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INDEX-AWARE MODEL ORDER REDUCTION FOR LINEAR INDEX-2 DAES WITH CONSTANT COEFFICIENTS∗

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Abstract. A model order reduction method for index-2 differential-algebraic equations (DAEs) is introduced, which is based on the intrinsic differential equations and on the remaining algebraic constraints. This extends the method introduced in a previous paper for index-1 DAEs. This procedure is implemented numerically and the results show numerical evidence of its robustness over the traditional methods.

Key words. differential algebraic equations, tractability index, model order reduction, modified decomposition of DAEs

AMS subject classifications. 78M34, 65L80

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1. Introduction. Consider a linear time invariant differential-algebraic equation (DAE) in descriptor form:

\begin{align*}
Ex' &= Ax + Bu, \quad x(0) = x_0, \\
y &= C^T x,
\end{align*}

(1.1a)
(1.1b)

with matrices \( E \in \mathbb{R}^{n,n} \), \( A \in \mathbb{R}^{n,n} \), \( C \in \mathbb{R}^{n,\ell} \), \( B \in \mathbb{R}^{n,m} \), state vector \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^\ell \), and initial value \( x_0 \in \mathbb{R}^n \). We assume that the matrix pencil \( (E,A) := \{ A - \lambda E | \lambda \in \mathbb{C} \} \) is regular, that is, \( \det(A - \lambda E) \neq 0 \) for at least one value of \( \lambda \in \mathbb{C} \). Moreover, the matrix \( E \) is assumed to be singular, so that (1.1a) is not an ordinary differential equation (ODE). We also assume that \( x_0 \) is a consistent initial value for the DAE and \( u \) is smooth enough. There exist many developed model order reduction (MOR) methods for reducing ODEs, i.e., when \( E \) is nonsingular, but little has yet been done to reduce DAEs. Some of these methods can be found in [17, 2]. Most recent attempts to reduce DAEs are given in [4, 13, 15, 6]. Usually, the balanced truncation method is used. This method involves solving a Lyapunov equation, which can be computationally very expensive. In [6], the authors reduce index-2 systems from electric power grids by first converting the DAE into an ODE. In [13] the reduction approach involves solving a projected Lyapunov equation. In [15] a passivity preserving approach for circuits has been developed based on certain projected Lyapunov equations. In practice, balanced truncation methods cannot be used on very large systems. Other attempts were made by using the Krylov-based subspace methods for descriptor systems such as PRIMA, SPRIM [14, 7], but these methods should not be used on DAEs with index greater than one, as illustrated in

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Example 1. Furthermore, these methods do not always preserve the index of a DAE, which can lead to loss of relevant information of the original DAE. The example below illustrates how using conventional MOR methods, such as Krylov-subspace-based methods, directly on higher index DAEs can lead to a good approximation of the transfer function but the resulting reduced-order models may be wrong or very difficult to solve.

Example 1. Consider an index-2 dynamical system with system matrices:

\[
E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 2 & -1 & 1 & 0.5 \\ 1 & -1 & 1 & 0 & -0.5 \\ -1 & 1 & 0 & 1 & 0 \\ 1.25 & 2.25 & 0 & -4 & 1 \\ -0.5 & -0.5 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

This system is solvable since the polynomial \( \det(\lambda E - A) = 2\lambda + 3 \) does not vanish identically and in addition, we assume that input function \( u \) is differentiable in the desired time interval and \( x(0) \) is a consistent initial condition. In this example we consider two different cases of control input matrix \( B \) with input data \( u(t) = \cos(t) \).

(i) If \( B = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T \), then the consistent initial condition is given by \( x(0) = [3 \ 1 \ -4 \ 2 \ -1]^T x_2(0) + [0 \ 0 \ -\frac{1}{2} \ 0 \ -\frac{1}{2}]^T u(0) \), where \( x_2(0) \) can be chosen arbitrarily. If we set \( x_2(0) = 0 \) then \( x(0) = [0 \ 0 \ -\frac{1}{2} \ 0 \ -\frac{1}{2}]^T \). We then apply the PRIMA method \([14]\) on the DAE. Using \( s_0 = 0 \) as the expansion point leads to the reduced-order model below:

\[
E_r = \begin{bmatrix} 0.73684 & 0.12114 & 0.42065 \\ 0.12114 & 0.019915 & 0.069155 \\ 0.42065 & 0.069155 & 0.25289 \\ -0.94737 & -0.32015 & -1.7338 \end{bmatrix},
\]

\[
A_r = \begin{bmatrix} -0.15575 & -0.052632 & -0.24306 \\ -0.5754 & -0.15246 & -1.3469 \end{bmatrix},
\]

\[
B_r = \begin{bmatrix} 0.68825 \\ 0.11315 \\ 0.41802 \end{bmatrix}, \quad C_r = \begin{bmatrix} -0.22942 \\ 0.67888 \\ -0.88987 \end{bmatrix}, \quad x_r(0) = \begin{bmatrix} 0.11471 \\ -0.69774 \\ -2.2204 \times 10^{-16} \end{bmatrix}.
\]

The reduced-order model is an ODE, that is, \( E_r \) is invertible. We compared the transfer function of the original model with that of the reduced-order model and we observed that the transfer functions coincide within a very small error. We then solved the reduced-order model and compare its solution with that of the original model. We observed that the solution of the original model coincides with that of the reduced-order model (PRIMA model). Thus the PRIMA model (1.2) is a good reduced-order model for the original system since the reduced-order model leads to accurate solutions.

(ii) If \( B = [0 \ 0 \ 0 \ 0 \ -1]^T \), then the consistent initial condition is given by \( x(0) = [3 \ 1 \ -4 \ 2 \ -1]^T x_2(0) + [2 \ 0 \ -\frac{5}{2} \ 2 \ -\frac{5}{2}]^T u(0) + [0 \ 0 \ 0 \ 0 \ 1]^T u'(0) \), where \( x_2(0) \) can be chosen arbitrarily. If we set \( x_2(0) = -0.5 \) then \( x(0) = [0.5 \ -0.5 \ 0.75 \ 1 \ 4]^T \). Using the same expansion point as before we obtain a
reduced-order model given by
\[
E_r = \begin{bmatrix}
0.043881 & 0.017375 & -0.1498 \\
0.017375 & 0.25518 & 0.24928 \\
-0.1498 & 0.24928 & 0.89488
\end{bmatrix},
\quad A_r = \begin{bmatrix}
0.20569 & -0.06545 & 0.052663 \\
-0.06545 & -0.15133 & -0.84866 \\
0.052663 & 0.86786 & -3.9271
\end{bmatrix},
\]
\[
B_r = \begin{bmatrix}
-0.92577 \\
0.29458 \\
-0.23702
\end{bmatrix}, \quad C_r = \begin{bmatrix}
0.46288 \\
0.66642 \\
-0.97969
\end{bmatrix}, \quad x_r(0) = \begin{bmatrix}
4.1613 \\
-0.78707 \\
-0.35553
\end{bmatrix}.
\]

(1.3)

The PRIMA method still leads to a ODE reduced-order model and also for this case the transfer function of the original model coincides with that of the reduced-order model (1.3) with a very small approximation error. We then solved the reduced-order model (1.3) and we observed that the reduced-order model leads to a good solution if the RelTol ≥ 0.1.

This example shows that solving the reduced-order model (1.3) is more difficult than solving (1.2), since we cannot achieve any better accuracy in the solution. This is due to the fact that the consistent initial condition \( x_0 \) in this example depends on \( u \) and its derivative, while in the former it only depends on \( u \). We know that conventional methods always assume that \( x(0) \) vanishes, but this assumption is not valid for DAEs, since we do not have enough freedom to choose the initial condition. We note that the conventional methods can reduce the DAE if its consistent initial condition \( x_0 \) only depends on \( u \), otherwise the resulting reduced-order model is not acceptable.

Thus, we cannot use conventional methods to reduce higher index DAEs in general. This motivated us to develop a new technique for reducing index-2 systems which eliminates this inconvenience, as discussed in section 3.

In this paper we propose a new strategy for reducing index-2 systems. This strategy is based on the ideas of März [10] of splitting a DAE into its differential and algebraic parts (possibly involving differentiations of other algebraic parts), by using appropriate projectors. This approach leads to a decoupled system of dimension \((\mu + 1)n\), where \( \mu \) is the tractability index. The spectrum of the decoupled system consists not only of the spectrum of the matrix pencil \((E, A)\) of the original system but also of additional infinite eigenvalues. This motivated us to do some modifications in the decomposition, using special basis vectors, which lead to a modified decoupled system of dimension \(n\). Moreover, this decoupling preserves the spectrum of the matrix pencil \((E, A)\) of the DAE. We can now apply MOR on both the differential and the algebraic parts. For the differential part, we use one of the existing MOR methods for ODEs, and for the algebraic parts we propose a new method, which is based on the reduction on the algebraic variables induced by the reduction on the differential variables. We call the resulting method the index-aware MOR (IMOR) method. This method leads to sparse reduced-order models and also always preserves the index of the original DAE.

