$. Without loss of generality, assume that $r_2(s_0) > r_1(s_0)$. The difference between r_2 and r_1 satisfies the differential equation

$$(5.8) \quad \frac{d}{ds} (r_2(s) - r_1(s)) = \int_{r_1(s)}^{r_2(s)} \frac{\partial f}{\partial r} dr > 0,$$

and so $r_2(s) - r_1(s) > r_2(s_0) - r_1(s_0) > 0$ for all $s > s_0$, specifically at $s = s^*$. $\square$

LEMMA 5.3. *Given $\xi \in (\xi_{min}, \xi_{max})$ and $s^* \in (0, s_{max}]$, there then exists a unique solution $r : (0, s^*] \rightarrow [0, 2]$ to the differential equation (5.2) satisfying $r(s^*) = G_\xi(s^*)$.*

Proof. We examine the differential equation

$$(5.9) \quad r'(s) = f(s, r(s)),$$

where

$$(5.10) \quad f(s, r) = \begin{cases} \sqrt{1 - \left(\frac{G_\xi(s)}{r}\right)^2} & \text{if } r \geq G_\xi(s) \text{ and } s \leq s_{max}, \\ 0 & \text{if } r \leq G_\xi(s) \text{ and } s \leq s_{max}, \\ \sqrt{1 - \left(\frac{2}{r}\right)^2} & \text{if } r \geq 2 \text{ and } s \geq s_{max}, \\ 0 & \text{if } r \leq 2 \text{ and } s \geq s_{max}. \end{cases}$$

The function f is continuous, so by Peano's existence theorem there exist solutions $s \mapsto r(s)$ satisfying $r(s^*) = G(s^*)$ defined in a neighborhood of s^* . Assume there is an $s_0 < s^*$ in this neighborhood such that $r(s_0) < G_\xi(s_0)$. Letting $s_1 = \min\{s \in (s_0, s^*] | r(s) \geq G_\xi(s)\}$, this definition makes sense since r and G_ξ are continuous and $r(s^*) = G_\xi(s^*)$. By the mean value theorem there exists an $s_2 \in (s_0, s_1)$ such that

$$(5.11) \quad r'(s_2) = \frac{r(s_1) - r(s_0)}{s_1 - s_0} > \frac{G_\xi(s_1) - G_\xi(s_0)}{s_1 - s_0} > 0,$$

since $G_\xi(s)$ is strictly monotone. Thus by (5.9), $r(s_2) \geq G_\xi(s_2)$, which is in contradiction with our definition of s_1 . Therefore $r(s) \geq G_\xi(s)$ for $s \leq s^*$ and the restriction of our solution to $s \leq s_{max}$ solves (5.2); repeating this argument enables us to extend this solution until $s = 0$. Uniqueness then follows from Lemma 5.2. $\square$

Setting $s^* = s_{max}$ in Lemma 5.3, we obtain a unique solution on $(0, s_{max}]$.

5.4. Matching. In the previous sections we have constructed solutions to the traveling wave ODE. The forward solution, which we denote as $r_{f,\xi}$, exists on an interval $[0, \bar{C}(\xi)]$, while the backward solution $r_{b,\xi}$ exists on the interval $(0, s_{max}]$. We will examine both solutions at the point $s = \bar{C}^{-1}$, where

$$(5.12) \quad \bar{C}^{-1} = \min_{\xi \in [\xi_{min}, \xi_{max}]} \{\bar{C}(\xi)^{-1}\}.$$

If we examine the solutions at $\xi = \xi_{min}$, we see that at some point s^* , $r_{f,\xi_{min}}(s^*) = G_{\xi_{min}}(s^*) \leq r_{b,\xi_{min}}(s^*)$. Since solutions to the same ODE cannot intersect,

$$(5.13) \quad r_{f,\xi_{min}}(\bar{C}^{-1}) \leq r_{b,\xi_{min}}(\bar{C}^{-1}).$$

Similarly, examining the solution at $\xi = \xi_{max}$, we see that $r_{f,\xi_{max}}(\bar{C}(\xi_{max})^{-1}) \geq 2 \geq r_{b,\xi_{max}}(\bar{C}(\xi_{max})^{-1})$ and so

$$(5.14) \quad r_{f,\xi_{max}}(\bar{C}^{-1}) \geq r_{b,\xi_{max}}(\bar{C}^{-1}).$$

LEMMA 5.4. *The forward and backward solutions $r_{f,\xi}$ and $r_{b,\xi}$ are strictly monotone in ξ for $s > 0$,*

$$(5.15) \quad \frac{\partial r_{f,\xi}}{\partial \xi} > 0, \quad \frac{\partial r_{b,\xi}}{\partial \xi} < 0.$$

Proof. If we differentiate the ODE (5.2) to ξ , we see that $u = \frac{\partial r_\xi}{\partial \xi}$ satisfies the differential equation

$$(5.16) \quad \frac{du}{ds}(s) = f(s)u(s) + g(s),$$

where

$$(5.17) \quad f(s) = \frac{G_\xi(s)^2}{r'_\xi(s)r_\xi(s)^3}, \quad g(s) = -\frac{G_\xi(s)}{r'_\xi(s)r_\xi(s)^2} \frac{\partial G_\xi}{\partial \xi}.$$

Since r_ξ , r'_ξ , and G_ξ are positive, $f(s) > 0$ for $s > 0$, and by Lemma 4.10, $g(s) > 0$ for $s > 0$.

