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Stabilization of polytopic delay difference inclusions via the Razumikhin approach[☆]

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ABSTRACT

Polytopic delay difference inclusions (DDIs) have received increasing attention recently, mostly due to their ability to model a wide variety of relevant processes, including networked control systems. One of the fundamental problems for DDIs that poses a non-trivial challenge is stabilization. This paper embraces the Razumikhin approach and provides several solutions to the stabilization problem as follows. Firstly, a method to synthesize a control Lyapunov–Razumikhin function (cLRF) is presented for unconstrained DDIs. Secondly, for constrained DDIs, a receding horizon controller based on the cLRF for the unconstrained system is proposed, along with a closed-loop stability analysis. Thirdly, it is shown that a tractable implementation of the developed control algorithm can be attained even for large delays, by means of an on-line Minkowski set addition. An advantageous feature of the developed methodology is that all the synthesis algorithms can be formulated as a low complexity semi-definite programming problem for quadratic cLRF candidates. A comparison with alternative synthesis methods demonstrates the advances provided by the developed theory.

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1. Introduction

Polytopic delay difference inclusions (DDIs) have the ability to model a wide variety of relevant processes, including certain types of networked control systems (Cloosterman, van de Wouw, Heemels, & Nijmeijer, 2009; Gielen et al., 2010). Therefore, stabilization of polytopic DDIs, possibly subject to constraints, is an important problem. Two issues that make stabilization of unconstrained DDIs inherently difficult are *time delays* and *uncertainty*, which are treated separately in most papers. A typical solution for dealing with time delays, see, e.g., Åström and Wittenmark (1990) and Hetel, Daafouz, and Jung (2008), is to augment the state vector with all the delayed states and inputs, which yields a polytopic difference inclusion of higher dimension. Thus, stabilizing controller synthesis methods for difference inclusions based on classical Lyapunov theory (Kellett & Teel, 2005) become available. The control Lyapunov function for the augmented system provides (Gielen, Lazar, & Kolmanovsky, 2010a) a control Lyapunov–Krasovskii function (cLKF) for the non-augmented system, which establishes stability in the presence

of delay. If the Krasovskii approach is employed, constraints on states/inputs can be handled via model predictive control (MPC), as was shown in Jeong and Park (2005). Also, the classical MPC scheme for polytopic difference inclusions without delay of Kothare, Balakrishnan, and Morari (1996) applies directly to the augmented system. Constraints handling within the Krasovskii approach was also considered in Fridman, Seuret, and Richard (2004). An alternative method for stabilization of systems with delays, which deals directly with the non-augmented system, is the Razumikhin approach, see, e.g., Hale (1977), which employs a type of Lyapunov function that is required to decrease only if a certain condition on the past state trajectory and the current state is satisfied. A control Lyapunov–Razumikhin function (cLRF) and static state-feedback controller synthesis solution for continuous-time systems with multiple state delays and actuator saturation was proposed in Cao, Lin, and Hu (2002). Therein, the controller and the cLRF are chosen such that the closed-loop region of attraction is maximized in the presence of symmetric input constraints. A translation of the Razumikhin approach to discrete-time systems was obtained in Liu and Marquez (2007) by requiring the candidate function to be less than the maximum over its past values within a time window. So far, the application of cLRFs has been limited to particular sub-classes of polytopic DDIs, see, e.g., Lin (2007), where a synthesis method is developed for linear time-invariant systems subject to input delays with a strictly stable matrix multiplying the current state.

An advantage of the Krasovskii approach, when compared to the Razumikhin approach, is the lack of conservatism, as shown

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in Kolmanovskii and Myshkis (1999) and Gielen et al. (2010a) for continuous- and discrete-time systems, respectively. However, the Razumikhin approach has a lower complexity when the size of the delay increases and it applies directly to the non-augmented system. As such, the sublevel sets of a cLRF provide a region of attraction in the state space of the original system.

The above presentation of existing synthesis methods for polytopic DDIs indicates that a comprehensive framework that can deal with both state and input delays, constraints and large delays, based on the Razumikhin approach, is missing. Therefore, in this paper, controller synthesis solutions for polytopic DDIs are developed within the Razumikhin framework. A method to obtain a cLRF and corresponding static state-feedback controller is presented for unconstrained DDIs. In the presence of state and input constraints the so-obtained cLRF and corresponding controller remain valid only locally. Therefore, a receding horizon control algorithm, which relaxes the cLRF conditions of the unconstrained case, is proposed, along with a closed-loop stability analysis. Then, it is shown that a tractable formulation of the corresponding optimization problem can be attained even for large delays, by means of an on-line Minkowski set addition. Due to the quadratic structure of the cLRF, all the control algorithms can be formulated as a low complexity semi-definite programming (SDP) problem. An extensive comparison with alternative synthesis methods demonstrates the advances provided by the developed theory.

The remainder of the paper is organized as follows. Section 2 provides some useful preliminaries regarding notation and stability notions. Section 3 presents the main results, divided into four subsections that focus on the unconstrained case, the constrained case, large delays and implementation as a SDP problem, respectively. An illustrative example is presented in Section 4, along with a comprehensive comparison with alternative existing synthesis methods. Conclusions are drawn in Section 5 and the Appendix contains the proof of a technical lemma.

2. Preliminaries

Necessary notation, basic definitions and stability definitions are introduced in what follows.

2.1. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$, define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$. Furthermore, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a vector $x \in \mathbb{R}^n$ let $[x]_i$, $i \in \mathbb{Z}_{[1,n]}$, denote the i -th component of x and let $\|x\| := \sqrt{\sum_{i=1}^n [x]_i^2}$. Let $\mathbf{x} := \{x(l)\}_{l \in \mathbb{Z}_+}$ with $x(l) \in \mathbb{R}^n$ for all $l \in \mathbb{Z}_+$ denote an arbitrary sequence and define $\|\mathbf{x}\| := \sup\{\|x(l)\| \mid l \in \mathbb{Z}_+\}$. Furthermore, $\mathbf{x}_{[c_1, c_2]} := \{x(l)\}_{l \in \mathbb{Z}_{[c_1, c_2]}}$, with $c_1, c_2 \in \mathbb{Z}$, denotes a sequence which is ordered monotonically with respect to the index $l \in \mathbb{Z}_{[c_1, c_2]}$. Let I_n denote the n -th dimensional identity matrix and let $\mathbf{0}_{n \times m} \in \mathbb{R}^{n \times m}$ denote a matrix with all elements equal to zero. Let $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ denote the smallest and largest absolute values over the eigenvalues of $Z \in \mathbb{R}^{n \times n}$, respectively. If Z is symmetric, let $Z > 0$ and $Z < 0$ denote that Z is positive definite and negative definite, respectively. Moreover, $*$ is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b^T \\ b & c \end{bmatrix} = \begin{bmatrix} a & * \\ b & c \end{bmatrix}$. Let $\text{co}(\cdot)$ denote the convex hull and let $\text{int}(\mathcal{S})$ denote the interior of an arbitrary set $\mathcal{S}$. Moreover, $\mathcal{S}^h := \mathcal{S} \times \dots \times \mathcal{S}$ for any $h \in \mathbb{Z}_{\geq 1}$. Let $\mathcal{S}_i \subset \mathbb{R}^n$, $i \in \mathbb{Z}_+$, denote arbitrary sets. $\mathcal{S}_1 \oplus \mathcal{S}_2 := \{x + y \mid x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$ denotes the Minkowski addition of $\mathcal{S}_1$ and $\mathcal{S}_2$. Moreover, define

$\bigoplus_{i=1}^p \mathcal{S}_i := \mathcal{S}_1 \oplus \dots \oplus \mathcal{S}_p$. A polyhedron is a set obtained as the intersection of a finite number of half-spaces. A polytope is a compact polyhedron. Let $f, g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Define $f \circ g(s) := f(g(s))$ and $f^k(s) := f \circ f^{k-1}(s)$ for all $k \in \mathbb{Z}_{\geq 1}$ and all $s \in \mathbb{R}_+$, where $f^0(s) := s$. A function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{K}$, i.e., $\varphi \in \mathcal{K}$, if it is continuous, strictly increasing and $\varphi(0) = 0$. Moreover, $\varphi \in \mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$. A function $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{KL}$, i.e., $\beta \in \mathcal{KL}$, if for each fixed $s \in \mathbb{R}_+$, $\beta(r, s) \in \mathcal{K}$ with respect to r and for each fixed $r \in \mathbb{R}_+$, $\beta(r, s)$ is continuous and decreasing with respect to s and $\lim_{s \rightarrow \infty} \beta(r, s) = 0$. The notation $f: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is used to indicate that f is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e., $f(x) \subseteq \mathbb{R}^m$, $\forall x \in \mathbb{R}^n$.