This paper is organized as follows. In section 2 we discuss the decoupling of index-2 DAEs into differential and algebraic parts. In particular, in subsection 2.1 we give an overview of März decomposition and its limitations. In subsection 2.2 we propose a modified decomposition which preserves the size of the original DAE. In section 3 we introduce the new IMOR method for reducing index-2 systems, which we call IMOR-2. This method uses the decoupled system derived in subsection 2.2 instead of the original DAE in order to obtain the reduced-order model. In subsection 3.3...
we compare the IMOR-2 method with the conventional methods based on Krylov subspaces. In section 4 we present some numerical examples, divided into simple and industrial examples. The simple examples are used to illustrate the idea of the method, and to show that the splitting of the DAE into differential and algebraic equations is beneficial also for the numerical solution of the system. The industrial examples show the feasibility of the method for real-life applications. The paper is concluded by some final remarks, in section 5.

2. Decomposition of index-2 systems. In this section, we discuss two ways of decoupling index-2 systems by using projectors. The first decomposition, which we call März decomposition, is achieved via canonical projectors. An extensive discussion of decoupling index-2 systems by using projectors. The first decomposition, which we call März decomposition, is simply a modification of the former decomposition which is suitable for numerical implementation.

2.1. März decomposition. Let us consider (1.1a), given by

\[ E x' = A x + B u. \]  
\[ (2.1) \]

Following [10, 12], it is possible to introduce the notion of tractability index of system (2.1). Roughly speaking, this index measures how many derivatives of the input \( u \) appear in the solution. In fact, if the index is \( \mu \), then at most \( \mu - 1 \) derivatives of \( u \) will appear in the solution of (2.1). It is possible to show that the tractability index is equivalent to the Kronecker index of the matrix pencil \((E,A)\) for constant matrices. In [10, 12], a different construction, based on geometrical concepts, making use of appropriate projectors, was proposed. This construction has been summarized in [1]. Here we specialize it briefly for index-2 systems, and we use it to decompose the system (2.1) into a differential part and two algebraic parts.

We assume that the tractability index of (2.1) is 2. We define \( E_0 := E, A_0 := A \). Let \( Q_0 \) be a projector on the nullspace of \( E_0 \), that is, \( Q_0^2 = Q_0, \im Q_0 = \ker E_0, \) and let \( P_0 = I - Q_0 \). Then we define \( E_1 := E_0 - A_0 Q_0, A_1 := A_0 P_0 \). Since we consider an index-2 system, \( E_1 \) is singular. Let \( Q_1 \) be a projector on the nullspace of \( E_1 \), that is, \( Q_1^2 = Q_1, \im Q_1 = \ker E_1 \). We can choose \( Q_1 \) so that it satisfies the additional condition

\[ Q_1 Q_0 = 0. \]  
\[ (2.2) \]

In fact, if \( \tilde{Q}_1 \) is any projector onto \( \ker E_1 \), then we can define \( Q_1 = -\tilde{Q}_1 (E_1 - A_1 \tilde{Q}_1)^{-1} A_1 \). It is possible to see that \( Q_1 \) is a new projector onto \( \ker E_1 \) which satisfies (2.2). Let \( P_1 = I - Q_1 \), and introduce the matrices \( E_2 = E_1 - A_1 Q_1, A_2 = A_1 P_1 \). The index-2 condition is equivalent to assuming \( E_2 \) nonsingular. It is possible to prove that the following equivalent form of (2.1) holds [10]:

\[ E_2 (P_1 P_0 x' + Q_1 x + Q_0 x) = A_2 x + B u. \]  
\[ (2.3) \]

Since \( E_2 \) is invertible, we can left multiply (2.3) by \( E_2^{-1} \), and then use the projectors \( P_0 P_1, P_0 Q_1, \) and \( Q_0 P_1 \). Then we can introduce the variables \( x_P = P_0 P_1 x, x_{Q,1} = P_0 Q_1 x, x_{Q,0} = Q_0 x, \) which satisfy the projected equations

\[ x'_P = A_P x_P + B_P u, \]  
\[ (2.4a) \]

\[ x_{Q,1} = A_{Q,1} x_P + B_{Q,1} u, \]  
\[ (2.4b) \]

\[ x_{Q,0} = A_{Q,0} x_P + B_{Q,0} u + A_{Q,01} x'_{Q,1}, \]  
\[ (2.4c) \]
with $A_P := P_0 P_1 E_2^{-1} A_2$, $B_P := P_0 P_1 E_2^{-1} B$, $A_{Q,1} := P_0 Q_1 E_2^{-1} A_2$, $B_{Q,1} := P_0 Q_1 E_2^{-1} B$, $A_{Q,0} := Q_0 P_1 E_2^{-1} A_2$, $B_{Q,0} := Q_0 P_1 E_2^{-1} B$, $A_{Q,01} := Q_0 Q_1$. It is possible to prove the following decomposition of the identity:

\begin{equation}
I_n = P_0 P_1 + P_0 Q_1 + Q_0.
\end{equation}

It is simple to verify that the three terms on the right-hand side of (2.5) are mutually orthogonal projectors, provided $Q_1 Q_0 = 0$. Then the desired solution $x$ of an index-2 system (2.1) can then be computed after solving the ODE (2.4a), by using the formula

\[ x = x_P + x_{Q,1} + x_{Q,0}. \]

This shows that the initial data $x_0$ must be consistent with the equations. In fact, if we decompose $x_0 = x_P,0 + x_{Q,1,0} + x_{Q,0,0} := P_0 P_1 x_0 + P_0 Q_1 x_0 + Q_0 x_0$, then the component $x_{P,0}$ determines uniquely the solution $x_P$ of the ODE (2.4a), by means of the initial condition $x_P(0) = x_{P,0}$, thus determining the other components $x_{Q,1}, x_{Q,0}$ by means of the algebraic constraints (2.4b) and (2.4c). It follows that for generic initial data $x_0$ there might be an initial boundary layer, if the constraints (2.4b), (2.4c) are not satisfied initially, for $t = 0$.

The previous decomposition is still valid if $P_0 P_1 = 0$, equality which is compatible with index-2 conditions. In this case the component $x_P$ vanishes, and the projected equations (2.4) reduce to

\begin{align}
(2.6a) & \quad x_{Q,1} = B_{Q,1} u, \\
(2.6b) & \quad x_{Q,0} = B_{Q,0} u + A_{Q,01} x_{Q,1}. \end{align}

which is a system without differential equations. Notice that $P_0 P_1 = 0$ implies $P_0 = P_0 Q_1$, so that $x_{Q,1} = P_0 Q_1 x = P_0 x$. The solution of the original system can be recovered immediately from (2.6),

\begin{equation}
(2.7) \quad x = x_{Q,1} + x_{Q,0} = B_{Q,1} u + B_{Q,0} u + A_{Q,01} B_{Q,1} u'.
\end{equation}

This special case corresponds to a matrix pencil $(E, A)$ with no finite eigenvalues. This statement will become clear at the end of the following section.

### 2.2. Modified März decomposition.

In the previous section, we have seen that we can write the index-2 system (2.1) in the projected form (2.4), where $x_P, x_{Q,1}, x_{Q,0} \in \mathbb{R}^n$. The projected system (2.4) is a decoupled system of total dimension $3n$, while the original index-2 system (2.1) has dimension $n$. This may be expensive in terms of storage and memory consumption, especially when solving systems in tens of thousands of degrees of freedom, thus making it even more difficult to apply MOR methods on such a system. In general, decoupling an index-$\mu$ system by using the projector approach leads to a decoupled system of dimension $n(1+\mu)$. In this section we come up with a strategy in order to eliminate this limitation.

We show how to represent system (2.4) in a simpler way by constructing new basis column matrices from the projectors $P_0, P_1, P_0 Q_1$, and $Q_0$. This procedure extends the construction presented in [1] for index-1 DAEs. We start from the projectors $Q_0$, $P_0$, $Q_1$, $P_1$ constructed in the previous section, with $Q_1 Q_0 = 0$.

Let $k_0 = \dim(\ker E_0)$, $n_0 = n - k_0$, and let us consider an orthonormal basis matrix $(p_0, q_0) = (p_0,1, \ldots, p_0,n_0, q_0,1, \ldots, q_0,k_0) \in \mathbb{R}^n$ which contains $k_0$ independent columns $q_{0,i}$ of $Q_0$, which span im $Q_0 = \ker E_0$, and $n_0$ independent columns $p_{0,i}$ of $P_0$, which span im $P_0 = \ker Q_0$. Since $(p_0, q_0)$ is a basis matrix, it is invertible, and
let \((p_0^*, q_0^*)^T\) be its inverse, with \(q_0^* \in \mathbb{R}^{n, k_0}\) and \(p_0^* \in \mathbb{R}^{n, n_0}\). Then, \((p_0^*, q_0^*)^T(p_0, q_0) = I_n = (p_0, q_0)(p_0^*, q_0^*)^T\), that is,

\[
\begin{align*}
q_0^T q_0 &= I_{k_0}, & q_0^T p_0 &= 0, & p_0^T q_0 &= 0, & p_0^T p_0 &= I_{n_0},
\end{align*}
\]

(2.8) \(q_0 q_0^* + p_0 p_0^* = I_n\).

