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Analysis of a Decentralized Supply Chain under Partial Cooperation

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In this article we analyze a decentralized supply chain consisting of a supplier and two independent retailers. In each order cycle retailers place their orders at the supplier to minimize inventory-related expected costs at the end of their respective response times. There are two types of lead-times involved. At the end of the supplier lead-time, retailers are given an opportunity to readjust their initial orders (without changing the total order size) so that both retailers can improve their expected costs at the end of respective retailer lead-times (the time it takes for items to be shipped from the supplier to the retailers). Because of the possibility of cooperation at the end of supplier lead-time, each retailer will consider the other’s order-up-to level in making the ordering decision. Under mild conditions we prove the existence of a unique Nash equilibrium for the retailer order-up-to levels, and show that they can be obtained by solving a set of newsboy-like equations. We also present computational analysis that provides valuable managerial insight for design and operation of decentralized systems under possibility of partial cooperation.

1 Introduction and Motivation

In this article we investigate a decentralized supply chain of two independent retailers (or manufacturers), and a supplier. In the system that we analyze retailers order a common product (or raw material) from the supplier to fulfill their own random customer demand. We consider a periodic review system where each retailer places an order at regular intervals to raise its inventory position to a predetermined level. The supplier has
ample capacity to satisfy the orders placed by the retailers, but there is a fixed lead-time (supplier lead-time) associated with order preparation (due to supplier’s manufacturing or ordering lead-time, packaging and loading times at the supplier’s plant). At the end of the supplier lead time, orders are shipped to retailers. We also assume that there’s a fixed shipment lead-time associated with each retailer (retailer lead-time). When the order is received by a retailer, it is used to satisfy the customer demand and accordingly inventory related holding and backorder costs are incurred.

The environment introduced above fits into two classical manufacturing/retailing schemes commonly considered in the literature:

(i) Consider two independently operated plants manufacturing electronic devices. Both utilizes the same computer chip in their production process, and chips are ordered from an overseas supplier. At the end of supplier lead-time chips are shipped to manufacturers.

(ii) Consider cross-docking operations in a retail business. Retailers (being outlets of the same retailing corporation, but having autonomy in their ordering decisions) order the same item from a common supplier. In this case, the supplier lead time consists of the duration of time that the orders are received at the cross-docking warehouse of the retailing corporation, plus the time it takes to load the trucks that are destined to individual retailers.

At the end of the supplier lead-time there is an ideal inventory position for retailers that they would like to attain, which minimizes their inventory related expected costs at the end of their respective response time (retailer lead time plus one period). However, as the demand observed during the supplier lead-time is random, their realized inventory positions will be either below or above that particular level (if demands are assumed to be continuous, being at the ideal level has zero probability). Therefore, a possible transfer of retailer orders at the supplier’s plant (or at the cross-docking warehouse), before they are shipped to retailers may improve the inventory related costs of the retailers. However, as the supply chain is decentralized, retailers will allow such a transaction only if the transfer yields an improvement for both of them. In Figure 1 we provide an illustration of the system. Our setting fits best for systems where the supplier-retailer chain can not enjoy the full benefits of centralized decision making, but a room for cooperation
still exists, if it improves the performance of both retailers. We detail the extend of cooperation between retailers by referring to the examples given above:

(i) Consider two manufacturing plants in Europe, operating under the same corporation but serving different markets. Managers of each plant are evaluated with respect to their own performance. They order computer chips from an Asian supplier and they agree on re-adjusting their original orders at the end of the supplier lead-time provided that it does not deteriorate their own performance. Since the re-adjustment will not affect the total quantity ordered from the supplier, no costs are associated with such a transaction.

(ii) Consider two retail stores for an apparel chain, supplied through a common regional cross-docking warehouse. Orders can be re-adjusted at the warehouse just before the trucks are loaded. Again, both stores would agree such a transaction as long as it improves their performances.

As depicted in Figure 1 an order cycle is divided into two subcycles. At the beginning of the order cycle each retailer places an order at the supplier. At the end of the first subcycle (at the end of the supplier lead-time) retailers re-assess their stock position, and decide whether a transfer of stock is to take place. Then, they inform the supplier (or the cross-docking warehouse) of their decision. Since no physical transshipment takes place (the transaction occurs either at the suppliers plant or at the cross-docking warehouse),
we assume that the cost of transfer is negligible. Each retailer starts the second subcycle with its new inventory position, and at the end of the order cycle associated costs are incurred.

The main objective of this article is to characterize the optimal order-up-to levels that the retailers base their ordering amounts from the supplier. Since, one retailer’s order-up-to level affects the order placed by the other retailer (consider the extreme case, where one retailer places an order of size infinity, and therefore it is always willing to transfer any required amount to the other retailer), the order-up-to levels should be found in a cooperative setting. We assume that the retailers share relevant information (such as costs, demand distributions, inventory levels) in order to jointly compute optimal order-up-to levels. We also would like to answer several questions pertaining to the behavior of the optimal solution:

(a) what is the degree of improvement (in terms of costs and safety stocks) gained by cooperation between retailers?

(b) which form of cooperation brings the most benefits (cooperation regarding the transfer of stock or using the cooperative solution of the order-up-to levels)?

(c) How far is the performance of independent cooperative system from the centralized solution?

In this article we make four major contributions: (1) We present and analyze a model of a decentralized supply chain where retailers make their ordering decision by taking into account the possibility of cooperation, (2) we prove the existence of a unique Nash equilibrium of retailer order-up-to levels, (3) we introduce a key random variable, the net change in the retailer stock position after re-adjustment of retailer orders, and provide its distribution function, which facilitates an explicit characterization of optimal order-up-to level through a newsboy-like equation, (4) we present our computational analysis that provides valuable managerial insight for design and operation of decentralized systems under possibility of partial cooperation.

The rest of the article is organized as follows. In Section 2 we review the related literature. In Section 3 we present the mathematical model and derive the expected cycle costs for the retailers. Our analysis relies on the characterization of a random variable denoting the net change in a retailer’s stock position after transfer of stock takes
In Section 4 we derive distribution function of that random variable, and discuss its various properties. We prove the existence of a unique cooperative solution in Section 5. Computational findings and discussion of results are presented in Section 6. We present our concluding remarks and discuss extensions in Section 7.

2 Related Literature

The underlying structure of the system that we investigate is essentially a two-echelon inventory model with transshipment. Such systems are widely studied in the literature when the decision making is centralized. Krishnan and Rao (1965), and Gross (1963) are early examples of models with transshipment. Das (1975) extends the single period model of Gross (1963) to allow transshipments in a certain epoch within the period. Das (1975) argues optimality of Base Stock Conserving (BSC) transfer rules. Under a BSC transshipment, the excess stock of one location (over its base stock level) is transshipped to other location (if it has a shortage with respect to its base stock level). Tagaras (1989) considers a similar model but restricts transshipment epoch to the end of a period. Robinson (1999), and Tagaras (1999) extend earlier studies to multi-location environments. Robinson (1999) shows the optimality of a myopic base stock policy for the case of identical retailers. Our basic difference from these articles is that the locations in our model make their decisions independently, rather than seeking to achieve a joint objective function.

Another related direction of research is the coordination issues of decentralized supply chains. Most of these articles consider incentive schemes between the supplier and the retailer(s) to achieve the centralized solution (see, for example, Cachon and Zipkin (1999), Chen (1999), Lee and Wang (1999), Cachon (2001)).

Game theoretic consumer choice models, as they impose an interaction among retailers, are related to stock transfer model that we investigate. Parlar (1988) considers two substitutable products, where excess demand for an item is directed to other item’s stock. Parlar (1988) proves the existence and uniqueness of the Nash solution in a single period setting. Wang and Parlar (1994) extend Parlar (1988) to a three-product case. Avsar and Gursoy (2002) consider a multi-period extension of Parlar (1988), and show the existence of a myopic Nash solution (within the class of stationary policies) for the infinite horizon case. In Lipman and McCardle (1997) industry demand for an item is
allocated across locations using specific splitting rules. After the allocation of demand, excess demand is re-allocated in a way similar to Parlar (1988). They characterize the Nash equilibrium of inventory levels and show that it is unique under certain conditions. Mahajan and van Ryzin (2001) analyze a system where consumers choose dynamically from the available products based on a utility maximization criterion. They show that competition among locations leads to overstocking, which in the limit becomes so excess that the individual profits of locations approach to zero. We differ from the consumer choice models reviewed above in two important aspects. In the above mentioned articles, excess demand is transferred across locations, whereas we consider the transfer of stock. Also, the transfer of demand from one location to another occurs after the realization of demand (with respect to a single order-up-to level computed at the beginning of a single period), whereas in our model we consider the transfer of stock after the occurrence of the supplier lead-time demand, but before the realization of retailer lead-time demand is observed. Therefore, in our analysis the stock position (right before the realization of retailer lead-time demand) of a retailer is not only affected by the excess demand in the other retailer, but also affected by the occurrence of shortage (with respect to retailer lead-time order-up-to level).