To study the forward solution, we examine the solution of this ODE with initial condition $u(0) = 0$. By (3.5) and Lemma 4.7 it is clear that f and g remain bounded as $s \rightarrow 0$, so by the variation of constants formula

$$(5.18) \quad \frac{\partial r_{f,\xi}}{\partial \xi}(s) = e^{-\mu(s)} \int_0^s e^{\mu(\sigma)} g(\sigma) d\sigma > 0,$$

where

$$(5.19) \quad \mu(s) = \int_0^s f(\sigma) d\sigma.$$

For the backward solution, $r_{b,\xi}(s_{max}) = G_\xi(s_{max}) = 2$ for all ξ . So a similar variation-of-constants argument, with $u(s_{max}) = 0$, yields that $\frac{\partial r_{b,\xi}}{\partial \xi} < 0$. $\square$

The inequalities for the forward and backward solutions at $s = \tilde{C}^{-1}$, together with the strict monotonicity established in the above lemma, yield that there must exist a unique $\xi^* \in [\xi_{min}, \xi_{max}]$ such that

$$(5.20) \quad r_{f,\xi^*}(\tilde{C}^{-1}) = r_{b,\xi^*}(\tilde{C}^{-1}).$$

We now define $r_{\xi^*}(s)$ to be equal to r_{f,ξ^*} if $s \leq \tilde{C}^{-1}$ and equal to r_{b,ξ^*} otherwise. Thus r_{ξ^*} solves (5.2) with the desired boundary conditions at $s = 0$ and $s = s_{max}$.

6. Fixed point of the Schauder map. In sections 4 and 5 we defined a map from $\Xi(M, A, C; s_{max})$ to $C^{1,\alpha} \times C^{0,\alpha}$. In this section we will choose the constants C , A , and M such that the image of this map is a subset of the original domain. We will then show that both domain and image are compact and the map is continuous in the $C^{1,\beta}$ topology for any $\beta < \alpha$, and thereby satisfies all the requirements of Schauder's fixed point theorem.

6.1. Determination of C . From Lemma 4.7 we have that

$$(6.1) \quad G_{\xi^*}(s) \leq \frac{1}{2} \tilde{C} s^2 \quad \text{for } 0 \leq s \leq \tilde{C}^{-1},$$

where $\tilde{C} = \max(\frac{16}{\xi_{min}^2}, \sqrt{\frac{C}{\xi_{min}}}, C)$. From Lemma 5.1 and the monotonicity of r_{ξ^*} we then have that

$$(6.2) \quad \begin{aligned} s - \frac{1}{9} \tilde{C}^2 s^3 &\leq r_{\xi^*}(s) \leq 2 \quad \text{for } 0 \leq s \leq \tilde{C}^{-1}, \\ \frac{8}{9} \tilde{C}^{-1} &\leq r_{\xi^*}(s) \leq 2 \quad \text{for } \tilde{C}^{-1} \leq s \leq s_{max}. \end{aligned}$$

If we choose $C \geq \frac{16}{\xi_{min}^2} = \frac{25600}{9}$, then $\tilde{C} = C$.

6.2. Determination of A . If we were to differentiate (5.2) to s , we would obtain

$$(6.3) \quad r_{\xi^*}'' = \frac{G_{\xi^*}^2}{r_{\xi^*}^3} - \frac{G_{\xi^*}}{r_{\xi^*}^2} \frac{G_{\xi^*}'}{r_{\xi^*}'}, \quad z_{\xi^*}'' = \frac{G_{\xi^*}}{r_{\xi^*}^2} r_{\xi^*}' - \frac{G_{\xi^*}'}{r_{\xi^*}}.$$

Since r_{ξ^*}' might be equal to zero, the negative contribution to r_{ξ^*}'' may be infinite and we cannot say that r is twice differentiable. The difference quotients of the first derivatives are, however, always defined, and equal to

$$(6.4) \quad r_{\xi^*}'(s_2) - r_{\xi^*}'(s_1) = \int_{s_1}^{s_2} \frac{G_{\xi^*}^2}{r_{\xi^*}^3} - \frac{G_{\xi^*}}{r_{\xi^*}^2} \frac{G_{\xi^*}'}{r_{\xi^*}'} ds \leq \int_{s_1}^{s_2} \frac{G_{\xi^*}^2}{r_{\xi^*}^3} ds,$$

where without loss of generality we can assume that $s_2 \geq s_1$. Now using (6.1), (6.2), and the fact that by the divergence theorem $G_{\xi^*}(s) \leq 2$, one can estimate

