Achievable Delay Performance in CSMA Networks

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Abstract—We explore the achievable delay performance in wireless CSMA networks. While relatively simple and inherently distributed in nature, suitably designed backlog-based CSMA schemes provide the striking capability to match the optimal throughput performance of centralized scheduling mechanisms in a wide range of scenarios. The specific type of activation rules for which throughput optimality has been established, may however yield excessive backlogs and delays. Motivated by that issue, we examine whether the poor delay performance is inherent to the basic CSMA operation of these schemes, or caused by the specific kind of activation rules. We first establish lower bounds for the delay in the case of fixed activation rates. The bounds indicate that the delay can dramatically grow with the load in certain topologies. We also discuss to what extent the bounds apply to backlog-based activation rules. Simulation experiments are conducted to illustrate and validate the analytical results.

I. INTRODUCTION

Emerging wireless mesh networks typically lack any centralized access control entity, and instead vitally rely on the individual nodes to operate autonomously and to efficiently share the medium in a distributed fashion. A popular mechanism for distributed medium access control is provided by the so-called Carrier-Sense Multiple-Access (CSMA) protocol, various incarnations of which are implemented in IEEE 802.11 networks. In the CSMA protocol each node attempts to access the medium after a certain back-off time, but nodes that sense activity of interfering nodes freeze their back-off timer until the medium is sensed idle.

While the CSMA protocol is fairly easy to understand at a local level, the interaction among interfering nodes gives rise to quite intricate behavior and complex throughput characteristics on a macroscopic scale. In recent years relatively parsimonious models have emerged that provide a useful tool in evaluating the throughput characteristics of CSMA-like networks. These models were originally considered by Boorstyn et al. [2], and pursued in the context of IEEE 802.11 systems by Wang & Kar [26], Durdy et al. [5], [6], and Garetto et al. [8]. Although the representation of the IEEE 802.11 back-off mechanism in the above-mentioned models is less detailed than in the landmark work of Bianchi [1], they accommodate a general interference graph and thus cover a broad range of topologies. Experimental results in Liew et al. [14] demonstrate that these models, while idealized, provide throughput estimates that match remarkably well with measurements in actual IEEE 802.11 systems.

Despite their asynchronous and distributed nature, CSMA-like algorithms have been shown to offer the capability of achieving the full capacity region and thus match the optimal throughput performance of centralized scheduling mechanisms operating in slotted time [12], [13], [15], [24]. More specifically, any throughput vector in the interior of the convex hull associated with the independent sets in the interference graph can be achieved through suitable back-off rates and/or transmission lengths. Based on this observation, various clever algorithms have been developed for finding the back-off rates that yield a particular target throughput vector or that optimize a certain concave throughput utility function in scenarios with saturated buffers [12], [13], [17]. In the same spirit, several powerful approaches have been devised for adapting the transmission lengths based on backlog information, and been shown to guarantee maximum stability [11], [19], [21].

Roughly speaking, the maximum-stability guarantees were established under the condition that the activity factors of the various nodes behave as logarithmic functions of the backlogs. Unfortunately, simulation experiments demonstrate that such activity factors can induce excessive backlogs and delays, which has triggered a strong interest in developing approaches for improving the delay performance [16], [18], [20]. Motivated by this issue, Ghaderi & Srikan [9] recently showed that it is in fact sufficient for the log functions of the activity factors to behave as logarithmic functions of the backlogs, divided by an arbitrarily slowly increasing, unbounded function. These results indicate that the maximum-stability guarantees are preserved for activity functions that are essentially linear for all practical values of the backlogs, although asymptotically the activity rate must grow slower than any positive power of the backlog.

In order to gain further insight in the root cause for the poor delay performance, we establish lower bounds for the average steady-state delay. The bounds indicate that the delay can dramatically grow with the load in certain topologies, even with low node degrees. To the best of our knowledge, the derivation of lower bounds for the average steady-state delay in CSMA networks has received little attention so far. An interesting paper by Shah et al. [22] showed that low-complexity schemes cannot be expected to achieve low delay in arbitrary topologies (unless P equals NP), since that would imply that certain NP-hard problems could be solved efficiently. However, the notion of delay in [22] is a transient one, and it is not exactly clear what the implications are for the average steady-state delay in specific networks, if any.

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Jiang et al. [10] derived upper bounds for the average steady-state delay based on mixing time results for Glauber dynamics. The bounds show that for sufficiently low load the delay only grows polynomially with the number of nodes in bounded-degree interference graphs. Recently, Subramanian & Alanyali [23] presented similar upper bounds for bounded-degree interference graphs with low load based on analysis of neighbor sets and stochastic coupling arguments. Bouman et al. [4] obtained delay bounds for backlog-based policies in complete interference graphs. While some of the conceptual notions in the present paper are similar (cliques, mixing times), we focus on lower rather than upper bounds, and exploit quite different techniques.

The remainder of the paper is organized as follows. In Section II, we present a detailed model description. We establish generic lower bounds for the delay in Section III. In Section IV, we apply the bounds to a canonical class of partite interference graphs, which includes several specific cases of interest such as grid topologies. In Section V, we make some concluding remarks and identify topics for further research.