In a recent article Rudi et al. (2001) consider a two-retailer decentralized model with transshipment of stock. The authors aim to find transshipment prices for which the joint decentralized profit achieves the centralized system profit. Anupindi et al. (2001) employ a general framework with $N$ retailers. Each location makes independent inventory decision (how much to stock), demands are observed, and then locations jointly determine the shipment decision (how to allocate the excess demand). For the inventory decision Anupindi et al. (2001) develop conditions for the existence of a pure strategy Nash equilibrium. They also show that there exists an allocation mechanism for the decentralized system that achieves the centralized solution. Both the motivation and the analysis of our article differ from these papers in the following respects. In our work we model a system differentiated by (possibly long) lead times that are observed both prior and after the transfer of stock takes place. Therefore, as well as the determination of the transfer (or allocation of stock) decision that takes place after the realization of supplier lead-time, and the determination of the inventory decision that precedes the supplier lead-time, we are also interested in the consequences of these decisions in the post-transfer periods (through the retailer lead-time). As mentioned in Section 1, a
cross-docking distribution system, where the replenishment cycle is decomposed into to-
warehouse and from-warehouse lead-times provides a good example. In such a case, not
only the inventory decision at the beginning of the cycle but also the revised decision
at some epoch within the cycle becomes important. The managers of the retailers are
interested in the impact of their inventory and transfer decisions with respect to the
order-up-to levels computed relative to the retailer shipment times. In our analysis we
derive and employ the distribution of net change in the stock position of a retailer as a
function of both the original order-up-to levels, and order-up-to levels relative to retailer
shipment times. This distribution enables us to analyze important system dynamics with
respect to essential system characteristics.

3 Description of the Mathematical Model

In this Section we present our mathematical model. We choose to describe the details
using a single order cycle in order to ease the exposition. Notation and basic definitions
are laid out in Section 3.1. In Section 3.2 we derive the expected cost function of a
retailer. In Section 3.3 we introduce the general periodic review model, and state and
discuss the conditions under which the single cycle analysis will hold for the stationary
problem. As will be observed in Section 3.3 these conditions are mild.

3.1 Preliminaries and Notation

The system is comprised of a supplier and two retailers. Suppose time is divided into
periods of unit length. Let $L$ be the length of the supplier lead-time, and $l_i$ be the length
of the retailer lead-time for retailer $i$ ($i = 1, 2$). For the cross-docking example of Section
1, $L$ represents the transportation time of shipping goods from the supplier to the cross-
docking warehouse, and $l_i$ represents the time it takes to ship goods from the warehouse
to the retailer. We define $D_{i,t}$ as the demand observed at retailer $i$ in period $t$, and we
let $D_i(k)$ as the generic continuous random variable denoting total demand occurring
at retailer $i$ ($i = 1, 2$) over $k$ periods ($k = 1, 2, \ldots$). Although we assume that retailer
demands are independent through time and across retailers, in Section 4 we argue that
this restriction can be relaxed. Let $F_i^{(k)}(x)$ be the distribution functions for $D_i(k)$. We
assume that $F_i^{(1)}(x)$ is continuously differentiable and strictly increasing on the support
(0, ∞) with no mass at x = 0. Our results would hold for distribution functions over a finite support as well. Let \( f_i^{(k)}(x) \) be the density function of \( D_i(k) \).

At the beginning of a period (denote as the first period for notational convenience) retailer \( i \) places an order at the supplier to raise its inventory position to level \( y_i, (i = 1, 2) \). When retailers jointly determine order-up-to levels, inventory decision of one retailer depends on the other. Therefore, for a given inventory decision \( y_2 \) of retailer 2, optimal inventory level of retailer 1 is denoted as \( S_1(y_2) \). In order to simplify the notation we suppress the dependency of \( S_1 \) (and of \( S_2 \)) on the other retailer's decision, and throughout the article we use \( S_i \) to denote the optimal value for \( y_i \).

For each retailer we focus on a replenishment cycle of length \( L + l_i + 1 \), as the impact of the inventory decision made at the beginning of the cycle affects inventory related costs of a retailer at the end of period \( L + l_i + 1 \). At the beginning of period \( L + 1 \), each retailer has its own ideal inventory position that it would like to achieve to minimize respective holding and backorder costs to be incurred at the end of period \( L + l_i + 1 \). Let \( Z_1 \) and \( Z_2 \) be these ideal levels for retailer 1 and retailer 2, respectively. We denote the holding and backorder costs associated with retailer \( i \) as \( h_i \) and \( b_i \) \((i = 1, 2)\), respectively.

### 3.2 Development of the Expected Cost Functions

We first derive the ideal stock position, \( Z_i \), that retailer \( i \) would like to attain at the beginning of period \( L + 1 \). In order to compute \( Z_i \), we concentrate on demands occurring in periods \( L + 1, L + 2, \ldots, L + l_i + 1 \): simply, \( Z_i \) is the base stock level for an inventory system where the response time are the periods \( L + 1, L + 2, \ldots, L + l_i + 1 \). Let \( G_i(y) \) be the expected single period cost observed at retailer \( i \), given that the inventory position at the beginning of period \( L + 1 \) (after any transfer transaction is realized) is \( y \):

\[
G_i(y) = h_i \int_0^y (y - x) f_i^{(l_i + 1)}(x) \, dx + b_i \int_y^\infty (x - y) f_i^{(l_i + 1)}(x) \, dx. \tag{1}
\]

Since \( G_i(y) \) is strictly convex (by assumptions imposed on \( F_i^{(l_i + 1)}(x) \)), the optimal inventory position for retailer \( i \) after any transfer transaction is uniquely given by:

\[
F_i^{(l_i + 1)}(Z_i) = \frac{b_i}{h_i + b_i}. \tag{2}
\]

For notational convenience, define

\[
\Delta_i = y_i - Z_i. \tag{3}
\]
Let $A_i$ and $B_i$ be the random variables denoting excess and shortfall inventory for retailer $i$ relative to $Z_i$ at the end of period $L$ (before any re-allocation decision):

$$A_i = (\Delta_i - D_i(L))^+$$
$$B_i = (D_i(L) - \Delta_i)^+,$$

where $(x)^+ = \max\{0, x\}$. Note that realizations of $A_i$ and $B_i$ can not be both positive.

We define random variables $\eta_1$ and $\eta_2$ as follows:

$$\eta_1 = \text{amount that can be transferred from retailer 2 to retailer 1}$$
$$\eta_1 = \min\{B_1, A_2\}$$

$$\eta_2 = \text{amount that can be transferred from retailer 1 to retailer 2}$$
$$\eta_2 = \min\{B_2, A_1\}.$$

We define $X_i$ as the net change in the stock position of retailer $i$ at the end of period $L$ after any transfer transaction is completed:

$$X_1 = D_1(L) - \eta_1 + \eta_2$$
$$X_2 = D_2(L) - \eta_2 + \eta_1.$$

We note that $\eta_1$ and $\eta_2$ can not be both positive and both $X_1$ and $X_2$ are functions of $y_1$ and $y_2$ (actually, of $\Delta_1$ and $\Delta_2$, but $Z_1$ and $Z_2$ can be found by equation (2)). We define $C_i(y_1, y_2)$ as the expected inventory holding and backorder costs of retailer $i$ at the end of period $L + l_i + 1$:

$$C_i(y_1, y_2) = E[G_i(y_i - X_i)], \quad (4)$$

where the expectation is taken over $X_i$. Since retailer $i$ would like to get as close to its $(l_i + 1)$-period optimal inventory position $Z_i$ after any transfer occurs, intuitively, for any given $y_j$ ($j \neq i$), $y_i$ that minimizes (4) should be bounded below by $Z_i$. We formalize this observation for retailer 1 in the following proposition (the argument for retailer 2 is the same).

**Proposition 1** Let $S_1$ be the minimizer of $C_1(y_1, y_2)$ over $y_1$ for a given value of $y_2$. Then, $S_1 \geq Z_1$.

**Proof:** It is sufficient to show that $C_1(Z_1, y_2) \leq C_1(Z_1 - \delta, y_2)$ for any $\delta > 0$. For $y_1 = Z_1$ we have $A_1 = 0$, $B_1 = D_1(L)$ and hence $\eta_2 = 0$ and $\eta_1 = \min\{D_1(L), A_2\}$. For $y_1 = Z_1 - \delta$
we have \( A_1 = 0, B_1 = D_1(L) + \delta \) and therefore \( \eta_2 = 0 \) and \( \eta_1 = \min\{D_1(L) + \delta, A_2\} \). Then

\[
X_1 = D_1(L) - \min\{D_1(L), A_2\}
\]

\[
X_1^\delta = D_1(L) - \min\{D_1(L) + \delta, A_2\},
\]

where \( X_1^\delta \) is the net change variable corresponding to \( y_1 = Z_1 - \delta \). It can easily be verified that \( 0 \leq X_1 \leq \delta + X_1^\delta \) for every point in the sample space generating \( (D_1(L), D_2(L)) \).

