On the properties of high temperature plasmas: a convergent collision integral and hybrid-kinetic stability theory

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ON THE PROPERTIES OF HIGH TEMPERATURE PLASMAS:
A CONVERGENT COLLISION INTEGRAL AND HYBRID-KINETIC STABILITY THEORY

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF. DR. P. VAN DER LEEDEN, VOOR EEN COMMISSIE AANGEWEZEN DOOR HET COLLEGE VAN DEKANEN, IN HET OPENBAAR TE VERDEDIGEN OP VRIJDAG 3 JUNI 1977 TE 16.00 UUR

DOOR

JOHANNES PETRUS MONDT

GEBORSEN TE AMSTERDAM
DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOREN

Dr.ir.P.P.J.M.Schart
Prof.dr.R.C.Davidson
"Ah God, we know that art
It long and short our life!
Often enough my analytical labours
Pester both brain and heart,
How hard it is to attain the means
By which one climbs to the fountain head;
Before a poor devil can reach the halfway house,
Like as not he is dead."

(Wagner in Goethe's "Faust")

Dedicated to my dear parents
and to my late grand-father.

Opgedragen aan mijn lieve ouders
en aan mijn overleden grootvader.
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SAMENVATTING

Voor een gematigd verdunnd moleculair gas kunnen de macroscopische transportcoëfficiënten worden berekend uitgaande van de Boltzmann vergelijking.\(^1\)\(^-\)\(^5\) De botsingsintegraal in deze vergelijking is slechts de invloed van binaire botsingen.

In een volledig geïoniseerd gas of plasma is de dracht van de wisselwerkinightspotentiaal dermate groot dat veel deeltjes gelijktijdig interacteren.\(^6\) Hierdoor is een vanuit een hoger standpunt ontwikkelde theorie van botsingen in een plasma noodzakelijk. Een uitgangspunt voor een dergelijke theorie is de zogenaamde Born-Bogolyubov-Green-Kirkwood-Yvon hierarchie, een oneindig stelsel van vergelijkingen welke de één-deeltjes verdelingsfunctie in verband brengt met alle multiple correlaties in de fase ruimte.\(^5\) Verscheidene botsingsintegrallen voor een plasma zijn aan de hand van deze hiërarchie verkregen.\(^6\)\(^-\)\(^8\) In dit proefschrift wordt echter uit de hiërarchie een ruimtelijk uniforme benadering voor de binaire correlatie functie afgeleid, met een expliciete evaluatie van de nauwkeurigheid van het resultaat.\(^9\)

Hierbij is geen gebruik gemaakt van ad hoc afkappingen van de hiërarchie vergelijkingen, of van inconsistentie machtreeks-ontwikkelingen in de plasma parameter. Evenmin zijn vormen van functies gebruikt welke ontleend zijn aan argumenten welke slechts geldig zijn in het geval van thermodynamisch evenwicht. De resulterende uitdrukking voor de vrije energie in thermodynamisch evenwicht blijkt eindig te zijn en in overeenstemming met het resultaat verkregen op grond van evenwichtstheorie. De verkregen botsingsintegraal convergeert volledig en is toegepast ter berekening van het elektrische geleidingsvermogen van ruimtelijk homogene plasmas onder invloed van hoogfrequentie velden.
Behalve het hierboven beschreven proces van verstrooiing door Coulomb interactie zijn er andere processen in plasmas welke aanleiding geven tot transportverschijnselen: golf-deeltjes en golf-golf interacties. Als het plasma slechts zwak turbulent is, hoeft in eerste instantie slechts de invloed van lineaire golf-deeltjes interacties op de ontwikkeling van de één-deeltjes verdelingsfunctie te worden beschouwd. Dit resulteert in een quasi-lineaire beschrijving van (anomaal) transport, welke beschrijving bestaat uit een kinetische vergelijking waarin de diffusietensor afhangt van de groeisnelheid verkregen uit lineaire stabiliteitsanalyse. Derhalve is lineaire stabiliteitsanalyse niet alleen nodig om de tijdschaal te bepalen waarin het plasma uiteenvalt ten gevolge van macro-instabiliteiten; het is evenzeer een noodzakelijke stap in het onderzoek naar anomale transportverschijnselen in zwak turbulente plasmas, veroorzaakt door micro-instabiliteiten.

In het bijzonder kan genoemde analyse dienen ter verklaring van anomal transport gedurende de implosie-fase en de fase welke daar onmiddellijk op volgt in een pinch experiment. In dit geval kan de aandacht niet beperkt blijven tot electrostatische verstoreingen en is juist essentieel een volledig electromagnetische behandeling noodzakelijk, daar de verhouding bèta van kinetische tot magnetische druk in een typisch pinch experiment vrij hoog is.

