A complex-like calculus for spherical vectorfields

by

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A Complex-like Calculus
for
Spherical Vectorfields

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Dedicated to Professor Bob Mattheij at his retirement

Abstract

First, \( \mathbb{R}^{1+\varnothing} \), \( \varnothing \in \mathbb{N} \), is turned into an algebra by mimicing the usual complex multiplication. Indeed the special case \( \varnothing = 1 \) reproduces \( \mathbb{C} \). For \( \varnothing > 1 \) the considered algebra is commutative, but non-associative and even non-alternative.

Next, the Dijkhuis class of mappings ('vectorfields') \( \mathbb{R}^{1+\varnothing} \rightarrow \mathbb{R}^{1+\varnothing} \), suggested by C.G. Dijkhuis for \( \varnothing = 3 \), \( \varnothing = 7 \), is introduced. This special class is then fully characterized in terms of analytic functions of one complex variable.

Finally, this characterization enables to show easily that the Dijkhuis-class is closed under pointwise \( \mathbb{R}^{0+1} \)-multiplication: It is a commutative and associative algebra of vector fields.

Previously it had not been observed that the Dijkhuis-class only contains vectorfields with a 'time-dependent' spherical symmetry. Such disappointment was to be expected!

The class of functions which are differentiable with respect to the algebraic structure, that we impose on \( \mathbb{R}^{1+\varnothing} \), contains only linear functions if \( \varnothing > 1 \).

The Dijkhuis-class does not appear this way either!

In our treatment neither quaternions nor octonions play a role.
1 Imitation of complex calculus in higher dimensions

On $\mathbb{R}^{1+\vartheta}$, with $\vartheta \in \mathbb{N}$, a commutative multiplication structure is introduced by

$$(\alpha; a) \cdot (\beta; b) = (\alpha \beta - a^\top b; \alpha b + \beta a), \quad \alpha, \beta \in \mathbb{R}, \quad a, b \in \mathbb{R}^\vartheta.$$

(1.1)

**Note 1.** This multiplication structure is non-associative (non-alternative) if $\vartheta > 1$.

Indeed

$$\left( (\alpha; a) \cdot (\beta; b) \right) \cdot (\gamma; c) - (\alpha; a) \cdot \left( (\beta; b) \cdot (\gamma; c) \right) = (0; b^\top c a - a^\top b c),$$

which may not vanish if for $(\lambda, \mu) \neq (0, 0)$ one has $\lambda a + \mu c \neq 0$.

Clearly, with suitable interpretation, $b^\top c a - a^\top b c = -b \times (c \times a)$. **Note 2.** If $\vartheta = 3$ or $\vartheta = 7$, the product $(\alpha; a) \cdot (\alpha; a)$ of equal elements corresponds, respectively, to the quaternion product and the octonion product.

**Note 3.** Symbolically, and sometimes conveniently, (1.1) can be written

$$(\alpha + i a) \cdot (\beta + i b) = (\alpha \beta - a^\top b) + i(\alpha b + \beta a).$$

**Note 4.** If for $v = v_1 + iv_2 \in \mathbb{C}$ and $\xi \in \mathbb{R}^\vartheta$ we introduce $v_\xi \in \mathbb{R}^{1+\vartheta}$ by

$$v_\xi = (v_1; v_2 \xi),$$

we have the multiplication rule

$$v_\xi \cdot w_\xi = (vw)_\xi.$$

Here $vw$ is the usual product of complex numbers.

**Note 5.** By induction one easily shows that, with $r = |\xi|$, one has for $n = 1, 2, \ldots$

$$(t; \xi)^n = (\text{Re}(t + ir)^n; \frac{\text{Im}(t + ir)^n}{r} \xi).$$

Some calculations

- $(\alpha; a) \cdot (t; \xi)^n = (\alpha \text{Re}(t + ir)^n - \frac{\text{Im}(t + ir)^n}{r} \xi^\top a; \alpha \frac{\text{Im}(t + ir)^n}{r} \xi + \text{Re}(t + ir)^n a)$
- $(t; \xi)^m \cdot ((\alpha; a) \cdot (t; \xi)^n) = ((\alpha; a) \cdot (t; \xi)^n) \cdot (t; \xi)^m = (t; \xi)^n \cdot ((\alpha; a) \cdot (t; \xi)^m) =
  = (\alpha \text{Re}(t + ir)^{m+n} - \frac{\text{Im}(t + ir)^{m+n}}{r} \xi^\top a; \alpha \frac{\text{Im}(t + ir)^m}{r} \xi \xi^\top a + \{ \text{Re}(t+ir)^m \text{Re}(t+ir)^n \} a).$
Definition 1.1 *(Dijkhuis: A special class of functions)*