The previous relations imply that we can represent the projectors \(Q_0\) and \(P_0\) as

\[
Q_0 = q_0 q_0^T, \quad P_0 = p_0 p_0^T.
\]

(2.10) Note that, by construction, we have \(Q_0 q_0 = q_0, Q_0 p_0 = 0, P_0 q_0 = 0, P_0 p_0 = p_0\).

We are now going to find a simple representation of the projectors \(P_0 P_1\) and \(P_0 Q_1\), which appear in (2.5) and are used for the decomposition of the variable \(x\). Recalling the identities (2.8), after multiplying (2.5) by \(p_0^T\) from the left, and by \(p_0\) from the right, we obtain

\[
I_{n_0} = P_0 | 1 + Q_0 := p_0^T P_1 p_0 + p_0^T Q_1 p_0.
\]

It is immediately clear that \(P_0\) and \(Q_0\) are mutually orthogonal projectors, acting on \(\mathbb{R}^{n_0}\). This leads to the following proposition.

**Proposition 2.1.** Let \(P_0 = p_0^T P_1 p_0, Q_0 = p_0^T Q_1 p_0\); then \(P_0, Q_0 \in \mathbb{R}^{n_0, n_0}\) are projectors in \(\mathbb{R}^{n_0}\) provided the constraint condition \(Q_1 Q_0 = 0\) holds. Moreover they are mutually orthogonal.

Let \(k_1 = \text{dim}(\text{im } Q_0)\), and \(n_{01} = n_0 - k_1\). Here we need to distinguish two cases: \(n_{01} > 0\) or \(n_{01} = 0\). The first case corresponds to the matrix pencil \((E, A)\) having at least one finite eigenvalue, while the second case corresponds to \((E, A)\) with no finite eigenvalues.

2.2.1. **Matrix pencil \((E, A)\) with finite eigenvalues \((n_{01} > 0)\).** If \(n_{01} > 0\), we can proceed for \(Q_0\) and \(P_0\) as we have done for \(Q_0\) and \(P_0\). Let us consider a basis matrix \((p_{01}, q_{01}) \in \mathbb{R}^{n_0}\) made of \(n_{01}\) independent columns of projection matrix \(P_0\) and \(k_1\) independent columns of the complementary projection matrix \(Q_0\). We denote by \((p_{01}^*, q_{01}^*)^T\) the inverse of \((p_{01}, q_{01})\), such that \(p_{01}^T p_{01} = I_{n_{01}}, p_{01}^T q_{01} = 0, q_{01}^T q_{01} = I_{k_1}, q_{01}^T p_{01} + q_{01}^T q_{01} = I_{n_0}\). Then, we can represent \(P_0, Q_0\) as \(P_0 = p_{01} p_{01}^T, Q_0 = q_{01} q_{01}^T\), and we have \(P_{01} p_{01} = p_{01}, P_{01} q_{01} = 0, Q_{01} p_{01} = 0, Q_{01} q_{01} = q_{01}\).

**Proposition 2.2.** Let \(n_{01} > 0, P_0 = p_{01} p_{01}^T, Q_0 = q_{01} q_{01}^T\), and let the constraint \(Q_1 Q_0 = 0\) hold. Then the projectors \(P_0 P_1, P_0 Q_1\) can be decomposed as follows:

\[
\begin{align*}
P_0 P_1 &= p_{01} p_{01}^* p_{01}^T p_0^T, & P_0 Q_1 &= p_{01} q_{01}^* p_{01}^T p_0^T.
\end{align*}
\]

(2.11) \(P_0 P_1 = p_{01}^* P_1 p_{01}, Q_0 = p_{01}^* Q_1 p_{01}\), we have \(p_{01} p_{01}^T = p_{01}^* P_1 p_{01}, q_{01} q_{01}^T = p_{01}^* Q_1 p_{01}\). Multiplying the above identities by \(p_0\) from the left, and by \(p_0^T\) from the right, and recalling that \(p_0 p_0^T = P_0\), we obtain \(p_{01} p_{01}^* p_{01}^T = P_0 P_1 p_{01}, p_{01} q_{01}^* p_{01}^T = P_0 Q_1 p_{01}\). Since \(Q_1 Q_0 = 0\), we have \(Q_1 P_0 = Q_1 (I - Q_0) = Q_1\), and \(P_0 P_1 P_0 = P_0^2 - P_0 Q_1 P_0 = P_0 - P_0 Q_1 = P_0 P_1\), hence the thesis. \(\square\)

We can now expand \(x\) with respect to the basis \((p_{001}, p_{001}, q_0)\), obtaining the decomposition

\[
x = p_{001} \xi_{q} + p_{001} \xi_{q, 1} + q_0 \xi_{q, 0}.
\]

(2.12) where \(\xi_{q} \in \mathbb{R}^{n_{01}}, \xi_{q, 1} \in \mathbb{R}^{k_1}, \xi_{q, 0} \in \mathbb{R}^{k_0}\), with inversion expressions \(\xi_{q} = p_{01}^* P_0^* x, \xi_{q, 1} = q_{01}^* p_{01}^* x, \xi_{q, 0} = q_0^* x\). We note that the variables \(\xi_{q}, \xi_{q, 1}, \xi_{q, 0}\) are related to the
variables \( x_P, x_{Q,1}, x_{Q,0} \) by the relations \( x_P = p_0 p_0^1 \xi_p, x_{Q,1} = p_0 q_0^1 \xi_{q,1}, x_{Q,0} = q_0 \xi_{q,0} \),
\( \xi_p = p_0^T p_0^T x_P, \xi_{q,1} = q_0^T p_0^T x_{Q,1}, \xi_{q,0} = q_0^T x_{Q,0} \).

The projected equations (2.4) can be written as

\begin{align*}
(2.13a) & \quad \xi_p' = A_p \xi_p + B_p u, \\
(2.13b) & \quad \xi_{q,1} = A_{q,1} \xi_p + B_{q,1} u, \\
(2.13c) & \quad \xi_{q,0} = A_{q,0} \xi_p + B_{q,0} u + A_{q,01} \xi_{q,1},
\end{align*}

with

\[
A_p := p_0^T p_0^T E_2^{-1} A_2 p_0 p_0^1 \in \mathbb{R}^{n_{01},n_{01}}, \quad B_p := p_0^T p_0^T E_2^{-1} B, \quad B_{q,1} := q_0^T p_0^T E_2^{-1} B, \quad B_{q,0} := q_0^T p_0^T E_2^{-1} B.
\]

We can see that the number of differential equations is equal to \( n_{01} \) and \( k_1 + k_0 \) is the total number of algebraic equations; thus the total system dimension is \( n_{01} + k_1 + k_0 = n_0 + k_0 = n \). This is illustrated in Example 1. We note that the rank of \( E \) is no longer equal to the number of differential equations as for the case of index-1 systems; rather it is equal to \( n_{01} + k_1 = n_0 \). If we apply initial condition \( \xi_p(0) = p_0^T p_0^T x_0 \), where \( x_0 \) is an initial condition, we can solve the differential part (2.13a), then solve algebraic parts (2.13b) and (2.13c). If the initial data are consistent, we obtain numerically stable solutions, otherwise we see the formation of an initial boundary layer.

In general, solving system (2.13) is computationally cheaper than solving system (2.4).

In order to gain some insight on the decomposed system (2.13), we write it in the descriptor form

\begin{equation}
\tilde{E} \xi' = \tilde{A} \xi + \tilde{B} u,
\end{equation}

where

\[
\tilde{E} = \begin{bmatrix} I_{n_{01}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -A_{q,01} & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_p & 0 & 0 \\ A_{q,1} & -I_{k_1} & 0 \\ A_{q,0} & 0 & -I_{k_0} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_p \\ B_{q,1} \\ B_{q,0} \end{bmatrix},
\]

and \( \xi = [\xi_p, \xi_{q,1}, \xi_{q,0}]^T \) is the projected state space. By construction, we see that \( \xi = V^{-1} x \), where \( V := [p_0 p_0^1 p_0 q_0 q_0] = [p_0^T p_0^T q_0^T p_0^T q_0^T]^{-T} \), and by comparison with the original system (2.1) we find \( (\tilde{E}, \tilde{A}) = W(E,A)V \), \( \tilde{B} = WB \), where \( W = M^{-1} V^T E_2^{-1} = (E_2 VM)^{-1} \),

\[
M = \begin{bmatrix} I_{n_{01}} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & A_{q,01} & I_{k_0} \end{bmatrix} = \begin{bmatrix} I_{n_{01}} & 0 & 0 \\ 0 & I_{k_1} & 0 \\ 0 & -A_{q,01} & I_{k_0} \end{bmatrix}^{-1}.
\]