In deel II van dit proefschrift wordt electromagnetische stabiliteit van botsingsloze hoog-bèta plasmas onderzocht op basis van laagste orde hybride-kinetische theorie (D’Ippolito en Davidson). De frequentie van de verstoreingen kan hierbij van de zelfde orde van grootte zijn.
als de ionencyclotronfrequentie of lagerv. Dit hybride-kinetische model bestaat uit een volledige Vlasov-beschrijving van de ionen en een drift-kinetische vergelijking voor de electronen. Twee eigenwaardevergelijkingen worden afgeleid: één voor verstoringen ten opzichte van een lineaire schroefpinch evenwichtsconfiguratie gerekend door een isotrope verdeling van de ionenergie, en één voor verstoringen ten opzichte van een mogelijkstevige door een anisotropie in de ionenergie verdeling gerekend. Dit hybride-kinetische model bestaat uit een volledige Vlasov-beschrijving van de ionen en een drift-kinetische vergelijking voor de electronen. Twee eigenwaardevergelijkingen worden afgeleid: één voor verstoringen ten opzichte van een lineaire schroefpinch evenwichtsconfiguratie gerekend door een isotrope verdeling van de ionenergie, en één voor verstoringen ten opzichte van een mogelijkstevige door een anisotropie in de ionenergie verdeling gerekend. Deze vergelijkingen tonen duidelijk de invloed van eindige electronentemperatuur, electronkinetische effecten langs het magnetische veld en anisotropie in de ionenergieverdeling aan op electromagnetische stabiliteit. 

In de limiet waarin de electronentemperatuur en de ionenergie-anisotropie naar null gaan, wordt de Vlasov-fluidum vergelijking van Freidberg als speciaal geval teruggevonden. Het blijkt numeriek tot de mogelijkheden te behoren de stabiliteitscriteria en groeisnelheden uit deze eigenwaardevergelijkingen te berekenen, zonder verdere beperkingen. Het behulp van analytische methoden zijn expliciete uitdrukkingen voor de groeisnelheden verkregen voor het geval van langgolvige verstoringen van een scherprandige evenwichtsconfiguratie. De frequenties liggen hierbij in het magnetohydrodynamische gebied en er is gebruik gemaakt van een additionele Eindige Larmor Straal ordening, en voor een ander gebied van parameters, van Freidberg's probeerfunctiemethode.

Tenslotte wordt aangetoond dat voor zeer langgolvige verstoringen in een scherprandige pinch met kleine helische magneetcomponent het hybride-kinetische model moet en kan worden veralgemeend ten einde de invloed op stabiliteit van effecten verband houdend met de eindige resistiviteit in rekening te brengen. Een eigenwaardevergelijking voor dit geval is afgeleid.
In a moderately dilute, molecular gas the macroscopic transport coefficients can be calculated from the Boltzmann equation. 5 The collision integral in this equation represents binary collisions only.

In a fully ionized gas or plasma the range of the inter-particle potential is so large that many particles simultaneously interact. Therefore, a more sophisticated theory of collisions in a plasma is needed. A point of departure for such a theory is the so-called Born-Bogolyubov-Green-Kirkwood-Yvon hierarchy, an infinite set of equations, relating the one-body distribution function to all multiple correlations in phase space. This hierarchy is formally equivalent with the Liouville equation, describing the exact dynamical evolution of a many-particle system. Several collision integrals for a plasma have been obtained from this hierarchy. In the present thesis, however, from this hierarchy a spatially uniform approximation for the binary correlation function is derived, with an explicit evaluation of the accuracy of the result. No use is made of ad hoc truncations of the open set of hierarchy equations, or of inconsistent power series expansions in the plasma parameter. No functional forms are used that are only valid in thermodynamic equilibrium. The resulting expression for the free energy in thermodynamic equilibrium is shown to be finite and in agreement with equilibrium theory. The resulting collision integral converges completely and is used to calculate the high-frequency electrical conductivity in a spatially homogeneous plasma.

In addition to the abovedescribed dissipational process of Coulomb-collisional scattering, there are other processes in plasmas that give rise to transport phenomena: wave-particle and wave-wave interactions. If the plasma is only weakly turbulent, then in leading order only the influence of linear wave-particle interactions on the
development of the one body distribution has to be considered.\textsuperscript{20-22} This results in a quasi-linear description of ("anomalous") transport, consisting of a kinetic equation in which the diffusion tensor depends on the growth rate obtained from linear stability analysis. Therefore, linear stability analysis is not only necessary to predict the time scale at which the plasma undergoes complete disruption due to gross instabilities; it is also a necessary step in the investigation of anomalous transport processes in weakly turbulent plasmas due to micro-instabilities.

In particular, this analysis may serve to explain the anomalous transport phenomena that are observed during the implosion and immediate post-implosion phases of a pinch experiment.\textsuperscript{23} In that case the attention cannot be restricted to electrostatic modes, but it is essential to give a fully electromagnetic treatment, since the ratio $\beta$ of kinetic versus magnetic pressure in a typical theta-pinch experiment is rather high.