Since \( Z_1 \) is the minimizer of \( G_1(y) \), and \( G_1(y) \) is convex,

\[
G_1(Z_1 - X_1) \leq G_1(Z_1 - \delta - X_1^\delta),
\]

and hence \( C_1(Z_1, y_2) \leq C_1(Z_1 - \delta, y_2) \).

By Proposition 1, \( \Delta_i \geq 0 \) and thus \( X_i \geq 0 \) for \( i = 1, 2 \). Let \( H_i(x) \) be the distribution function of \( X_i \), \( i = 1, 2 \). Equation (4) can be rewritten as:

\[
C_i(y_1, y_2) = \int_0^\infty G_i(y_i - x) dH_i(x),
\]

where \( H_i(x) \) is a function of both \( y_1 \) and \( y_2 \). In Section 5 we demonstrate that if retailer 1 and retailer 2 employ the proposed cooperation strategy, then unique Nash equilibrium can be obtained by simultaneously solving:

\[
H_1 * F_1^{(r_1+1)}(y_1) = \frac{b_1}{b_1 + h_1}
\]

\[
H_2 * F_2^{(r_2+1)}(y_2) = \frac{b_2}{b_2 + h_2},
\]

where \( * \) is the convolution operator.

### 3.3 Multiple Order Cycles

Although in this article we consider a single replenishment cycle, our development can be used as an approximation for the stationary multi-cycle problem. Let \( y_1 \) and \( y_2 \) be stationary order-up-to levels for retailer 1 and 2, respectively. Every period the following sequence of events takes place: (1) retailers receive any outstanding orders scheduled to arrive, (2) transfer of stock, if any, is realized at the cross-docking warehouse, (3) retailers place their orders at the supplier, (4) retailer demands are observed and inventory related costs are incurred.
Consider an arbitrary period \( t \). Let \( I_{i,t} \) be the inventory position of retailer \( i \) before any transfer of stock between retailers. Note that \( I_{i,t} \) includes net inventory at retailer \( i \) at time \( t \), outstanding orders from the supplier, and in-transit inventory from the cross-docking warehouse to retailer \( i \). Therefore, \( I_{i,t} = y_i - D_{i,t-1} \). Let \( \eta_{i,t} \) be the amount transferred from retailer \( j \) (\( j = 1, 2, j \neq i \)) to retailer \( i \) in period \( t \). After transfer of stock, inventory position of retailer \( i \) becomes \( I_{i,t} + \eta_{i,t} - \eta_{j,t} \), where \( \eta_{i,t} \) and \( \eta_{j,t} \) can not be both positive. Then, retailer \( i \) orders from the supplier an amount

\[
O_{i,t} = y_i - I_{i,t} - \eta_{i,t} + \eta_{j,t} = D_{i,t-1} - \eta_{i,t} + \eta_{j,t}. \tag{8}
\]

In period \( t + L \) a possible transfer of stock is realized to minimize inventory related expected costs at the end of period \( t + L + 1 \). Retailer \( i \) bases its transfer decision on its net inventory at time \( t + L \) plus in-transit inventory from the cross-docking warehouse. Notice that, orders that are placed in periods \( t+1, t+2, \ldots, t+L-1 \) (\( O_{i,t+1}, O_{i,t+2}, \ldots, O_{i,t+L-1} \)) do not have a relevance on the transfer decision in period \( t+L \). Let \( I'_{i,t+L} \) be the inventory position of retailer \( i \) at the beginning of period \( t+L \), prior to transfer decision, excluding orders placed in periods \( t+1, t+2, \ldots, t+L-1 \):  

\[
I'_{i,t+L} = I_{i,t+L} - \sum_{k=1}^{L-1} O_{i,t+k} = y_i - D_{i,t+L-1} - \sum_{k=1}^{L-1} D_{i,t+k-1} + \sum_{k=1}^{L-1} \eta_{i,t+k} - \sum_{k=1}^{L-1} \eta_{j,t+k} = y_i - D_i(L) + \sum_{k=1}^{L-1} \eta_{i,t+k} - \sum_{k=1}^{L-1} \eta_{j,t+k}. 
\]

The second equality follows from equation (8), and the third equality is due to the definition of \( D_i(L) \). In period \( t + L \) the amount that can be transferred from retailer 2 to retailer 1 is given by \( \eta_{1,t+L} = \min \{ (I'_{2,t+L} - Z_1) +, (Z_2 - I'_{1,t+L}) + \} \). In order the development in Section 3.2 to be valid, we require two assumptions. First, in order to be able to extend our analysis to multiple order cycles, we should ensure the feasibility of stock transfer. That is, \( O_{1,t} \geq \eta_{2,t+L} \) and \( O_{2,t} \geq \eta_{1,t+L} \). This assumption is similar to the balance assumption, commonly employed in multi-echelon inventory literature (see, for example Eppen and Schrage 1981). The second assumption is required for correct accounting of the single period expected cost function. In order equation (4) to be valid we should have \( \sum_{k=1}^{L-1} \eta_{i,t+k} - \sum_{k=1}^{L-1} \eta_{j,t+k} \approx 0 \). These assumptions would be valid if retailers have similar
cost and demand characteristics, or if the standard deviations of retailer demands are small. The second assumption can also be represented by changing the order size $O_{i,t}$ in any period $t$ by the factor $\eta_{i,t} - \eta_{j,t}$. In other words, by simply accounting for the most recent transfer of stock in the current order, one can assure that the single period expected cost function given in (4) will be exact. Let modified order size be $O'_{i,t} = O_{i,t} + \eta_{i,t} - \eta_{j,t}$. By equation (8) we have $O'_{i,t} = D_{i,t-1}$ and hence $I_{i,t+L} = y_t - D_i(L)$.

We can also incorporate order cycles of length $R > 1$. Suppose each retailer orders from the supplier using a common order interval of length $R$. If $R > L$, then our development in Section 3.2 will still be valid for a stationary multiple-order cycle analysis without requiring the second assumption above (since in that situation there is only one possible transfer of stock in every order cycle).

4 Distribution of the Net Change in Inventory Position

As stated by equations (5), (6), and (7), if we can derive the distribution of $X_i$, then the expected cost functions and the optimal policy parameters can be obtained. The following proposition characterizes the distribution function for $X_i$.

Proposition 2 The distribution function of $X_1$ has the following form:

$$H_1(x) = \begin{cases} 
\int_r^{x} F_2(L)(\Delta_2 + x - y)f_1(L)(y)dy & x < \Delta_1 \\
F_1(L)(x) + \int_r^{x+\Delta_2} F_2(L)(\Delta_2 + x - y)f_1(L)(y)dy & x \geq \Delta_1
\end{cases}$$

Proof: The result is established by first conditioning on $D_1(L)$ and then by using the distributions of $A_i$ and $B_i$. Details are provided in the Appendix. $\square$

The distribution of $X_2$, $H_2(x)$, is symmetric and obtained by inter-changing indices 1 and 2 wherever possible in Proposition 2. For simplifying the notation we define

$$R_{11}(x) = \int_0^{x} F_2(L)(\Delta_2 + x - y)f_1(L)(y)dy$$

$$R_{12}(x) = F_1(L)(x) + \int_x^{x+\Delta_2} F_2(L)(\Delta_2 + x - y)f_1(L)(y)dy,$$

so that $H_1(x) = R_{11}(x)$ on $x < \Delta_1$, and $H_1(x) = R_{12}(x)$, otherwise. Let $r_{11}(x)$ and $r_{12}(x)$ be the derivatives of $R_{11}(x)$ and $R_{12}(x)$, respectively ($r_{11}(x)$ and $r_{12}(x)$ are well defined by the assumptions imposed on $F_1(L)(x)$).
Closer inspection of $H_1(x)$ reveals that $H_1(x)$ is continuous for $x < \Delta_1$ and $x > \Delta_1$, but has a jump at $x = \Delta_1$ with probability

$$P_1(\Delta_1) = R_{12}(\Delta_1) - R_{11}(\Delta_1)
= \int_{\Delta_1}^{\Delta_1+\Delta_2} F_2(L)(\Delta_2 + \Delta_1 - y)f_1(L)(y)dy
+ \int_0^{\Delta_1} (1 - F_2(L)(\Delta_2 + \Delta_1 - y))f_1(L)(y)dy.$$  

The jump at $x = \Delta_1$ characterizes the mass for which retailer 1 can achieve its best $(l_1+1)$-period order-up-to level $Z_1$ through transfer of retailer orders at the cross-docking warehouse. We note that the sum $P(\Delta_1, \Delta_2) = P_1(\Delta_1) + P_2(\Delta_2)$ gives the probability that a transfer of stock occurs between retailer 1 and retailer 2. It can easily be shown that this probability can be rewritten as:

$$P(\Delta_1, \Delta_2) = F_1(L)(\Delta_1)(1 - F_2(L)(\Delta_2)) + F_2(L)(\Delta_2)(1 - F_1(L)(\Delta_1)).$$

If retailers are identical, that is $F_1(L)(\Delta_1) = F_2(L)(\Delta_2)$, then the probability that a transfer of stock takes place can not exceed 0.5 (in this case $P(\Delta_1, \Delta_2)$ is concave and attains its maximum at 0.5).