Since the matrices \( V \) and \( W \) are invertible, it follows that the matrix pencil \( (\tilde{E}, \tilde{A}) \) is equivalent to \( (E,A) \), so they have the same spectrum. It is simple to check that \( \det(\tilde{A} - \lambda \tilde{E}) = (-1)^{k_0 + k_1} \det(A_p - \lambda I_{n_{01}}). \) This identity shows that the finite eigenvalues of the matrix pencil \( (E,A) \) coincide with the (possibly complex) eigenvalues of the matrix \( A_p \) of the ordinary differential system (2.13), which are exactly \( n_{01} \), counting their multiplicity. This also shows that the stability of the DAE (2.1) is equivalent to the stability of the ODE system (2.13a).
2.2.2. Matrix pencil \((E, A)\) with no finite eigenvalues \((n_{01} = 0)\). If \(n_{01} = 0\), then \(\text{im} Q_{01} = \mathbb{R}^{k_{0}}\), and thus \(P_{0} = 0\). It follows that \(P_{0} P_{1} = 0\), since by definition \(0 = p_{0} P_{01} p_{0}^{*} T = P_{0} P_{1} P_{0} = P_{0} P_{1}\). Then, we have also \(P_{0} Q_{1} = P_{0}\), so the decomposition (2.12) reduces to
\[(2.15)\]
\[x = x_{Q,1} + x_{Q,0} = p_{0} \xi_{q,1} + q_{0} \xi_{q,0},\]
where \(\xi_{q,1} \in \mathbb{R}^{k_{1}}, \xi_{q,0} \in \mathbb{R}^{k_{0}}, k_{1} = n_{0},\) and with inversion expressions \(\xi_{q,1} = p_{0}^{*} T x_{Q,1}, \xi_{q,0} = q_{0}^{*} T x_{Q,0}\). The projected system (2.6) becomes
\[(2.16a) \quad \xi_{q,1} = B_{q,1} u,\]
\[(2.16b) \quad \xi_{q,0} = B_{q,0} u + A_{q,01} \xi_{q,1}\]
with \(B_{q,1} := p_{0}^{*} T E_{2}^{-1} B \in \mathbb{R}^{n_{0},m}, B_{q,0} := q_{0}^{*} T P_{1} E_{2}^{-1} B \in \mathbb{R}^{k_{0},m}, A_{q,01} := q_{0}^{*} T Q_{1} p_{0} \in \mathbb{R}^{k_{0},n_{0}}\). We can see that this system does not involve differential equations, and the total number of algebraic equations is equal to \(k_{1} + k_{0} = n_{0} + k_{0} = n\), which is the dimension of the DAE. In order to solve (2.16), we first solve the algebraic part (2.16a), then (2.16b), and the solution is:
\[x = p_{0} \xi_{q,1} + q_{0} \xi_{q,0} = p_{0} B_{q,1} u + q_{0} B_{q,0} u + q_{0} A_{q,01} B_{q,1} u'.\]

As in the previous case, this system (2.16) can also be written in the descriptor form
\[(2.17) \quad \tilde{E} \dot{\xi}' = \tilde{A} \xi + \tilde{B} u,\]
where \(\tilde{E} = \begin{bmatrix} 0 & 0 \\ -A_{q,01} & 0 \end{bmatrix}, \tilde{A} = \begin{bmatrix} -I_{k_{1}} & 0 \\ 0 & -I_{k_{0}} \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_{q,1} \\ B_{q,0} \end{bmatrix},\) and \(\xi = \begin{bmatrix} \xi_{q,1} \\ \xi_{q,0} \end{bmatrix}\) is the projected state space. By construction, we see that \(\xi = V^{-1} x,\) where
\[V := \begin{bmatrix} p_{0} & q_{0} \end{bmatrix} = \begin{bmatrix} p_{0}^{*} T \\ q_{0}^{*} T \end{bmatrix}^{-1},\]
and by comparison with the original system (2.1) we find \((\tilde{E}, \tilde{A}) = W(E, A)V, \tilde{B} = W B,\) with \(W = M^{-1} V^{-1} E_{2}^{-1} = (E_{2} VM)^{-1},\) and
\[M = \begin{bmatrix} I_{k_{1}} & 0 \\ A_{q,01} & I_{k_{0}} \end{bmatrix} = \begin{bmatrix} I_{k_{1}} & 0 \\ -A_{q,01} & I_{k_{0}} \end{bmatrix}^{-1}.\]

The matrices \(V\) and \(W\) are nonsingular, so the matrix pencil \((\tilde{E}, \tilde{A})\) is equivalent to \((E, A)\), and thus they have the same spectrum. It is simple to check that \(\text{det}(\tilde{A} - \lambda \tilde{E}) = (-1)^{k_{0} + k_{1}} \neq 0\). This identity shows that the matrix pencil \((E, A)\), equivalent to \((\tilde{E}, \tilde{A})\), has no finite eigenvalues. From system (2.14) and (2.17), we observe that this form also reveals the interconnection structure of the DAE (1.1a).

3. Index-aware MOR for index-2 systems (IMOR-2). In subsection 2.2, we saw that an index-2 system can be decoupled in two ways depending on the spectrum of the matrix pencil \((E, A)\). In this section, we propose a new technique of reducing index-2 systems depending on the nature of its matrix pencil. We call this technique “index-aware MOR” for index-2 systems (IMOR-2).

In short the idea is the following: attempt to reduce only the differential part, by using any MOR method; then, make explicit the limitation to the algebraic variables due to their explicit expression, obtained by the projection procedure. If there is no differential part, the second part of the above reduction procedure can still be applied.

For definiteness, and for its wide application and simplicity of implementation, we concentrate on Krylov-based MOR methods.
3.1. Matrix pencil \((E, A)\) with finite eigenvalues. Assume that the DAE in descriptor form \((1.1)\) is an index-2 dynamical system and its matrix pencil \((E, A)\) has at least one finite eigenvalue. Then, recalling \((2.12)\) and \((2.13)\), \((1.1)\) can be written as

\[
\begin{align*}
(3.1a) & \quad \xi' = A_p \xi_p + B_p u, \\
(3.1b) & \quad \xi_{q,1} = A_{q,1} \xi_p + B_{q,1} u, \\
(3.1c) & \quad \xi_{q,0} = A_{q,0} \xi_p + B_{q,0} u + A_{q,01} \xi_{q,1}' , \\
(3.1d) & \quad y = C^T_p \xi_p + C^T_{q,1} \xi_{q,1} + C^T_{q,0} \xi_{q,0} .
\end{align*}
\]

where \(C_p = p_0^T \rho_0^T C \in \mathbb{R}^{n_{0,1}, \ell} , C_{q,1} = q_0^T \rho_0^T C \in \mathbb{R}^{k_1, \ell} ,\) and \(C_{q,0} = q_0^T C \in \mathbb{R}^{k_0, \ell} .\) In descriptor form, this system can be written as

\[
\begin{align*}
(3.2a) & \quad \tilde{E} \xi' = \tilde{A} \xi + \tilde{B} u, \\
(3.2b) & \quad y = C^T \xi
\end{align*}
\]

with \((\tilde{E}, \tilde{A}) = W(E, A)V, \tilde{B} = WB, \tilde{C} = V^T C,\) and \(V, W\) defined as in \((2.14)\).

We can rewrite \((3.1)\) in three blocks, strictly separating the differential and algebraic parts:

\[
\begin{align*}
(3.3a) & \quad \xi' = A_p \xi_p + B_p u, \\
(3.3b) & \quad y_p = C^T_p \xi_p , \\
(3.4a) & \quad \xi_{q,1} = A_{q,1} \xi_p + B_{q,1} u , \\
(3.4b) & \quad y_{q,1} = C^T_{q,1} \xi_{q,1} ,
\end{align*}
\]

and

\[
\begin{align*}
(3.5a) & \quad \xi_{q,0} = A_{q,0} \xi_p + B_{q,0} u + A_{q,01} \xi_{q,1}' , \\
(3.5b) & \quad y_{q,0} = C^T_{q,0} \xi_{q,0} .
\end{align*}
\]

We observe that the subsystem \((3.3)\) is an ODE, while \((3.4)\) and \((3.5)\) are algebraic subsystems. We can also see that using the output equations of these subsystems, we can reconstruct the output equation of \((3.1)\) as,

\[
(3.6) \quad y = y_p + y_{q,1} + y_{q,0} .
\]

1. Reduction of the differential part \(\xi_p\). To reduce the differential part \((3.3)\) we use a Krylov-subspace-based method, which preserves the first \(r\) moments of the transfer function. The transfer function measures the sensitivity of the output with respect to the input, in the frequency domain. Taking the Laplace transform of \((3.3)\) and simplifying, we obtain

\[
\begin{align*}
(3.7) & \quad \Xi_p(s) = (sI - A_p)^{-1} B_p U(s) + (sI - A_p)^{-1} \xi_p(0) , \\
(3.8) & \quad Y_p(s) = C_p^T \Xi_p(s) ,
\end{align*}
\]

which yields \(Y_p(s) = C_p^T (sI - A_p)^{-1} B_p U(s) + C_p^T (sI - A_p)^{-1} \xi_p(0) .\) Thus, the transfer function restricted to the ODE part is given by \(H_p(s) = C_p^T (sI - A_p)^{-1} B_p .\) We choose
\[ s_0 \in \mathbb{C} \setminus \sigma(A_p), \text{ and we consider the subspace } V_p := \mathcal{K}_r(M_p(s_0), R_p(s_0)), \quad r \leq n_{01}, \]

where \( M_p(s_0) = (s_0 I - A_p)^{-1} \), and \( R_p(s_0) = (s_0 I - A_p)^{-1} B_p \), and \( \mathcal{K}_r(M_p, R_p) \) is the order-\( r \) Krylov subspace generated by \( M_p \) and \( R_p \).