In part II of the present thesis electromagnetic stability of collisionless high $\beta$ plasmas is investigated on basis of the lowest-order hybrid-kinetic model of D'Ippolito and Davidson.\textsuperscript{24} The frequency of the perturbations is allowed to be of the order of the ion cyclotron frequency or lower. This hybrid-kinetic model consists of a complete Vlasov description of the ions and a drift-kinetic equation for the electrons. Two eigenvalue equations are derived: one for perturbations around a linear screw-pinch equilibrium configuration characterized by an isotropic ion energy distribution,\textsuperscript{25} and another for the case of a linear theta-pinch equilibrium configuration allowing for an anisotropic ion energy distribution.\textsuperscript{26} In both derivations the density profile is arbitrary. These eigenvalue equations clearly exhibit the influence of finite electron temperature, electron kinetic effects parallel to the magnetic field and ion energy anisotropy on electromagnetic stability. In the limit of zero electron temperature and isotropic ion energy, the Vlasov-fluid eigenvalue equation of Freidberg is recovered.\textsuperscript{27} It seems numerically tractable to obtain the instability thresholds and growth rates from these eigenvalue equations, without further restrictions.\textsuperscript{25-27} Analytically, explicit expressions for the growth rates are obtained for the case of long-wavelength perturbations.
around a sharp-boundary equilibrium configuration. The frequencies are in the magnetohydrodynamic range and use is made of an additional Finite-Larmor-Radius ordering and, for another parameter region, of Preidberg's trial function method.

Finally, it is shown that for extremely long wavelength perturbations in a near-theta-pinch plasma with sharp boundaries, the hybrid-kinetic model should and can be generalized to include finite resistivity effects on stability. An eigenvalue equation for this case is derived.
List of References of Summary


4. Cf. Ref. 20 of Part I.


10-18. Cf. Refs. 9, 11-17 and 19 of Part I.


23. Cf. Ref. 8 of Part II.

24. Cf. Ref. 35 of Part II.


27. Cf. Ref. 1 of Part II.

28. Cf. Ref. 45 of Part II.


30. Cf. Ref. 33 of Part II.
DERIVATION FROM THE B.B.G.K.Y.-HIERARCHY

OF A COMPLETELY CONVERGENT COLLISION INTEGRAL

FOR A FULLY IONIZED PLASMA

(through an evaluation of the binary correlation function, with application to the free energy and to the high-frequency electrical conductivity)
CHAPTER I: INTRODUCTION

1.1. Basic equations.

Considerations are restricted to a homogeneous stable classical electron plasma in the electrostatic approximation, moving in a continuous homogeneous background of immobile positively charged ions. The system as a whole is assumed to be electrically neutral and infinitely large. Therefore its Hamiltonian is

\[ H_N = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i\neq j}^{N} \frac{Q_i Q_j}{r_{ij}} \]  \hspace{1cm} (1)

\( N \) being the total number of particles and \( Q \) is the Coulomb energy. Several methods \(^{1-7}\) for investigation of transport properties of this system have been constructed starting from the Liouville equation for the distribution function \( D_N \) of the dynamical states of the whole system:

\[ \frac{\partial D_N}{\partial t} = \{ H_N ; D_N \} \]  \hspace{1cm} (2)

where \( \{ ; \} \) denotes the familiar Poisson bracket.

Perhaps the most systematic method \(^{1-5}\) is based upon a set of equations for the so-called reduced distribution functions

\[ F_s = V^s/d\xi^s d\chi^s d\nu^s d\pi^s D^s \]  \hspace{1cm} (3)

\( V \) denoting the system's volume in configuration space. These equations directly result from Liouville's theorem [Eq. (2)] under the usual symmetry condition for \( D_N \). In the thermodynamic limit \( N \to \infty, V \to \infty, N/V = n \) constant this set of equations, known as the B.B.G.K.Y.-hierarchy, reads

\[ \frac{D(s) F_s}{D_t} = I_s F_s + L_s F_{s+1} \]  \hspace{1cm} (4)
where

$$D(s)/Dt = \partial/\partial t + \sum_{i=1}^{s} \partial/\partial x_i^s$$

is the time-derivative taken along the free s-particle trajectories;

$$I_s = m^{-1} \sum_{i<j}^{s} \left( \partial/\partial x_i^s - \partial/\partial y_i^s \right) \ (\partial^2/\partial x^2_{ij}$$

is the operator representing the direct interactions between s particles;

$$L_s = \frac{n}{m} \sum_{i=1}^{s} \frac{\partial}{\partial x_i^s} \int \frac{d\phi_{s+1}^s d\phi_{s+1}^s}{\partial x_i^s} \cdot \frac{\partial}{\partial y_i^s}$$

is the "phase-mixing" operator representing the deviation of the evolution of $F_s$ from that of $D_s$, caused by the interactions with the remainder of the system.