2.2. Stabilization of delay difference inclusions

Consider the DDI

$$x(k+1) \in f(\mathbf{x}_{[k-h, k]}, \mathbf{u}_{[k-h, k]}), \quad k \in \mathbb{Z}_+, \quad (1)$$

where $\mathbf{x}_{[k-h, k]} \in \mathbb{X}^{h+1}$ is a sequence of (past) states, $\mathbb{X} \subseteq \mathbb{R}^n$ is the state space, $\mathbf{u}_{[k-h, k]} \in \mathbb{U}^{h+1}$ is a sequence of (past) control inputs, $\mathbb{U} \subseteq \mathbb{R}^m$ is the input space and $h \in \mathbb{Z}_+$ is the maximal delay. The DDI (1) is called polytopic if

$$f(\tilde{\mathbf{x}}_{[-h, 0]}, \tilde{\mathbf{u}}_{[-h, 0]}) := \left\{ \sum_{i=-h}^0 A_i \tilde{x}(i) + B_i \tilde{u}(i) \mid A_i \in \mathcal{A}_i, B_i \in \mathcal{B}_i, i \in \mathbb{Z}_{[-h, 0]} \right\}, \quad (2)$$

where

$$\mathcal{A}_i := \text{co}(\{\hat{A}_{i, l_i}\}_{l_i \in \mathbb{Z}_{[1, L_i]}}) \subset \mathbb{R}^{n \times n}$$

and

$$\mathcal{B}_i := \text{co}(\{\hat{B}_{i, l'_i}\}_{l'_i \in \mathbb{Z}_{[1, L'_i]}}) \subset \mathbb{R}^{n \times m}, \quad i \in \mathbb{Z}_{[-h, 0]}.$$

Moreover, $\hat{A}_{i, l_i} \in \mathbb{R}^{n \times n}$ and $\hat{B}_{i, l'_i} \in \mathbb{R}^{n \times m}$ are generators of polytopic sets in the matrix spaces $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{n \times m}$, respectively. Furthermore, $L_i \in \mathbb{Z}_{\geq 1}$ and $L'_i \in \mathbb{Z}_{\geq 1}$ denote the numbers of generators of these sets. In what follows it is assumed that (1) is a polytopic DDI. Also, above and in what follows, $\tilde{\mathbf{x}}_{[-h, 0]} \in \mathbb{X}^{h+1}$ and $\tilde{\mathbf{u}}_{[-h, 0]} \in \mathbb{U}^{h+1}$ are used to denote arbitrary sequences in $\mathbb{X}^{h+1}$ and $\mathbb{U}^{h+1}$, respectively, and to distinguish from the initial conditions $\mathbf{x}_{[-h, 0]}$ and $\mathbf{u}_{[-h, -1]}$ of (1).

For the stabilization of the DDI (1) a control law $\pi: \mathbb{X}^{h+1} \rightrightarrows \mathbb{U}$ will be used. The polytopic DDI (1) in closed-loop with this control law yields

$$x(k+1) \in F_\pi(\mathbf{x}_{[k-h, k]}), \quad k \in \mathbb{Z}_+, \quad (3)$$

where

$$F_\pi(\tilde{\mathbf{x}}_{[-h, 0]}) := \{f(\tilde{\mathbf{x}}_{[-h, 0]}, \tilde{\mathbf{u}}_{[-h, 0]}) \mid \tilde{u}(0) \in \pi(\tilde{\mathbf{x}}_{[-h, 0]})\}$$

and $\tilde{\mathbf{u}}_{[-h, -1]}$ is assumed to be known. In (3) and throughout the remainder of the paper it is assumed that each input in the initial sequence $\mathbf{u}_{[-h, -1]} \in \mathbb{U}^h$ is dependent on the initial state sequence $\mathbf{x}_{[-h, 0]} \in \mathbb{X}^{h+1}$ only. Moreover, without loss of generality, it can be assumed that, at time $k \in \mathbb{Z}_+$, $\mathbf{u}_{[k-h, k-1]}$ is dependent on $\mathbf{x}_{[k-h, k]}$ only. Therefore, the dependence of F_π on $\mathbf{u}_{[k-h, k-1]}$ can be omitted, as done above in (3).

Let $\mathcal{S}(\mathbf{x}_{[-h, 0]})$ denote the set of all trajectories of (3) that correspond to an initial sequence of states $\mathbf{x}_{[-h, 0]} \in \mathbb{X}^{h+1}$. Furthermore, let $\Phi(\mathbf{x}_{[-h, 0]}) := \{\phi(k, \mathbf{x}_{[-h, 0]})\}_{k \in \mathbb{Z}_{\geq -h}} \in \mathcal{S}(\mathbf{x}_{[-h, 0]})$ denote a trajectory of (3) such that $\phi(k, \mathbf{x}_{[-h, 0]}) = x(k)$ for all $k \in \mathbb{Z}_{[-h, 0]}$ and $\phi(k+1, \mathbf{x}_{[-h, 0]}) \in F_\pi(\Phi_{[k-h, k]}(\mathbf{x}_{[-h, 0]}))$ for all $k \in \mathbb{Z}_+$, where $\Phi_{[c_1, c_2]}(\mathbf{x}_{[-h, 0]}) := \{\phi(k, \mathbf{x}_{[-h, 0]})\}_{k \in \mathbb{Z}_{[c_1, c_2]}}$, $c_1, c_2 \in \mathbb{Z}$. Next, the stability of the closed-loop system (3) is discussed. For this purpose, we will restrict our attention to strong properties, i.e., properties that hold for all $\Phi(\mathbf{x}_{[-h, 0]}) \in \mathcal{S}(\mathbf{x}_{[-h, 0]})$.

Definition 1. (i) The origin of (3) is called *attractive in $\mathbb{X}$* if for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ and all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ it holds that $\lim_{k \rightarrow \infty} \|\phi(k, \mathbf{x}_{[-h,0]})\| = 0$; (ii) the origin of (3) is called *Lyapunov stable (LS)* if for every $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta(\varepsilon) \in \mathbb{R}_{>0}$ such that, if $\|\mathbf{x}_{[-h,0]}\| < \delta$, then $\|\phi(k, \mathbf{x}_{[-h,0]})\| < \varepsilon$ for all $\Phi(\mathbf{x}_{[-h,0]}) \in \mathcal{S}(\mathbf{x}_{[-h,0]})$ and all $k \in \mathbb{Z}_+$; (iii) system (3) is *asymptotically stable in $\mathbb{X}$* (AS($\mathbb{X}$)) if its origin is both attractive in $\mathbb{X}$ and LS.

Theorem 2. Suppose that $F_\pi(\tilde{\mathbf{x}}_{[-h,0]}) \subseteq \mathbb{X}$ for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$. Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and let $\rho \in \mathbb{R}_{(0,1)}$. If there exists a function $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|), \quad \forall \mathbf{x} \in \mathbb{X}, \quad (4a)$$

$$V(\mathbf{x}^+) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(\tilde{\mathbf{x}}(\theta)), \quad (4b)$$

for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$ and all $\mathbf{x}^+ \in F_\pi(\tilde{\mathbf{x}}_{[-h,0]})$, then the DDI (3) is AS($\mathbb{X}$).