II. MODEL DESCRIPTION

Network, interference graph, and traffic model

We consider a network of several nodes sharing a wireless medium according to a random-access mechanism. The network is represented by an undirected graph \( G = (V, E) \) where the set of vertices \( V = \{1, \ldots, N\} \) correspond to the various nodes and the set of edges \( E \subseteq V \times V \) indicate which pairs of nodes interfere. Nodes that are neighbors in the interference graph are prevented from simultaneous activity, and thus the independent sets correspond to the feasible joint activity states of the network. A node is said to be blocked whenever the node itself or any of its neighbors is active, and unblocked otherwise. Define \( S \subseteq \{0,1\}^N \) as the set of incidence vectors of all the independent sets of the interference graph, and denote by \( \mathcal{C} = \text{conv}(S) \) the capacity region.

Packets arrive at node \( i \) as a Poisson process of rate \( \lambda_i \). The packet transmission times at node \( i \) are independent and exponentially distributed with mean \( 1/\mu_i \). Denote by \( \rho_i = \lambda_i/\mu_i \), the traffic intensity of node \( i \).

Let \( U(t) \in S \) represent the joint activity state of the network at time \( t \), with \( U_i(t) \) indicating whether node \( i \) is active at time \( t \) or not. Denote by \( Q_i(t) \) the number of packets waiting for transmission at node \( i \) at time \( t \) (excluding any packet that may be in the process of being transmitted).

Random-access mechanism

As mentioned above, the various nodes share the medium in accordance with a random-access mechanism. When a node ends an activity period (consisting of possibly several back-to-back packet transmissions), it starts a back-off period. The back-off times of node \( i \) are independent and exponentially distributed with mean \( 1/\nu_i \). The back-off period of a node is suspended whenever it becomes blocked by activity of any of its neighbors, and only resumed once the node becomes unblocked again. Thus the back-off period of a node can only end when none of its neighbors are active. Now suppose a back-off period of node \( i \) ends at time \( t \). Then the node starts a transmission with probability \( \phi_i(Q_i(t)) \), and begins a next back-off period otherwise. We allow for \( \phi_i(0) > 0 \), so a node may activate even when its buffer is empty, and transmit a dummy packet. When a transmission of node \( i \) ends at time \( t \), it releases the medium and begins a back-off period with probability \( \psi_i(Q_i(t)) \), or starts the next transmission otherwise. Equivalently, node \( i \) may be thought of as activating at an exponential rate \( f_i(Q_i(t)) \), with \( f_i(\cdot) = \nu_i \phi_i(\cdot) \), whenever it is unblocked at time \( t \), and de-activating at rate \( g_i(Q_i(t)) \), with \( g_i(\cdot) = \mu_i \psi_i(\cdot) \), whenever it is active at time \( t \). For conciseness, the functions \( f_i(\cdot) \) and \( g_i(\cdot) \) will be referred to as activation and de-activation functions, respectively.

Network dynamics

Under the above-described backlog-based schemes, the process \( \{(U(t), Q(t))\}_{t \geq 0} \) evolves as a continuous-time Markov process with state space \( S \times \mathbb{N}^V \). Transitions (due to arrivals) from a state \( (U, Q) \) to \( (U, Q + e_i) \) occur at rate \( \lambda_i \). Transitions (due to activations) from a state \( (U, Q) \) with \( U_i = 0 \) and \( U_j = 0 \) for all neighbors \( j \) of node \( i \) to \( (U + e_i, Q - e_i1_{\{Q_i > 0\}}) \) occur at rate \( f_i(Q_i) \), transitions (due to transmission completions followed back-to-back by a subsequent transmission) from a state \( (U, Q) \) with \( U_i = 1 \) to \( (U, Q - e_i1_{\{Q_i > 0\}}) \) occur at rate \( \mu_i - g_i(Q_i) \) (due to transmission completions followed by a back-off period) from a state \( (U, Q) \) with \( U_i = 1 \) to \( (U - e_i, Q) \) occur at rate \( g_i(Q_i) \).

For any \( u \in S \), define \( \pi(u) = \lim_{t \to \infty} P\{U(t) = u\} \). Note that for fixed activation and de-activation rates, i.e., \( \phi_i(q_i) = \phi_i \) and \( \psi_i(q_i) = \psi \), the process \( \{U(t)\} \) does not depend on the process \( \{Q(t)\} \), and in fact constitutes a reversible Markov process, with a product-form stationary distribution

\[
\pi(u) = Z^{-1} \prod_{i=1}^{N} \sigma_i^{\mu_i}, \quad u \in S,
\]

and normalization constant

\[
Z = \sum_{u \in S} \prod_{i=1}^{N} \sigma_i^{\mu_i},
\]

with \( \sigma_i = \nu_i \phi_i/\mu_i \psi_i \). In that case \( \rho_i < \theta_i = \sum_{u \in S} \pi(u) \mu_i \), \( i = 1, \ldots, N \), is a simple necessary and sufficient condition for stability. It is in general difficult to establish under what conditions the system is stable, i.e., the process \( \{(U(t), Q(t))\}_{t \geq 0} \) is positive-recurrent. It is easily seen that \( (\rho_1, \ldots, \rho_N) < (\eta_1, \ldots, \eta_N) \in \mathcal{C} \) is a necessary condition for that to be the case. In [9], [11], [19], [21], it is shown that this condition is in fact also sufficient for activation and de-activation functions \( f_i(q) = r_i(q)/(1 + r_i(q)) \) and \( g_i(q) = 1/(1 + r_i(q)) \) with suitably chosen \( r(\cdot) \), e.g., \( r(q) = \log(q + 1) \). For \( f_i(q) \equiv r, \quad q \geq 1, \quad f_i(0) = 0, \quad g_i(q) \equiv 1, \quad [25] \) derives necessary and sufficient stability conditions for full interference graphs, and shows that an exact characterization of the stability region is difficult for all other topologies.
III. DELAY BOUNDS