Limiting cases (which will be utilized later) for $H_1(x)$ can be obtained for $\Delta_2 \rightarrow \infty$ and for $\Delta_2 = 0$. For $\Delta_2 \rightarrow \infty$,

$$H_1(x) = \begin{cases} F_1(L)(x) & x < \Delta_1, \\ 1 & x \geq \Delta_1 \end{cases},$$  

and for $\Delta_2 = 0$,

$$H_1(x) = \begin{cases} F_1(L) * F_2(L)(x) & x < \Delta_1, \\ F_1(L)(x) & x \geq \Delta_1 \end{cases}.$$  

In the former case, where retailer 2 has ample stock that can be transferred to retailer 1, the stock position of retailer 1 will be greater than $Z_1$ ($x < \Delta_1$) only if its demand over supplier lead time is below $\Delta_1$ (with probability $F_1(L)(x)$). On the other hand in the latter case, retailer 2 is willing to receive any excess stock from retailer 1. Therefore, retailer 1 can only end up below $Z_1$ if its own lead-time demand exceeds $\Delta_1$.

In deriving $H_i(x)$ we assumed that the demands occurring at retailer 1 and 2 are independent. This restriction can easily be relaxed. When retailer demands are correlated, $H_1(x)$ is affected only through conditional distribution of $D_2(L)$. Define $F_2(L)(x|y)$ as the
distribution of $D_2(L)$ conditioned on $D_1(L) = y$. Then, using similar steps as in proof of Proposition 2 we can obtain:

$$H_1(x) = \begin{cases} 
\int_0^x F_2^{(L)}(\Delta_2 + x - y|y)f_1^{(L)}(y)dy & x < \Delta_1 \\
F_1^{(L)}(x) + \int_{x+\Delta_2}^{y} F_2^{(L)}(\Delta_2 + x - y|y)f_1^{(L)}(y)dy & x \geq \Delta_1
\end{cases}.$$

5 Optimal Order-up-to Levels under Transfer of Stock

In Section 3.2 we argued that the expected cost function of retailer $i$ can be written as $C_i(y_1, y_2) = \int_0^\infty G_i(y_i - X_i)dH_i(x)$, where $H_i(x)$ is given by Proposition 2. Given that there is an opportunity of stock transfer at the cross-docking warehouse, the retailers would determine their initial order-up-to levels $y_1$ and $y_2$ jointly by cooperation. Our aim in this section is to show the existence of the unique equilibrium solution.

We can outline major results that we present in this section as follows: By Theorem 1.2 of Fudenberg and Tirole (1996), a Nash equilibrium for retailer 1 and retailer 2 exists whenever (i) the action space for retailers (that is, set of values $(y_1, y_2)$) is non-empty and compact, (ii) $C_i(y_1, y_2)$ is continuous in $(y_1, y_2)$, and (iii) $C_i(y_1, y_2)$ is convex in $y_i$ for a given $y_j$, ($j \neq i$). Our first result in this section establishes the continuity of $C_i(y_1, y_2)$ on a compact subset of $\mathbb{R}^2$ (Proposition 3). Then, we show that for a given $y_j$ ($j \neq i$) $C_i(y_1, y_2)$ is strictly convex in $y_i$ (Proposition 4). This proves the existence of a Nash solution. We obtain reaction curves for retailer 1 and retailer 2 as solutions of the following equations in $(y_1, y_2)$ plane:

$$U_1(y_1, y_2) := \frac{\partial C_1(y_1, y_2)}{\partial y_1} = 0 \tag{12}$$

$$U_2(y_1, y_2) := \frac{\partial C_2(y_1, y_2)}{\partial y_2} = 0. \tag{13}$$

In Proposition 5 we show that $U_1(y_1, y_2) = 0$ and $U_2(y_1, y_2) = 0$ form decreasing curves in $(y_1, y_2)$ plane with finite upper and lower bounds. Furthermore, we show that the slope of the curve $J_1(y_1) := \{y_2 : U_1(y_1, y_2) = 0\}$ is less than the slope of the curve $J_2(y_1) := \{y_2 : U_2(y_1, y_2) = 0\}$ for all $y_1$, which establishes the uniqueness of the Nash solution.

**Proposition 3** $C_i(y_1, y_2)$ is continuous in $(y_1, y_2)$ on the action space $S = [Z_1, \tilde{S}_1] \times [Z_2, \tilde{S}_2]$ where $\tilde{S}_i$ is obtained by solving $F_j^{(L)} \ast F_i^{(L+h_{i-1})}(\tilde{S}_i) = b_i/(b_i + h_i)$, $i, j = 1, 2$, $i \neq j$.\
Finally, we can rewrite \( C_1(y_1, y_2) \) by using Proposition 4. Noting that \( G \) is provided in the Appendix.

**Proof:** We show the convexity of \( C_1(y_1, y_2) \) for a given \( y_2 \); the argument for \( C_2(y_1, y_2) \) is basically the same by symmetry. The Proposition is established by conditioning \( C_1(y_1, y_2) \) on \( D_2(L) \). The details are provided in the Appendix.\( \square \)

By using Proposition 4, we can characterize the optimal order-up-to level for a retailer, given the order-up-to level decision of the other. For \( i = 1 \) (the case for \( i = 2 \) is the same by symmetry), equation (5) can be written as:

\[
C_1(y_1, y_2) = \int_0^\Delta G_1(y_1 - w)r_{11}(x)dx + \int_{\Delta_1}^\infty G_1(y_1 - w)r_{12}(x)dx + G'(y_1 - \Delta_1)P_1(\Delta_1).
\]

Then, \( U_1(y_1, y_2) \) becomes:

\[
U_1(y_1, y_2) = \frac{\partial C_1(y_1, y_1)}{\partial y_1} = \frac{\partial}{\partial y_1}\{\int_0^\Delta G_1(y_1 - w)r_{11}(x)dx + \int_{\Delta_1}^\infty G_1(y_1 - w)r_{12}(x)dx + G'(y_1 - \Delta_1)P_1(\Delta_1)\}
= G_1(y_1 - \Delta_1)\{r_{11}(\Delta_1) - r_{12}(\Delta_1)\} + G_1(Z_1)P'_1(\Delta_1)
+ \int_0^\Delta G'_1(y_1 - w)r_{11}(x)dx + \int_{\Delta_1}^\infty G'_1(y_1 - w)r_{12}(x)dx.
\]

Noting that

\[
P'_1(\Delta_1) = r_{12}(\Delta_1) - r_{11}(\Delta_1),
\]

and \( y_1 - \Delta_1 = Z_1, U_1(y_1, y_2) \) simplifies to:

\[
U_1(y_1, y_2) = \int_0^\Delta G'_1(y_1 - w)r_{11}(x)dx + \int_{\Delta_1}^\infty G'_1(y_1 - w)r_{12}(x)dx. \tag{14}
\]

By equations (1) and (2), we also know that \( G'_1(Z_1) = 0 \). Hence,

\[
U_1(y_1, y_2) = \int_0^\Delta G'_1(y_1 - w)r_{11}(x)dx + \int_{\Delta_1}^\infty G'_1(y_1 - w)r_{12}(x)dx + G'(Z_1)P_1(\Delta_1)
= \int_0^\infty G'_1(y_1 - w)dH_1(w). \tag{15}
\]

Finally, we can rewrite (15) by using \( G'_1(x) = -b_1 + (h_1 + b_1)F_1^{(l_1+1)}(x) \) for \( x \geq 0 \), and \( G'_1(x) = 0 \) for \( x < 0 \):

\[
U_1(y_1, y_2) = -b_1 + (h_1 + b_1)\int_0^y_1 F_1^{(l_1+1)}(y_1 - w)dH_1(w)
= -b_1 + (h_1 + b_1)H_1 * F_1^{(l_1+1)}(y_1) \tag{16}
= -b_1 + (h_1 + b_1)\int_0^{y_1} H_1(y_1 - w)f_1^{(l_1+1)}(w)dw. \tag{17}
\]
Therefore, for a given value of \( y_2 \), one can obtain the optimal order-up-to level of the second retailer, \( S_1(y_2) \) by solving (16). This key result is summarized in the following theorem.

**Theorem 1** Optimal order-up-to level for retailer 1, \( S_1 \) for a given \( y_2 \) can be obtained by solving

\[
U_1(y_1, y_2) = \frac{\partial C_1(y_1, y_1)}{\partial y_1} = \int_0^\infty G_1'(y_1 - w) dH_1(w) = -b_1 + (h_1 + b_1) H_1 * F_1^{(h_1+1)}(y_1) = 0.
\]

Since \( H_1(y_1 - w) < 1 \) in (17), \( y_1 \) that solves \( U_1(y_1, y_2) = 0 \) for a given value of \( y_2 \) is greater than or equal to \( Z_1 \) (as proved in Proposition 1). Therefore, whatever inventory decision retailer 2 chooses, the optimal order-up-to level for retailer 1 is bounded from below by \( Z_1 \). In what follows we state an intuitive result:

**Corollary 1** The optimal order-up-to level \( S_1(y_2) \) for retailer 1, is decreasing in the inventory decision, \( y_2 \), of retailer 2.