The above theorem is proven in [Gielen et al. \(2010a\)](#) and is an extension of the similar result for delay difference equations in [Liu and Marquez \(2007\)](#).

Definition 3. Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and let $\rho \in \mathbb{R}_{(0,1)}$. Suppose that $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfies (4a). Moreover, suppose that there exists a control law $\pi: \mathbb{X}^{h+1} \rightarrow \mathbb{U}$ such that $F_\pi(\tilde{\mathbf{x}}_{[-h,0]}) \subseteq \mathbb{X}$ and (4b) hold for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$ and all $\mathbf{x}^+ \in F_\pi(\tilde{\mathbf{x}}_{[-h,0]})$. Then, V is a *control Lyapunov–Razumikhin function with respect to $\mathbb{X}$ and $\mathbb{U}$* (or, in short, cLRF($\mathbb{X}, \mathbb{U}$)) for the DDI (1).

Given a cLRF($\mathbb{X}, \mathbb{U}$) the stability of the closed-loop system (3) follows from [Theorem 2](#).

Definition 4. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is called *constrained control invariant* with respect to $\mathbb{U}$ (CCI($\mathbb{X}, \mathbb{U}$)) for system (1) if there exists a control law $\pi: \mathbb{X}^{h+1} \rightarrow \mathbb{U}$ such that $F_\pi(\tilde{\mathbf{x}}_{[-h,0]}) \subseteq \mathbb{X}$ for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$.

3. Main results

In this section several solutions to obtain a stabilizing controller for polytopic DDIs are presented.

3.1. Stabilization of unconstrained DDIs

The first problem of interest in this paper is the synthesis of a cLRF($\mathbb{R}^n, \mathbb{R}^m$) and corresponding control law π for the polytopic DDI (1). Let $P \in \mathbb{R}^{n \times n}$ and consider the candidate quadratic cLRF

$$V(\mathbf{x}) := \mathbf{x}^\top P \mathbf{x}, \quad (5)$$

and the corresponding control law

$$\pi(\tilde{\mathbf{x}}_{[-h,0]}) := \{K\tilde{\mathbf{x}}(0)\}. \quad (6)$$

Above π is denoted as a function of the complete sequence $\tilde{\mathbf{x}}_{[-h,0]}$ for consistency with further results. The main synthesis result for unconstrained polytopic DDIs is stated next.

Theorem 5. Let $\rho \in \mathbb{R}_{(0,1)}$. Suppose that there exist a symmetric matrix $Z \in \mathbb{R}^{n \times n}$, a matrix $Y \in \mathbb{R}^{m \times n}$ and $\gamma_i \in \mathbb{R}_+$, $i \in \mathbb{Z}_{[-h,0]}$, such that $\sum_{i=-h}^0 \gamma_i \leq 1$ and

$$\begin{bmatrix} \gamma_0 \rho Z & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \gamma_{-h} \rho Z & * \\ \hat{A}_{0,l_0} Z + \hat{B}_{0,l'_0} Y & \cdots & \hat{A}_{-h,l_{-h}} Z + \hat{B}_{-h,l'_{-h}} Y & Z \end{bmatrix} > 0, \quad (7)$$

for all $(l_i, l'_i) \in \mathbb{Z}_{[1,l_i]} \times \mathbb{Z}_{[1,l'_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Let $P := Z^{-1}$ and $K := YZ^{-1}$. Then, (5) is a cLRF($\mathbb{R}^n, \mathbb{R}^m$), with corresponding control law (6), for the DDI (1).

Proof. Let $\tilde{A}_{i,l_i,l'_i} := \hat{A}_{i,l_i} + \hat{B}_{i,l'_i} K$ for all $i \in \mathbb{Z}_{[-h,0]}$. Substituting $Y = KZ$, applying a congruence transform with a matrix that has Z^{-1} on its diagonal and zero elsewhere, and substituting $Z^{-1} = P$ yields

$$\begin{bmatrix} \gamma_0 \rho P & \cdots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \gamma_{-h} \rho P & * \\ P \tilde{A}_{0,l_0,l'_0} & \cdots & P \tilde{A}_{-h,l_{-h},l'_{-h}} & P \end{bmatrix} > 0, \quad (8)$$

for all $(l_i, l'_i) \in \mathbb{Z}_{[1,l_i]} \times \mathbb{Z}_{[1,l'_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. Applying the Schur complement to (8) yields $P > 0$ and

$$\begin{bmatrix} \tilde{A}_{0,l_0,l'_0}^\top \\ \vdots \\ \tilde{A}_{-h,l_{-h},l'_{-h}}^\top \end{bmatrix} P \begin{bmatrix} \tilde{A}_{0,l_0,l'_0}^\top \\ \vdots \\ \tilde{A}_{-h,l_{-h},l'_{-h}}^\top \end{bmatrix}^\top - \begin{bmatrix} \gamma_0 \rho P & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{-h} \rho P \end{bmatrix} < 0.$$

Next, pre-multiplying with $[\tilde{\mathbf{x}}(0)^\top \cdots \tilde{\mathbf{x}}(-h)^\top]$ and post-multiplying with $[\tilde{\mathbf{x}}(0)^\top \cdots \tilde{\mathbf{x}}(-h)^\top]^\top$ yields

$$\left(\sum_{i=-h}^0 \tilde{A}_{i,l_i,l'_i} \tilde{\mathbf{x}}(i) \right)^\top P \left(\sum_{i=-h}^0 \tilde{A}_{i,l_i,l'_i} \tilde{\mathbf{x}}(i) \right) - \rho \sum_{i=-h}^0 \gamma_i \tilde{\mathbf{x}}(i)^\top P \tilde{\mathbf{x}}(i) < 0, \quad (9)$$

for all $(l_i, l'_i) \in \mathbb{Z}_{[1,l_i]} \times \mathbb{Z}_{[1,l'_i]}$ and all $i \in \mathbb{Z}_{[-h,0]}$. The observation that all $A_i \in \mathcal{A}_i$ and $B_i \in \mathcal{B}_i$ are convex combinations of $\hat{A}_{i,l_i}$ and $\hat{B}_{i,l'_i}$, $i \in \mathbb{Z}_{[-h,0]}$, respectively, and noticing that $\sum_{i=-h}^0 \gamma_i \tilde{\mathbf{x}}(i)^\top P \tilde{\mathbf{x}}(i) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \tilde{\mathbf{x}}(\theta)^\top P \tilde{\mathbf{x}}(\theta)$, yields that (4b) holds for all $\tilde{\mathbf{x}}_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$ and all $\mathbf{x}^+ \in F_\pi(\tilde{\mathbf{x}}_{[-h,0]})$, with π as defined in (6). Moreover, as $P > 0$, (4a) holds with $\alpha_1(s) := \lambda_{\min}(P)s^2$ and $\alpha_2(s) := \lambda_{\max}(P)s^2$, which completes the proof. $\square$

The matrix inequality (7) is bilinear in the scalars γ_i and the matrix Z . The set $\mathbb{R}_{[0,1]}^{h+1}$, where the sequence of scalar variables $\{\gamma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ is allowed to take values, can be discretized using a gridding technique. Then, solving (7) for each point in the resulting grid amounts to solving a linear matrix inequality. Thus, a feasible solution to (7) can be obtained by solving a sequence of SDP problems. Observe that if Z , Y and $\{\gamma_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ satisfy (7) with $\sum_{i=-h}^0 \gamma_i < 1$, then there exist $\{\hat{\gamma}_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ such that $\sum_{i=-h}^0 \hat{\gamma}_i = 1$ and that Z , Y and $\{\hat{\gamma}_i\}_{i \in \mathbb{Z}_{[-h,0]}}$ also satisfy (7). Therefore, it suffices to consider only those points in the grid such that $\sum_{i=-h}^0 \gamma_i = 1$. Alternatively, also after defining a grid for the scalars γ_i , $i \in \mathbb{Z}_{[0,h]}$, branch and bound optimization algorithms ([Land & Doig, 1960](#)) can be used to obtain a computationally more efficient solution to the matrix inequality (7).