$$(6.5) \quad \begin{aligned} \frac{G_{\xi^*}^2}{r_{\xi^*}^3} &\leq \frac{1}{4} \frac{C^2}{(1 - \frac{1}{9}C^2s^2)^3} s \leq \frac{1}{4} \left(\frac{9}{8}\right)^3 C \quad \text{for } 0 \leq s \leq C^{-1}, \\ \frac{G_{\xi^*}^2}{r_{\xi^*}^3} &\leq 4 \left(\frac{9}{8}\right)^3 C^3 \quad \text{for } C^{-1} \leq s \leq s_{max}, \\ \frac{G_{\xi^*}}{r_{\xi^*}^2} &\leq \frac{1}{2} \frac{C}{(1 - \frac{1}{9}C^2s^2)^2} \leq \frac{1}{4} \left(\frac{9}{8}\right)^2 C \quad \text{for } 0 \leq s \leq C^{-1}, \\ \frac{G_{\xi^*}}{r_{\xi^*}^2} &\leq 2 \left(\frac{9}{8}\right)^2 C^2 \quad \text{for } C^{-1} \leq s \leq s_{max}. \end{aligned}$$

All four of these estimates are bounded and depend only on the constant C determined previously, and so we can choose a suitable A such that we can estimate

$$(6.6) \quad r_{\xi^*}'(s_2) - r_{\xi^*}'(s_1) \leq A(s_2 - s_1).$$

Similar arguments yield the estimate for the difference quotient on z_{ξ^*}' .

6.3. Hölder continuity. We have now established enough bounds on r_{ξ^*} and G_{ξ^*} to establish a uniform $C^{1,\alpha}$ Hölder norm. Since r_{ξ^*}' is the square root of a C^1 function, we expect it to be bounded for Hölder exponent $\alpha \leq \frac{1}{2}$.

LEMMA 6.1. *There is an M , independent of s_{max} , such that the solutions r_{ξ^*} and z_{ξ^*}' have the following Hölder norms for Hölder exponent $\alpha \leq \frac{1}{2}$:*

$$(6.7) \quad \begin{aligned} \|r_{\xi^*}\|_{1,\alpha} &\leq M, \\ \|z_{\xi^*}'\|_{0,\alpha} &\leq M. \end{aligned}$$

Proof. We estimate the difference in first derivatives at points s_1 and s_2 :

$$(6.8) \quad \begin{aligned} |r_{\xi^*}'(s_2) - r_{\xi^*}'(s_1)| &= \left| \sqrt{1 - \left(\frac{G_{\xi^*}(s_2)}{r_{\xi^*}(s_2)}\right)^2} - \sqrt{1 - \left(\frac{G_{\xi^*}(s_1)}{r_{\xi^*}(s_1)}\right)^2} \right| \\ &\leq \sqrt{\left| \left(\frac{G_{\xi^*}(s_2)}{r_{\xi^*}(s_2)}\right)^2 - \left(\frac{G_{\xi^*}(s_1)}{r_{\xi^*}(s_1)}\right)^2 \right|} \\ &\leq \left| \max \left(\frac{d}{ds} \left(\frac{G_{\xi^*}^2}{r_{\xi^*}^2} \right) \right) \right|^{\frac{1}{2}} \cdot |s_2 - s_1|^{\frac{1}{2}}. \end{aligned}$$

Now,

$$(6.9) \quad \frac{d}{ds} \left(\frac{G_{\xi^*}(s)^2}{r_{\xi^*}(s)^2} \right) = \frac{2G_{\xi^*}G'_{\xi^*}}{r_{\xi^*}^2} - \frac{2G_{\xi^*}^2 r'_{\xi^*}}{r_{\xi^*}^3},$$

and using Lemma 4.3 and estimates (6.5) we see all these terms are uniformly bounded by constants depending only on C , A , $\xi_{\min}$, and $\xi_{\max}$. Similarly, by (6.3) and (6.5), z'' can be bounded from above by A , and from below by a constant depending only on C , A , and $F_{\max}$, and so the Hölder norm of z' can be bound similarly. The constants C , A , $\xi_{\max}$, and $\xi_{\min}$ have been determined in the previous sections; $F_{\max}$ depends only on these constants, and thus M can be expressed in terms of these known constants. Specifically, M does not depend on $s_{\max}$. $\square$

6.4. Continuity of $(r, z) \rightarrow G_{\xi}$. We wish to show that, for a sequence $(r_n, z'_n) \in C^{1,\alpha} \times C^{0,\alpha}$ uniformly, describing boundaries $\partial\Omega_n$ such that $(r_n, z_n) \rightarrow (\bar{r}, \bar{z})$ in the $C^{1,\beta}$ topology, the associated fluxes $F_n \rightarrow \bar{F}$ in the $C^{0,\beta}$ topology. In order to do this we need to create bijections between the domains Ω_n and $\bar{\Omega}$.