\[ \mathcal{K}_r(M_p, R_p) = \text{span} \{ R_p, M_p R_p, \ldots, M_p^{r-1} R_p \}. \]

We denote by \( V_p \in \mathbb{R}^{n_{01} \times \tau_m} \) the matrix of an orthonormal basis of \( V_p \), so that we have \( V_p^T V_p = I \). We seek an approximate solution of the form \( \xi_p = V_p \hat{\xi}_p \), that is, we replace (3.3) with

\begin{align*}
(3.9a) & \quad \hat{\xi}_p^T = \hat{A}_p \hat{\xi}_p + \hat{B}_p u, \\
(3.9b) & \quad \hat{y}_p = \hat{C}_p \hat{\xi}_p,
\end{align*}

where \( \hat{A}_p = V_p^T A_p V_p, \hat{B}_p = V_p^T B_p, \) and \( \hat{C}_p = V_p^T C_p \). By construction, this reduced system has a transfer function whose first \( r \) moments around \( s_0 \) coincide with the first \( r \) moments of the original transfer function \( H_p(s) \).

2. **Reduction of the algebraic part \( \xi_{q,1} \).** The reduction of the differential part induces a reduction of the algebraic parts. First we consider the algebraic part \( \xi_{q,1} \).

We denote by \( \xi_{q,1}^* \) the expression obtained from (3.4a) by using the approximation \( \xi_p = V_p \hat{\xi}_p \), that is, \( \xi_{q,1}^* = A_{q,1} V_p \hat{\xi}_p + B_{q,1} u \). This expression shows that \( \xi_{q,1}^* \) belongs to the subspace

\[ V_{q,1} = \text{span} \{ B_{q,1}, A_{q,1} V_p \} = \text{span} \{ B_{q,1} \} + A_{q,1} \mathcal{K}_r(M_p, R_p). \]

We denote by \( n_{q,1} \) the dimension of \( V_{q,1} \), and by \( V_{q,1} \in \mathbb{R}^{k_{1},q_{1}} \) the matrix of an orthonormal basis of \( V_{q,1} \), so that \( V_{q,1}^T V_{q,1} = I \). Then we can approximate the algebraic solution in the form \( \xi_{q,1}^* = V_{q,1} \hat{\xi}_{q,1} \), that is, we replace (3.4) with

\begin{align*}
(3.11a) & \quad \hat{\xi}_{q,1}^T = \hat{A}_{q,1} \hat{\xi}_{q,1} + \hat{B}_{q,1} u, \\
(3.11b) & \quad \hat{y}_{q,1} = \hat{C}_{q,1} \hat{\xi}_{q,1},
\end{align*}

with \( \hat{A}_{q,1} = V_p^T A_{q,1} V_p, \hat{B}_{q,1} = V_p^T B_{q,1}, \), and \( \hat{C}_{q,1} = V_p^T C_{q,1} \).

3. **Reduction of the algebraic part \( \xi_{q,0} \).** Finally, we consider the reduction of the algebraic part \( \xi_{q,0} \), which involves differentiations of \( \xi_{q,1} \). We denote by \( \xi_{q,0}^* \) the expression obtained from (3.5a) by writing \( \xi_p = V_p \hat{\xi}_p, \xi_{q,1} = V_{q,1} \hat{\xi}_{q,1}, \xi_{q,0}^* = A_{q,0} V_p \hat{\xi}_p + B_{q,0} u + A_{q,01} V_{q,1} \hat{\xi}_{q,1}^T \). This expression shows that \( \xi_{q,0}^* \) belongs to the subspace

\[ V_{q,0} = \text{span} \{ B_{q,0}, A_{q,01} V_{q,1}, A_{q,0} V_p \}, \]

\[ \equiv \text{span} \{ B_{q,0}, A_{q,01} B_{q,1}, A_{q,01} A_{q,1} V_p, A_{q,0} V_p \}. \]

We denote by \( n_{q,0} \) the dimension of \( V_{q,0} \), and by \( V_{q,0} \in \mathbb{R}^{k_{0},n_{q,0}} \) the matrix of an orthonormal basis of \( V_{q,0} \), so that \( V_{q,0}^T V_{q,0} = I \). Then we can approximate the algebraic solution in the form \( \xi_{q,0}^* = V_{q,0} \hat{\xi}_{q,0} \), that is, we replace (3.5) with

\begin{align*}
(3.13a) & \quad \hat{\xi}_{q,0} = \hat{A}_{q,0} \hat{\xi}_p + \hat{B}_{q,0} u + \hat{A}_{q,01} \hat{\xi}_{q,1}^T, \\
(3.13b) & \quad \hat{y}_{q,0} = \hat{C}_{q,0} \hat{\xi}_{q,0},
\end{align*}
where \( \hat{A}_{q,0} = V_{q,0}^{T} A_{q,0} V_{p} \), \( \hat{A}_{q,01} = V_{q,0}^{T} A_{q,01} V_{q,1} \), \( \hat{B}_{q,0} = V_{q,0}^{T} B_{q,0} \), and \( \hat{C}_{q,0} = V_{q,0}^{T} C_{q,0} \). Combining subsystems (3.9), (3.11), and (3.13), we obtain a reduced-order model of the DAE (2.1) given by:

(3.14a) \[ \dot{\xi} = \hat{A} \xi + \hat{B} u, \]
(3.14b) \[ \hat{y} = \hat{C}^{T} \xi \]

with

(3.15a) \[ (\hat{A}, \hat{E}) = \hat{V}^{T} (\hat{A}, \hat{E}) \hat{V}, \]
(3.15b) \[ \hat{B} = \hat{V}^{T} \hat{B}, \]
(3.15c) \[ \hat{C} = \hat{V}^{T} \hat{C}, \]
\[ \hat{\xi} = \begin{bmatrix} \xi_{p} \\ \hat{\xi}_{q,1} \\ \hat{\xi}_{q,0} \end{bmatrix}, \]
\[ \hat{\xi} = \begin{bmatrix} V_{p} \\ V_{q,1} \\ V_{q,0} \end{bmatrix}. \]

Here, the matrices \( \hat{A}, \hat{E} \) are as in (3.2).

### 3.2. Matrix pencil \((E, A)\) has no finite eigenvalues.

Next we assume that the DAE (1.1) is an index-2 dynamical system and that its matrix pencil \((E, A)\) has no finite eigenvalues. Then, recalling (2.15) and (2.16), (1.1) can be written as

(3.16a) \[ \dot{\xi} = \hat{A} \xi + \hat{B} u, \]
(3.16b) \[ \hat{y} = \hat{C}^{T} \xi \]

with \( (\hat{E}, \hat{A}) = W (E, A) V, \)
\[ \hat{B} = W B, \]
\[ \hat{C} = V^{T} C, \]
and \( V, W \) as in (2.17).

We can rewrite (3.15) in two blocks, strictly separating the algebraic parts,

(3.17a) \[ \xi_{q,1} = B_{q,1} u, \]
(3.17b) \[ y_{q,1} = C_{q,1}^{T} \xi_{q,1}, \]

and

(3.18a) \[ \xi_{q,0} = B_{q,0} u + A_{q,01} \xi_{q,1}, \]
(3.18b) \[ y_{q,0} = C_{q,0}^{T} \xi_{q,0}. \]

Using the output equations of these algebraic subsystems, we can reconstruct the output equation of (3.15) as,

(3.19) \[ y = y_{q,1} + y_{q,0}. \]

1. **Reduction of the algebraic part \(\xi_{q,1}\).** In this case the algebraic variable \(\xi_{q,1}\) can be computed directly from (3.17a). This expression shows that \(\xi_{q,1}\) belongs to the subspace \(V_{q,1} = \text{span} B_{q,1}\). We denote by \(n_{q,1}\) the dimension of \(V_{q,1}\), and by \(V_{q,1} \in \mathbb{R}^{k_{2} 	imes n_{q,1}}\) the matrix of an orthonormal basis of \(V_{q,1}\), so that \(V_{q,1}^{T} V_{q,1} = I\). Then we can approximate the algebraic solution in the form \(\xi_{q,1}^{*} = V_{q,1} \hat{\xi}_{q,1}\), that is, we
replace (3.17) with

\begin{align}
(3.20a) & \quad \dot{\xi}_{q,1} = \hat{B}_{q,1} u,\\
(3.20b) & \quad \dot{y}_{q,1} = \hat{C}_q^T \dot{\xi}_{q,1},
\end{align}

where \( \hat{B}_{q,1} = V_{q,1}^T B_{q,1} \), \( \hat{C}_q = V_{q,1}^T C_{q,1} \).