On open sets in $\mathbb{R}^{1+\varnothing}$ we introduce the class of functions

\[ (t; x) \mapsto (T(t; x); X(t; x)) \in \mathbb{R}^{1+\varnothing}, \tag{1.2} \]

where $T$ and $X = \text{column}[X_1, \ldots, X_8]$ are supposed to satisfy

\[
\begin{align*}
\nabla T &= -\frac{\partial X}{\partial t} \\
\nabla \times X &= 0 \\
x \times X &= 0 \\
(x \cdot \nabla) X &= \frac{\partial T}{\partial t} x.
\end{align*}
\tag{1.3}
\]

Here we denote

\[ \nabla T = \text{column}[\partial_1 T, \ldots, \partial_6 T], \]

$x \times X$ stands for the anti-symmetric matrix $[x X^T - X x^T]_{k\ell} = [x_k X_\ell - x_\ell X_k] \in \mathbb{R}^{\varnothing \times \varnothing}$,

$\nabla \times X$ stands for the anti-symmetric matrix $[(D X)^T - D X]_{k\ell} = [\partial_k X_\ell - \partial_\ell X_k] \in \mathbb{R}^{\varnothing \times \varnothing}$.

If $\varnothing = 3$ the identities in (1.3) correspond with the usual interpretation!

**Note 6.** From $[x X^T - X x^T] = [0]$ it immediately follows that $X$ can only be a multiple of $x$.

**Theorem 1.2** Suppose (1.3). Then the function

\[ (t; x) \mapsto (t; x) \cdot (T(t; x); X(t; x)) \tag{1.4} \]

also satisfies (1.3).

**Proof** In index notation the conditions (1.3) read

\[
\partial_\ell T = -\partial_0 X_\ell, \quad \partial_\ell X_j - \partial_j X_\ell = 0, \quad x_i (\partial_\ell X_k) = (\partial_0 T) x_k, \quad 1 \leq i, j, k \leq \varnothing.
\]

The product (1.4) reads $(tT - x^T X; tX + T x)$. We list the components of all derivatives needed. Summation over repeated indices.

\[
\begin{align*}
\nabla (tT - x^T X) &= \partial_k (tT - x_i X_i) = t(\partial_k T) - \delta_{ki} x_i - x_i (\partial_k X_i) = \\
&= t(\partial_k T) - X_k - x_i (\partial_i X_k) + x_i (\partial_i X_k - \partial_k X_i) \\
\n\partial_\ell (tT - x^T X) &= T + t\partial_\ell T - x_i \partial_\ell X_i = T + t\partial_\ell T + x_i \partial_\ell T \\
\n\nabla \times (tX + T x) &= \partial_k (tX_\ell + T x_\ell) - \partial_\ell (tX_\ell + T x_\ell) = \\
&= t(\partial_k X_\ell - \partial_\ell X_k) + (\partial_0 T) x_\ell - (\partial_\ell T) x_k = \\
&= t(\partial_k X_\ell - \partial_\ell X_k) + \partial_0 (x_\ell x_k - X_k x_\ell) \\
\n\partial_\ell (tX + T x) &= X_k + t(\partial_\ell X_k) + (\partial_0 T) x_k \\
(x \cdot \nabla) (tX + T x) &= x_i \partial_i (tX_\ell + T x_\ell) = tx_i (\partial_\ell X_k) + x_i (\partial_0 T) x_k + x_i T \delta_{ik}
\end{align*}
\]

\footnote{Introduced by G.C. Dijkhuis for $\mathbb{R}^{1+3}$ and $\mathbb{R}^{1+7}$. Private communication.}
Taking into account (1.3) leads to the desired result.

Corollary 1.3 Convergent power series with real coefficients $c_n$

$$ (T(t; \bar{x}); X(t; \bar{x}) = \sum_{m=0}^{\infty} c_n(t; \bar{x})^m, \tag{1.5} $$

all lead to functions which satisfy (1.3).

Note 7. The 'vectorial part' of the sum of such power series is always a multiple of $\bar{x}$.

Note 8. If $d = 3$ or $d = 7$ these series correspond to quaternion and octonion power series, respectively. It is emphasized again that the coefficients are real!

Inspired by Note 5. we come to a full description of functions (1.2) that satisfy (1.3).