LEMMA 6.2. *There exist surjective mappings $\phi_n : \bar{\Omega} \rightarrow \Omega_n \subset \mathbb{R}^3$ such that $\phi_n \in C^{1,\alpha}$ uniformly, and $\phi_n \rightarrow \text{Id}$ in the $C^{1,\beta}$ topology.*

Proof. We first define ϕ_n on the boundary $\partial\bar{\Omega}$ by mapping the point at pathlength s on $\partial\bar{\Omega}$ to the point at the same pathlength on $\partial\Omega_n$. For $x = (x_1, x_2, x_3) = (\bar{r}(s) \cos \theta, \bar{r}(s) \sin \theta, \bar{z}(s)) \in \partial\bar{\Omega}$,

$$(6.10) \quad \begin{aligned} \phi_n^{(1)}(x) &= r_n(s) \cos \theta, \\ \phi_n^{(2)}(x) &= r_n(s) \sin \theta, \\ \phi_n^{(3)}(x) &= z_n(s). \end{aligned}$$

We treat these as boundary conditions and extend $\phi_n^{(i)}$ to the interior by solving $\Delta\phi_n^{(i)} = 0$ for $i = 1, 2, 3$. By Schauder's boundary estimates for $C^{1,\alpha}$ domains [7, Theorem 8.33], $\|\phi_n^{(i)}\|_{1,\alpha} \leq C(\|r_n\|_{1,\alpha} + \|z_n\|_{1,\alpha})$, where the constant C depends only on the boundary $\partial\bar{\Omega}$. As such, the functions $\phi_n^{(i)}$ are uniformly bounded in the $C^{1,\alpha}$ Hölder norm. Furthermore, $\phi_n(\bar{\Omega}) = \Omega_n$; if there were a point in Ω_n which was not mapped to, the image would have a different topological class than $\bar{\Omega}$, which is impossible for a continuous map. If we were to apply the same procedure to construct a map $\bar{\phi} : \bar{\Omega} \rightarrow \bar{\Omega}$, then on the boundary $\bar{\phi}^{(1)}(x)|_{\partial\bar{\Omega}} = \bar{r}(s) \cos \theta = x_1$. The harmonic extension of this boundary would be $\bar{\phi}^{(1)}(x) = x_1$, and similarly for the x_2 and x_3 coordinates, so $\bar{\phi}$ is the identity mapping Id . We now examine $\phi_n^{(1)} - x_1$; this function is harmonic and assumes values $(r_n(s) - \bar{r}(s)) \cos \theta$ on the boundary. We apply Schauder's boundary estimates again to see that $\|\phi_n - \text{Id}\|_{1,\beta} \leq C(\|r_n - \bar{r}\|_{1,\beta} + \|r_n - \bar{z}\|_{1,\beta})$, with the constant C depending only on $\partial\bar{\Omega}$. Since $r_n \rightarrow \bar{r}$ and $z_n \rightarrow \bar{z}$ in $C^{1,\beta}$, $\phi_n \rightarrow \text{Id}$ in $C^{1,\beta}$. $\square$

We now examine the Dirichlet problem on Ω_n and $\bar{\Omega}$. For simplicity we place the source at the origin; the tip of each domain lies at $(r, z) = (0, z_n(0))$. We write $u_n(x) = \frac{1}{|x|} - w_n$, $\bar{u} = \frac{1}{|x|} - \bar{w}$, where w_n and $\bar{w}$ are the solutions of

$$(6.11) \quad \begin{aligned} \Delta w_n &= 0 \quad \text{in } \Omega_n, \quad w_n = \frac{1}{|x|} \quad \text{for } x \in \partial\Omega_n, \\ \Delta \bar{w} &= 0 \quad \text{in } \bar{\Omega}, \quad \bar{w} = \frac{1}{|x|} \quad \text{for } x \in \partial\bar{\Omega}. \end{aligned}$$

Now $\frac{1}{|x|}$ is smooth away from the origin, and all its derivatives can be bounded by a function of $d_{\min}$, the minimal distance between $\partial\Omega_n$ and the origin for all n . So $\|\frac{1}{|x|}\|_{1,\alpha;\partial\Omega_n} \leq C(\|r_n\|_{1,\alpha} + \|z_n\|_{1,\alpha})$, where the constant C depends only on $d_{\min}$. By the boundary estimates, $\|w_n\|_{1,\alpha;\Omega_n} \leq C(\|\frac{1}{|x|}\|_{1,\alpha;\partial\Omega_n})$, where the constant C depends on the $C^{1,\alpha}$ norms of r_n and z_n . Since these are uniformly bounded, there is a C independent of n such that $\|w_n\|_{1,\alpha;\Omega_n} \leq C$. We now examine the compositions $w_n \circ \phi_n : \bar{\Omega} \rightarrow \mathbb{R}$. Since, by the above lemma, the maps ϕ_n are uniformly bounded in $C^{1,\alpha}$, these compositions are uniformly bounded in $C^{1,\alpha}$. On the boundary $w_{n_j} \circ \phi_{n_j}|_{\partial\Omega} = \frac{1}{|\phi_{n_j}|}$ and clearly converges to $\bar{w}|_{\partial\Omega}$.