Remark 6. SDP based synthesis solutions for quadratic cLRFs were also proposed in, e.g., [Lin \(2007\)](#) (discrete-time) and [Cao et al. \(2002\)](#) (continuous-time). The results in [Lin \(2007\)](#) apply to systems where $\mathcal{A}_0$ is a singleton, A_0 is strictly stable and $\mathcal{A}_i = \{0_{n \times n}\}$ for all $i \in \mathbb{Z}_{[-h,-1]}$, i.e., there are no state delays. Furthermore, the synthesis algorithm in ([Cao et al., 2002](#)) considers systems without input delays, i.e., $\mathcal{B}_i = \{0_{n \times m}\}$, for all $i \in \mathbb{Z}_{[-h,-1]}$.

3.2. Stabilization of constrained DDIs

The second problem of interest in this paper is the stabilization of DDIs in the presence of state and/or input constraints, specified by some sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$.

In what follows we establish how a cLRF($\mathbb{R}^n, \mathbb{R}^m$), e.g., obtained via [Theorem 5](#), can be used to stabilize the DDI (1) in the presence

of constraints. The following standing assumption will be used in the remainder of the paper.

Assumption 7. The set $\mathbb{X} \subseteq \mathbb{R}^n$ is CCI($\mathbb{X}, \mathbb{U}$) for the polytopic DDI (1).

The fact that $\mathbb{X}$ is CCI($\mathbb{X}, \mathbb{U}$) for system (1) does not provide a controller such that (1) is AS($\mathbb{X}$). Assumption 7 merely implies that there exists a control law π such that all solutions that start in $\mathbb{X}^{h+1}$ remain in $\mathbb{X}$ for all time. If $\mathbb{X}$ is not CCI($\mathbb{X}, \mathbb{U}$), the results apply for any subset of $\mathbb{X}$ with this property.

Problem 8. At time $k \in \mathbb{Z}_+$, let $\mathbf{x}_{[k-h,k]}$ and $\mathbf{u}_{[k-h,k-1]}$ be known and minimize $\lambda(k)$ subject to

$$u(k) \in \mathbb{U}, \quad f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}) \subseteq \mathbb{X}, \quad \lambda(k) \in \mathbb{R}_+, \quad (10a)$$

$$V(x^+) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(x(k+\theta)) + \lambda(k), \quad (10b)$$

for all $x^+ \in f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]})$.

The role of the variable $\lambda(k)$, $k \in \mathbb{Z}_+$, is to introduce additional flexibility in (4b) and enhance the feasibility in the presence of constraints (Lazar, 2009), as will be made clear later in this section. Note that Problem 8 defines the set-valued control law

$$\bar{\pi}(\mathbf{x}_{[k-h,k]}) := \{u(k) \in \mathbb{R}^m \mid \exists \lambda(k) \in \mathbb{R}_+ \text{ s.t. Eq. (10) holds}\}, \quad (11)$$

where $k \in \mathbb{Z}_+$. Similarly to in Section 2.2, $\bar{\pi}$ is defined to be a function of $\mathbf{x}_{[k-h,k]}$ only. Without loss of generality, it is assumed that, at time $k \in \mathbb{Z}_+$, $\mathbf{u}_{[k-h,k-1]}$ is dependent on $\mathbf{x}_{[k-h,k]}$ only. Therefore, the dependence of $\bar{\pi}$ on $\mathbf{u}_{[k-h,k-1]}$ can also be omitted. Thus, the DDI (1) in closed-loop with (11) yields

$$x(k+1) \in F_{\bar{\pi}}(\mathbf{x}_{[k-h,k]}), \quad k \in \mathbb{Z}_+, \quad (12)$$

where $F_{\bar{\pi}}(\tilde{\mathbf{x}}_{[-h,0]}) := \{f(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,0]}) \subseteq \mathbb{R}^n \mid \tilde{u}(0) \in \bar{\pi}(\tilde{\mathbf{x}}_{[-h,0]})\}$ and $\tilde{\mathbf{u}}_{[-h,-1]}$ is assumed to be known. In what follows, it will be proven that Problem 8 is recursively feasible in $\mathbb{X}$ and sufficient conditions for stability of the closed-loop system (12) will be presented.

Let $\lambda^*(k)$ denote the optimum in Problem 8 at time $k \in \mathbb{Z}_+$ and let $\bar{\delta}(\mathbf{x}_{[-h,0]})$ denote the set of all trajectories of (12) that correspond to initial condition $\mathbf{x}_{[-h,0]}$. Furthermore, let $\bar{\Phi}(\mathbf{x}_{[-h,0]}) := \{\bar{\phi}(k, \mathbf{x}_{[-h,0]})\}_{k \in \mathbb{Z}_{\geq -h}} \in \bar{\delta}(\mathbf{x}_{[-h,0]})$ denote a trajectory of system (12) such that for all $k \in \mathbb{Z}_{[-h,0]}$, $\bar{\phi}(k, \mathbf{x}_{[-h,0]}) = x(k)$ and that for all $k \in \mathbb{Z}_+$, $\bar{\phi}(k+1, \mathbf{x}_{[-h,0]}) \in F_{\bar{\pi}}(\bar{\Phi}_{[k-h,k]}(\mathbf{x}_{[-h,0]}))$.

Theorem 9. Suppose² that there exist a $P \in \mathbb{R}^{n \times n}$ and a $K \in \mathbb{R}^{m \times n}$ such that (5), with the control law (6), is a cLRF($\mathbb{R}^n, \mathbb{R}^m$) for the DDI (1). Let $\mathbb{X}$ and $\mathbb{U}$ denote arbitrary bounded sets with the origin in their interior that satisfy Assumption 7. Then:

- (i) there exists a neighborhood of the origin $\mathcal{N}$ such that V is a cLRF($\mathcal{N}, \mathbb{U}$);
- (ii) Problem 8 is feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ and remains feasible for all $k \in \mathbb{Z}_+$;
- (iii) if $\lim_{k \rightarrow \infty} \lambda^*(k) = 0$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$, the DDI (12) is AS($\mathbb{X}$).

The following lemma, which is proven in the Appendix, is required to prove Theorem 9.

Lemma 10. Let $\rho \in \mathbb{R}_{[0,1]}$ and consider a sequence $\{\lambda(i)\}_{i \in \mathbb{Z}_+}$ with $\lambda(i) \in \mathbb{R}_+$ and bounded for all $i \in \mathbb{Z}_+$. If $\lim_{i \rightarrow \infty} \lambda(i) = 0$, then $\lim_{k \rightarrow \infty} \sum_{i=0}^k \rho^{k-i} \lambda(i) = 0$.

Next, we proceed with the proof of the above theorem.

Proof (Proof of Theorem 9). As (6) is a continuous function of the state and as by assumption $0 \in \text{int}(\mathbb{X})$ and $0 \in \text{int}(\mathbb{U})$, there exists an $\varepsilon \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$ satisfying $\|x\| \leq \varepsilon$ it holds that $x \in \mathbb{X}$ and $Kx \subseteq \mathbb{U}$. Thus, letting $\mathbb{X}_\pi := \{x \in \mathbb{X} \mid Kx \subseteq \mathbb{U}\}$ it follows that $0 \in \text{int}(\mathbb{X}_\pi)$. Next, from (4a) it follows that there exists a $\gamma \in \mathbb{R}_{>0}$ such that $\mathbb{V}_\gamma := \{x \in \mathbb{X} \mid V(x) \leq \gamma\}$ satisfies $\mathbb{V}_\gamma \subseteq \mathbb{X}_\pi$. Thus, letting $\mathcal{N} := \mathbb{V}_\gamma$ it follows that (5) is a cLRF($\mathcal{N}, \mathbb{U}$) with corresponding control law (6), which asserts claim (i).