Several authors 1,8,9,15 solved this hierarchy by expansion of $F_s$ into a power series in the plasma parameter $\varepsilon$.

Depending on which terms in the hierarchy equations were considered small perturbations, the results were inadequate for small 1,8,9,15 or large interparticle distances, giving cause to short- and long-range divergences in the collision integral, respectively. 10

The aim of the present investigation is to develop a perturbation approach to the B.B.G.K.Y. hierarchy that takes into account the spatial inhomogeneity of the ratio of magnitudes of the terms $I_s F_s$ and $L_s F_{s+1}$ in Eq. (4). Besides, a separate assumption about the order of magnitude of the irreducible correlations, as made in one of the approaches 7,9,15 should be avoidable in any systematic perturbation theory.

Furthermore, any modification of the potential function or pre-assumed
form of the two-body distribution function, as made in Refs. (11,12) is avoided in the present work.

Finally, no matching procedure, as put forward either phenomenologically 14, or based on the form of the hierarchy equations in thermodynamic equilibrium, 13,17,18,27, is needed to obtain a completely convergent collision integral. In contrast with, e.g., the result of Ref. (16), the occurrence of a logarithmic dependence on the plasma parameter does not reveal an inconsistency in the method, and unlike the result of Ref. (19), the obtained collision integral is explicit.

1.2. Outline of the method

As functions of $\chi^s_i(s) \equiv (\chi^s_1, \ldots, \chi^s_s)$ the operators $I_s$ and $L_s$ have the scaling properties [Eqs. (6), (7)]

$$I_s(\lambda \chi^s) = \lambda^{-2} I_s(\chi^s),$$

$$L_s(\lambda \chi^s) = \lambda^1 L_s(\chi^s).$$

These properties give rise to a scale dependent ordering $^{10,20}$ in the nondimensionalized hierarchy, depending on the unit of length chosen, the unit of velocity being invariantly $(2E/mN)^{1/2}$, where $E$ is the total kinetic energy of the system.

The smallest length unit is the Landau-length $r_L = e^2/T$, i.e. the characteristic length of the "collision region" in space of interparticle coordinates where direct interactions dominate.

The second unit of length considered is the Debye-length

$$r_D = \left(\frac{T}{4\pi Ne^2}\right)^{1/2},$$

(9)
i.e., the characteristic length for processes related to the interaction with the remainder of the system. In the case of a plasma the Debye-length is much greater than the Landau-length, i.e.

$$\epsilon = \frac{r_L}{r_D} \ll 1,$$  \hspace{1cm} (10)

where $\epsilon$ is called the plasma parameter.

At the scale of the mean free path for strong collisions, $r_{mfp} = \epsilon^{-1}r_D$, the ordering of terms as suggested by the corresponding nondimensional B.E.G.K.Y. hierarchy leaves some doubt for the possibility of truncation at such large interparticle distances. The problem of how to give a good description of this region is beyond the scope of the present work which only aims to obtain a spatially uniform solution of the hierarchy equations for interparticle distances much smaller than $r_{m.f.p.}$. 

The dimensionless hierarchy corresponding with $r_L$ as unit of length reads:

$$D^{(s)}\frac{F_s}{Dt} = I_s F_s + \epsilon L_s F_{s+1},$$  \hspace{1cm} (11)

and the one corresponding with $r_D$ as unit of length reads:

$$D^{(s)}\frac{\gamma^{(s)}}{Dt} = \epsilon L^{(s)} \frac{\gamma^{(s)}}{s+1},$$  \hspace{1cm} (12)

with

$$\gamma^{(s)}(x^{(s)}; \tau; y^{(s)}; \epsilon) = F^{(s)}_{s}(x^{(s)}; t; y^{(s)}; \epsilon),$$

where $x^{(s)} = \epsilon x^{(s)}$; $\tau = \epsilon t$. 

Eqs. (11) and (12) then serve as a starting point for the perturbation analysis. They are supplemented by the well-known initial condition of Sandri:\textsuperscript{8,22}

\[ F_s(\varepsilon=0) = \prod_{i=1}^{s} F_i(\varepsilon) \]  

(13)

related to the point of view that the relevant part of the correlations is caused by interactions. The well-known adiabatic assumption\textsuperscript{1,20} will be adopted, i.e., the correlations reach their time-asymptotic form in a time very short compared with the relaxation time of $F_1$. Putting $\varepsilon = 0$ in Eq. (11) yields approximations $g_2^{(0)}$ and $g_3^{(0)}$ for the binary and ternary correlations for interparticle distances not exceeding $r_L$ by order of magnitude.