**Proof:** By differentiating \( H_1(x) \) with respect to \( y_2 \), we can easily establish from Proposition 2 that \( H_1(x) \) is increasing in \( y_2 \) (implying that \( X_1 \) is stochastically decreasing in \( y_2 \)). The corollary then follows by this property of \( H_1(x) \) and equation (16). □

By letting \( \Delta_2 = y_2 - Z_2 = 0 \), we can obtain an upper bound on the optimal order-up-to level for retailer 1. Let \( H_1(x) \) to assume equation (11) in \( U_1(y_1, Z_2) = 0 \). Then, the upper bound on \( S_1 \) is found by solving:

\[
U_1(y_1, Z_2) = -b_1 + (h_1 + b_1) \{ \int_0^{Z_1} F_1^{(L)}(y_1 - w) f_1^{(h_1+1)}(w) dw + \int_{Z_1}^{y_1} F_1^{(L)} * F_2^{(L)}(y_1 - w) f_1^{(h_1+1)}(w) dw \} = 0.
\]

In the following Proposition we examine the monotonicity of the reaction curves in \((y_1, y_2)\) plane.

**Proposition 5** Define

\[
J_1(y_1) = \{ y_2 : U_1(y_1, y_2) = 0 \}, \; y_1 \geq Z_1
\]

\[
J_2(y_1) = \{ y_2 : U_2(y_1, y_2) = 0 \}, \; y_1 \geq 0.
\]
i. $U_1(y_1, y_2) = 0$ and $U_2(y_1, y_2) = 0$ form strictly decreasing curves in $(y_1, y_2)$ plane. That is,

\[ \frac{dJ_1(y_1)}{dy_1} < 0 \]
\[ \frac{dJ_2(y_1)}{dy_1} < 0. \]

ii. $dJ_1(y_1)/dy_1 < dJ_2(y_1)/dy_1$.

**Proof:** We use implicit differentiation to obtain $dJ_1(y_1)/dy_1$ and $dJ_2(y_1)/dy_1$. Part (ii) follows by direct comparison of derivatives. Details are presented in the Appendix. □

Existence and uniqueness of the Nash equilibrium are obtained by combining Propositions 4 and 5 together with the bounds on the optimal order-up-to levels. Convexity of $C_1(y_1, y_2)$ and $C_2(y_1, y_2)$ in respective arguments $y_1$ and $y_2$, together with the monotonicity of the reaction curves are sufficient for the existence of a unique Nash solution (given by equations (6) and (7)):

**Theorem 2** Optimal order-up-to levels for the retailers, $(S_1, S_2)$, is found as the unique solution of the following equations:

\[
H_1 \ast F_1^{[l_1+1]}(y_1) = \frac{b_1}{b_1 + h_1} \\
H_2 \ast F_2^{[l_2+1]}(y_2) = \frac{b_2}{b_2 + h_2}.
\]

Using Proposition 5 and Theorem 2 we can present an algorithm that converges to $(S_1, S_2)$. Let $S_2^{(0)} = S_2(Z_1)$. Define $S_1^{(i)} = S_1(S_2^{(i)})$ and $S_2^{(i+1)} = S_2(S_1^{(i)})$ for $i = 1, 2, \ldots$. Then $S_1^{(i)}$ converges from below to $S_1$, and $S_2^{(i)}$ converges from above to $S_2$.

6 Computational Results

In this section we present and discuss our numerical findings. Our main focus of inquiry will span:

1. the behavior of the Nash solution with respect to essential system parameters,

2. improvement (in terms of costs and safety stocks) relative to non-cooperative decentralized solution,

3. improvement relative to decentralized solution with transfer opportunity,
4. System performance relative to completely centralized model.

In our numerical study, the random demand faced by retailer $i$ in a period is normally distributed with mean $\mu_i$ and variance $\sigma_i^2$. We set $h_1 = h_2 = 1$ and use $b_1 = b_2 \in \{4, 9, 19\}$ (corresponding to service levels $\{0.8, 0.9, 0.95\}$), respectively. The mean demand for retailer 1 is fixed at $\mu_1 = 100$, and $\mu_2$ is set to either 100 or 200. We let $cv_i = \sigma_i / \mu_i$ to assume either 0.05 or 0.25, $i = 1, 2$. The supplier lead-time is set to $L = 5$, and the retailer lead-time takes either $l_i = 1$ or $l_i = 3$ for $i = 1, 2$. We used numerical integration procedures of Maple 8 for solving the optimal order-up-to levels and evaluating expected costs.

Through our experiments we computed and recorded the following policy parameters and performance measures:

- $S_i^{\text{ind}}$: order-up-to level for retailer $i$, given that they place their orders independently. $S_i^{\text{ind}}$ is found from equation
  \[ F_i^{(L+L_i+1)}(S_i^{\text{ind}}) = \frac{b_i}{b_i + h_i}. \]

- $C_i^{\text{ind}}$: expected cost incurred by retailer $i$ if it operates with the order-up-to level $S_i^{\text{ind}}$
  \[ C_i^{\text{ind}} = b_i E[(S_i^{\text{ind}} - D_i(L + l_i + 1))^+] + b_i E[(D_i(L + l_i + 1) - S_i^{\text{ind}})^+]. \]

- $S_i$: Nash equilibrium order-up-to level for retailer $i$.

- $C_i(S_1, S_2)$: expected cost incurred at retailer $i$ by employing the Nash solution.

- $C_i(S_1^{\text{ind}}, S_2^{\text{ind}})$: expected cost incurred at retailer $i$, if retailers cooperate by allowing transfer of stock at the end of supplier lead-time, but operate with independent order-up-to levels
  \[ C_i(S_1^{\text{ind}}, S_2^{\text{ind}}) = E[G_i(S_i^{\text{ind}} - X_i)]. \]

This cost term is used to understand the benefits of the transfer policy. Roughly, the benefit of employing Nash solution can be decomposed into two parts: benefits obtained by using the transfer policy over the complete decentralized model (policy improvement)

\[ C_i^{\text{ind}} - C_i(S_1^{\text{ind}}, S_2^{\text{ind}}), \]
and the benefit of cooperative solution of the order-up-to levels (Nash improvement)

\[ C_i(s_1^{\text{ind}}, s_2^{\text{ind}}) - C_i(S_1, S_2). \]

- \( %C_i \): percentage improvement gained for retailer \( i \) over the independent solution

\[ %C_i = \frac{C_i^{\text{ind}} - C_i(S_1, S_2)}{C_i^{\text{ind}}} \times 100. \]

- \( %TC \): percent improvement gained over the total cost of the independent solution

\[ %TC = \frac{\sum_i C_i^{\text{ind}} - \sum_i C_i(S_1, S_2)}{\sum_i C_i^{\text{ind}}} \times 100. \]

- \( %POL_i \): percent policy improvement

\[ %POL_i = \frac{C_i^{\text{ind}} - C_i(S_1^{\text{ind}}, S_2^{\text{ind}})}{C_i^{\text{ind}} - C_i(S_1, S_2)} \times 100. \]

- \( TC^{\text{cen}} \): total cost incurred for a centralized solution. Essentially, this is the cost of a centralized two-echelon model (Eppen and Schrage (1981)).

- \( %D \): percent deviation of total cooperative decentralized model from the centralized cost.

\[ %D = \frac{\sum_i C_i(S_1, S_2) - TC^{\text{cen}}}{TC^{\text{cen}}} \times 100. \]

- The safety stock for retailer \( i \) under Nash solution can be defined as \( S_i - (L + l_i + 1)\mu_i \). Similarly, \( S_i^{\text{ind}} - (L + l_i + 1)\mu_i \) is the safety stock for retailer \( i \) in the decentralized model. Then the percentage improvement in terms of safety stocks can be calculated as

\[ %SS_i = \frac{S_i^{\text{ind}} - S_i}{S_i^{\text{ind}} - (L + l_i + 1)\mu_i} \times 100. \]

Similarly, total safety stock improvement is calculated as

\[ %SS = \frac{\sum_i S_i^{\text{ind}} - \sum_i S_i}{\sum_i S_i^{\text{ind}} - \sum_i (L + l_i + 1)\mu_i} \times 100. \]

- \( P(\Delta_1, \Delta_2) \): Probability that a transfer of stock takes place between retailers.

Tables containing all the computational results can be provided by the authors upon request. Main observations drawn from our experiments can be summarized as follows:
1. For the parameter set that we use, we always observed that $C_i(S_1, S_2) < C_i^{ind}$ and $S_i < S_i^{ind}$, for $i = 1, 2$. Percent cost improvement gained by employing our model, $\%C_i$, can be as high as 22% for a retailer. On the other hand, total cost improvement over the independent solution, $\%TC$, can be as high as 14%. Percent improvement in the safety stocks for a retailer, $\%SS_i$, can be as high as 33%. Similarly, total safety stock improvement can be as high as 17%.