Consider any $k \in \mathbb{Z}_+$. By Assumption 7 the set $\mathbb{X}$ is CCI($\mathbb{X}, \mathbb{U}$) for system (1). Therefore, the constraints in (10a) are feasible for all $\mathbf{x}_{[k-h,k]} \in \mathbb{X}^{h+1}$. Moreover, setting

$$\lambda(k) = \sup_{\substack{\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}, \\ \tilde{\mathbf{u}}_{[-h,0]} \in \mathbb{U}^{h+1}, \\ x^+ \in f(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,0]})}} \left(V(x^+) - \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(\tilde{x}(\theta)) \right)$$

in (10b) yields that (10b) is feasible for all $\mathbf{x}_{[k-h,k]} \in \mathbb{X}^{h+1}$. Note that the supremum exists due to boundedness of $\mathbb{X}, \mathbb{U}$, compactness of the map (1), (4a) and continuity of the class $\mathcal{K}_\infty$ functions in (4a). Therefore, Problem 8 is feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ and remains feasible for all $k \in \mathbb{Z}_+$ and, hence, claim (ii) is proven.

To prove the third claim, consider any $k \in \mathbb{Z}_+$. Let $\hat{\rho} := \rho^{\frac{1}{h+1}}$, $\theta^*(k) := \arg \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+\theta)} V(x(k+\theta))$ and

$$U(k) := \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+\theta)} V(x(k+\theta)). \quad (13)$$

We will prove that $U(k+1) \leq U(k) + \hat{\rho}^{-(k+1)} \lambda^*(k)$ for all $\mathbf{x}_{[k-h,k]} \in \mathbb{X}^{h+1}$. If $\theta^*(k+1) = 0$ then, by (10b), it holds that

$$\begin{aligned} U(k+1) &= \hat{\rho}^{-(k+1)} V(x^+) \\ &\leq \hat{\rho}^{-(k+1)} \left(\max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{h+1} V(x(k+\theta)) + \lambda^*(k) \right) \\ &\leq U(k) + \hat{\rho}^{-(k+1)} \lambda^*(k). \end{aligned} \quad (14)$$

Furthermore, if $\theta^*(k+1) \in \mathbb{Z}_{[-h,-1]}$ it holds that

$$\begin{aligned} U(k+1) &= \max_{\theta \in \mathbb{Z}_{[-h,-1]}} \hat{\rho}^{-(k+\theta+1)} V(x(k+\theta+1)) \\ &\leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \hat{\rho}^{-(k+\theta)} V(x(k+\theta)) = U(k). \end{aligned} \quad (15)$$

Therefore, as $k \in \mathbb{Z}_+$ was arbitrary, from (14) and (15) it follows that $U(k+1) \leq U(k) + \hat{\rho}^{-(k+1)} \lambda^*(k)$ for all $\mathbf{x}_{[k-h,k]} \in \mathbb{X}^{h+1}$, $x^+ \in F_{\bar{\pi}}(\mathbf{x}_{[k-h,k]})$ and all $k \in \mathbb{Z}_+$. From claim (ii) it follows that Problem 8 is feasible for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ and all $k \in \mathbb{Z}_+$ and, hence, it is recursively feasible. As such, the inequality $U(k+1) \leq U(k) + \hat{\rho}^{-(k+1)} \lambda^*(k)$ can be applied recursively, which yields

$$U(k) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta)) + \sum_{i=0}^{k-1} \hat{\rho}^{-(i+1)} \lambda^*(i), \quad (16)$$

for all $k \in \mathbb{Z}_+$. Combining (13) and (16) yields

$$\begin{aligned} V(\bar{\phi}(k, \mathbf{x}_{[-h,0]})) &\leq \hat{\rho}^k U(k) \\ &\leq \hat{\rho}^k \max_{\theta \in \mathbb{Z}_{[-h,0]}} V(x(\theta)) + \hat{\rho}^{-1} \sum_{i=0}^{k-1} \hat{\rho}^{k-i} \lambda^*(i), \end{aligned} \quad (17)$$

² The matrices $P \in \mathbb{R}^{n \times n}$ and $K \in \mathbb{R}^{m \times n}$ can be obtained using the techniques presented in Theorem 5.

which holds for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$, $\bar{\Phi}(\mathbf{x}_{[-h,0]}) \in \bar{\mathcal{S}}(\mathbf{x}_{[-h,0]})$ and all $k \in \mathbb{Z}_+$. Next, from (4a), (17) and the inequality $\alpha_1^{-1}(y+z) \leq \alpha_1^{-1}(2 \max\{y, z\}) \leq \alpha_1^{-1}(2y) + \alpha_1^{-1}(2z)$ it follows that

$$\|\bar{\Phi}(k, \mathbf{x}_{[-h,0]})\| \leq \alpha_1^{-1}(2\hat{\rho}^k \max_{\theta \in \mathbb{Z}_{[-h,0]}} \alpha_2(x(\theta))) + \alpha_1^{-1} \left(2\hat{\rho}^{-1} \sum_{i=0}^{k-1} \hat{\rho}^{k-i} \lambda^*(i) \right),$$

with $\alpha_1^{-1} \in \mathcal{K}_\infty$. Moreover, it follows from the proof of claim (ii) that $\lambda(k)$ is bounded for all $k \in \mathbb{Z}_+$. Therefore, as $\hat{\rho} \in \mathbb{R}_{(0,1)}$ and $\lim_{k \rightarrow \infty} \lambda^*(k) = 0$, Lemma 10 yields that $\lim_{k \rightarrow \infty} \|\bar{\Phi}(k, \mathbf{x}_{[-h,0]})\| = 0$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$ and all $\bar{\Phi}(\mathbf{x}_{[-h,0]}) \in \bar{\mathcal{S}}(\mathbf{x}_{[-h,0]})$, which establishes that the DDI (12) is attractive in $\mathbb{X}$.

Attractivity of the origin for the DDI (12) further implies that there exists some finite $k^*(\mathbf{x}_{[-h,0]}) \in \mathbb{Z}_+$ such that $\bar{\Phi}_{[k^*-h, k^*]}(\mathbf{x}_{[-h,0]}) \in \mathcal{N}^{h+1}$ for all $\bar{\Phi}(\mathbf{x}_{[-h,0]}) \in \bar{\mathcal{S}}(\mathbf{x}_{[-h,0]})$ and all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$. As V is a cLRF($\mathcal{N}$, $\mathbb{U}$), it follows from (4b) that there exists a feasible solution to Problem 8 with $\lambda(k^*) = 0$ for all $\bar{\Phi}_{[k^*-h, k^*]}(\mathbf{x}_{[-h,0]}) \in \mathcal{N}^{h+1}$. Hence, it follows by optimality that solving Problem 8 yields $\lambda^*(k) = 0$ for all $k \in \mathbb{Z}_{\geq k^*}$. Thus, it follows from Theorem 2 that the origin of (3) is LS, which completes the proof of claim (iii). $\square$

Remark 11. The property $\lim_{k \rightarrow \infty} \lambda^*(k) = 0$ can be guaranteed by augmenting Problem 8 with the constraint

$$0 \leq \lambda(k) \leq \rho \max_{i \in \mathbb{Z}_{[1,M]}} \lambda^*(k-i), \quad \forall k \in \mathbb{Z}_{\geq M}, \quad (18)$$

for some $M \in \mathbb{Z}_{\geq 1}$. The constraint (18) is non-conservative in the sense that a non-monotone evolution of $\lambda^*(k)$ is allowed, while $\lambda^*(k) \rightarrow 0$ as $k \rightarrow \infty$.

When (18) is added to Problem 8, claim (ii) of Theorem 9 remains inherently valid only locally. Claims (ii) and (iii) of Theorem 9 can then be reformulated as typically done in sub-optimal MPC (Lazar & Heemels, 2009; Scokaert, Mayne, & Rawlings, 1999), i.e., as a result of the type “feasibility implies stability”. However, it would be desirable to identify sufficient conditions under which Problem 8 yields a stabilizing control law, without explicitly restricting the evolution of $\{\lambda(k)\}_{k \in \mathbb{Z}_+}$, and thus removing the recursive feasibility guarantee that Problem 8 has.