The remainder of interparticle space can be divided into three regions which partly overlap:

Region $R_1(1,2)$: A cylinder parallel to $\mathcal{H}_{12}$ with the origin on its axis and with radius $\lambda$ equal to

\[ \lambda = \varepsilon^{-R_1} r_L, \quad 0 < R_1 < 1. \]  

Region $R_2(1,2)$: The complement of the union of $R_1$ and the sphere around the origin with radius $\mu$ equal to:

\[ \mu = \varepsilon^{-R_2} r_L, \quad 0 < R_1 < R_2 < 1. \]  

Region $R_3(1,2)$: Sphere around the origin with radius $\mu$ as given above.
The radii $\lambda$ and $\mu$ shall be determined in the sequel such as to make the approximation for the binary correlation optimal, i.e. the accuracy of the resulting collision integral as high as possible.
CHAPTER II: SOME NONUNIFORMLY VALID APPROXIMATIONS

In deriving a spatially uniform approximation for the binary correlation function as part of a solution of the B.B.G.K.Y. hierarchy equations, use shall be made of expressions for and properties of some nonuniformly valid approximations to be discussed in this chapter.

II.1 The binary correlation function in accordance with the Boltzmann theory.

On the scale of the Landau-length, \( r_L \equiv \frac{e^2}{\lambda} \), and in lowest order in \( \epsilon \), the asymptotic two-body distribution function \( F^{(0)}_{2\alpha} \) is the solution for \( \epsilon \to \infty \) of [Eq. (11)]

\[
\frac{2}{3t} + n_{12} \cdot \frac{3}{3x_{12}} - \frac{1}{m} \phi \langle x_{12} \rangle \cdot \left( \frac{2}{3x_{11}} - \frac{3}{3x_{12}} \right) F^{(0)}_{2\alpha} = 0,
\]

with [Eq. (13)] \( F^{(0)}_{2\alpha}(t = 0) = F(x_{11})F(x_{12}) \) as initial condition.

The characteristics of Eq. (16) are the trajectories in a two-particle system. The initial velocities of the trajectories are determined by the conservation laws, i.e., those of energy, momentum, angular momentum and an additional vector quantity, being conserved only in the case of Coulomb interaction.

(a) Conservation of energy yields:

\[
\frac{1}{2} \nu \left( \frac{e^2}{\lambda} \right) = \text{constant}, \quad \text{say } E,
\]

with \( \nu = m/2 \) the reduced mass, \( e^2/\lambda = \phi \) is the Coulomb energy,

\[
\chi_{12} = \chi_1 - \chi_2, \quad \chi_{12} = \chi_1 - \chi_2.
\]
(b) Conservation of momentum implies that the mean velocity

\( \frac{1}{2} (\chi_1 + \chi_2) \) is conserved.

(c) Conservation of angular momentum implies that

\[ \mathbf{L} = \mu \chi_{12} \times \chi_{12} \]

is conserved. \( (18) \)

(d) The additional conserved quantity is the vector defined by

\[ \mathbf{\zeta} = \chi_{12} \times \mathbf{k} + e \frac{\mathbf{\hat{z}}_{12}}{\chi_{12}} \]

(19)

In order to make use of these conservation laws we put Eq. (16) in the following form:

\[ \frac{d}{dt} F_2^{(0)} \left[ \chi_{120}(\tau), \chi_{10}(\tau), \chi_{20}(\tau), \tau \right] = 0, \]  

(20)

with:

\[ \frac{d}{dt} \chi_{120}(\tau) = \chi_{10}(\tau) - \chi_{20}(\tau), \]

\[ \frac{d}{dt} [\chi_{10}(\tau) - \chi_{20}(\tau)] = - \frac{1}{m} \Phi \left[ \chi_{120}(\tau) \right]. \]  

(21)

Let \( \chi_{120}(\tau), \chi_{10}(\tau), \chi_{20}(\tau) \) form the solution of Eq. (21) subjected to the boundary conditions:

\[ \chi_{120}(\tau = t) = \chi_{12}; \chi_{10}(\tau = t) = \chi_1; \chi_{20}(\tau = t) = \chi_2. \]  

(22)

Then Eq. (20) implies that:

\[ F_2^{(0)} (\chi_{12}, \chi_1, \chi_2, \tau) = F_2^{(0)} [\chi_{120}(0), \chi_{10}(0), \chi_{20}(0), 0]. \]  

(23)

In view of the molecular chaos condition [Eq. (13)] we get:
In general:

\[ \chi_{i_0}(0) = \chi_{i_0}(\xi_{12}; \chi_1, \chi_2; t) \]  

(25)

since \( \chi_{i_0}(0) \) is determined by tracing back the trajectories along the characteristics given by Eq. (21).

In particular it follows from Eq. (24) and Eq. (25) that

\[ \lim_{t \to \infty} F_2(0, \xi_{12}; t, \chi_1, \chi_2) = F_1(\chi_{i_0}(\xi_{12}; \chi_1, \chi_2; t \to \infty)) \]

(26)

or, using momentum conservation:

\[ F_2(0, \xi_{12}; \chi_1, \chi_2; \to \infty) = F_1(\chi_{i_0}(\xi_{12}; \chi_1, \chi_2; \infty)) \]

(27)

Remains the problem of finding \( v_{12}^{(\infty)} \equiv \chi_{12_0}^{(\infty)}(\xi_{12}; \chi_1, \chi_2; \infty) \), i.e. the relative velocity of the particles in the extreme past.