2. We observe that as retailers become more symmetric with respect to the standard deviation of demand (as $\sigma_1$ gets close to $\sigma_2$), $\%TC$ increases. Intuitively, given that one retailer ends up above and the other retailer ends up below its respective $Z_i$ value at the end of supplier lead-time, respective excess and shortfall quantities will be close to each other. Therefore, the likelihood that both of the retailers will get close to their respective $Z_i$ values will increase. We observe the same behavior for the safety stock improvements. Our model leads to higher improvements in total safety stocks whenever retailers are more balanced.

On the other hand, highest improvement in the expected cost of a retailer, $\%C_i$, occurs when retailers are highly unbalanced (for example, $\%C_1 = 22.4\%$ and $\%C_2 = 2.4\%$ for $\mu_1 = 100$, $\mu_2 = 200$, $\sigma_1 = 5$, $\sigma_2 = 50$, $b_1 = b_2 = 4$, $l_1 = l_2 = 1$). Moreover, in these cases we observed that the retailer with lower standard deviation of demand gains higher improvement. Similarly, as one would expect, these cases also coincide with the cases where one of the retailers gain high saving in terms of safety stocks (for the example above, $\%SS_1 = 33.4\%$, $\%SS_2 = 1.9\%$). The retailer with higher standard deviation of demand is expected to show higher deviation from its $Z_i$ value at the end of the supplier lead-time. Consequently, the retailer with lower standard deviation has higher chance of correcting its shortfall or excess by transfer of stock from the former, and hence can retain lower stock levels.

We summarize our conclusions in Table 1 and Table 2 for the case $\mu_1 = 100$, $\mu_2 = 200$, and $b_1 = b_2 = 4$. In Table 1, one can observe that highest total percent improvements are obtained for the row corresponding to $(\sigma_1, \sigma_2) = (5, 10)$ (where $\sigma_2/\sigma_1 = 2$) and lower percent improvements are obtained for the row corresponding to $(\sigma_1, \sigma_2) = (5, 50)$ (where $\sigma_2/\sigma_1 = 10$). However, as illustrated in Table 2, the row where $(\sigma_1, \sigma_2) = (5, 50)$ yields higher percent improvements for retailer 1.
Table 1: Total Percent Improvement Values

<table>
<thead>
<tr>
<th>$(\sigma_1, \sigma_2)$</th>
<th>$%TC$</th>
<th>$(l_1, l_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5,10)$</td>
<td>12.4</td>
<td>10.4 10.9 9.1</td>
</tr>
<tr>
<td>$(5,50)$</td>
<td>4.2</td>
<td>3.5   3.9 3.1</td>
</tr>
<tr>
<td>$(25,10)$</td>
<td>11.3</td>
<td>5.9   9.4 8.2</td>
</tr>
</tbody>
</table>

Table 2: Percent Improvement Values for Retailer 1

<table>
<thead>
<tr>
<th>$(\sigma_1, \sigma_2)$</th>
<th>$%C_1$</th>
<th>$(l_1, l_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5,10)$</td>
<td>18.2</td>
<td>18.0 13.3 13.1</td>
</tr>
<tr>
<td>$(5,50)$</td>
<td>22.3</td>
<td>22.0 16.2 16.0</td>
</tr>
<tr>
<td>$(25,10)$</td>
<td>8.1</td>
<td>8.0   6.1 6.0</td>
</tr>
</tbody>
</table>

3. Total cost improvement, $\%TC$, also depends on the retailer lead-time parameters. As retailer lead-times increase $\%TC$ decreases. This effect is expected as in the cases of higher retailer lead-times, the relative length of risk-pooling period, $L$, over the length of the cycle decreases. These observations are illustrated in Table 1 and 2. Notice that both $\%TC$ and $\%C_1$ decrease as we move from $(l_1, l_2) = (1, 1)$ to $(l_1, l_2) = (1, 3)$ and $(l_1, l_2) = (3, 3)$.

Obviously, the supplier lead-time, $L$ also has an effect on the performance of our policy. In our experiments supplier lead-time is fixed as $L = 5$. We also computed cases where $L = 7$, and observed that the performance of our model improves, as expected.

4. We observe a decrease in $\%TC$ and $\%SS$ as unit backorder cost increases. This is due to the fact that an increase in the cost of backordering increases order-up-to levels. Consequently, it becomes more likely for the retailers to end up above their respective $Z_i$ values at the end of supplier lead-time, and occurrence of transfer of stock becomes less likely.

5. We compared the performance of our model against the performance of the centralized supply chain. In centralized version of our model, at the end of supplier lead-time the supplier makes an allocation to the retailers. We only considered cases where
$\sigma_1 = \sigma_2$ and $l_1 = l_2$. Essentially, this corresponds to the two-echelon model of Eppen and Schrage (1981). In Table 3 we present total expected cost of the system for centralized model (CM), partial cooperation model that we propose (PCM), and decentralized model with no cooperation (DM), for cases where $\sigma_1 = \sigma_2$ and $l_1 = l_2$. It can be observed that, for the cases that we considered, the deviation of the total cost of our model from the centralized solution is no more than 7.4%. This deviation decreases as $l_i$ increases, and increases as $b$ increases. Last column of Table 3 illustrates that approximately 70% of the benefits of complete centralization over complete decentralization can be obtained by employing our model. Therefore, a decentralized model operating under the prescribed optimal order-up-to levels and the transfer policy proposed can explain almost 70% of the improvement that would have been obtained by a centralized model.

An interesting observation in Table 3 is that for a fixed $b$ and $l$ value, $\%D$ figures for $\sigma_i = 5$ and $\sigma_i = 25$ are approximately the same. We observed a similar behaviour for $\%C_i$ and $\%SS_i$ values as well. This observation leads us to believe that there is a linear relation (as in a single stock point model) between the standard deviation of demand and performance measures of a retailer (such as $C_i(S_1, S_2) \approx \sigma_i \times$ constant and safety stock, $i \approx \sigma_i \times$ constant).

<table>
<thead>
<tr>
<th>$\sigma_i$</th>
<th>$l_i$</th>
<th>Model</th>
<th>%D = $\frac{PCM-DM}{CM} \times 100$</th>
<th>$\frac{DM-PCM}{DM-CM} \times 100$</th>
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<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>CM</td>
<td>29.7</td>
<td>43.8</td>
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</tr>
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<td>CM</td>
<td>148.5</td>
<td>186.1</td>
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<td>199.5</td>
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<td></td>
<td>DM</td>
<td>185.2</td>
<td>232.2</td>
</tr>
<tr>
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<td>35.7</td>
<td>44.7</td>
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</tbody>
</table>
6. In Table 4 we present $P(\Delta_1, \Delta_2)$ values for the parameter sets used in our computations. For the parameter set that we used these probabilities range from 0.395 to 0.492. We observe from Table 4 that the probability of a stock transfer increases as $l_i$ increases. Moreover, as $b$ increases (for both retailers), order-up-to levels increase, and it becomes more likely for the retailer inventory levels to be above their respective $Z_i$ values at the end of supplier lead-time. Therefore $P(\Delta_1, \Delta_2)$ decreases.

Table 4: Transfer of Stock Probabilities

<table>
<thead>
<tr>
<th>$cv_1 = 0.05$</th>
<th>$b_1 = 4$</th>
<th>$b_1 = 9$</th>
<th>$b_1 = 19$</th>
</tr>
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<tbody>
<tr>
<td>$l_1 = 1$</td>
<td>0.471</td>
<td>0.477</td>
<td>0.434</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.477</td>
<td>0.481</td>
<td>0.449</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.474</td>
<td>0.482</td>
<td>0.440</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.476</td>
<td>0.484</td>
<td>0.444</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.478</td>
<td>0.487</td>
<td>0.449</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.476</td>
<td>0.484</td>
<td>0.446</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.482</td>
<td>0.492</td>
<td>0.455</td>
</tr>
<tr>
<td>$l_1 = 3$</td>
<td>0.477</td>
<td>0.485</td>
<td>0.448</td>
</tr>
</tbody>
</table>

7. Finally, our computational analysis reveals that most of the system improvement over the completely decentralized model is due to the transfer of stock at the cross-docking warehouse (or supplier’s plant). In Table 5 we present $\%POL_1$ values (part of percent improvement in retailer 1’s expected costs that are accounted by the transfer of stock) for the case $\mu_1 = 100, \mu_2 = 200, b_1 = b_2 = 19$. Then, $1 - \%POL_1$ is the percent improvement due to joint determination of the stock levels. As can be seen from Table 5, $1 - \%POL_1$ values are approximately in $15\% - 20\%$ range. $1 - \%POL_1$ increases as retailer lead-time $l_1$ decreases, and $1 - \%POL_1$ increases with a decrease in $\sigma_1$.