LEMMA 6.3. *Let $\phi_n : \bar{\Omega} \rightarrow \Omega_n \subset \mathbb{R}^3$ be a sequence of mappings, let $w_n : \Omega_n \rightarrow \mathbb{R}$ be a sequence of harmonic functions, and let $\bar{w} : \bar{\Omega} \rightarrow \mathbb{R}$ be a harmonic function, such that*

- *the mappings $\phi_n \rightarrow \text{Id}$ in the $C^{1,\beta}(\bar{\Omega}; \mathbb{R}^3)$ norm;*
- *the compositions $w_n \circ \phi_n$ are uniformly bounded in the $C^{1,\alpha}(\bar{\Omega}; \mathbb{R})$ norm;*
- *on the boundary, $w_n \circ \phi_n|_{\partial\bar{\Omega}} \rightarrow \bar{w}|_{\partial\bar{\Omega}}$ in the $C^{1,\beta}(\partial\bar{\Omega}; \mathbb{R})$ norm.*

Then we have convergence on the full set $\bar{\Omega}$, the compositions $w_n \circ \phi_n \rightarrow \bar{w}$ in the $C^{1,\beta}(\bar{\Omega}; \mathbb{R})$ norm.

Proof. Since $w_n \circ \phi_n$ is bounded in $C^{1,\alpha}$ any subsequence has a convergent subsequence in $C^{1,\beta}$. Let us examine such a convergent subsequence, which we will also denote as $w_{n_j} \circ \phi_{n_j}$, and let the limit be denoted as $\tilde{w}$. Each w_{n_j} is a weak solution of the Dirichlet problem, so for all $v_{n_j} \in C_0^1(\Omega_{n_j}; \mathbb{R})$

$$(6.12) \quad \int_{\Omega_{n_j}} \nabla w_{n_j} \cdot \nabla v_{n_j} dx = 0.$$

Now $\Omega_{n_j} = \phi_{n_j}(\bar{\Omega})$, and so we can perform a coordinate transform. Furthermore, for sufficiently large n the Jacobian $D\phi_{n_j}$ is sufficiently close to the identity matrix that it can be inverted, and ϕ_{n_j} is a bijection. So writing $v_{n_j} = v \circ \phi_{n_j}^{-1}$, the above statement holds for all $v \in C_0^1(\bar{\Omega}; \mathbb{R})$. Now

$$(6.13) \quad \begin{aligned} \int_{\phi_{n_j}(\bar{\Omega})} \nabla w_{n_j} \cdot \nabla v_{n_j} &= \int_{\bar{\Omega}} ((\nabla w_{n_j}) \circ \phi_{n_j}) \cdot ((\nabla v_{n_j}) \circ \phi_{n_j}) |\det D\phi_{n_j}| dx \\ &= \int_{\bar{\Omega}} (D\phi_{n_j})^{-1} \nabla(w_{n_j} \circ \phi_{n_j}) \cdot (D\phi_{n_j})^{-1} \nabla v |\det D\phi_{n_j}| dx \\ &= 0. \end{aligned}$$

We now take the limit as $n \rightarrow \infty$ and see that for all v

$$(6.14) \quad \int_{\bar{\Omega}} \nabla \tilde{w} \cdot \nabla v dx = 0,$$

and so $\tilde{w}$ is weakly harmonic. Since $w_n \circ \phi_n|_{\partial\bar{\Omega}} \rightarrow \bar{w}|_{\partial\bar{\Omega}}$ and there is only one harmonic function with a given boundary value, $\tilde{w} = \bar{w}$.

Now assume that $w_n \circ \phi_n$ does not converge to $\bar{w}$; then there exist $\varepsilon > 0$ and a subsequence $w_{n'} \circ \phi_{n'}$ such that $\|w_{n'} \circ \phi_{n'} - \bar{w}\|_{1,\beta} \geq \varepsilon$. This is in contradiction with the fact that $w_{n'} \circ \phi_{n'}$ has a subsequence converging to $\bar{w}$ in $C^{1,\beta}$. $\square$

We can now prove the convergence of the fluxes F_n . We consider each flux to be a function on $\partial\bar{\Omega}$ parametrized by s . Then

$$(6.15) \quad F_n = |(\nabla u_n) \circ \phi_n| = \left| (D\phi_n)^{-1} \left(\nabla \left(\frac{1}{|\phi_n|} \right) - \nabla(w_n \circ \phi_n) \right) \right|,$$

and clearly

$$(6.16) \quad F_n \rightarrow \left| \nabla \left(\frac{1}{|x|} \right) - \nabla \bar{w} \right| = \bar{F}$$

in $C^{0,\beta}$.