Energy conservation [Eq. (17)] yields:

\[ \frac{1}{2} \mu v_{12}^2 + \frac{e^2}{\epsilon_{12}} = \frac{1}{2} \mu v_{12}^{(\infty)} \]

with \( \mu = \frac{m}{2} \).
so that \[ |\psi_{20}|^2 = (v_{12}^2 + \frac{4e^2}{m_1 x_{12}})^{\frac{1}{2}}. \] (28)

By means of (28) we can write (24) as:

\[
F(0) = F_1 \left[ \frac{X_1 + X_2}{2} + \frac{\sqrt{V_{12}^2 + \frac{4e^2}{m_1 x_{12}}}}{x_{12}} \hat{\xi} \right] F_1 \left[ \frac{X_1 + X_2}{2} - \frac{\sqrt{V_{12}^2 + \frac{4e^2}{m_1 x_{12}}}}{x_{12}} \hat{\xi} \right],
\] (29)

where \( \hat{\xi}(X_{12}; X_1, X_2) \) is the unit vector in the direction of \( X_{12} \).

This vector is determined in Appendix A, by making use of conservation of angular momentum and of \( \mathcal{C} \) [Eqs. (18) and (19)]. Thereby the asymptotic two-body distribution function is completely determined.

Making use of Eq. (A.11) for \( \mathcal{C} \) and Eq. (29) for \( F_{2A}^{(0)} \) we obtain for the asymptotic binary correlation \( \mathcal{C}_{2A}^{(0)} = F_{2A}^{(0)} - F_1(X_1)F_1(X_2) \) the following form:

\[
\mathcal{C}_{2A}^{(0)} = \prod_{\pm} F_1 \left[ \frac{X_1 + X_2}{2} \pm \frac{1}{2} \left( \sqrt{V_{12}^2 + \frac{4e^2}{m_1 x_{12}}} \right) \hat{\xi} \right] - F_1(X_1)F_1(X_2),
\] (30)

where

\[
\hat{\xi} = -e^2 \frac{\mathcal{C}}{C^2} + \sqrt{V_{12}^2 + \frac{4e^2}{m_1 x_{12}}} \mathcal{L} \left[ \frac{\hat{\xi} \times \mathcal{C}}{|\hat{\xi} \times \mathcal{C}|} \right]^2,
\] (31)

\[
\mathcal{C} = X_{12} \times \hat{\xi} + e^2 \frac{X_{12}}{x_{12}}, \quad \hat{\xi} = \frac{m}{2} \hat{X}_{12} \times \hat{X}_{12},
\]

and where the product \( \prod_{\pm} \) is over the plus and minus sign in front of the square root.
The approximation (30) for the binary correlation function breaks down on the scale of the Debye-length and can be shown to lead to Boltzmann's collision integral \( I \), when substituted into the first B.B.G.K.Y. hierarchy equation. Accordingly, we refer to the solutions \( g_{2A}^{(0)} \) and \( g_{3A}^{(0)} \) of Eq. (11) to lowest order in \( \epsilon \) as the "Boltzmann correlations".

In the next section, series representations for \( g_{2A}^{(0)} \) and \( g_{3A}^{(0)} \) are derived in order to determine how these functions behave for large impact parameters.

II.2 The straight path approximations for the Boltzmann correlations.

Consider Eq. (11) in lowest order in \( \epsilon \), for \( s = 2 \) and for \( s = 3 \). The "straight path approximation" consists of considering the force convective part \( L_s \) \( g_s \) to be smaller than the source term \( L_s \) \( \int_1^{\infty} F_1(\chi) \), yielding in leading order and for \( s = 2 \):

\[
\chi_{12} \cdot \frac{a}{\partial \chi_{12}} \Delta_A^{(0)} (\chi_{12}; \chi_1, \chi_2) = I_2(\chi_{12}; \chi_1, \chi_2) F_1(\chi_1) F_1(\chi_2),
\]

(32)

where \( \Delta_A^{(0)} (\chi_{12}; \chi_1, \chi_2) \) is the leading order straight path approximation of \( g_A^{(0)} \); \( \Delta_A^{(0)} \) yields, when substituted into the first hierarchy equation, the "Fokker-Planck" or "Landau" collision integral and will therefore henceforth be referred to as the Landau correlation (cf. II.2 for an explicit expression).