7 Conclusions and Extensions for Further Research

In this article we considered a decentralized supply chain of two independent retailers (or manufacturers), and a supplier. In the system that we analyzed retailers order a common
product (or raw material) from the supplier to fulfill their own random customer demand. The supplier has ample capacity to satisfy the orders placed by the retailers, but there is a fixed lead-time (supplier lead-time) associated with order preparation (due to supplier’s manufacturing or ordering lead-time, packaging and loading times at the supplier’s plant). At the end of the supplier lead time, orders are shipped to retailers. Before retailer orders are shipped they are given an opportunity to re-adjust their orders (by transferring part of an order quantity from one retailer to the other), as long as this transaction improves expected costs of both retailers. Under this setting we derived unique equilibrium order-up-to levels for the retailers. The derivation is based on a single cycle analysis, but can be generalized under some mild conditions, as given in Section 3.3.

Our computational results gave us important managerial insights on design and operation of such systems, some of which are summarized as follows: (1) this scheme may lead to considerable benefits in terms of expected costs and safety stocks, (2) the benefits may be much higher for a retailer with relatively smaller standard deviation of demand, (3) benefits of cooperation increase whenever the supplier lead-time is long and the retailer lead-time is short, (4) expected total cost of the decentralized system under cooperation does not show a considerable deviation from the centralized system, (5) approximately 70% of the expected cost benefits of a centralized model can be gained from the model that we propose, and (6) around 80% of the cost improvement is due to the transfer of stock at the cross-docking warehouse, and the remaining 20% is attributed to the joint determination of the order-up-to levels (Nash improvement). The last point is important, as joint determination of order-up-to levels may be much more difficult than implementing the transfer policy at the cross-docking warehouse.

Our work can be extended in several directions. One possible extension is to consider more than two, say $N$ retailers. Main difficulty is in defining a policy for the transfer

<table>
<thead>
<tr>
<th>$(\sigma_1, \sigma_2)$</th>
<th>$P(L_1)$</th>
<th>$(l_1, l_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1,1)</td>
<td>(1,3)</td>
</tr>
<tr>
<td>(5,10)</td>
<td>75.7</td>
<td>51.9</td>
</tr>
<tr>
<td>(5,50)</td>
<td>51.9</td>
<td>79.8</td>
</tr>
<tr>
<td>(25,10)</td>
<td>100</td>
<td>99.8</td>
</tr>
<tr>
<td>(25,50)</td>
<td>75.7</td>
<td>99.8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(\sigma_1, \sigma_2)$</th>
<th>$P(L_2)$</th>
<th>$(l_1, l_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(3,1)</td>
<td>(3,3)</td>
</tr>
<tr>
<td>(5,10)</td>
<td>76.5</td>
<td>67.6</td>
</tr>
<tr>
<td>(5,50)</td>
<td>72.6</td>
<td>80.4</td>
</tr>
<tr>
<td>(25,10)</td>
<td>99.0</td>
<td>92.5</td>
</tr>
<tr>
<td>(25,50)</td>
<td>76.6</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 5: Percent Cost Improvement for Retailer 1 Due to Transfer of Stock
of stock at the cross-docking warehouse. A simple transfer rule is to index retailers in a
cyclic order in such a way that whenever retailer $i$ has an excess, it is used to satisfy the
shortage of retailer $i + 1 \mod N$. In fact, under such well-defined transfer rules one can
derive the distribution of net change in the stock position, $H_i(x)$, much like the derivation
in Section 4.

Another potential extension is to consider fixed retailer shipment and transfer costs.
The form of the optimal order-up-to levels under fixed costs associated with supplier-to-
retailer shipments, and fixed transfer costs is worthwhile to investigate. In our analysis
we assumed an uncapacitated supplier. Apparently, if the supply is limited, a rationing
game at the supplier, on top of the transfer game that we considered should also be
incorporated.

Appendix

Proof of Proposition 2
We condition $X_1$ on $D_1(L)$:

$$\Pr\{X_1 \leq x \mid D_1(L) \leq \Delta_1\} = \Pr\{X_1 \leq x, D_1(L) \leq \Delta_1\} + \Pr\{X_1 \leq x, D_1(L) > \Delta_1\}. \quad (A.1)$$

For the first probability term in (A.1):

$$\Pr\{X_1 \leq x, D_1(L) \leq \Delta_1\} = \int_{0}^{\Delta_1} \Pr\{X_1 \leq x \mid D_1(L) = y\} f_1^{(L)}(y) dy$$

$$= \int_{0}^{\Delta_1} \Pr\{y + \eta_2 \leq x \mid D_1(L) = y\} f_1^{(L)}(y) dy$$

$$= \int_{0}^{\Delta_1} [1 - \Pr\{\eta_2 > x - y \mid D_1(L) = y\}] f_1^{(L)}(y) dy$$

$$= \int_{0}^{\Delta_1} [1 - \Pr\{\min(\Delta_1 - y, B_2) > x - y \mid D_1(L) = y\}] f_1^{(L)}(y) dy$$

$$= \int_{0}^{\Delta_1} [1 - \Pr\{B_2 > x - y\}_{\Delta_1 > x}] f_1^{(L)}(y) dy$$

$$= \left\{ \begin{array}{ll}
          f_1^{(L)}(\Delta_1) & x < \Delta_1 \\
          F_1^{(L)}(\Delta_1) & x \geq \Delta_1
        \end{array} \right.$$

The second and the fourth equalities are obtained by noting that $\eta_1 = 0$ and $A_1 = \Delta_1 - y$
on $D_1(L) \leq \Delta_1$. For the second probability term in (A.1):

$$\Pr\{X_1 \leq x, D_1(L) > \Delta_1\} = \int_{\Delta_1}^{\infty} \Pr\{X_1 \leq x \mid D_1(L) = y\} f_1^{(L)}(y) dy$$
\[\int_{\Delta_1} \Pr\{y - \eta_1 \leq x | D_1(L) = y\} f_1^{(L)}(y) dy = \int_{\Delta_1} \Pr\{\eta_1 \geq y - x | D_1(L) = y\} f_1^{(L)}(y) dy = \int_{\Delta_1} \Pr\{\min(y - \Delta_1, A_2) \geq y - x | D_1(L) = y\} f_1^{(L)}(y) dy = \int_{\Delta_1} \Pr\{A_2 > y - x\} 1_{\{\Delta_1 \leq x\}} f_1^{(L)}(y) dy\]

\[= \left\{ \begin{array}{ll}
0 & x < \Delta_1 \\
\int_{\Delta_1} \Pr\{A_2 \geq y - x\} f_1^{(L)}(y) dy & x \geq \Delta_1
\end{array} \right.\]

The second and fourth equalities are obtained by noting that \(\eta_1 = 0\) and \(B_1 = y - \Delta_1\) on \(D_1(L) > \Delta_1\). Combining two probability terms yields

\[\Pr\{X_1 \leq x\} = \left\{ \begin{array}{ll}
\int_{0}^{\Delta_1} \Pr\{B_2 \leq x - y\} f_1^{(L)}(y) & x < \Delta_1 \\
F_1^{(L)}(\Delta_1) + \int_{\Delta_1}^{\infty} \Pr\{A_2 \geq y - x\} f_1^{(L)}(y) dy & x \geq \Delta_1
\end{array} \right.\]  \hspace{1cm} (A.2)

We can obtain the probability distributions for \(A_2\) and \(B_2\) as:

\[\Pr\{A_2 \geq y - x\} = \left\{ \begin{array}{ll}
1 & y < x \\
F_2^{(L)}(\Delta_2 + x - y) & x \leq y < \Delta_2 \\
0 & y \geq x + \Delta_2
\end{array} \right.,\]  \hspace{1cm} (A.3)

\[\Pr\{B_2 \leq x - y\} = \left\{ \begin{array}{ll}
0 & x < y \\
F_2^{(L)}(\Delta_2 + x - y) & x \geq y
\end{array} \right..\]  \hspace{1cm} (A.4)

By inserting equations (A.3) and (A.4) in equation (A.2) and simple algebraic manipulation we obtain Proposition 2. \(\Box\)

**Proof of Proposition 3**

We prove the continuity of \(C_1(y_1, y_2)\). The case for \(i = 2\) is the same. In Proposition 1 we showed that \(S_1 \geq Z_1\). On the other hand, since \(\eta_1 \geq 0\) and \(\eta_2 = \min\{A_1, B_2\} \leq D_2(L)\),

\[X_1 = D_1(L) - \eta_1 + \eta_2 \leq D_1(L) + D_2(L).\]

Therefore \(S_1 \leq \tilde{S}_1\) where \(\tilde{S}_1\) is obtained as

\[F_2^{(L)} \ast F_1^{(L+i_1+1)}(\tilde{S}_1) = \frac{b_1}{b_1 + h_1}.\]
Similarly, \( S_2 \leq \tilde{S}_2 \) where \( \tilde{S}_2 \) is given by

\[
F_1^{(L)} * F_2^{(L+L_2+1)}(\tilde{S}_2) = \frac{b_2}{b_2 + h_2}.
\]