6.5. Continuity of $G_\xi \rightarrow (\xi^*, r_{\xi^*}, z'_{\xi^*})$. We wish to show that for a sequence $G_{n,\xi}$ of families of cumulative fluxes parametrized by ξ , uniformly bounded in $C^{1,\alpha}$, and converging to $\bar{G}_\xi$ in $C^{1,\beta}$ for each value of ξ , the associated solutions (r_n, z'_n, ξ_n^*) converge to $(\bar{r}, \bar{z}', \bar{\xi}^*)$. Each r_n satisfies

$$(6.17) \quad \begin{aligned} \int_0^s \sqrt{r_n(\sigma)^2 - G_{n,\xi_n^*}(\sigma)^2} d\sigma &= \frac{1}{2} r_n(s)^2, \\ \int_0^{s_{max}} \sqrt{r_n(\sigma)^2 - G_{n,\xi_n^*}(\sigma)^2} d\sigma &= 2. \end{aligned}$$

Since r_n is uniformly bounded in $C^{1,\alpha}$ and $\xi_n \in [\xi_{min}, \xi_{max}]$ we examine a subsequence of solutions r_{n_j} and ξ_{n_j} , such that $r_{n_j} \rightarrow \tilde{r}$ and $\xi_{n_j}^* \rightarrow \tilde{\xi}^*$. Now

$$(6.18) \quad \begin{aligned} \|G_{n_j, \xi_{n_j}^*} - \bar{G}_{\tilde{\xi}^*}\|_{1,\beta} &\leq \|G_{n_j, \xi_{n_j}^*} - G_{n_j, \tilde{\xi}^*}\|_{1,\beta} + \|G_{n_j, \tilde{\xi}^*} - \bar{G}_{\tilde{\xi}^*}\|_{1,\beta} \\ &\leq \left\| \frac{\partial G_{n_j, \xi}}{\partial \xi} \right\|_{1,\beta} |\xi_{n_j} - \tilde{\xi}| + \|G_{n_j, \tilde{\xi}^*} - \bar{G}_{\tilde{\xi}^*}\|_{1,\beta}. \end{aligned}$$

By Lemma 4.10 the derivative of the flux to ξ is uniformly bounded in $C^{1,\alpha}$, so $G_{n_j, \xi_{n_j}^*} \rightarrow \bar{G}_{\tilde{\xi}^*}$ in $C^{1,\beta}$. Therefore we can pass through the limit and see that $\tilde{r}$, $\tilde{\xi}^*$, and $\bar{G}_{\tilde{\xi}^*}$ satisfy (6.17). Since, given a family of cumulative fluxes, solutions to the traveling wave ODE are unique, $\tilde{r} = \bar{r}$ and $\tilde{\xi}^* = \bar{\xi}^*$. Now assume that r_n and ξ_n^* do not converge to $\bar{r}$ and $\bar{\xi}$. Then there exist ϵ and N such that for all $n \geq N$, $\|r_n - \bar{r}\|_{1,\beta} \geq \epsilon$ or $|\xi_n^* - \bar{\xi}^*| \geq \epsilon$. This is in contradiction with the fact that there is a subsequence such that $r_{n_j} \rightarrow \bar{r}$ and $\xi_{n_j}^* \rightarrow \bar{\xi}^*$.

The quotients $z'_n = \frac{G_{n,\xi_n^*}}{r_{n,\xi_n^*}}$ are uniformly bounded in $C^{1,\alpha}$ and thus have a convergent subsequence z'_{n_j} which converges to $\bar{z}'$. Now clearly $\bar{z}'(0) = \bar{z}'(0)$, and for $s > 0$,

$$(6.19) \quad |\bar{z}'(s) - z'(s)| \leq \left| \bar{z}'(s) - z'_{n_j}(s) \right| + \frac{1}{r_{n_j}(s)} \left| G_{n_j, \xi_{n_j}^*}(s) - \bar{G}_{\bar{\xi}^*}(s) \right| + \frac{\bar{z}'(s)}{r_{n_j}(s)} |\bar{r}(s) - r_{n_j}(s)|.$$

Since the right-hand side converges to zero, $\bar{z}'(s) = z'(s)$ for all s . An argument similar to that used for the convergence of r_n and ξ_n now yields that $z'_n \rightarrow \bar{z}'$ in $C^{0,\beta}$.

6.6. Fixed point. We now have all the ingredients to use Schauder's fixed point theorem. The necessary constants C , A , M , ξ_{min} , and ξ_{max} have been determined in sections 6.1, 6.2, 6.3, and section 5.2. Since $\Xi(M, A, C; s_{max})$ is bounded in the $C^{1,\alpha} \times C^{0,\alpha}$ norm, it is compact in the $C^{1,\beta} \times C^{0,\beta}$ norm. In sections 4 and 5 we define a map from $\Xi(M, A, C; s_{max})$ to itself, and in sections 6.4 and 6.5 we prove this map is continuous in the $C^{1,\beta} \times C^{0,\beta}$ norm. Therefore this map has a fixed point, and this fixed point solves the bounded traveling wave problem as defined in section 2.6.

7. The limit as $s_{max} \rightarrow \infty$. In the previous sections we have shown that, given a sufficiently large value of s_{max} , one can find a solution to the traveling wave problem with a bounded domain, as described in section 2.6. In this section we will show that as $s_{max} \rightarrow \infty$ the solution profiles approach a limit profile which solves the traveling wave problem on an unbounded domain.