In \( n \) th order, by iteration, we get corrections \( \Delta_A^{(n)} \) defined by:

\[
\chi_{12} \cdot \frac{a}{\partial \chi_{12}} \Delta_A^{(n)} (\chi_{12}; \chi_1, \chi_2) = I_2(\chi_{12}; \chi_1, \chi_2) \Delta_A^{(n-1)} (\chi_{12}; \chi_1, \chi_2),
\]

(33)
such that

\[ s_{2A}^{(0)}(\xi_{12}; \xi_1, \xi_2) = \sum_{n=0}^{\infty} s_{A}^{(n)}(\xi_{12}; \xi_1, \xi_2). \tag{34} \]

As will be shown in appendix A, the function \( s_{A}^{(0)}(\xi_{12}) \) can be obtained by a Taylor expansion of \( g(0) \) with respect to the projection \( \xi_{12} \) of \( \xi_{12} \) perpendicular to \( \xi_{12} \). Eq. (34) can therefore be expected to be valid outside a cylinder with \( \xi_{12} \) as axis, the axis going through the origin.

Since \( g(0) \) is completely independent of \( \xi_{12} \), the radius of this cylinder must be of order unity.

By means of scaling property (8) for the operator \( I_2 \) it is readily verified that

\[ s_{A}^{(n)}(\xi_{12}; \xi_1, \xi_2) = \lambda^{-1-n}s_{A}^{(n)}(\xi_{12}; \xi_1, \xi_2), \tag{35} \]

for any \( \lambda \neq 0 \).

In an analogous way, starting from the equation for \( s_{3A}^{(0)} \) [Eq. (11) for \( s = 3, \epsilon = 0 \)], i.e.,

\[ \frac{D^{(3)}}{Dt} s_{3A}^{(0)}(\xi_{13}, \xi_{23}; \xi_1, \xi_2, \xi_3) = \]

\[ \sum_{p} I_2(\xi_{ij}, \xi_i, \xi_j) [ f_{1}(\xi_i) s_{A}^{(0)}(\xi_k; \xi, \xi_j) + f_{1}(\xi_j) s_{A}^{(0)}(\xi_k; \xi, \xi_i) ] \]

\[ + I_3(\xi_{13}, \xi_{23}; \xi_1, \xi_2, \xi_3) s_{3A}^{(0)}(\xi_{13}, \xi_{23}; \xi_1, \xi_2, \xi_3), \tag{36} \]
the summation \( \sum \) being over all permutations of \( \{1, 2, 3\} \), iterative straight path approximations are now made by means of the function sequence \( \{D_A(n)\}_{n=0}^{\infty} \) defined by:

\[
\frac{D_A(3)}{D_A(2)} D_A(0)(\chi_{13}, \chi_{23}; \chi_1, \chi_2, \chi_3) =
\]

\[
\sum_{P} I_2(\chi_{ij}; \chi_i, \chi_j) \left[ F_1(\chi_i) D_A(0)(\chi_{jk}; \chi_j, \chi_k) +
\right.
\]

\[
+ F_1(\chi_j) D_A(0)(\chi_{ik}; \chi_i, \chi_k) \right],
\]

yielding:

\[
D_A(0) = \sum_{o} D_A(n)(\chi_{13}, \chi_{23}; \chi_1, \chi_2, \chi_3).
\]

If \( x_{12} \gg 1 \), then at least two particles are far apart so that \( 1_3 e_3 \) in Eq. (36) can be considered smaller than

\[
\sum_{P} I_2(\chi_{ij}) \left[ F_1(i) g_{2A}^{(0)}(j,k) + F_1(j) g_{2A}^{(0)}(i,k) \right].
\]

If, moreover, \( |\chi_{13}| > 1, |\chi_{12}| > 1 \) and \( |\chi_{23}| > 1 \), the aforementioned
straight path approximation for $\xi^{(0)}_{ij}(i,j)$ holds for every $i$ and $j$.

Series (39) can therefore be expected to hold, provided

$$x_{12} \gg 1 \cap |\xi_{12} | > 1 \cap |\xi_{13} | > 1 < |\xi_{23} |.$$  

Making use of scaling property (8) and of Eq. (38) it is readily verified that

$$D^{(n)}_{\lambda} (\lambda x_{13}, \lambda x_{23}; x_1, x_2, x_3) = \lambda^{-2-n} D^{(n)}_{\lambda} (\xi_{13}, \xi_{23}; x_1, x_2, x_3),$$  

(40)

for any $\lambda \neq 0$, and for $n \geq 0$.

From series expansions (34), (39) and scaling laws (35), (40) it follows that the binary Boltzmann correlation falls off inversely proportional with the impact parameter $\xi_{12}$ whereas the ternary Boltzmann correlation falls off quadratically when all impact parameters increase indefinitely, a difference important for the justification of truncating the B.B.C.K.Y. hierarchy equations, as will be discussed later.

II.2c The Landau correlation function.