Therefore \((y_1, y_2) \in S = [Z_1, \tilde{S}_1] \times [Z_2, \tilde{S}_2] \), a compact subset of \( R^2 \). Let \((y_1^k, y_2^k)\) be a sequence in \( S \) converging to \((y_1, y_2)\) (define all relevant random variables \( X_l^k, A_l^k, B_l^k \) associated with \( y_l^k \)). We can write \( C_1(y_1^k, y_2^k) \) as:

\[
C_1(y_1^k, y_2^k) = (h_1 + b_1)E[(y_1^k - X_1^k - D_1(l_1 + 1))^+] + b_1 E[X_1^k + D_1(l_1 + 1) - y_1^k]. \quad \text{(A.5)}
\]

We first note that \((y_1^k - X_1^k - D_1(l_1 + 1))^+ \leq \tilde{S}_1\), and \( y_1^k \) and \( X_1^k \) converge to \( y_1 \) and \( X_1 \), respectively. Therefore, the first expectation in (A.5) converges to \( E[(y_1 - X_1 - D_1(l_1 + 1))^+] \) by bounded convergence theorem (Billingsley 1986). Since \( X_1^k \leq D_1(L) + D_2(L) \) the second expectation in (A.5) converges to \( E[X_1 + D_1(l_1 + 1) - y_1] \) by dominated convergence theorem (Billingsley 1986). Therefore, \( C_1(y_1^k, y_2^k) \) converges to \( C_1(y_1, y_2) \) which proves continuity. \( \square \)

**Proof of Proposition 4**

Recall that \( y_1 - X_1 = y_1 - D_1(L) + \eta_1 - \eta_2 \). For a given \( y_2 \) and \( D_2(L) \) we either have \( A_2 > 0 \) and \( D_2 = 0 \) or \( A_2 = 0 \) and \( D_2 > 0 \). Hence:

\[
C_1(y_1, y_2) = E[G_1(y_1 - X_1)] = E[G_1(y_1 - X_1)1_{\{A_2 > 0\}}] + E[G_1(y_1 - X_1)1_{\{A_2 = 0\}}]. \quad \text{(A.6)}
\]

**Case 1:** \( A_2 > 0 \)

On \( A_2 > 0 \) we can write \( y_1 - X_1 \) as:

\[
y_1 - X_1 = \begin{cases} 
    y_1 - D_1(L) & \text{if } D_1(L) < y_1 - Z_1 \\
    Z_1 & \text{if } y_1 - Z_1 \leq D_1(L) \leq y_1 - Z_1 + A_2 \\
    y_1 - D_1(L) + A_2 & \text{if } D_1(L) \geq y_1 - Z_1 + A_2
\end{cases}.
\]

(A.7)

Therefore, on \( A_2 > 0 \), \( C_1(y_1, y_2) \) becomes:

\[
E[G_1(y_1 - X_1)1_{\{A_2 > 0\}}] = \int_{\{y: A_2 > 0\}} \int_0^{y_1 - Z_1} G_1(y_1 - w)f_1^{(L)}(w)dw \\
+ \int_{y_1 - Z_1 + A_2}^{y_1 - Z_1} G_1(Z_1)f_1^{(L)}(w)dw \\
+ \int_{y_1 - Z_1 + A_2}^{\infty} G_1(y_1 - w + A_2)f_2^{(L)}(w)dy. \quad \text{(A.8)}
\]

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The second derivative of (A.8) with respect to $y_1$ is:

$$E[G''_1(y_1 - X_1)1_{\{A_2 > 0\}}] = \int_{\{y, A_2 > 0\}} \left\{ \int_0^{y_1 - Z_1} G''_1(y_1 - w) f_1^{(L)}(w) dw \right\} f_2^{(L)}(y) dy > 0 \quad (A.9)$$

Since $Z_1$ is the unique minimizer of $G_1(x)$ both $G''_1$ terms in (A.9) are strictly positive.

**Case 2:** $A_2 = 0$

On $A_2 = 0$ we can write $y_1 - X_1$ as:

$$y_1 - X_1 = \begin{cases} 
    y_1 - D_1(L) - B_2 & D_1(L) < y_1 - Z_1 - B_2 \\
    Z_1 & y_1 - Z_1 - B_2 \leq D_1(L) \leq y_1 - Z_1 \\
    y_1 - D_1(L) & D_1(L) \geq y_1 - Z_1 
\end{cases} \quad (A.10)$$

Therefore, on $A_2 = 0$, $C_1(y_1, y_2)$ becomes:

$$E[G_1(y_1 - X_1)1_{\{A_2 = 0\}}] = \int_{\{y, A_2 = 0\}} \left\{ \int_0^{y_1 - Z_1 - B_2} G_1(y_1 - w - B_2) f_1^{(L)}(w) dw \right\} f_2^{(L)}(y) dy > 0 \quad (A.11)$$

The second derivative of (A.11) with respect to $y_1$ is:

$$E[G''_1(y_1 - X_1)1_{\{A_2 = 0\}}] = \int_{\{y, A_2 = 0\}} \left\{ \int_0^{y_1 - Z_1 - B_2} G''_1(y_1 - w - B_2) f_1^{(L)}(w) dw \right\} f_2^{(L)}(y) dy > 0 \quad (A.12)$$

Again, as $Z_1$ is the unique minimizer of $G_1(x)$ both $G''_1$ terms in (A.12) are strictly positive.\]

**Proof of Proposition 5**

Let $y_2^1$ and $y_2^2$ be values of $y_2$ solving $U_1(y_1, y_2) = 0$ and $U_2(y_1, y_2) = 0$ for a given value of $y_1$, respectively. Let $\partial y_2^1 / \partial y_1$ and $\partial y_2^2 / \partial y_1$ be derivatives of $U_1(y_1, y_2) = 0$ and $U_2(y_1, y_2) = 0$ at $(y_1, y_2)$. We use implicit differentiation of $U_1(y_1, y_2) = 0$ with respect to $y_1$ to obtain:

$$(h_1 + b_1) a_1(y_1, y_2) + (h_1 + b_1)(1 + \frac{\partial y_2^1}{\partial y_1}) d_1(y_1, y_2) = 0,$$

where

$$a_1(y_1, y_2) = 1 - F_2^{(L)}(\Delta_2) \int_0^{y_1} f_1^{(L)}(y_1 - x) f_1^{(d_1 + 1)}(x) dx + F_2^{(L)} \int_{Z_1}^{y_1} f_1^{(L)}(y_1 - x) f_1^{(d_1 + 1)}(x) dx > 0,$$
\[ d_1(y_1, y_2) = \int_0^{Z_1} \int_{y_1 - x}^{y_1 - x + \Delta_2} f_1^{(L)}(y) f_2^{(L)}(y_2 - x) f_1^{(t_1+1)}(x)\,dx\,dy \]
\[ + \int_{Z_1}^{y_2} \int_{y_1 - x}^{y_1 - x + \Delta_2} f_1^{(L)}(y) f_2^{(L)}(y_2 - x) f_1^{(t_1+1)}(x)\,dx\,dy > 0. \]

Therefore,
\[ \frac{\partial y_2}{\partial y_1} = \frac{-\{a_1(y_1, y_2) + d_1(y_1, y_2)\}}{d_1(y_1, y_2)} < 0. \tag{A.13} \]

Similarly, we use implicit differentiation of \( U_2(y_1, y_2) = 0 \) with respect to \( y_1 \) and find:
\[ (h_2 + b_2) \frac{\partial y_2}{\partial y_1} a_2(y_1, y_2) + (h_2 + b_2) (1 + \frac{\partial y_2}{\partial y_1}) d_2(y_1, y_2) = 0, \]
where
\[ a_2(y_1, y_2) = (1 - F_1^{(L)}(\Delta_1)) \int_0^{Z_2} f_1^{(L)}(y_2 - x) f_2^{(t_2+1)}(x)\,dx \]
\[ + F_1^{(L)} \int_{Z_2}^{y_2} f_2^{(L)}(y_2 - x) f_2^{(t_2+1)}(x)\,dx > 0, \]
\[ d_2(y_1, y_2) = \int_0^{Z_2} \int_{y_2 - x}^{y_2 - x + \Delta_1} f_2^{(L)}(y) f_1^{(L)}(\Delta_1 - y + y_2 - x) f_2^{(t_2+1)}(x)\,dx\,dy \]
\[ + \int_{Z_2}^{y_2} \int_{y_2 - x}^{y_2 - x + \Delta_1} f_2^{(L)}(y) f_1^{(L)}(\Delta_1 - y + y_2 - x) f_2^{(t_2+1)}(x)\,dx\,dy > 0. \]

Then,
\[ \frac{\partial y_2}{\partial y_1} = \frac{-d_2(y_1, y_2)}{a_2(y_1, y_2) + d_2(y_1, y_2)} < 0, \tag{A.14} \]

which proves part (i). For part (ii) we compare (A.13) with (A.14) and conclude that:
\[ \frac{\partial y_2}{\partial y_1} > \frac{\partial y_2}{\partial y_1}. \square \]

References


Tagaras, G., 1999, 'Pooling in Multi-location Periodic Inventory Distribution Systems',