To this end, we will firstly establish that the set of feasible choices for $\{\lambda(k)\}_{k \in \mathbb{Z}_+}$ can be upper bounded by a function of the state trajectory. Let $\alpha_i \in \mathcal{K}_\infty$, $i \in \mathbb{Z}_{[1,4]}$, and let $\rho, \rho_1 \in \mathbb{R}_{(0,1)}$. Moreover, let $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$ be such that

$$\alpha_3(\|x\|) \leq V(x) \leq \alpha_4(\|x\|), \quad \forall x \in \mathbb{X}. \quad (19)$$

Notice that a function V calculated as in Theorem 5 trivially satisfies the above inequalities.

Assumption 12. The DDI (1) admits a cLRF($\mathbb{X}$, $\mathbb{U}$), i.e., V_1 , with corresponding control law π_1 , satisfying (4a) with α_1, α_2 and (4b) with ρ_1 .

The control law π_1 in closed-loop with the DDI (1) yields a system of the form (3), denoted by F_{π_1} .

Assumption 13. There exists a $\sigma \in \mathcal{K}_\infty$ such that $V(x^+) \leq \sigma(V_1(x_1^+))$ for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$, $x^+ \in F_{\tilde{\pi}}(\tilde{\mathbf{x}}_{[-h,0]})$ and all $x_1^+ \in F_{\pi_1}(\tilde{\mathbf{x}}_{[-h,0]})$.

Note that the existence of σ follows from the fact that both V and V_1 enjoy upper and lower $\mathcal{K}_\infty$ bounds.

Lemma 14. Suppose that Assumptions 12 and 13 hold. Then, there exists a $\lambda: \mathbb{X}^{h+1} \rightarrow \mathbb{R}_+$ that is bounded on bounded sets, $\lambda(\mathbf{0}_{[-h,0]}) = 0$, and such that

$$V(x^+) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(\tilde{x}(\theta)) + \lambda(\tilde{\mathbf{x}}_{[-h,0]}), \quad (20)$$

for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$ and all $x^+ \in F_{\tilde{\pi}}(\tilde{\mathbf{x}}_{[-h,0]})$.

Proof. Let

$$\lambda(\tilde{\mathbf{x}}_{[-h,0]}) := \max\{0, \sigma(\max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho_1 V_1(\tilde{x}(\theta))) - \max_{\theta' \in \mathbb{Z}_{[-h,0]}} \rho V(\tilde{x}(\theta'))\}.$$

Using (4a) for V_1 and (19) for V yields that

$$\lambda(\tilde{\mathbf{x}}_{[-h,0]}) \leq \max\{0, (\sigma \circ (\rho_1 \alpha_2) - \rho \alpha_3)(\|\tilde{\mathbf{x}}_{[-h,0]}\|)\}, \quad (21)$$

and hence that $\lambda(\mathbf{0}_{[-h,0]}) = 0$ and $\lambda(\tilde{\mathbf{x}}_{[-h,0]})$ is bounded on bounded sets. Moreover, using (4b) for V_1 yields

$$V(x^+) \leq \sigma(V_1(x_1^+)) \leq \sigma\left(\max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho_1 V_1(\tilde{x}(\theta))\right) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(\tilde{x}(\theta)) + \lambda(\tilde{\mathbf{x}}_{[-h,0]}),$$

for all $\tilde{\mathbf{x}}_{[-h,0]} \in \mathbb{X}^{h+1}$, $x^+ \in F_{\tilde{\pi}}(\tilde{\mathbf{x}}_{[-h,0]})$ and all $x_1^+ \in F_{\pi_1}(\tilde{\mathbf{x}}_{[-h,0]})$. $\square$

Notice that the result of Lemma 14 establishes that if the DDI (1) admits a cLRF($\mathbb{X}$, $\mathbb{U}$), i.e., V_1 , then any candidate cLRF($\mathbb{R}^n$, $\mathbb{R}^m$), e.g., V , can be employed to “approximate” the evolution of V_1 via a suitable sequence of variables $\{\lambda(k)\}_{k \in \mathbb{Z}_+}$. The next result makes use of Lemma 14 to obtain sufficient conditions under which Problem 8 yields a stabilizing control law and hence under which the DDI (12) is AS($\mathbb{X}$).

Theorem 15. Suppose that Assumptions 7, 12 and 13 hold. Furthermore, suppose that

$$\beta(r, s) := (\alpha_3^{-1} \circ (\rho(\alpha_4 - \alpha_3) + \sigma \circ (\rho_1 \alpha_2)))^s(r) \quad (22)$$

belongs to class $\mathcal{KL}$. Then, the DDI (12) is AS($\mathbb{X}$).

Proof. Let $k \in \mathbb{Z}_+$ and $\mathbf{x}_{[k-h, k]} \in \mathbb{X}^{h+1}$. By optimality it follows from Lemma 14 that $\lambda^*(k)$ satisfies the upper bound in (21). Using (10b), the above observation and the bounds in (4a) for V yield

$$V(x^+) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(x(k+\theta)) + \lambda^*(k) \leq (\rho(\alpha_4 - \alpha_3) + \sigma \circ (\rho_1 \alpha_2))(\|\mathbf{x}_{[k-h, k]}\|),$$

for all $x^+ \in F_{\tilde{\pi}}(\mathbf{x}_{[k-h, k]})$. As Problem 8 is recursively feasible by Theorem 9(ii), inequality (10b) can be applied recursively and, as such, the above inequality can also be applied recursively. This together with (22) yields $\|\bar{\Phi}(k, \mathbf{x}_{[-h,0]})\| \leq \beta(\|\mathbf{x}_{[-h,0]}\|, k)$ for all $\mathbf{x}_{[-h,0]} \in \mathbb{X}^{h+1}$, $\bar{\Phi}(\mathbf{x}_{[-h,0]}) \in \bar{\mathcal{S}}(\mathbf{x}_{[-h,0]})$ and all $k \in \mathbb{Z}_+$. Hence, as $\beta \in \mathcal{KL}$ it follows that the DDI (12) is AS($\mathbb{X}$). $\square$

An inherent consequence of Theorem 15 is that $\lim_{k \rightarrow \infty} \lambda^*(k) = 0$, which is in accordance with the hypothesis of Theorem 9(iii).

Remark 16. An alternative³ set of sufficient conditions for establishing stability of the closed-loop DDI (12) corresponding to Problem 8 can be obtained using the recent article (Grüne, 2009) on recursive feasibility and stability of MPC. The results in Grüne (2009)

³ Classical stabilization conditions employed in MPC (Rawlings & Mayne, 2009), which rely on a sufficiently large prediction horizon N , are not suitable for DDIs, as increasing N leads to an exponential increase of the complexity of the optimization problem.

also rely on the construction of a $\mathcal{KL}$ bound on the closed-loop trajectories from an unknown control Lyapunov function for the constrained system. The application of these results to the setting of [Problem 8](#) indicates that the conditions proposed within [Theorem 15](#) are less conservative.

