A complex-like calculus for spherical vectorfields

by

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A Complex-like Calculus
for
Spherical Vectorfields

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Dedicated to Professor Bob Mattheij at his retirement

Abstract

First, $\mathbb{R}^{1+d}$, $d \in \mathbb{N}$, is turned into an algebra by mimicing the usual complex multiplication. Indeed the special case $d = 1$ reproduces $\mathbb{C}$. For $d > 1$ the considered algebra is commutative, but non-associative and even non-alternative. Next, the Dijkhuis class of mappings (‘vectorfields’) $\mathbb{R}^{1+d} \rightarrow \mathbb{R}^{1+d}$, suggested by C.G. Dijkhuis for $d = 3, d = 7$, is introduced. This special class is then fully characterized in terms of analytic functions of one complex variable.

Finally, this characterization enables to show easily that the Dijkhuis-class is closed under pointwise $\mathbb{R}^{d+1}$-multiplication: It is a commutative and associative algebra of vector fields.

Previously it had not been observed that the Dijkhuis-class only contains vectorfields with a ‘time-dependent’ spherical symmetry. Such disappointment was to be expected!

The class of functions which are differentiable with respect to the algebraic structure, that we impose on $\mathbb{R}^{1+d}$, contains only linear functions if $d > 1$. The Dijkhuis-class does not appear this way either!

In our treatment neither quaternions nor octonions play a role.
1 Imitation of complex calculus in higher dimensions

On $\mathbb{R}^{1+\mathfrak{d}}$, with $\mathfrak{d} \in \mathbb{N}$, a commutative multiplication structure is introduced by

$$
(\alpha; a) \cdot (\beta; b) = (\alpha\beta - a^\top b; \alpha b + \beta a), \quad \alpha, \beta \in \mathbb{R}, \ a, b \in \mathbb{R}^\mathfrak{d}.
$$

(1.1)

**Note 1.** This multiplication structure is non-associative (non-alternative) if $\mathfrak{d} > 1$. Indeed

$$
( (\alpha; a) \cdot (\beta; b)) \cdot (\gamma; c) - (\alpha; a) \cdot ( (\beta; b) \cdot (\gamma; c)) = (0; b^\top c - a^\top b c),
$$

which may not vanish if for $(\lambda, \mu) \neq (0, 0)$ one has $\lambda a + \mu c \neq 0$.

Clearly, with suitable interpretation, $b^\top c a - a^\top b c = -b \times (c \times a)$. **Note 2.** If $\mathfrak{d} = 3$ or $\mathfrak{d} = 7$, the product $(\alpha; a) \cdot (\alpha; a)$ of equal elements corresponds, respectively, to the quaternion product and the octonion product.

**Note 3.** Symbolically, and sometimes conveniently, (1.1) can be written

$$
(\alpha + ia) \cdot (\beta + ib) = (\alpha\beta - a^\top b) + i(\alpha b + \beta a).
$$

**Note 4.** If for $v = v_1 + iv_2 \in \mathbb{C}$ and $\xi \in \mathbb{R}^\mathfrak{d}$ we introduce $v\xi \in \mathbb{R}^{1+\mathfrak{d}}$ by

$$
v\xi = (v_1; \frac{v_2}{|\xi|^2})
$$

we have the multiplication rule

$$
v\xi \cdot w\xi = (vw)\xi.
$$

Here $vw$ is the usual product of complex numbers.

**Note 5.** By induction one easily shows that, with $r = |\xi|$, one has for $n = 1, 2, \ldots$

$$
(t; \xi)^n = (\text{Re}(t + ir)^n; \frac{\text{Im}(t + ir)^n}{r} \xi).
$$

Some calculations

- $(\alpha; a) \cdot (t; \xi)^n = (\alpha \text{Re}(t + ir)^n - \frac{\text{Im}(t + ir)^n}{r} x^\top a; \alpha \frac{\text{Im}(t + ir)^n}{r} x + \text{Re}(t + ir)^n a)$

- $(t; \xi)^m \cdot ((\alpha; a) \cdot (t; \xi)^n) = ((\alpha; a) \cdot (t; \xi)^n) \cdot (t; \xi)^m = (t; \xi)^n \cdot ((\alpha; a) \cdot (t; \xi)^m) =

= (\alpha \text{Re}(t + ir)^{m+n} - \frac{\text{Im}(t + ir)^{m+n}}{r} x^\top a; \frac{\alpha \text{Im}(t + ir)^n}{r} x^\top a) x + \{\frac{\text{Re}(t+ir)^m}{r} \text{Re}(t+ir)^n a\}$. 