7.1. The limit profile r_∞, z_∞ . We examine a sequence $s_{max,n}$ such that $s_{max,n} \rightarrow \infty$ as $n \rightarrow \infty$ and such that $s_{max,n+1} - s_{max,n} > 2$. The estimates established in (3.8) then imply that $|z(s_{max})|$ increases monotonically to infinity. Previously we showed that for each value of $s_{max,n}$ one can find a distance ξ_n^* and two functions $r_n : [0, s_{max,n}] \rightarrow [0, 2]$ and $z_n : [0, s_{max,n}] \rightarrow \mathbb{R}^-$ solving the traveling wave problem on a bounded domain. Since $\xi_n^* \in [\xi_{min}, \xi_{max}]$, it is possible to choose our sequence $s_{max,n}$ such that ξ_n^* converges to some value ξ_∞^* in the same interval.

Since the functions $r_{\xi_n^*}, z'_{\xi_n^*}$ are uniformly bound in $C^{1,\alpha} \times C^{0,\alpha}$, by the Arzelà–Ascoli theorem it is always possible to find a subsequence such that the restrictions to a compact interval converge in $C^{1,\beta} \times C^{0,\beta}$. We use this to construct r_∞ and z'_∞ .

Let $n_j^{(0)}$ be the sequence $1, 2, 3, \dots$. For each $i \geq 1$ we define the subsequence $n_j^{(i)}$ of $n_j^{(i-1)}$ such that the restrictions of $r_{n_j^{(i)}}$ and $z'_{n_j^{(i)}}$ to the interval $[0, s_{max,i}]$ converge. We now examine the diagonal sequence $n_j^{(j)}$. Let $I \subset \mathbb{R}^+$ be an arbitrary compact interval. For sufficiently large j , $r_{n_j^{(j)}}$ and $z'_{n_j^{(j)}}$ are defined on this interval and their restrictions converge to functions r_∞ and z'_∞ . Since I was arbitrary, the functions r_∞ and z'_∞ are defined on $\mathbb{R}^+$. Integrating z'_∞ with respect to s yields the function z_∞ . To simplify notation, for the remainder of section 7 we will denote the sequences of functions $r_{n_j^{(j)}}$ and $z_{n_j^{(j)}}$ as r_n and z_n .

7.2. The cumulative flux G_∞ . The functions r_∞ and z_∞ define a boundary $\partial\Omega_\infty$ of a domain Ω_∞ in the same manner as described previously. In this case, the domain is unbounded and there is no zero flux boundary condition at $z = z(s_{max})$. We place a source at a distance ξ_∞ from the tip and solve the Dirichlet problem (2.1) to obtain a vesicle density u_∞ and the associated flux F_∞ and cumulative flux G_∞ . By Lemma 4.1, Ω_∞ is of class $C^{1,\alpha}$ and so $G_\infty \in C^{1,\alpha}(\mathbb{R})$. By the divergence theorem, $G_\infty(s) \rightarrow 2$ as $s \rightarrow \infty$.

The curvature of the limit profile $\partial\Omega_\infty$ has the same upper bound as for the bounded profiles. This again gives an outer sphere condition which enables us to give an upper bound for the flux F_∞ .

LEMMA 7.1. *The flux F_∞ passing through the point $(r_\infty(s), z_\infty(s))$ on the boundary is bounded,*

$$(7.1) \quad F_\infty \leq F_{max},$$

where F_{max} is the same as in Lemma 4.3. This implies we can estimate $G'_\infty(s) \leq sF_{max}$ and $G'_\infty(s) \leq 2F_{max}$.

Proof. The proof is essentially the same as the first part of the proof of Lemma 4.3. In that proof, the reflected source gives a positive contribution and thus can be neglected. $\square$

The comparison principle established in Theorem 4.4 allows us to calculate the asymptotics of G_∞ as $s \rightarrow \infty$.

LEMMA 7.2. *The cumulative flux G_∞ approaches its limit value of 2 algebraically.*

For sufficiently large s ,

$$(7.2) \quad 0 < 2 - G_\infty(s) \leq \frac{4}{(s - 2 - \xi_\infty)^2}.$$

Proof. We assume $s > \xi_\infty + 2$; then $z(s) < -\xi$. We use arguments similar to those in Lemma 4.7, except we compare with a half space on the other side of the VSC. Let $\tilde{\Omega}$ be the half space $\{(r, z) | z \geq z(s)\}$. Let

$$(7.3) \quad \tilde{u}(r, z) = (r^2 + (z + \xi)^2)^{-\frac{1}{2}} - (r^2 + (z - 2z(s) - \xi)^2)^{-\frac{1}{2}}.$$

Then $\tilde{u}$ solves the Dirichlet problem on $\tilde{\Omega}$ with a source at $z = -\xi_\infty$. The comparison principle now states that

$$(7.4) \quad \begin{aligned} 2\pi(2 - G_\infty(s)) &= \int_{\partial\Omega \setminus \tilde{\Omega}} F_u dS \leq \int_{\partial\tilde{\Omega} \cap \Omega} F_{\tilde{u}} dS \\ &\leq 4\pi \left[1 - \left(1 + \left(\frac{r(s)}{z(s) + \xi_\infty} \right)^2 \right)^{-\frac{1}{2}} \right], \\ &\leq 2\pi \left(\frac{r(s)}{z(s) + \xi_\infty} \right)^2 \leq 2\pi \left(\frac{2}{s - 2 - \xi_\infty} \right)^2, \end{aligned}$$

where we used the inequalities $r(s) < 2$ and $-s < z(s) < -s + 2$. $\square$

In fact it is possible, by comparing with a capped cylinder instead of a half plane, to show that G_∞ approaches 2 exponentially fast; however, the above result is sufficient for our purposes.