3.3. Large delays

The computational complexity of virtually all existing controller synthesis algorithms for polytopic DDIs increases exponentially with the maximum value of the delay, i.e., $h \in \mathbb{Z}_+$. This increase is mainly due to the fact that all possible combinations of vertices of $\mathcal{A}_i$ and $\mathcal{B}_i$, $i \in \mathbb{Z}_{[-h,0]}$, have to be taken into account. Moreover, the dimension of the augmented state vector increases linearly with the value of h . In particular, for optimization based controllers, which include MPC schemes in general and [Problem 8](#) in particular, this is not acceptable. As [Problem 8](#) does not employ the augmented state vector, part of this increase in complexity is inherently avoided. To make the complexity of the computation of the control update independent of h , let

$$\begin{aligned} & \hat{f}(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,0]}) \\ & := \left\{ B_0 \tilde{\mathbf{u}}(0) + v \mid B_0 \in \mathcal{B}_0, v \in \mathcal{V}(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,-1]}) \right\}, \end{aligned} \quad (23)$$

where

$$\mathcal{V}(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,-1]}) := \mathcal{A}_0 \tilde{\mathbf{x}}(0) \oplus \left(\bigoplus_{i=-h}^{-1} \mathcal{A}_i \tilde{\mathbf{x}}(i) \oplus \mathcal{B}_i \tilde{\mathbf{u}}(i) \right).$$

Lemma 17. Let f and $\hat{f}$ be defined as in (2) and (23), respectively. Then,

$$\begin{aligned} & \hat{f}(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,0]}) \equiv f(\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,0]}), \\ & \text{for all } (\tilde{\mathbf{x}}_{[-h,0]}, \tilde{\mathbf{u}}_{[-h,0]}) \in \mathbb{X}^{h+1} \times \mathbb{U}^{h+1}. \end{aligned}$$

Proof. The claim follows straightforwardly from the properties ([Schneider, 1993](#)) that the Minkowski addition is commutative and associative. $\square$

In [Problem 8](#), at time $k \in \mathbb{Z}_+$, $\mathbf{x}_{[k-h,k]}$ and $\mathbf{u}_{[k-h,k-1]}$ are known before computation of the control update and, hence, the set $\mathcal{V}$ can be computed at each time instant. The computation of $\mathcal{V}$ can be performed efficiently using, for example, the tools for performing Minkowski addition of polytopes available within the Multi-Parametric Toolbox for Matlab. Moreover, recent research ([Barki, Denis, & Dupont, 2009](#)) has led to much faster algorithms for performing Minkowski additions of polytopes. As the number of vertices spanning a polytope resulting from a Minkowski addition is likely to be much smaller than all possible combinations of vertices spanning the original polytopes, the control of a system of the form (23) is far simpler than one of the form (2). However, determining an exact upper bound on the number of vertices spanning a polytope resulting from a Minkowski addition is a non-trivial problem that has attracted much interest. The interested reader is referred to [Sanyal \(2009\)](#), [Weibel \(2007\)](#) and the references therein for further reading.

Remark 18. The structure that is exploited in [Lemma 17](#) to obtain a reduction in complexity of the control algorithm is particular to [Problem 8](#). Other optimization based control schemes, such as the ones in [Jeong and Park \(2005\)](#) and [Kothare et al. \(1996\)](#), cannot benefit from this structure to obtain a similar reduction in complexity.

3.4. SDP implementation of [Problem 8](#)

Next, we show how [Problem 8](#) can be formulated as an SDP problem. Suppose that the set $\mathcal{V}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k-1]})$ is known at each time $k \in \mathbb{Z}_+$ and let $\hat{v}_{l_v}(k) \in \mathbb{R}^n$, $l_v \in \mathbb{Z}_{[1,L_v(k)]}$ denote the corresponding vertices with $L_v(k) \in \mathbb{Z}_{\geq 1}$ the number of vertices at time $k \in \mathbb{Z}_+$. Note that the set $\mathcal{V}$ can be computed at each time $k \in \mathbb{Z}_+$, as explained in [Section 3.3](#).

Proposition 19. Let $P \in \mathbb{R}^{n \times n}$ be known. At time $k \in \mathbb{Z}_+$, consider the following set of inequalities in the unknowns $\lambda(k)$ and $u(k)$:

$$\hat{B}_{0,l'_0} u(k) + \hat{v}_{l_v}(k) \in \mathbb{X}, \quad (24a)$$

$$u(k) \in \mathbb{U}, \quad \lambda(k) \in \mathbb{R}_+, \quad (24b)$$

$$\begin{bmatrix} \rho \max_{\theta \in \mathbb{Z}_{[-h,0]}} x(k+\theta)^\top P x(k+\theta) + \lambda(k) & * \\ P(\hat{B}_{0,l'_0} u(k) + \hat{v}_{l_v}(k)) & P \end{bmatrix} \succ 0, \quad (24c)$$

for all $(l'_0, l_v) \in \mathbb{Z}_{[1,L'_0]} \times \mathbb{Z}_{[1,L_v(k)]}$. Then, any feasible solution $(\lambda(k), u(k))$ of (24) satisfies the inequalities (10) with $V(x) := x^\top P x$.

Proof. Note that all $B_0 \in \mathcal{B}_0$ are convex combinations of $\hat{B}_{0,l'_0}$. Moreover, note that all $v(k) \in \mathcal{V}(\mathbf{x}_{[k-h,k-1]}, \mathbf{u}_{[k-h,k-1]})$ are convex combinations of $\hat{v}_{l_v}(k)$. These two observations yield that (24a) implies $\hat{f}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}) \subseteq \mathbb{X}$. From [Lemma 17](#) it then follows that $f(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]}) \subseteq \mathbb{X}$. Thus, inequalities (24a) and (24b) imply (10a). Using the same convexity argument as above and applying the Schur complement to (24c) yields $V(x^+) \leq \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho V(x(k+\theta)) + \lambda(k)$ for all $x^+ \in \hat{f}(\mathbf{x}_{[k-h,k]}, \mathbf{u}_{[k-h,k]})$ with $V(x) = x^\top P x$. From [Lemma 17](#) it then follows that (10b) holds. $\square$

If $\mathbb{X}$ and $\mathbb{U}$ are polytopes or ellipsoids, the set of inequalities (24) is a set of linear matrix inequalities. As such, a solution to [Problem 8](#) can be obtained via SDP.

4. Illustrative example

To illustrate the developed theory, consider the polytopic DDI (1) with $h = 3$,

$$\mathcal{A}_0 := \text{co} \left(\left[\begin{array}{cc} 0.5 & 1.3 \\ -1.1 & 1 \end{array} \right], \left[\begin{array}{cc} 0.5 & 1 \\ -1.1 & 1 \end{array} \right], \left[\begin{array}{cc} 0.5 & 1 \\ -1.1 & 0.8 \end{array} \right], \left[\begin{array}{cc} 0.8 & 1 \\ -1.1 & 0.8 \end{array} \right] \right),$$

$\mathcal{B}_0 = \left\{ \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \right\}$, $\mathcal{A}_i := 0.05 \mathcal{A}_0$ and $\mathcal{B}_i := \{0_{n \times m}\}$, $i \in \mathbb{Z}_{[-3,-1]}$. The system is constrained, i.e., $\mathbb{X} := \mathbb{R}_{[-0.9,0.9]} \times \mathbb{R}_{[-1.6,1.6]}$ and $\mathbb{U} := \mathbb{R}_{[-0.8,0.8]}$. To grid the set $\mathbb{R}_{[0,1]}^4$, each interval $\mathbb{R}_{[0,1]}$ was split into 15 subintervals, which results in 16^4 points. However, when the constraint $\sum_{i=-3}^0 \gamma_i = 1$ is added, only 793 points remain. Then, solving [Theorem 5](#) with $\rho = 0.9$ for those points in the grid, a feasible solution was obtained for the point $\gamma_0 = 0.601$ and $\gamma_i = 0.133$ for all $i \in \mathbb{Z}_{[-3,-1]}$ which yielded

$$P = \begin{bmatrix} 1 & -0.1993 \\ -0.1993 & 0.2109 \end{bmatrix}, \quad K = \begin{bmatrix} -0.8068 \\ -1.0909 \end{bmatrix}^\top. \quad (25)$$

[Fig. 1](#) shows the sets $\mathbb{X}$, $\mathbb{X}_\pi$ and $\mathcal{N}$, as defined in the proof of [Theorem 9](#). In what follows, several control strategies are applied to stabilize the DDI (1) for initial condition $x(i) = [0.9 \ 0.6]^\top$ for all $i \in \mathbb{Z}_{[-3,0]}$. Note that, as $x(0) \notin \mathbb{X}_\pi$, $Kx(0) \notin \mathbb{U}$ and hence the control law (6) is not feasible. Therefore, the following designs are considered:

Fig. 1. Left: the state constraints $\mathbb{X}$ (---), the set $\mathbb{X}_\pi$ (·····), the neighborhood $\mathcal{N}$ (- · - · -) and the state trajectory (—). Right: the control signal (—) and the input constraints (---).