2. Reduction of the algebraic part \( \xi_{q,0} \). Finally, we consider the reduction of the algebraic part \( \xi_{q,0} \). We denote by \( \xi^*_{q,0} \) the expression obtained from (3.18a) by using the approximation \( \xi_{q,1} = V_{q,1}^T \xi_{q,1} \), that is, \( \xi^*_{q,0} = B_{q,0} u + A_{q,01} V_{q,1} \xi^q_{q,1} \). This expression shows that \( \xi^*_{q,0} \) belongs to the subspace

\begin{equation}
V_{q,0} = \text{span} \{ B_{q,0}, A_{q,01} V_{q,1} \}.
\end{equation}

We denote by \( n_{q,0} \) the dimension of \( V_{q,0} \), and by \( V_{q,0} \in \mathbb{R}^{k,q_{0},n_{q,0}} \) the matrix of an orthonormal basis of \( V_{q,0} \), so that \( V_{q,0}^T V_{q,0} = I \). Then we can approximate the algebraic solution in the form \( \xi^*_{q,0} = V_{q,0} \hat{\xi}_{q,0} \), that is, we replace (3.18) with

\begin{align}
(3.22a) & \quad \dot{\xi}_{q,0} = \hat{B}_{q,0} u + \hat{A}_{q,01} \dot{\xi}_{q,1},\\
(3.22b) & \quad \dot{y}_{q,0} = \hat{C}_{q,0}^T \dot{\xi}_{q,0},
\end{align}

where \( \hat{A}_{q,01} = V_{q,0}^T A_{q,01} V_{q,1} \), \( \hat{B}_{q,0} = V_{q,0}^T B_{q,0} \), \( \hat{C}_{q,0} = V_{q,0}^T C_{q,0} \).

Also combining (3.17) and (3.18), we obtain a reduced-order model in descriptor form given by:

\begin{align}
(3.23a) & \quad \dot{E} \dot{\xi} = \hat{A} \xi + \hat{B} u,\\
(3.23b) & \quad \dot{y} = \hat{C}^T \dot{\xi},
\end{align}

with \( \hat{A}, \hat{E} = V^T (\hat{A}, \hat{E}) V, \hat{B} = V^T \hat{B}, \hat{C} = V^T \hat{C}, \hat{\xi} = \begin{bmatrix} \xi_{q,1} \\ \xi_{0,0} \end{bmatrix}, \hat{V} = \begin{bmatrix} V_{q,1} & 0 \\ 0 & V_{q,0} \end{bmatrix} \). Here, the matrices \( \hat{A}, \hat{E} \) are as in (3.16).

We note that the orthonormal basis matrices \( V_{q,1} \) and \( V_{q,0} \) can be computed numerically either using the modified Gram–Schmidt or SVD-based methods. In this paper, we use the SVD method; in this way the dimension of the reduced parts can be determined by truncating the columns of either \( V_{q,1} \) or \( V_{q,0} \) which correspond to the largest singular values. The approach gives a further reduction of the order of the reduced algebraic parts, as shown in Example 2.

3.3. Comparison with traditional projection methods. In the previous subsection, we have proposed a new MOR procedure, based on the decomposition of a DAE into differential and algebraic components. Starting from the system in decoupled form, we reduce first the ODE part, and then observe that this reduction also induces a reduction on the other parts. In this section we compare the new, index-aware MOR method with traditional MOR methods. In order to make the comparison more effective, we concentrate on a specific class of MOR methods, namely, projection methods.

In traditional projection methods, the starting point is the state space system (1.1). The main idea is to find a reduction procedure which preserves the first moments of the transfer function of the system. The transfer function is defined by taking the Laplace transform of the previous expression and computing explicitly the Laplace transform \( \hat{Y}(s) \) of \( y(t) \) as a function of the data, that is, the Laplace transform \( U(s) \).
of the input $u(t)$, and the initial datum $x_0$. Explicitly, we find $Y(s) = C^T R(s) U(s) + C^T M(s)x_0$, with $M(s) := (sE - A)^{-1} E$, $R(s) := (sE - A)^{-1} B$. Then, the transfer function $H(s)$ is the term in front of $U(s)$, which measures the dependence of the output on the input, that is, $H(s) := C^T R(s)$. It is simple to see that, for any $s_0$ which is not in the spectrum of $(E, A)$, we can write
\begin{equation}
R(s) = [I + M(s_0)(s - s_0)]^{-1} R(s_0).
\end{equation}

By using the Neumann expansion, we find
\begin{equation}
R(s) = \sum_{k=0}^{\infty} R^{(k)}(s_0)(s - s_0)^k \quad \text{with} \quad R^{(k)}(s_0) := (-1)^k M(s_0)^k R(s_0).
\end{equation}

Thus the transfer function can be expanded around $s = s_0$ as
\begin{equation}
H(s) = \sum_{k=0}^{\infty} h^{(k)}(s_0)(s - s_0),
\end{equation}
where the $k$th moment $h^{(k)}(s_0)$ of $H(s)$ around $s_0$ is given by the formula $h^{(k)}(s_0) := C^T R^{(k)}(s_0) = (-1)^k C^T M(s_0)^k R(s_0)$. One wishes to find a subspace such that the projection of the original system into this subspace is a reduced system which preserves the first $r$ moments. We consider an orthonormal basis of this hypothetical subspace, which we can write in the columns of a matrix $\hat{V}$, with $\hat{V}^T \hat{V} = I$. Then the reduced system is
\begin{equation}
\begin{aligned}
\hat{E} \hat{x} &= \hat{A} \hat{x} + \hat{B} u, \\
\hat{y} &= \hat{C}^T \hat{x},
\end{aligned}
\end{equation}
with $(\hat{E}, \hat{A}) = \hat{V}^T (E, A) \hat{V}$, $\hat{B} = \hat{V}^T B$, $\hat{C} = \hat{V}^T C$. Again, we find the formal expansion of $\hat{R}(s)$ around $s = s_0$,
\begin{equation}
\hat{R}(s) = \sum_{k=0}^{\infty} \hat{R}^{(k)}(s_0)(s - s_0)^k \quad \text{with} \quad \hat{R}^{(k)}(s_0) := (-1)^k \hat{M}(s_0)^k \hat{R}(s_0),
\end{equation}
where $\hat{M}(s) := (s\hat{E} - \hat{A})^{-1} \hat{E}$, $\hat{R}(s) := (s\hat{E} - \hat{A})^{-1} \hat{B}$. The transfer function of the reduced system can be written as $\hat{H}(s) = \hat{C}^T \hat{R}(s) = \sum_{k=0}^{\infty} \hat{h}^{(k)}(s_0)(s - s_0)$, with $\hat{h}^{(k)}(s_0) := \hat{C}^T \hat{R}^{(k)}(s_0) = (-1)^k \hat{C}^T M(s_0)^k \hat{R}(s_0)$. It follows that the first $r$ moments of $H(s)$ are preserved if and only if
\begin{equation}
\hat{C}^T \hat{R}^{(k)}(s_0) = C^T R^{(k)}(s_0), \quad k = 0, 1, \ldots, r - 1.
\end{equation}

It can be proved that this condition is satisfied if $\hat{V}$ is chosen so that $\text{span}\{\hat{V}\} = K_r(M(s_0), R(s_0))$, where $K_r(M, R)$ is the order-$r$ Krylov space generated by $M$, $R$, that is, the subspace spanned by $R, MR, \ldots, M^{r-1} R$. This choice defines the traditional projection methods.

We wish to give some insight into the method. Since $\hat{V} \hat{V}^T$ is a projector onto $K_r(M(s_0), R(s_0))$, by construction we have
\begin{equation}
\hat{V} \hat{V}^T M(s_0)^j R(s_0) = M(s_0)^j R(s_0), \quad j = 0, 1, \ldots, r - 1,
\end{equation}
which yields $\hat{V} \hat{V}^T R^{(j)}(s_0) = R^{(j)}(s_0), j = 0, 1, \ldots, r - 1$. It is possible to prove that
condition (3.26) implies \( \bar{M}(s)\bar{R}(s) = \bar{V}^T M(s)\bar{R}(s) \), \( j = 0, 1, \ldots, r - 1 \), that is, 
\( \bar{R}^{(j)}(s) = \bar{V}^T R^{(j)}(s), j = 0, 1, \ldots, r - 1 \). Then, for \( k = 0, 1, \ldots, r - 1 \), we have 
\[ \hat{h}^{(k)}(s) = \bar{C}^T \hat{R}^{(k)}(s) = C^T \bar{V}^T R^{(k)}(s) = C^T R^{(k)}(s) = \hat{h}^{(k)}(s), \]
which shows that the chosen moments are preserved.