Theorem 1.4 The functions (1.2) satisfy (1.3) if and only if, locally, there exists an analytic function $t + ir \mapsto \mathbf{F}(t, r) = \text{Re} \mathbf{F}(t, r) + i \text{Im} \mathbf{F}(t, r)$, such that

$$ (t; \bar{x}) \mapsto (T(t; \bar{x}); X(t; \bar{x})) = (\text{Re} \mathbf{F}(t, r); \frac{\text{Im} \mathbf{F}(t, r)}{r} \bar{x}). $$

For convenience in the proof I first summarize

Some properties of analytic functions
- A function $f : \mathbb{C} \to \mathbb{C}$ is analytic iff $f(z) = f(x + iy) = \text{Re} f(x, y) + i \text{Im} f(x, y)$ satisfies the Cauchy-Riemann identities

$$ \frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2}(\partial_x + i\partial_y) f(x + iy) = \frac{1}{2}(\partial_x + i\partial_y) \left( \text{Re} f(x, y) + i \text{Im} f(x, y) \right) = 0, $$

which corresponds to

$$ \partial_x \text{Re} f - \partial_y \text{Im} f = 0, \quad \partial_y \text{Re} f + \partial_x \text{Im} f = 0. $$

- For the 'complex' derivative we have

$$ \frac{\partial}{\partial z} f(z) = f'(z) = \frac{1}{2}(\partial_x - i\partial_y) f(x + iy) = \frac{1}{2}(\partial_x - i\partial_y) \left( \text{Re} f(x, y) + i \text{Im} f(x, y) \right) = \frac{1}{2} \{ \partial_x \text{Re} f + \partial_y \text{Im} f \} + \frac{i}{2} \{ \partial_x \text{Im} f - \partial_y \text{Re} f \} = \partial_x \text{Re} f - i \partial_y \text{Re} f. $$

- Analytic functions are harmonic, indeed

$$ \Delta (\text{Re} f(x, y) + i \text{Im} f(x, y)) = 4 \frac{1}{2}(\partial_x - i\partial_y) \frac{1}{2}(\partial_x + i\partial_y) \left( \text{Re} f(x, y) + i \text{Im} f(x, y) \right) = 0 $$
\[ \Delta \text{Re} f(x, y) + i \Delta \text{Im} f(x, y) = 0. \]

- \[ z \frac{d}{dz} f = zf'(z) = (x \partial_x + y \partial_y) \text{Re} f - i (x \partial_y - y \partial_x) \text{Re} f \]

- If \((x, y) \mapsto h(x, y)\) is harmonic, that means \(\Delta h(x, y) = 0\), then the function \(z = x + iy \mapsto \partial_x h(x, y) - i \partial_y (x, y)\), is analytic.

**Proof of Theorem 1.4** (\(\iff\)) If \(T = \text{Re} F\) and \(X = \frac{\text{Im} F}{r} x\), the 2nd and 3rd property in (1.3) follow from the symmetry of
\[ x_i x_j = \frac{x_i x_j}{r} F \quad \text{and} \quad \partial_i X_j = \frac{x_i x_j}{r^2} (\partial_i F) + \left( \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right) F. \]
Substitution in the 1st condition leads to
\[ \partial_r (\text{Re} F) \frac{1}{r} x = - \partial_r \text{Im} F \frac{1}{r} x, \]
which is OK because of one of the Cauchy-Riemann properties. Substitution in the 4th condition, because of \((x \cdot \nabla) (\frac{1}{r} x) = 0\), leads to
\[ r (\partial_r \text{Im} F) \frac{1}{r} x = (\partial_r \text{Re} F) x, \]
which is also OK because of the other Cauchy-Riemann property.

\((\Rightarrow)\) Since \(X\) has rotation 0 it has a potential. Write \(X(t; x) = - \nabla G(t, x)\). Further, from \([x X^T - X x^T] = [0]\) it follows that \(X\) can only be a multiple of \(x\). It follows that there exists a scalar function \((t; x) \mapsto \alpha(t; x)\), such that \(\nabla G(t, x) = \alpha(t; x) x\). We want to show that, for all fixed \(t\), the function \(x \mapsto G(t, x)\) is constant on spheres \(|x| = r\). Take \(a, b\) with \(|a| = |b| = r\). Let \(C\) be an oriented curve \(s \to x(s)\) which runs from \(a\) to \(b\) and which lies entirely on the sphere \(|x| = r\). Then \(G(t, b) - G(t, a) = \int_C \nabla G(t, x(s)) \cdot \dot{x}(s) ds.\) The integrand vanishes at all points of the curve because \(\nabla G\) is orthogonal to the sphere at all points of it. From now on we write \(G(t, x) = G(t, r)\). Therefore \(X(t; x) = - (\partial_r G(t, r)) \frac{1}{r} x.\) Put \(T(t, r) = \partial_t G(t, r)\) and we only have to satisfy the final condition in (1.3). Substitute our \(T\) and \(G\). The condition reads
\[ -r (\partial_r \partial_r G) \frac{1}{r} x = \partial_r \partial_t G x. \]
It follows that \(G\) has to be harmonic: \(\Delta G = 0\). We now define
\[ F(t + ir) = \partial_t G(t, r) - i \partial_r G(t, r), \]
and we are done. ■

**Examples** The analytic functions \(F(t, r) = (t + ir)^m, m \in \mathbb{N}\), represent the polynomial vectorfields \((t; x)^m\).