- CS1: Problem 8 with Minkowski addition;
- CS2: Problem 8 without Minkowski addition;
- CS3: the method proposed in Kothare et al. (1996);
- CS4: minimization of the cLRF candidate (5), i.e., $\min_{u \in \mathbb{U}} x(k+1)^\top P x(k+1)$ with Minkowski addition;
- CS5: minimization of the cost function

$$\sum_{i=1}^N x(k+i)^\top Q_i x(k+i) + u(k+i)^\top R_i u(k+i),$$

with parameters $N = 2$, $Q_1 = I_2$, $Q_2 = P$, $R_i = 0.01$, $i \in \mathbb{Z}_{[1,2]}$, and with Minkowski addition;

- CS6: CS5 with parameters $N = 3$, $Q_1 = Q_2 = I_2$, $Q_3 = P$ and $R_i = 0.01$, $i \in \mathbb{Z}_{[1,3]}$;
- CS7: the terminal constraint $x(k+N) \in \mathcal{N}$, $N \in \mathbb{Z}_{\geq 1}$, see, e.g., Rawlings and Mayne (2009), in combination with some cost function, e.g., the one used in CS5.

The methods proposed in Fridman et al. (2004) and Lin (2007) can be used to obtain a stabilizing controller for polytopic DDIs in the presence of constraints. Unfortunately, as both methods do not consider state delays they cannot be applied to the system under consideration. Notice that CS4 is a direct application of the standard control Lyapunov function (cLF) approach (Artstein, 1983) to time-delay systems. Moreover, CS4 corresponds to CS1 without the constraint $\lambda(k) \geq 0$ in the sense that the same optimal cLRF value will be obtained. Hence, the value of the cLRF will coincide for CS4 and CS1 if $\lambda^*(k) > 0$. However, for $\lambda^*(k) = 0$ CS4 may be computationally more expensive than CS1, since CS4 obtains the control input that results in the lowest possible value for the cLRF. Contrary to that, for CS1 any solution that attains $\lambda^*(k) = 0$ can be selected. Furthermore, the methods CS5 and CS6 correspond to a typical terminal cost MPC design approach, where one selects, e.g., based on the results in Reble and Allgöwer (2010), a suitable cost function and pursues minimization of the cost, while taking N sufficiently large. Recursive feasibility of such a scheme is guaranteed if a terminal constraint is added, as is the case for the design CS7.

Fig. 1 shows the state trajectory and the control signal as a function of time for CS1. Fig. 2 shows the values of $V(x(k))$ and $\lambda^*(k)$ as a function of time. Also shown in Fig. 2 are the computation time⁴ for the different methods. Note that the computation time of CS1 is a factor 15 smaller than that of CS2, thus indicating the advantage of performing the Minkowski addition.

For the considered initial condition and realization of the uncertain matrices A_i , $i \in \mathbb{Z}_{[-3,0]}$, CS4 led to an infeasible problem at time $k = 4$. This is indicated in the plot of Fig. 2, i.e., the time needed for solving a specific problem is shown until the first sampling instant when the problem becomes infeasible. Notice that infeasibility of CS4 demonstrates that the “maximum decrease” approach of the standard cLF design (Artstein, 1983) is not always the feasible choice in the presence of constraints, which demonstrates the effectiveness of the control design method developed in this paper (see also Lazar (2009)).

CS5 led to an infeasible problem at time $k = 3$. Again, this can also be observed in Fig. 2. Furthermore, CS3 led to an infeasible optimization problem for $k = 0$ and CS6 led to an untractable optimization problem for $k = 0$, i.e., the solver⁵ did not return a solution, even after a very large period of time. Finally, at time $k = 0$, CS7 led to an infeasible optimization problem for $N \in \mathbb{Z}_{[1,3]}$ and an untractable optimization problem for $N \in \mathbb{Z}_{\geq 4}$, which indicates the conservativeness of the terminal constraint set method and, for that matter, of any other MPC design that requires a sufficiently large N for recursive feasibility, see, e.g., Rawlings and Mayne (2009).

5. Concluding remarks

A framework for stabilizing controller synthesis, which can deal with both state and input delays, constraints and large delays, was developed in this paper via the Razumikhin approach. For unconstrained delay difference inclusions, a semi-definite programming based solution for synthesizing control Lyapunov–Razumikhin functions that allows for general delays was presented. Furthermore, it was shown how the control Lyapunov–Razumikhin function can be used to set up a stabilizing receding horizon scheme that can handle constraints. It was then demonstrated that by exploiting properties of the Minkowski addition of polytopes and the structure of the developed control law, an efficient implementation can be attained even for large delays.

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⁴ Simulations were performed using Matlab 7.5.0 on an Intel Q9400, 2.66 GHz desktop PC.

⁵ Simulations were performed using SeDuMi, LMILab and SDPT3. Similar results were obtained for all solvers. The results shown in Figs. 1 and 2 were obtained using SeDuMi.

Fig. 2. Left: the values of the cLRF (–) and λ (•) as a function of time. Right: the computation time of the control update in logarithmic scale, i.e., CS1 (▲), CS2 (◆), CS4 (●) and CS5 (■).

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Appendix. Proof of Lemma 10

An arbitrary sequence $\{s(k)\}_{k \in \mathbb{Z}_+}$ of real numbers is convergent (Stewart, 2003) and has the limit s if, for all $\varepsilon \in \mathbb{R}_{>0}$, there exists $N(\varepsilon) \in \mathbb{Z}_+$ such that $|s(k) - s| < \varepsilon$ for all $k \in \mathbb{Z}_{\geq N(\varepsilon)}$. Therefore, fix any $\varepsilon \in \mathbb{R}_{>0}$. Observe that the boundedness of each $\lambda(i) \geq 0$ implies the existence of an $M \in \mathbb{R}_+$ such that $\lambda(i) \leq M$ for all $i \in \mathbb{Z}_+$. Moreover, $\lim_{i \rightarrow \infty} \lambda(i) = 0$ implies that for all ε there exists an $i^*(\varepsilon) \in \mathbb{Z}_+$ such that $\lambda(i) < m := \frac{\varepsilon}{2}(1 - \rho)$ for all $i > i^*(\varepsilon)$. Consider now the sequence $s(k) := \sum_{i=0}^k \rho^{k-i} \lambda(i)$ and the desired limit $s = 0$. Then, for all $\varepsilon \in \mathbb{R}_{>0}$ and for any $k > i^*(\varepsilon)$ it holds that

$$\begin{aligned} |s(k) - 0| &= \left| \sum_{i=0}^k \rho^{k-i} \lambda(i) \right| \\ &< \left| \sum_{i=0}^{i^*(\varepsilon)} \rho^{k-i} M + \sum_{i=i^*(\varepsilon)+1}^k \rho^{k-i} m \right|. \end{aligned}$$

As $(\rho^{-1} - 1) \sum_{i=0}^{i^*(\varepsilon)} \rho^{-i} = \rho^{-i^*(\varepsilon)-1} - 1$, we further obtain

$$|s(k) - 0| < \left| \frac{\rho^{k-i^*(\varepsilon)} - \rho^{k+1}}{1 - \rho} M + \frac{1 - \rho^{k-i^*(\varepsilon)+1}}{1 - \rho} m \right|.$$

Choosing $N(\varepsilon)$ such that $\frac{\rho^{N(\varepsilon)-i^*(\varepsilon)} - \rho^{N(\varepsilon)+1}}{1 - \rho} M \leq \frac{\varepsilon}{2}$ yields that the series $s(k)$ converges to 0. $\square$

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