Next, we observe that the transfer function is invariant for equivalent systems. The system
\[ \hat{E} \xi' = \hat{A} \xi + \hat{B} u, \]
\[ y = \hat{C}^T \xi, \]
is said to be equivalent to system (1.1) if there exist invertible matrices \( V, W \) such that \( x = V \xi \) and
\[ (\hat{E}, \hat{A}) = W(E, A)V, \quad \hat{B} = WB, \quad \hat{C} = V^T C. \]
The transfer function is then 
\[ \hat{h}(s) = \hat{C}^T \hat{R}(s) = \hat{C}^T (s \hat{E} - \hat{A})^{-1} \hat{B} \]
\[ = C^T V[W(sE - A)V]^{-1}WB = C^T (sE - A)^{-1}B \]
\[ = C^T R(s) = H(s). \]
In particular, we can use the structure of the system obtained after the decoupling, that is, we use the descriptor form (3.2) of the projected index-2 system, with transformation matrices \( V, W \) as in (2.14):
\[ \bar{E} = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -A_{q,01} & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A_p & 0 & 0 \\ A_{q,1} & -I_{k_1} & 0 \\ A_{q,0} & 0 & -I_{k_0} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_p \\ B_{q,1} \\ B_{q,0} \end{bmatrix}, \]
\[ \hat{C} = [C_p \quad C_{q,1} \quad C_{q,0}]^T. \]

Using these matrices we obtain,
\[ (s \hat{E} - \hat{A})^{-1} = \begin{bmatrix} M_p(s) & 0 & 0 \\ A_{q,1} M_p(s) & I_{k_1} & 0 \\ (sA_{q,01}A_{q,1} + A_{q,0}) M_p(s) & 0 & I_{k_0} \end{bmatrix}, \]
where \( M_p(s) := (sI_{n_0} - A_p)^{-1} \). Then we can write
\[ \hat{R}(s) = \begin{bmatrix} R_p(s) \\ R_{q,1}(s) \\ R_{q,0}(s) \end{bmatrix} := \begin{bmatrix} R_p(s) \\ A_{q,1} R_p(s) + B_{q,1} \\ A_{q,0} R_p(s) + sA_{q,01} R_{q,1}(s) + B_{q,0} \end{bmatrix}, \]
where \( R_p(s) := (sI_{n_0} - A_p)^{-1} B_p \).

Since the transfer function is the same for equivalent systems, we find
\[ H(s) = \hat{C}^T \hat{R}(s) = H_p(s) + H_{q,1}(s) + H_{q,0}(s), \]
with \( H_p(s) := C_p^T R_p(s), \ H_{q,1}(s) := C_{q,1}^T R_{q,1}(s), \ H_{q,0}(s) := C_{q,0}^T R_{q,0}(s) \). The IMOR-2 method amounts to considering separately the three parts of the transfer function. It can easily be proved that, if the first \( r \) moments of \( H_p(s) \) around \( s = s_0 \) are preserved, then we can ensure that the first \( r \) moments of \( H_{q,1}(s) \) and \( H_{q,0}(s) \) around \( s = s_0 \) are preserved by using the projection spaces \( \mathcal{V}_q, \mathcal{V}_q \) prescribed by the IMOR method, which leads to the reduced-order model (3.14).
4. Numerical experiments. In this section, we present simple and industrial problems of index-2 DAEs to demonstrate the effectiveness of our IMOR approach. In subsection 4.2, we compare IMOR with the traditional method (PRIMA) using industrial examples. All the results are computed under MATLAB environment version 2010a on a laptop with 2.53 GHz Intel(R) Core(TM) 2 Duo CPU and 4-GB RAM.

4.1. Simple problem. We consider again Example 1, applying IMOR-2 instead of PRIMA.

Example 2. In this example we use system matrices $E, A, B, C$ from Example 1. We have $\det(\lambda E - A) = 2\lambda + 3 \neq 0, \forall \lambda \in \mathbb{C}$, thus this system is solvable and its matrix pencil $(E, A)$ has one finite eigenvalue $\sigma_f(E, A) = \{-\frac{3}{2}\}$. This implies that we can use the IMOR-2 approach discussed in subsection 3.1.

(i) Here we consider the case when $B = [-1 0 0 0]^T$. The transfer function of the DAE is given by:

\[
H(s) = C^T (sE - A)^{-1} B = \frac{1}{2s + 3} - \frac{1}{2}.
\]

In order to apply the IMOR-2 we need to first decompose the DAE into differential and algebraic parts. This leads to a decoupled system of the form (3.1) with system matrices $A_p = -\frac{3}{2}, B_p = -\frac{3}{2}, A_{q,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_{q,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, A_{q,0} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 0 \\ 0 \end{bmatrix}, B_{q,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, A_{q,01} = \begin{bmatrix} -\frac{3}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, C_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C_{q,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$ and $C_{q,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The decoupled system can be written in descriptor form as,

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{3}{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \xi' = \begin{bmatrix}
-\frac{3}{2} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{2} & 0 & 0 & 0 \\
-\frac{3}{2} & 0 & 0 & -1 & 0
\end{bmatrix} \xi + \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} u,
\]

\[
y = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \xi.
\]

Using formula (3.31) the transfer function (4.1) can be decomposed as

\[
H(s) = -\frac{1}{2s + 3} + \frac{0}{H_{q,1}(s)} + \frac{2}{2s + 3} - \frac{1}{2}.
\]

The next step is to apply the proposed MOR technique (IMOR-2) for the index-2 system discussed in subsection 3.1 on the projected DAE (4.2). Choosing $s_0 = 0$ as the expansion point, we can construct orthonormal bases for the differential and algebraic parts of the decoupled system given by

\[
V_p = -1,
\quad V_{q,1} = \begin{bmatrix}
-0.72015 & -0.69382 \\
0.69382 & -0.72015
\end{bmatrix},
\quad V_{q,0} = \begin{bmatrix}
-0.90749 & -0.42008 \\
-0.42008 & 0.90749
\end{bmatrix}.
\]

We computed $V_{q,1}$ and $V_{q,0}$ using the SVD method and the corresponding singular values are given by

\[
S_{q,1} = \begin{bmatrix}
3.1 \times 10^{-16} & 0 \\
0 & 5.0 \times 10^{-17}
\end{bmatrix}
\quad \text{and} \quad
S_{q,0} = \begin{bmatrix}
1.9114 & 0 \\
0 & 0.9327
\end{bmatrix}.
\]
The singular values can now give us the number columns of $V_{q,1}$ and $V_{q,0}$ we can truncate. We observe that all columns of $V_{q,1}$ can be truncated since their corresponding singular values are close to zero, while $V_{q,0}$ remains unchanged. Thus the first algebraic part can be ignored, and the system reduces to an index-1 system. Using (4.3), after truncation the reduced-order model can be written as

\begin{align}
(4.4a) & 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \hat{\xi} =
\begin{bmatrix}
-1.5 & 0 & 0 \\
-1.3672 & -1 & 0 \\
-0.14048 & 0 & -1 \\
\end{bmatrix} \hat{\xi} +
\begin{bmatrix}
0.75 \\
0.64035 \\
-0.29992 \\
\end{bmatrix} u,
\end{align}

\begin{align}
(4.4b) & 
\hat{y} =
\begin{bmatrix}
-0.66667 \\
-0.94027 \\
-0.34043 \\
\end{bmatrix} \hat{\xi}.
\end{align}

Thus the original DAE is reduced from dimension 5 to 3 using the IMOR-2 method. We then compared the magnitude of the transfer functions of the reduced-order model (IMOR-2 model) with that of the original model and observed that their transfer functions coincide within a very small error. The reduced-order model also leads to more accurate solutions compared to the PRIMA model.

(ii) We now consider the case $B = [0 \ 0 \ 0 \ 0 \ -1]^T$. If we use the popular formula for the transfer function we then have

\begin{align}
(4.5) & 
H(s) = C^T(sE - A)^{-1}B = \frac{1}{2s + 3} + \frac{7}{4}.
\end{align}

As in the previous case, in order to apply the IMOR-2 method we need to first decompose the DAE into the differential and algebraic parts. This leads to a decoupled system with system matrices: $A_p = -\frac{3}{4}$, $B_p = -\frac{3}{4}$, $A_{q,1} = [0]$, $B_{q,1} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ \end{bmatrix}$, $A_{q,0} = \begin{bmatrix} -\frac{3}{4} \\ 0 \\ \end{bmatrix}$, $B_{q,0} = \begin{bmatrix} \frac{3}{4} \\ 0 \\ \end{bmatrix}$, $A_{q,01} = -\frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ \end{bmatrix}$, $C_p = \begin{bmatrix} 2 \\ -1 \\ 4 \\ \end{bmatrix}$, $C_{q,1} = [1]$, and $C_{q,0} = [1]$.