We now proceed to construct lower bounds for the expected aggregate weighted queue lengths and delays in the case of fixed activation and deactivation rates. The bounds revolve around three simple observations: (i) nodes in cliques with large load require high activation rates for stability; (ii) high activation rates cause long mixing times, in particular slow transitions between dominant activity states; (iii) slow transitions between dominant states imply long starvation periods for some nodes, and hence large queue lengths and delays. In this section we formalize (iii), and establish lower bounds for the expected aggregate weighted queue lengths and delays in terms of the expected return times of the process \( \{U(t)\} \). In order to lower bound these return times, we will build in the next section on insights (i) and (ii) for a canonical class of partite interference graphs.

We first introduce some useful notation. For any \( w \in \mathbb{R}_+ \), \( V_0 \subseteq V \), \( S_0 \subseteq S \), define

\[
E(w, V_0, S_0) = \max_{u \in S_0} \sum_{i \in V_0} w_i \mu_i u_i,
\]

and denote

\[
D(w, V_0, S_0) = \sum_{i \in V_0} w_i \lambda_i - E(w, V_0, S_0).
\]

The coefficient \( E(w, V_0, S_0) \) represents the maximum aggregate weighted service rate of the nodes in \( V_0 \) when the system resides in one of the activity states in the subset \( S_0 \). Noting that \( \sum_{i \in V_0} w_i \lambda_i \) is the weighted arrival rate of the nodes in \( V_0 \), the coefficient \( D(w, V_0, S_0) \) may thus be interpreted as the minimum drift in the aggregate weighted queue lengths of the nodes in \( V_0 \) when the system resides in one of the activity states in the subset \( S_0 \).

With minor abuse of notation, define \( \pi(S_0) = \sum_{u \in S_0} \pi(u) \) as the fraction of time that the system resides in one of the activity states in the subset \( S_0 \). Denote by \( T_{S_0} \) a random variable representing the stationary return time to the subset of activity states \( S \setminus S_0 \). Denote by \( T_{S_0}^e \) a random variable representing the elapsed equilibrium lifetime of \( T_{S_0} \), i.e.,

\[
P\{T_{S_0}^e < t\} = \frac{1}{E\{T_{S_0}\}} \int_0^t P\{T_{S_0} > s\} \, ds.
\]

Now observe that when the system resides in one of the activity states in \( S_0 \), which is the case with probability \( \pi(S_0) \), the aggregate weighted queue lengths of the nodes in \( V_0 \) have experienced a drift no less than \( D(w, V_0, S_0) \) for an expected amount of time \( E\{T_{S_0}^e\} \). This observation indicates that the expected aggregate weighted queue lengths of the nodes in \( V_0 \) is bounded from below by \( \pi(S_0)D(w, V_0, S_0)E\{T_{S_0}^e\} \) for any choice of \( S_0 \), as formalized in the next proposition.

**Proposition 1.** For any \( w \in \mathbb{R}_+ \), \( V_0 \subseteq V \),

\[
\sum_{i \in V_0} w_i \mathbb{E}\{Q_i\} \geq \max_{S_0 \subseteq S} D(w, V_0, S_0)\pi(S_0)E\{T_{S_0}^e\}.
\]

Using Little’s law, we deduce

\[
\sum_{i \in V_0} w_i \lambda_i \mathbb{E}\{W_i\} \geq \max_{S_0 \subseteq S} D(w, V_0, S_0)\pi(S_0)E\{T_{S_0}^e\},
\]

with \( W_i \) a random variable representing the waiting time of an arbitrary packet at node \( i \).

In general it is difficult to calculate the expected elapsed equilibrium return time \( E\{T_{S_0}^e\} \). However, a simple lower bound is provided by

\[
E\{T_{S_0}^e\} \geq \frac{2}{\mathbb{E}\{T_{S_0}\}} \geq \frac{2}{E\{T_{S_0}\}}.
\]

Kac’s formula yields \( E\{T_{S_0}^e\} = \frac{\pi(S_0)}{Q(S_0)} \), with

\[
Q(S_0) = \sum_{u \in S_0} \sum_{u' \in S \setminus S_0} \pi(u)q(u, u') = \sum_{u \in S \setminus S_0} \sum_{u' \in S \setminus S_0} \pi(u')q(u', u),
\]

and \( q(u, u') \) denoting the transition rate from state \( u \) to state \( u' \) of the component \( \{U(t)\} \) of the Markov process as specified in the previous section, i.e., \( q(u, u + e_i) = \nu_i \phi_i \) and \( q(u + e_i, u) = \mu_i \psi_i \), \( u + e_i \in S_0 \). Define

\[
\partial S_0 = \{u \in S_0 : \sum_{u' \in S_0} q(u, u') > 0\}
\]

as the ‘boundary’ of \( S_0 \). Now suppose that \( S_0 \) is such that \( u + e_i \notin S \setminus S_0 \) for all \( u \in \partial S_0 \). In case \( \phi_i \equiv 1 \), \( \psi_i \equiv 1 \), \( \mu_i \equiv 1 \) and \( \nu_i \equiv \nu \geq 1 \), we then have \( Q(S_0) \leq N \pi(\partial S_0) \), and thus

\[
\pi(S_0) \leq \sum_{u \in S_0} \pi(u) = \frac{1}{N} \sum_{u \in S_0} \nu \sum_{i=1}^N u_i = \frac{1}{N} \sum_{i=1}^N M(S) - M(\partial S_0),
\]

with \( M(S') = \max_{u \in S'} \sum_{i=1}^N u_i \).