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Definition 1.1 (Dijkhuis: A special class of functions)

On open sets in $\mathbb{R}^{1+\mathfrak{d}}$ we introduce the class of functions

$$(t ; \underline{x}) \mapsto (T(t ; \underline{x}) ; \underline{X}(t ; \underline{x})) \in \mathbb{R}^{1+\mathfrak{d}},$$

where $T$ and $\underline{X} = \text{column}[X_1, \ldots, X_\mathfrak{d}]$ are supposed to satisfy

$$\begin{align*}
\nabla T &= -\frac{\partial X}{\partial t} \\
\nabla \times \underline{X} &= 0 \\
x \times \underline{X} &= 0 \\
(x \cdot \nabla)\underline{X} &= \frac{\partial T}{\partial t} \underline{x}.
\end{align*}$$

Here we denote

$$\nabla T = \text{column}[\partial_1 T, \ldots, \partial_\mathfrak{d} T],$$

$$x \times X$$

stands for the anti-symmetric matrix $[x X^T - X x^T]_{k\ell} = [x_k X_\ell - x_\ell X_k] \in \mathbb{R}^{1+\mathfrak{d}} \times \mathbb{R}^{1+\mathfrak{d}},$

$$\nabla \times \underline{X}$$

stands for the anti-symmetric matrix $[(\mathcal{D}X)^T - \mathcal{D}X]_{k\ell} = [\partial_k X_\ell - \partial_\ell X_k] \in \mathbb{R}^{1+\mathfrak{d}} \times \mathbb{R}^{1+\mathfrak{d}}.$

If $\mathfrak{d} = 3$ the identities in (1.3) correspond with the usual interpretation!

Note 6. From $[x X^T - X x^T] = [0]$ it immediately follows that $\underline{X}$ can only be a multiple of $\underline{x}$.

Theorem 1.2 Suppose (1.3). Then the function

$$(t ; \underline{x}) \mapsto (t ; \underline{x}) \cdot (T(t ; \underline{x}) ; \underline{X}(t ; \underline{x}))$$

also satisfies (1.3).

Proof In index notation the conditions (1.3) read

$$\partial_\ell T = -\partial_0 X_\ell, \quad \partial_\ell X_j - \partial_j X_\ell = 0, \quad x_i(\partial_\ell X_\ell) = (\partial_0 T)x_k, \quad 1 \leq i, j, k \leq \mathfrak{d}.$$

The product (1.4) reads $(t T - \underline{x}^T \underline{X}; t \underline{X} + T \underline{x})$. We list the components of all derivatives needed. Summation over repeated indices.

$$\begin{align*}
\nabla(t T - \underline{x}^T \underline{X}) : \quad &\partial_k(t T - x_i X_i) = t(\partial_k T) - \delta_k i X_i - x_i(\partial_k X_i) = \\
&= t(\partial_0 T) - X_k - x_i(\partial_i X_k) + x_i(\partial_\ell X_\ell - \partial_\ell X_k) \\
\partial_\ell(t T - \underline{x}^T \underline{X}) : \quad &T + t\partial_\ell T - x_i \partial_\ell X_i = T + t\partial_\ell T + x_i \partial_\ell T \\
\nabla \times (t \underline{X} + T \underline{x}) : \quad &\partial_k(t X_\ell + T x_\ell) - \partial_\ell(t X_k + T x_k) = \\
&= t(\partial_\ell X_k - \partial_\ell X_\ell) + (\partial_0 T)x_\ell - (\partial_\ell T)x_k = \\
&= t(\partial_\ell X_k - \partial_\ell X_\ell) + \partial_0(X_x x_k - X_k x_\ell) \\
\partial_\ell(t \underline{X} + T \underline{x}) : \quad &X_k + t(\partial_0 X_k) + (\partial_0 T)x_k \\
(x \cdot \nabla)(t \underline{X} + T \underline{x}) : \quad &x_i \partial_\ell(t X_k + T x_k) = tx_i(\partial_\ell X_k) + x_i(\partial_\ell T)x_k + x_i T \delta_{ik}
\end{align*}$$

$^1$Introduced by G.C. Dijkhuis for $\mathbb{R}^{1+3}$ and $\mathbb{R}^{1+7}$. Private communication.
Taking into account (1.3) leads to the desired result.

**Corollary 1.3** Convergent power series with **real** coefficients \( c_n \)

\[
(T(t; x); X(t; x)) = \sum_{m=0}^{\infty} c_n(t; x)^m, \quad (1.5)
\]

all lead to functions which satisfy (1.3).