5
Theorem 1.5 Endowed with pointwise multiplication the Dijkhuis class of vectorfields, defined by (1.3), is a commutative and associative algebra.

Proof For analytic \( F, G \) we only have to check the multiplication

\[
\left( \text{Re } F; \frac{\text{Im } F}{r}; \mathbf{z} \right) \cdot \left( \text{Re } G; \frac{\text{Im } G}{r}; \mathbf{z} \right) = \left( \text{Re } FG; \frac{\text{Im } FG}{r}; \mathbf{z} \right).
\]

Associativity follows because all vectorial parts are multiples of \( \mathbf{z} \).

Further Consequences It will be clear by now that operations on the Dijkhuis class can be represented fully by operations on analytic functions. We mention some examples

- Multiplication by \((t; \mathbf{z})\) corresponds to \( F \mapsto \{z \mapsto zF(z)\}\).
- The Kelvin transform corresponds to \( F \mapsto \{z \mapsto F \left( \frac{1}{z} \right) \}\).
- The harmonic conjugate corresponds to \( F \mapsto \{z \mapsto iF(z)\}\).
- The Euler operator corresponds to \( F \mapsto \{z \mapsto zdzF(z)\}\).
- Meaningful derivatives are given by \( F \mapsto \{z \mapsto \frac{dz}{dz}mF(z)\}\).

2 Differentiability with respect to the algebra

A mapping

\[
\mathbb{R}^{1+d} \rightarrow \mathbb{R}^{1+d} : \left[ \begin{array}{c} t \\ \mathbf{z} \end{array} \right] \mapsto \left[ \begin{array}{c} T(t; \mathbf{z}) \\ X(t; \mathbf{z}) \end{array} \right], \quad (2.1)
\]

is differentiable (in the usual sense) at \( \left[ \begin{array}{c} t \\ \mathbf{z} \end{array} \right] \in \mathbb{R}^{1+d} \), if for any \( \left[ \begin{array}{c} h \\ k \end{array} \right] \in \mathbb{R}^{1+d} \), we have

\[
\left[ \begin{array}{c} T(t + h; \mathbf{z} + \mathbf{k}) \\ X(t + h; \mathbf{z} + \mathbf{k}) \end{array} \right] = \left[ \begin{array}{c} T(t; \mathbf{z}) \\ X(t; \mathbf{z}) \end{array} \right] + \left[ \begin{array}{c} \partial_t T(t; \mathbf{z}) \\ \partial_t X(t; \mathbf{z}) \end{array} \right] \nabla T(t; \mathbf{z}) + \left[ \begin{array}{c} \partial_x T(t; \mathbf{z}) \\ \partial_x X(t; \mathbf{z}) \end{array} \right] \mathcal{D}X(t; \mathbf{z}) \left[ \begin{array}{c} h \\ k \end{array} \right] + o(\sqrt{h^2 + |k|^2}). \quad (2.2)
\]

Here, \( \nabla T = \text{row}(\partial_1 T, \ldots, \partial_{10} T) \) and \( \mathcal{D}X = \text{matrix} \left[ \partial_j X_\ell \right], \ 1 \leq j, \ell \leq 10 \).
For left/right differentiability with respect to the algebraic structure imposed on $\mathbb{R}^{1+\varnothing}$, it is required that the linearization term in (2.2) has the form

$$\begin{bmatrix}
\partial_t T(t; x) & \nabla T(t; x) \\
\partial_t X(t; x) & D_X(t; x)
\end{bmatrix} \begin{bmatrix}
h \\
k
\end{bmatrix} = \begin{bmatrix}
\alpha(t; x) & -a^\top(t; x) \\
\alpha(t; x) & \alpha(t; x)I
\end{bmatrix} \begin{bmatrix}
h \\
k
\end{bmatrix}.$$

(2.3)

As a consequence the conditions for differentiability, with respect to the algebra, are

$$\partial_j X_\ell = 0, \text{ if } j \neq \ell, \quad \alpha = \partial_t T = \partial_j X_j, \quad 1 \leq j, \ell \leq \varnothing, \quad a = -\nabla T = \partial_t X.$$

(2.4)

It follows that, for $\varnothing > 1$, the only differentiable functions are

$$T = Bt - A \cdot x + D, \quad X = Bx + tA, \quad B, D \in \mathbb{R}, A \in \mathbb{R}^\varnothing,$$

(2.5)

which does not look very exciting.

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J. de Graaf, May 2011.
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