The question arises how to choose \( S_0 \) such that the maximum and thus the tightest possible lower bound is obtained. Evidently, the more \( S_0 \) includes states with some of the nodes in \( V_0 \) active, the larger the potential aggregate weighted service rates of the nodes in \( V_0 \), i.e., the larger \( E(w, V_0, S_0) \), and the smaller \( D(w, V_0, S_0) \). In other words, we need to ensure that \( S_0 \) excludes some of the states with nodes in \( V_0 \) active. Indeed, if \( S_0 \) includes all states with maximal subsets of the nodes in \( V_0 \) active, then \( E(w, V_0, S_0) = \max_{u \in S} \sum_{i=1} w_i \mu_i u_i \), with \( w_i = w_i \) if \( i \in V_0 \) and \( w_i = 0 \) otherwise. The fact that \( \lambda \pm \eta \) are necessary for stability in a heavy-traffic regime as we will show, the above arguments suggest that we should choose \( S_0 \) so that \( M(S_0) \) is large, but \( M(\partial S_0) \) is small so as to maximize the difference, i.e., \( S_0 \) contains a state with many active nodes, while the boundary of \( S_0 \) only contains states with few active nodes.
IV. PARTITE GRAPHS

In the previous section we derived generic lower bounds for the expected aggregate weighted queue lengths and delays in terms of the expected return times of the activity process \{U(t)\}. In this section we describe an approach to lower bound these return times for a broad class of \(K\)-partite interference graphs. We additionally assume that each of the nodes belongs to at least one clique of size \(K\) (where the other \(K - 1\) nodes necessarily belong to \(K - 1\) different components). The above class of graphs covers a wide range of network topologies with nearest-neighbor interference, e.g., linear topologies, ring networks with an even number of nodes, two-dimensional grid networks, toruses (two-dimensional grid networks with a wrap-around boundary) with an even number of nodes in both directions, and complete \(K\)-partite graphs, where all nodes are connected except those that belong to the same component.

A. Preliminaries

We first introduce some further notation and state a few preparatory lemmas. Denote by \(V_k \subseteq V\) the subset of nodes that belong to the \(k\)-th component and \(M_k = |V_k|\), \(k = 1, \ldots, K\). For compactness, define

\[
\Upsilon_k = \prod_{i \in V_k} (1 + \sigma_i) - 1 - \sum_{U \subseteq V_k, i \notin U} \prod_{i \notin U} \sigma_i.
\]

Denote by \(\theta_i = \mathbb{P}\{U_i = 1\} = \sum_{u \in S} \pi(u)u_i\) the fraction of time that node \(i\) is active and by \(\mathcal{N}_1\) the set of neighbors of node \(i\) in the graph \(G\), along with \(i\) itself.

**Lemma 1.** For any clique \(\mathcal{N} \subseteq V\) containing node \(i \in V_k\),

\[
\theta_i \leq \sigma_i [1 - \sum_{j \in \mathcal{N}} \theta_j],
\]

or equivalently,

\[
\theta_i \leq \frac{\sigma_i}{1 + \sigma_i}[1 - \sum_{j \in \mathcal{N}\setminus\{i\}} \theta_j].
\]

Also,

\[
\theta_i \geq Z^{-1} \frac{\sigma_i}{1 + \sigma_i}(\Upsilon_k + 1),
\]

\[
\Upsilon_k \leq Z[1 - \sum_{j \in \mathcal{N}\setminus\{i\}} \theta_j] - 1.
\]

**Proof:** Note that (4) is an immediate consequence of (2) and (3).

In order to prove (3), denote

\[
P(u) = \prod_{j=1}^{N} \sigma_j^{u_j}.
\]

Then,

\[
\theta_i = Z^{-1} \sum_{u \in S, u_i = 1} P(u) \geq Z^{-1} \sigma_i \prod_{i \in V_k \setminus\{i\}} (1 + \sigma_i) = Z^{-1} \frac{\sigma_i}{1 + \sigma_i}(\Upsilon_k + 1).
\]

In order to prove (1), we may write

\[
\theta_i = \nu_i \frac{\sigma_i}{1 + \sigma_i} \mathbb{P}\{U_j = 0 \text{ for all } j \notin \mathcal{N}_i^+\} \mathbb{P}\{U_j = 0 \text{ for all } j \in \mathcal{N}_i^+\}.
\]

Since the events \(\{U_j = 1\}\) are mutually exclusive for all \(j \in \mathcal{N}\), it follows that

\[
\mathbb{P}\{U_j = 0 \text{ for all } j \in \mathcal{N}\} \leq \mathbb{P}\{U_j = 0 \text{ for all } j \notin \mathcal{N}^+\}
\]

\[
= 1 - \sum_{j \in \mathcal{N}} \mathbb{P}\{U_j = 1\} = 1 - \sum_{j \in \mathcal{N}} \theta_j.
\]

Substitution into the above expression for \(\theta_i\) then yields (1).