**Note 7.** The 'vectorial part' of the sum of such power series is always a multiple of \( x \).

**Note 8.** If \( d = 3 \) or \( d = 7 \) these series correspond to **quaternion** and **octonion** power series, respectively. It is emphasized again that the coefficients are real!

Inspired by Note 5, we come to a full description of functions (1.2) that satisfy (1.3).

**Theorem 1.4** The functions (1.2) satisfy (1.3) if and only if, locally, there exists an analytic function \( t + ir \mapsto F(t, r) = \text{Re} F(t, r) + i \text{Im} F(t, r) \), such that

\[
(t; x) \mapsto (T(t; x); X(t; x)) = (\text{Re} F(t, r); \frac{\text{Im} F(t, r)}{r} x).
\]

For convenience in the proof I first summarize

**Some properties of analytic functions**

- A function \( f: \mathbb{C} \to \mathbb{C} \) is analytic iff \( f(z) = f(x + iy) = \text{Re} f(x, y) + i \text{Im} f(x, y) \) satisfies the Cauchy-Riemann identities

\[
\frac{\partial}{\partial z} f(z) = \frac{1}{2} (\partial_x + i \partial_y) f(x + iy) = \frac{1}{2} (\partial_x + i \partial_y) (\text{Re} f(x, y) + i \text{Im} f(x, y)) = 0,
\]

which corresponds to

\[
\partial_x \text{Re} f - \partial_y \text{Im} f = 0, \quad \partial_y \text{Re} f + \partial_x \text{Im} f = 0.
\]

- For the 'complex' derivative we have

\[
\frac{\partial}{\partial z} f(z) = f'(z) = \frac{1}{2} (\partial_x - i \partial_y) f(x + iy) = \frac{1}{2} (\partial_x - i \partial_y) (\text{Re} f(x, y) + i \text{Im} f(x, y)) = \frac{1}{2} \{\partial_x \text{Re} f + \partial_y \text{Im} f\} + i \frac{1}{2} \{\partial_x \text{Im} f - \partial_y \text{Re} f\} = \partial_x \text{Re} f - i \partial_y \text{Re} f.
\]

- Analytic functions are harmonic, indeed

\[
\Delta (\text{Re} f(x, y) + i \text{Im} f(x, y)) = 4 \frac{1}{2} (\partial_x - i \partial_y) \frac{1}{2} (\partial_x + i \partial_y) (\text{Re} f(x, y) + i \text{Im} f(x, y)) = 0
\]
\[ = \Delta \text{Re } f(x, y) + i\Delta \text{Im } f(x, y) = 0. \]

\[ z \frac{d}{dz} f = zf'(z) = (x \partial_x + y \partial_y) \text{Re } f - i(x \partial_y - y \partial_x) \text{Re } f \]

- If \((x, y) \mapsto h(x, y)\) is harmonic, that means \(\Delta h(x, y) = 0\), then the function
  \[ z = x + iy \mapsto \partial_x h(x, y) - i\partial_y (x, y), \]
  is analytic.

**Proof of Theorem 1.4** \((\Leftarrow)\) If \(T = \text{Re } F\) and \(X = \frac{\text{Im } F}{r} x\), the 2nd and 3rd property in (1.3) follow from the symmetry of

\[ x_i x_j = \frac{x_i x_j}{r} F \quad \text{and} \quad \partial_i X_j = \frac{x_i x_j}{r^2} (\partial_i F) + \left( \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3} \right) F. \]

Substitution in the 1st condition leads to

\[ \partial_r (\text{Re } F) \frac{1}{r} x = -\partial_r \text{Im } F \frac{1}{r} x, \]

which is OK because of one of the Cauchy-Riemann properties. Substitution in the 4th condition, because of \((x \cdot \nabla)\left( \frac{1}{r} x \right) = 0\), leads to

\[ r(\partial_r \text{Im } F) \frac{1}{r} x = (\partial_r \text{Re } F) x, \]

which is also OK because of the other Cauchy-Riemann property.

\((\Rightarrow)\) Since \(X\) has rotation 0 it has a potential. Write \(X(t; x) = -\nabla G(t; x)\). Further, from \([xX^\top - XX^\top] = [0]\) it follows that \(X\) can only be a multiple of \(x\). It follows that there exists a scalar function \((t; x) \mapsto \alpha(t; x)\), such that \(\nabla G(t; x) = \alpha(t; x) x\). We want to show that, for all fixed \(t\), the function \(x \mapsto G(t; x)\) is constant on spheres \(|x| = r\). Take \(a, b\) with \(|a| = |b| = r\). Let \(C\) be an oriented curve \(s \mapsto x(s)\) which runs from \(a\) to \(b\) and which lies entirely on the sphere \(|x| = r\). Then \(G(t, b) - G(t, a) = \int_a^b \nabla G(t, x(s)) : \hat{x}(s) \, ds\). The integrand vanishes at all points of the curve because \(\nabla G\) is orthogonal to the sphere at all points of it. From now on we write \(G(t, x) = G(t, r)\). Therefore \(X(t; x) = -(\partial_r G(t, r)) \frac{1}{r} x\).