This decoupled system can also be written in descriptor form, given by

\begin{align}
(4.6a) & 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \dot{\xi} =
\begin{bmatrix}
-\frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\end{bmatrix} \dot{\xi} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} u,
\end{align}

\begin{align}
(4.6b) & 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
\end{bmatrix} =
\begin{bmatrix}
\frac{3}{4} & 0 & 1 & 0 & 0 \\
\frac{3}{4} & 0 & 1 & 0 & 0 \\
\frac{3}{4} & 0 & 1 & 0 & 0 \\
\frac{3}{4} & 0 & 1 & 0 & 0 \\
\frac{3}{4} & 0 & 1 & 0 & 0 \\
\end{bmatrix} \xi.
\end{align}

Thus the transfer function (4.5) can also be decomposed as

\begin{align}
H(s) = -\frac{1}{2s + 3} + \frac{1}{H_{q,1}(s)} + \frac{2}{2s + 3} + \frac{3}{4} + \frac{1}{H_{q,0}(s)}.
\end{align}

The next step is to apply the proposed MOR technique (IMOR-2) on the projected DAE (4.6). Choosing $s_0 = 0$ as the expansion point, we can construct orthonormal bases for differential and algebraic parts of the decoupled system given by

\begin{align}
(4.7) & 
V_p = -1, V_{q,1} = \begin{bmatrix} -0.34425 & -0.93888 \\ -0.93888 & 0.34425 \end{bmatrix}, V_{q,0} = \begin{bmatrix} -0.21798 & -0.97595 \\ -0.97595 & 0.21798 \end{bmatrix}.
\end{align}

The corresponding singular values for $V_{q,1}$ and $V_{q,0}$ are given by

\begin{align}
S_{q,1} = \begin{bmatrix} 1.0651 & 0 \\ 0 & 2.7341 \times 10^{-16} \end{bmatrix} \quad \text{and} \quad S_{q,0} = \begin{bmatrix} 4.2145 & 0 \\ 0 & 1.2534 \end{bmatrix}.
\end{align}

respectively. We can see that the last column of $V_{q,1}$ can be truncated since its
corresponding singular value is close to zero. Thus the block diagonal orthonormal basis matrix can be written as
\[
\hat{V} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -0.34425 & 0 & 0 \\
0 & -0.93888 & 0 & 0 \\
0 & 0 & -0.21798 & -0.97595 \\
0 & 0 & -0.97595 & 0.21798
\end{bmatrix}.
\]

If we substitute \( \xi = \hat{V} \hat{\xi} \) into (4.6) it leads to the reduced-order model
\[
(4.8a) \quad \hat{\xi}' = \begin{bmatrix}
-1.5 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-0.61596 & 0 & -1 & 0 \\
-1.2286 & 0 & 0 & -1
\end{bmatrix} \hat{\xi} + \begin{bmatrix}
0.75 \\
-1.0651 \\
-4.0673 \\
0.13996
\end{bmatrix} u,
\]
\[
(4.8b) \quad \hat{y} = \begin{bmatrix}
-0.66667 & -0.93888 & -0.21798 & -0.97595
\end{bmatrix} \hat{\xi}.
\]

We can see that the dimension of the original model is reduced to 4, and the index of the original system is preserved. When we compared the magnitude of the transfer function, we observe that the transfer function of the IMOR-2 model coincides with that of the original model with very small error. The reduced-order model also leads to accurate solutions and its easier to solve numerically than the PRIMA model. Hence the IMOR-2 method is a reliable method for reducing index-2 systems.

4.2. Industrial problems. In this section we test the IMOR-2 method on large scale problems.

Example 3. This is an MNA model that originates from [5]. It is an index-2 system with dimension 578. The sparsity of its matrices \( E \) and \( A \) are shown in Figure 4.1. Using the procedure in subsection 2.2, we decouple the system into differential and algebraic parts, as shown in the third row of Table 4.1. We then reconstruct the projected DAE in the descriptor form, and the sparsity of its matrix pencil is shown in Figure 4.2.

We used \( s_0 = 0 \) as the expansion point and, we were able to reduce the decoupled system of dimension 578 to a reduced system of total dimension 58 as shown in the fourth row of Table 4.1.

We compared the results with that of the PRIMA method applied directly on the original DAE. Unfortunately, we could not obtain a reduced-order model of the
Figure 4.2. Example 3: sparsity of the projected matrices $\tilde{E}, \tilde{A}$.

**Table 4.1**

<table>
<thead>
<tr>
<th>Models</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td># differential eqns</td>
</tr>
<tr>
<td>Original model</td>
<td>4</td>
</tr>
<tr>
<td>IMOR-2 model</td>
<td>4</td>
</tr>
</tbody>
</table>

Figure 4.3. Example 3: sparsity of the IMOR-2 model ($n = 58$).

same dimension as the one obtained with IMOR-2. Figure 4.3 shows the sparsity of the matrix pencil of the IMOR-2 model, while Figure 4.4 shows that of the PRIMA model. If we compare the two figures you can see that the IMOR-2 method leads to a sparse model, while the PRIMA method leads to a dense reduced-order model.

We observe that the PRIMA model is an ODE, thus it does not always preserve the index of the DAE, while the IMOR-2 model does. In Figure 4.5, we compare the transfer function of the original model and that of the reduced-order models. We can observe that the magnitude of the transfer function of the original model coincides with that of both reduced-order models. But when we solve both reduced-order models, we observe that the PRIMA model leads to wrong solutions while the IMOR-2 model leads to good solutions. Due to space limitations in Figure 4.6, we present only the third and last solutions, i.e., $y_3$ and $y_9$.
Example 3: sparsity of the PRIMA model ($n = 63$).

Fig. 4.4. Example 3: sparsity of the PRIMA model ($n = 63$).

Fig. 4.5. Example 3: comparison of the magnitude of the transfer function.

Fig. 4.6. Example 3: solutions of the reduced-order model, $u(t) = \text{ones}(9,1)\sin(2\pi \times 10^6t)$.

Example 4. This is an electric power grid system [6] which can be found in [16]. It is a SISO system of dimension 4182 with the sparsity of matrix pencil $(E,A)$ shown in Figure 4.7. We were able to decouple the system into differential and algebraic parts as shown in the third row of Table 4.2.

Figure 4.8 shows the sparsity of the matrix pencil $(\tilde{E}, \tilde{A})$ of the projected system. Using $s_0 = 1$ as the expansion point we were able to reduce the differential and algebraic parts as shown in the fourth row of Table 4.2. We reduced the dimension of
the algebraic parts in the same way as in the previous example, using singular values as shown in Figure 4.9. Also in this example we can eliminate all the equations of the first algebraic part since its corresponding singular values are close to zero as shown in Figure 4.9(a). Thus the power system is reduced to a dimension of 254 and the sparsity of its matrix pencil is shown in Figure 4.10 which is also sparse.

Figure 4.11 shows the sparsity of the reduced system using the PRIMA method and we can observe that it leads to a dense model. We also observe that the PRIMA method leads to an ODE while the IMOR method preserves the index of the system.

In Figure 4.12, we compare the magnitude of the transfer function of the original model with that of the reduced-order models of both methods. We observe that both reduced-order models coincide with that of the original model at low frequencies within a small error as shown in Figure 4.12. In Figure 4.13, we compare the output solutions and their respective errors of the reduced-order models. We can observe that both reduced-order models lead to accurate solutions.
In Table 4.3, we compare the computational cost of solving the reduced-order models. We observe that the IMOR-2 model is easier to solve than the PRIMA model since it requires less time, which is not a surprise. This is because IMOR-2 leads to sparse reduced-order models while PRIMA leads to very dense reduced-order models.
5. Conclusion. We extended a new MOR method developed for linear index-1 DAEs [1] with constant coefficients to linear index-2 DAEs with constant coefficients. In contrast to conventional approaches treating the DAE systems as a whole, the presented IMOR-2 method first splits the index-2 DAE into three parts: the inherent differential equation part, the pure algebraic part, and one part including differentiations of the algebraic part. Then, the PRIMA method is used to reduce the differential part whereas the algebraic and differentiation parts are treated by adapted projections.

We have discussed that conventional methods based on Krylov subspaces (like the PRIMA method) may lead to wrong reduced-order models, or the reduced-order models may be difficult to solve, if the consistent initial data depend on the derivatives of the input vector \( u \). It is caused by the fact that—in a conventional approach—such methods are applied to the whole DAE system. Additionally, we have seen that the conventional PRIMA method may lead to reduced-order models with dense matrices and a DAE index which differs from the one of the original system.
The IMOR-2 approach has the advantage that it leads to reduced-order models which are sparse, always solvable, and index preserving. An interesting additional feature of this method is that it can also be applied to index-2 systems without inherent differential equations. For the reduction of the differential part one could also use other methods based on Krylov subspace instead of the PRIMA method. Furthermore, an extension to DAEs with an index greater than 2 is naturally given by exploiting an index-adapted decoupling approach as given in [9]. The development of IMOR methods for higher index DAEs is not straightforward, and will be the topic of a forthcoming paper [3].

The decoupling technique developed could also be used to solve DAEs different from existing methods. However, one has to be aware that the numerical computation of the bases for the decoupling may involve serious difficulties because of the accuracy sensitive rank decisions. But it is expected to be profitable if the bases functions can be computed in a robust way, for example by a pure network topological analysis for RLC circuits. The decoupling idea can be extended to nonlinear systems or linear systems with time varying coefficients; see [11]. Although this is in general computationally expensive and highly sensitive with respect to perturbations we may exploit it in a robust manner for model order reductions if the system to reduce has a time and state independent structure, i.e., if one can find bases functions that are time and solution independent as is the case for circuit parts without controlled sources.

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