An alternative proof argument proceeds as follows. Since \(\sum_{j \in \mathcal{N}} u_j \leq 1\) for all \(u \in S\), we may write

\[
Z = \sum_{u \in S} P(u)
\]

\[
= \sum_{j \in \mathcal{N}} \sum_{u \in S, u_j = 1} P(u) + \sum_{u \in S, \sum_{j \notin \mathcal{N}^+} u_j = 0} P(u)
\]

\[
\geq \sum_{j \in \mathcal{N}} \sum_{u \in S, u_j = 1} P(u) / \sigma_i \sum_{u \in S, u_i = 1} P(u)
\]

\[
= \sum_{j \in \mathcal{N}} \theta_j Z + \frac{1}{\sigma_i} Z
\]

\[
\Upsilon_k < Z[1 - \sum_{j \in \mathcal{N}\setminus\{i\}} \rho_j] - 1
\]

for any clique \(\mathcal{N} \subseteq V\) containing node \(i \in V_k\).

**Proof:** The proof follows from (1) and (4) and the fact that \(\rho_i > \theta_i\) for all \(i = 1, \ldots, N\) is a necessary condition for stability of the system.

B. Complete partite graphs

In order to gain some useful intuition, we first focus on complete \(K\)-partite graphs, where all nodes are connected except those that belong to the same component. In other words, the complement of the graph consists of \(K\) fully connected components. Thus, transmission activity is mutually exclusive across the various components.

In this case, we have

\[
Z = 1 + \sum_{l=1}^{K} \sum_{i \notin U \subseteq V_i} \prod_{i \notin U} \sigma_i = 1 + \sum_{l=1}^{K} \Upsilon_l.
\]

Throughout this subsection we assume that \(\rho_i = \hat{\rho}_k\) for all \(i \in V_k\), and denote \(\rho = \sum_{k=1}^{K} \hat{\rho}_k\), and \(\rho_{\min} = \min_{k=1,\ldots,K} \hat{\rho}_k\).

For convenience, we also assume \(\phi_i = 1, \psi_i = 1, \mu_i = 1, \nu_i = 1\),

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so that \( \sigma_i = \nu_i \) for all \( i = 1, \ldots, N \). Define \( M = \max_{k=1,\ldots,K} M_k \) as the maximum component size.

The next proposition provides a lower bound for the expected aggregate weighted queue lengths at some subset of nodes in \( V_0 \subseteq V_k \).

**Proposition 2.** For any \( w \in \mathbb{R}^N_+ \), \( V_0 \subseteq V_k \).

\[
\sum_{i \in V_0} w_i \mathbb{E} \{ Q_i \} \geq \frac{1}{2M} (\rho_{\min})^{M+1} \sum_{i \in V_0} w_i \lambda_i \left( \frac{1}{1 - \rho} \right)^{M-1},
\]

and for a symmetric scenario \((M_k \equiv M \) and \( \hat{\rho}_k \equiv \rho / K \) for all \( k = 1, \ldots, K \)),

\[
\mathbb{E} \{ Q_i \} \geq \frac{(K - 1)^2 \rho^M + 2}{2MK^{M+1}(K - (K - 1)\rho)} \left( \frac{1}{1 - \rho} \right)^{M-1}
\]

for all activation rate vectors \((\nu_1, \ldots, \nu_N)\).

The above lower bound implies that the expected delay grows as \( 1 / (1 - \rho)^{M-1} \) as the load \( \rho \) approaches 1. Based on observations (i)–(iii) in the previous section, this may be heuristically explained as follows. In order for the system to be stable, each node must at least have an activation rate of the order \( \nu = O(1 / (1 - \rho)) \). In turn, the transition times between the various activity states as governed by the maximum-size component occur on a time scale of the order \( O(\nu^{M-1}) \).

For \( M = 1 \) (full interference graph), the lower bound is loose, reflecting that it is not the slow transitions between the various components that cause the delays to be long in that case, but the sheer load. For \( M = 2 \), the lower bound is tight, but could also have been obtained by treating cliques as single-resource systems. For \( M = 3 \), the lower bound is particularly relevant, and reflects that the slow transitions between the various components cause the delays to be exponentially larger than can be explained from sheer load considerations alone.

**Proof:** For any \( l = 1, \ldots, K \), define \( S_l = \{ u \in S : \sum_{i \in V_l} u_i \geq 1 \} \) as the set of activity states where at least one of the nodes in \( V_l \) is active. For any \( W \subseteq V \), denote by \( p(W) = \sum_{u \in S} \pi(u) \max_{i \in W} u_i \) the probability that at least one of the nodes in \( W \) is active. Note that \( p(V_k) = \lambda_k / Z \), and \( \hat{\rho}_k < p(V_k) \) for stability.

The proof relies on applying Proposition 1, taking \( S_0 \) to be \( S \setminus S_k \) and \( S_l, l \neq k \). In either case, \( u_i = 0 \) for all \( i \in V_0, u \in S_0 \), so that \( E(w, V_0, S_0) = 0 \), i.e.,

\[
D(w, V_0, S_0) = \sum_{i \in V_0} \lambda_i w_i.
\]

We first consider case (i). Then

\[
\pi(S_0) = \pi(S \setminus S_k) = 1 - p(V_k) = p(\emptyset) + \sum_{l \neq k} p(V_l) \geq \sum_{l \neq k} \hat{\rho}_l,
\]

and

\[
Q(S_0) = \pi(\emptyset)\sigma^{(k)} = p(\emptyset)\sigma^{(k)} = \sigma^{(k)} / Z,
\]

with \( \sigma^{(k)} = \sum_{i \in V_k} \sigma_i \).