Put \(T(t, r) = \partial_t G(t, r)\) and we only have to satisfy the final condition in (1.3). Substitute our \(T\) and \(G\). The condition reads

\[ -r(\partial_t \partial_r G) \frac{1}{r} x = \partial_t \partial_r G x. \]

It follows that \(G\) has to be harmonic: \(\Delta G = 0\). We now define

\[ F(t + ir) = \partial_t G(t, r) - i\partial_r G(t, r), \]

and we are done. 

**Examples** The analytic functions \(F(t, r) = (t + ir)^m, m \in \mathbb{N}\), represent the polynomial vectorfields \((t; x)^m\).
Theorem 1.5  Endowed with pointwise multiplication the Dijkhuis class of vectorfields, defined by (1.3), is a commutative and associative algebra.

Proof  For analytic $F, G$ we only have to check the multiplication

$$(\text{Re} F; \frac{\text{Im} F}{r}z) \cdot (\text{Re} G; \frac{\text{Im} G}{r}z) = (\text{Re} FG; \frac{\text{Im} FG}{r}z).$$

Associativity follows because all vectorial parts are multiples of $z$. ■

Further Consequences
It will be clear by now that operations on the Dijkhuis class can be represented fully by operations on analytic functions. We mention some examples

- Multiplication by $(t; x)$ corresponds to $F \mapsto \{z \mapsto zF(z)\}$.
- The Kelvin transform corresponds to $F \mapsto \{z \mapsto F\left(\frac{1}{z}\right)\}$.
- The harmonic conjugate corresponds to $F \mapsto \{z \mapsto iF(z)\}$.
- The Euler operator corresponds to $F \mapsto \{z \mapsto zd\frac{d}{dz}F(z)\}$.
- Meaningful derivatives are given by $F \mapsto \{z \mapsto \frac{d^n}{dz^n}F(z)\}$.

2 Differentiability with respect to the algebra

A mapping

$$\mathbb{R}^{1+\varnothing} \to \mathbb{R}^{1+\varnothing} : \begin{bmatrix} t \\ x \end{bmatrix} \mapsto \begin{bmatrix} T(t; x) \\ X(t; x) \end{bmatrix},$$

is differentiable (in the usual sense) at $\begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{1+\varnothing}$, if for any $\begin{bmatrix} h \\ k \end{bmatrix} \in \mathbb{R}^{1+\varnothing}$, we have

$$\begin{bmatrix} T(t + h; x + k) \\ X(t + h; x + k) \end{bmatrix} = \begin{bmatrix} T(t; x) \\ X(t; x) \end{bmatrix} + \begin{bmatrix} \partial_t T(t; x) & \nabla T(t; x) \\ \partial_t X(t; x) & \mathcal{D}X(t; x) \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} + o(\sqrt{h^2 + |k|^2}).$$

Here, $\nabla T = \text{row}(\partial_1 T, \ldots, \partial_\varnothing T)$ and $\mathcal{D}X = \text{matrix}[\partial_j X_\ell], 1 \leq j, \ell \leq \varnothing$. 

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For left/right differentiability with respect to the algebraic structure imposed on $\mathbb{R}^{1+\varnothing}$, it is required that the linearization term in (2.2) has the form

$$\begin{bmatrix} \partial_t T(t; x) & \nabla T(t; x) \\ \partial_t X(t; x) & \mathcal{D}X(t; x) \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} \alpha(t; x) & -a^\top(t; x) \\ a(t; x) & \alpha(t; x)I \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}. \quad (2.3)$$

As a consequence the conditions for differentiability, with respect to the algebra, are

$$\partial_j X_\ell = 0, \text{ if } j \neq \ell, \quad \alpha = \partial_t T = \partial_j X_j, \quad 1 \leq j, \ell \leq d, \quad a = -\nabla T = \partial_t X. \quad (2.4)$$

It follows that, for $d > 1$, the only differentiable functions are

$$T = Bt - A \cdot x + D, \quad X = Bx + tA, \quad B, D \in \mathbb{R}, \quad A \in \mathbb{R}^d, \quad (2.5)$$

which does not look very exciting.

**Acknowledgement** This note has been triggered by arguments and a dispute between its author and Dr. C.G. Dijkhuis.

J. de Graaf, May 2011.
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