7.3. The limit of G_n . Let $I \subset \mathbb{R}^+$ be an arbitrary compact interval. For sufficiently large n , G_n is defined on I . In this section we wish to show that, restricted to I , $G_n \rightarrow G_\infty$ in the $C^{1,\beta}(I)$ topology. It is sufficient to show this in the case that $I = [0, s^*]$ for some arbitrary, but sufficiently large value of s^* . This can be shown using arguments similar to those used in section 6.4 when showing the continuity of the Schauder map.

LEMMA 7.3. *Let $I = [0, s^*]$ for some arbitrary but sufficiently large value of s^* . Then*

$$(7.5) \quad G_n|_I \rightarrow G_\infty|_I$$

in the $C^{1,\beta}(I)$ norm.

Proof. We cut off Ω_∞ at the plane $z = z_\infty(s^*)$. Let $\bar{\Omega}_\infty = \{(r, z) \in \Omega_\infty | z > z_\infty(s^*)\}$. Similarly we cut off Ω_n at the plane $z = z_n(s^*)$, so $\bar{\Omega}_n = \{(r, z) \in \Omega_n | z > z_n(s^*)\}$. Since the restrictions of r_n and z_n to I approach r_∞ and z_∞ in $C^{1,\beta}$ and are uniformly bounded in $C^{1,\alpha}$, by Lemma 6.2 there exist mappings $\phi_n : \bar{\Omega} \rightarrow \bar{\Omega}_n \subset \mathbb{R}^3$ which are uniformly bounded in $C^{1,\alpha}$ and converge to the identity in the $C^{1,\beta}$ norm. Let u_n be the solution of the Dirichlet problem (with a zero flux condition at the plane $z = z_n(s_{max,n})$) on Ω_n , restricted to $\bar{\Omega}_n$. Similarly, let u_∞ be the solution to the (unbounded) Dirichlet problem, restricted to $\bar{\Omega}_n$. On $\bar{\Omega}_n$, respectively, $\bar{\Omega}_\infty$, we define

$$(7.6) \quad \begin{aligned} w_n(r, z) &= \frac{1}{\sqrt{r^2 + (z + \xi_n)^2}} - u_n, \\ w_\infty(r, z) &= \frac{1}{\sqrt{r^2 + (z + \xi_\infty)^2}} - u_\infty. \end{aligned}$$

We examine the compositions $w_n \circ \phi_n$ viewed as functions of s and restricted to I . So $w_n \circ \phi_n|_I(s) = w_n(r_n(s), z_n(s))$. By Lemma 3.3 the terms in the square roots are bounded from below. Since $r_n \rightarrow r_\infty$, $z_n \rightarrow z_\infty$ in $C^{1,\beta}$ and $\xi_n \rightarrow \xi_\infty$, the compositions $w_n \circ \phi_n|_I \rightarrow w_\infty(r_\infty, z_\infty)|_I$ in the $C^{1,\beta}$ norm. By Lemma 6.2, $w_n \circ \phi_n \rightarrow w_\infty$ on $\bar{\Omega}$. The same arguments as those used at the end of section 6.4 now yield that $G_n \rightarrow G_\infty$ in $C^{1,\beta}$. $\square$

7.4. Solution to the unbounded traveling wave problem. For sufficiently large n , r_n is defined on the interval I , and for $s \in I$,

$$(7.7) \quad r_n(s)^2 = 2 \int_0^s \sqrt{r_n(\sigma)^2 - G_n(\sigma)^2} d\sigma.$$

Since $r_n \rightarrow r_\infty$ and $G_n \rightarrow G_\infty$, we pass through the limit to see that r_∞ and G_∞ satisfy the same equation. Differentiating to s and dividing by $r_\infty(s)$ shows that r_∞ satisfies the traveling wave ODE on any arbitrary interval I , and thus it satisfies it for all s . Hence r_∞ , z'_∞ , and ξ_∞^* solve the unbounded traveling wave problem. Scaling back, as described in section 2.4, results in a solution for arbitrary values of the parameters P and c yielding the proof of our main result, Theorem 2.1.

8. Conclusions. Using a Schauder fixed point argument, we have shown the existence of traveling solutions to the diffusive vesicle supply center model. Note that Schauder's fixed point theorem does not guarantee uniqueness, and we have not been able to show that this solution is unique by other means. However, we conjecture that this solution is indeed unique, as is the case in the ballistic model in [9]. Possibly one can prove this using the comparison principle (Theorem 4.4) along with the equation for traveling solutions (2.11). This is a direction for future research.

Since the equations in section 2.5 hold for any flux, not only for the diffusive case, the Schauder map Ψ defined in section 3.1 will have the same form for many models. Therefore the method described in this article should yield an existence proof for various related models, provided estimates similar to those in sections 4 through 6 can be derived.

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