Invoking (6), we derive

\[
\sigma^{(k)} \leq M_k \max_{i \in V_k} \sigma_i = M_k \max_{i \in V_k} \frac{\sigma_i}{\lambda_i} \frac{\lambda_i}{\hat{\rho}_l} (\lambda_k + 1) = M_k \frac{\sigma^{(k)} \lambda_k + 1}{1 + \sigma^{(k)} \hat{\rho}_l}.
\]

Exploiting (5), we deduce

\[
\min_{i \in V_k} \frac{\lambda_i}{\hat{\rho}_l} \geq \frac{\rho / K}{1 - \rho}.
\]

Thus

\[
\frac{1}{Q(S_0)} = \frac{Z / \sigma^{(k)}}{1 + \min_{i \in V_k} \frac{\lambda_i}{\hat{\rho}_l}} \geq \frac{1}{M_k (1 - \sum_{l \neq k} \hat{\rho}_l)} \frac{\rho / K}{1 - \rho}.
\]

We obtain the lower bound

\[
\sum_{i \in V_0} w_i \mathbb{E} \{ Q_i \} \geq \frac{\lambda_i}{\hat{\rho}_l} (\lambda_k + 1) \frac{\rho / K}{1 - \rho}.
\]

Taking \( V_0 = V_k \) yields the second statement of the lemma for a symmetric scenario.

In order to complete the proof of the first part of the lemma, we now turn to case (ii). Then

\[
\pi(S_0) = \pi(S_l) = p(V_l) \geq \hat{\rho}_l,
\]

and

\[
Q(S_0) = \pi(\emptyset)\sigma^{(l)} = p(\emptyset)\sigma^{(l)} = \sigma^{(l)} / Z,
\]

with \( \sigma^{(l)} = \sum_{i \in V_l} \sigma_i \).

Just like in the previous case, we arrive at

\[
\frac{1}{Q(S_0)} = \frac{Z / \sigma^{(l)}}{1 + \min_{i \in V_l} \frac{\lambda_i}{\hat{\rho}_l}} \geq \frac{1}{M_l (1 - \sum_{m \neq l} \hat{\rho}_m)} \frac{\rho / K}{1 - \rho}.
\]

We obtain the lower bound

\[
\sum_{i \in V_0} w_i \mathbb{E} \{ Q_i \} \geq \frac{\rho / K}{1 - \rho}.
\]

Combining the above two lower bounds yields the first part of the lemma.

In order to validate the result of Proposition 2, we simulated a system that is represented by a symmetric bipartite \((K = 2)\) complete interference graph. Figure 1 shows the average total number of packets for a symmetric bipartite complete graph with \( M = 4 \) for several activation rate vectors.

**Fig. 1.** Average total number of packets for a symmetric bipartite complete graph with \( M = 4 \) for several activation rate vectors.

\[
M_k \max_{i \in V_k} \frac{\sigma_i}{\lambda_i} (\lambda_k + 1) \frac{\rho / K}{1 - \rho} \leq M_k Z (1 - \sum_{l \neq k} \hat{\rho}_l).
\]
the activation rate vector with \( \nu_i \equiv \nu \). Note that we used a log-lin scale.

We see that the simulated curves lie well above the lower bound of Proposition 2 for all chosen values of \( \nu \). Note that the system is not stable for all values of \( \rho \), e.g., for \( \nu = 1 \) the system is unstable if \( \rho \geq 20I_i = \frac{10}{\pi} \), explaining the jumps in the simulation result. Further note that the expected time between activation of nodes in different components is smaller for small values of \( \nu \) than for large values of \( \nu \). This explains why small values of \( \nu \) tend to perform better in case \( \rho \) is small, i.e., for large values of \( \nu \) the nodes in one component will often be transmitting dummy packets while the nodes in the other component do have packets waiting to be transmitted.

C. Extensions

We now return to the broader class of (not necessarily complete) \( K \)-partite graphs. Thus, transmission activity is no longer mutually exclusive across the various components. However, we make the next assumption implying that joint activity across various components is relatively inefficient. Denote by \( v^{(k)} = 1_{V_k} \) the incidence vector of \( V_k \), i.e., \( v_i^{(k)} = 1 \) if \( i \in V_k \) and \( v_i^{(k)} = 0 \) otherwise, and define \( S^* = \{ v^{(1)}, \ldots, v^{(K)} \} \).

**Assumption 1.** For any \( u \in S \),
\[
H(u) = \sum_{k=1}^K \sum_{i \in V_k} \frac{u_i}{M_k} \leq 1,
\]
with strict inequality for any \( u \not\in S^* \).

Based on the above assumption, we define \( \zeta = 1 - \max_{u \in S} H(u) > 0 \). An illustrative example is provided by a \( 2L \times 2L \) grid with nodes labeled as \( \{(i, j) \} \), \( i, j = 1, \ldots, 2L \), and nearest-neighbor interference. The two components are \( V_1 = \{(i, j) : (i + j) \mod 2 = 1 \} \) and \( V_2 = \{(i, j) : (i + j) \mod 2 = 0 \} \), with \( M_1 = M_2 = 2L^2 \). In order for \( m \) \geq 1 \) nodes in \( V_1 \) to be active, at least \( m + 1 \) or \( m + 3 \) nodes in \( V_2 \) must be inactive (depending on whether or not we assume a wrap-around boundary). Thus \( \sum_{i=1}^{2L} \sum_{j=1}^{2L} u_i \leq 2L^2 - 1 \) (or \( 2L^2 - 3 \)) for all \( u \not\in S^* \), and \( \zeta = \frac{1}{2} \) (or \( \frac{3}{2} \)).

In order to state a lower bound for the expected aggregate weighted queue lengths at some subset of nodes \( V_0 \subseteq V \), we continue to assume that \( \rho_i = \hat{\rho} \) for all \( i \in V_k \), and first introduce some further notation and concepts. Denote \( U(V_0) = \{ u \in S : \sum_{i=1}^N w_i \lambda_i \leq \sum_{i=1}^N w_i \mu_i \} \), and define
\[
\delta = \sum_{i=1}^N w_i \lambda_i - \max_{u \in U(V_0)} \sum_{i=1}^N w_i \mu_i > 0.
\]

A sequence of states \( (u^{(0)}, u^{(1)}, \ldots, u^{(l)}) \), with \( u^{(k)} \in S \), \( k = 0, \ldots, l \), is called a path from \( u^{(0)} \) to \( u^{(l)} \) if \( (u^{(k)}, u^{(k+1)}) \) are feasible transitions, i.e., \( q(u^{(k)}, u^{(k+1)}) > 0 \) for all \( k = 0, \ldots, l - 1 \). For a given path \( p = (u^{(0)}, u^{(1)}, \ldots, u^{(l)}) \), denote by \( m(p) := \min_{k=0,1,..,l} H(u^{(k)}) \) the minimum value of the function \( H(\cdot) \) along the path. For given states \( u, v \in S \), denote by \( P(u, v) \) the collection of all paths from \( u \) to \( v \). Define \( M(u, v) := \max_{p \in P(u, v)} m(p) \) as the maximum of the minimum value of the function \( H(\cdot) \) along any path from state \( u \) to state \( v \), with the convention that \( M(u, u) = \infty \).

For all \( l \neq k \), define \( m_l = \max_{u \in U(V_0)} M(v^{(l)}), u \) and
\[
S_l = \{ u \in S : M(v^{(l)}, u) > m_l \}
\]
as the set of states that can be reached from \( v^{(l)} \) via a path \( p \) with \( m(p) > m_l \). By definition \( S_l \cap U(V_0) = \emptyset \), and thus \( D(w, V_0, S_l) \geq \delta, l \neq k \).

Also, define \( m_k = \max_{u \not\in S_k} m_l \), and
\[
S_k = \{ u \in S : \max_{u \in U(V_0)} M(u, v) > m_k \}
\]
as the set of states that can be reached from \( U(V_0) \) via a path \( p \) with \( m(p) > m_k \). Note that \( U(V_0) \subseteq S_k \), so that \( D(w, V_0, S_k) \geq \delta \), and \( S_k \cap S_l = \emptyset \), i.e., \( S_l \subseteq S \setminus S_k, l \neq k \).

Define \( H_l = \max_{u \in S_k} H(u) \) and \( H^* = \min_{k=1, \ldots, K} H_l \). Denote \( k^* = \arg \max_{k=1, \ldots, K} M_k \) and \( Q = \delta_{k^*} \delta_{M_{k^*}} = \delta_{M_k}^* \).

**Proposition 3.** Assume \( \rho \geq \rho_0 = 1 - \epsilon \rho_{\min}^2 \) for \( \epsilon > 0 \) sufficiently small. For any \( w \in \mathbb{R}^{2L} \), \( V_0 \subseteq V \),
\[
\sum_{i=1}^L w_i 1 \{ Q_{i} \} \geq \frac{1}{2} \delta C \left( \frac{1}{1 - \rho} \right)^{M(1-H^*)}
\]
for all activation rate vectors \( (\nu_1, \ldots, \nu_N) \), with \( C \) denoting a positive constant that only depends on the \( \rho_k \) values through \( \rho_{\min} \).

The value of the coefficient \( H^* \) depends strongly on the specific properties of the interference graph \( G \). For a complete bipartite graph, the sets \( S_l \) coincide with those in the previous subsection, and we have \( \partial S_l = \bigcup_{i \in V_l} \{ e_i \} \), so that \( H_1 = 1/M_1 \), and \( H^* = 1/M \), recovering the result of Proposition 2. On the other hand, when the graph consists of \( N/K \) fully connected components, we have \( H_1 \equiv 1 \), and the result trivializes. An interesting intermediate situation is the \( 2L \times 2L \) grid mentioned earlier with \( M_1 = M_2 = 2L^2 \), for which we conjecture that \( H^* = H_1 = H_2 = 1 - 1/L \) or \( 1 - 1/(2L) \) if \( L \geq 2 \), depending on whether or not we assume a wrap-around boundary, suggesting that the mean queue lengths would grow as \( 1/(1 - \rho)^L \) or \( 1/(1 - \rho)^{2L} \).

D. Backlog-based strategies

The lower bounds in Propositions 2 and 3 for the expected aggregate weighted queue lengths and delays assume fixed activation and de-activation rates. Over the last few years, significant interest has emerged in backlog-based activation and de-activation strategies. As mentioned earlier, such strategies, when suitably designed, provide the remarkable capability of achieving throughput optimality without requiring explicit knowledge of the arrival rates. However, the type of activation rules for which throughput optimality has been established, may yield excessive delay and backlogs.

We now investigate to what extent the lower bounds for fixed activation and de-activation rates apply to backlog-based
activation rules as well. Let $N \subseteq V$ be a clique in the interference graph $G$. As shown earlier,

$$P\{U_j = 0 \text{ for all } j \in N\} = 1 - \sum_{j \in N} \theta_j,$$

since the events $\{U_j = 1\}$ are mutually exclusive for all $j \in N$. Also, $\rho_j = \theta_j$ when node $j$ is stable and $\phi_j(0) = 1 - \psi_j(0) = 0$.

Observing that the mean number of activations at node $i$ equals the mean number of de-activations at node $i$ per unit of time, provided node $i$ is stable, we obtain

$$E \left\{ f_i(Q_i) I\{U_j = 0 \text{ for all } j \in N_i^+\} \right\} = \rho_i E \left\{ g_i(Q_i) \right\}, \quad (7)$$

where $Q_i^t$ denotes the number of packets waiting for transmission at node $i$ at a departure epoch. Note that $Q_i^t$ is, in distribution, equal to $Q_i$.

A non-serving interval for the clique is a time interval during which none of the nodes $i \in N$ is active. Denoting by $Q_{i,t}$ the number of packets at node $i$ at an arbitrary epoch during a non-serving interval for the clique $N$, we find

$$E \left\{ f_i(Q_i) I\{U_j = 0 \text{ for all } j \in N_i^+\} \right\} \leq E \left\{ f_i(Q_i) I\{U_j = 0 \text{ for all } j \in N\} \right\} = E \left\{ f_i(Q_{i,t}) \right\} P\{U_j = 0 \text{ for all } j \in N\}. \quad (8)$$

For compactness, define $\lambda_N = \sum_{j \in N} \lambda_j$, $\nu_N = \sum_{j \in N} \nu_j$, and $\rho_N = \sum_{j \in N} \rho_j$. Assuming the system is stable, we obtain the following proposition.

**Proposition 4.** (i) If $f_i(\cdot) \equiv f(\cdot)$, with $f(0) = 0$ and $f : [0, \infty) \to [0, \infty)$, is a strictly increasing concave function and $g_i(\cdot) = \mu_i$, then

$$\sum_{i \in N} E\{Q_i\} \geq \lambda_N \sum_{i \in N} \lambda_i / \rho_N^2 + |N| f^{-1}\left(\frac{\lambda_N}{|N| - \rho_N}\right),$$

for any clique $N \subseteq V$.

For a symmetric $K$-partite graph,

$$E\{Q_i\} \geq \frac{\rho_i^2}{1 - \rho_i} + f^{-1}\left(\frac{\lambda_i}{1 - \rho_i}\right).$$

(ii) If $f_i(0) = 0$, $f_i(\cdot) = \nu_i$ for all $q_i \geq 1$, $\mu_i \equiv \mu$ and $g_i(\cdot) \equiv g(\cdot)$, with $g(0) = \mu$ and $g : [0, \infty) \to [0, \mu]$, is a strictly decreasing convex function, then

$$\sum_{i \in N} \rho_i E\{Q_i\} \geq \rho_N g^{-1}\left(\frac{(1 - \rho_N)\nu_N}{\rho_N}\right),$$

for any clique $N \subseteq V$.

For a symmetric $K$-partite graph,

$$E\{Q_i\} \geq g^{-1}\left(\frac{(1 - \rho)\nu_i}{\rho_i}\right).$$

**Proof:** For case (i) we find using equations (7) and (8) that

$$(1 - \rho_N) \sum_{i \in N} E\{f_i(Q_{i,t})\} \geq \lambda_N.$$
We illustrate the above through a numerical experiment. We again take a system represented by a symmetric bipartite complete interference graph with $M = 4$ and take the de-activation function $g(q) \equiv 1$. Figures 2 and 3 show the average total number of packets in the time interval $[0, 5 \cdot 10^6]$ starting from an empty configuration for the activation functions $f(q) = q$ and $f(q) = \log(q + 1)$ respectively.

In both figures we see that the lower bound of Proposition 4 is remarkably close to the simulation result for small values of $\rho$. Further, for larger values of $\rho$ the bound and simulation result are farther apart. One explanation for this lies in the approximation made in (8), for small values of $\rho$ this approximation is relatively good while for large values of $\rho$ this approximation is off by a factor of about 2 in this case. Finally note that the simulation result for $f(q) = q$ lies below the lower bound found in Proposition 2 for large values of $\rho$, suggesting that the activation function $f_i(q) \equiv q$ performs better in heavy-traffic than $f_i(q) = \nu_i$ for any choice of the activation rate vector $(\nu_1, \ldots, \nu_N)$.

V. CONCLUSION

We have established lower bounds for the expected queue lengths and delays in wireless CSMA networks. The bounds indicate that the delay for fixed activation rates can dramatically grow with the load in certain topologies. We further showed that the delay for backlog-based activation and deactivation strategies exceeds this lower bound when the degree of aggressiveness grows slowly as function of the backlog. An interesting issue for further research is to determine whether better delay performance can be achieved when the aggressiveness grows faster as function of the backlog.

REFERENCES