The asymptotic Landau correlation $\Delta^{(0)}_A (\xi_{12}, t; x_1, x_2)$ can be determined from equation (32), i.e.

$$\langle \frac{\partial}{\partial x_{12}} + \frac{\partial}{\partial x_{12}} \rangle \Delta^{(0)}_A (\xi_{12}, t; x_1, x_2) = \frac{1}{m} \nabla \phi (\xi_{12}) \cdot \left( \frac{\partial}{\partial x_1} \right) f_1 (x_1) f_1 (x_2).$$  

(41)
supplemented by the initial condition [Eq. (13)]
\[ \delta^{(0)}(t = 0) = 0. \] (42)

By integration along the (straight) characteristics we find:
\[ \Delta^{(0)}(x_{12}, t; x_1, x_2) = \frac{1}{m} \int ds \psi(x_{12} - x_{12}^0) \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) F_1(x_1) F_1(x_2) \]
and in particular:
\[ \Delta^{(0)}_A(x_{12}; x_1, x_2) = \frac{1}{m} \int ds \psi(x_{12} - x_{12}^0) \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) F_1(x_1) F_1(x_2) = \]
\[ \frac{a^2}{m} k(x_{12}; x_{12}) \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) F_1(x_1) F_1(x_2), \] (44)
the last equality defining \( k \).

In order to evaluate \( k \), we choose a Cartesian frame with \( x_{12} \)-axis along \( x_{12} \). Then
\[ k = v_{12} \int_0^\infty ds \left( s - \frac{x_{12}}{v_{12}} \right) \left( v^2_{12} \left( s - \frac{x_{12}}{v_{12}} \right)^2 + \rho_{12}^2 \right)^{-3/2} g_x - g_{12} \int_0^\infty ds \left( s - \frac{x_{12}}{v_{12}} \right) \left( v^2_{12} \left( s - \frac{x_{12}}{v_{12}} \right)^2 + \rho_{12}^2 \right)^{-3/2} g_z, \] (45)
where \( g_x, g_y \) and \( g_z \) are unit vectors along the Cartesian axes.

The integration is readily performed to yield
\[ k = \frac{1}{v_{12}^2} \frac{x_{12}^2}{|x_{12}|} \left( 1 + \frac{x_{12}^2}{|x_{12}|^2} \right), \] (46)
where \( x_{12}^2 \) is the projection of \( x_{12} \) in a plane perpendicular to \( x_{12} \)
and \( x_{12} \) the component of \( x_{12} \) parallel to \( x_{12} \).

Eqs. (44) and (46) completely determine \( \Delta^{(0)}_A \). Notice that the Landau-
correlation is singular around zero impact parameter (i.e., for \( |\chi_{12} | + 0 \)), except if \( \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_2} F_1(\chi_1) F_1(\chi_2) \) only has a component along the \( \chi_{12}- \) direction, as is the case in thermodynamic equilibrium.

II,3. The binary correlation function in accordance with the Balescu-Guerney-Lenard theory.

In order to construct a collision integral describing particle interactions at the scale of the Debye-length, Balescu, Guernsey, and Lenard formulated approaches equivalent with adopting ordering (12) with the supplementary assumption that the irreducible s-particle correlation is of order \( \xi^{s-1} \) times the uncorrelated part \( \prod_{i=1}^{s} P_1(\chi_i) \) of the s-body distribution \( F_s \), for all \( s > 1 \).

This leads to the following equation for the binary correlation function, first put forward by Bogolyubov:

\[
\frac{D g_{2}(1,2)}{Dt} = L_2 F_1(\chi_1) F_1(\chi_2) + L_1^{(1)} [g_2(2,3) F_1(\chi_1)] + \\
+ L_1^{(2)} [g_2(1,3) F_1(\chi_2)] ,
\]

where the operators \( L_1^{(1)} \) and \( L_1^{(2)} \) are defined by

\[
L_1^{(j)} = \frac{n}{m} \int d^3x_3 \int d^3v_3 \, \frac{\partial}{\partial v_j} \left( \frac{\partial}{\partial \chi_j} \right)
\]

and where we wrote \( g_2(i,j) = g_2(\chi_{ij}, \chi_i, \chi_j, t) \) as a shorthand. Although the collision integral can be obtained without explicitly solving for
The asymptotic binary correlation function $g_2$ determined by Eq. (47) and initial condition (13) has been evaluated in Fourier space,

$$g_2(k; \chi_1, \chi_2) \equiv \frac{1}{8\pi^3} \int d^3x \exp(-ik \cdot x) g_B(k; \chi_1, \chi_2). \tag{49}$$

We only give here the final result:

$$
\begin{align*}
g_2(k; \chi_1, \chi_2) &= \lim_{\delta \to 0} \left( \frac{u_2^2}{u_1 - u_2 - k \delta} \right) \frac{1}{8\pi^3 n} \\
&= \left[ F_1(\chi_2) \frac{3}{3u_1} F_1(\chi_1) - F_1(\chi_1) \frac{3}{3u_2} F_1(\chi_2) \right] + \\
&+ \left[ \frac{\partial F_1(\chi_2)}{\partial u_1} \left( \frac{3}{3u_2} F_1(\chi_2) \right) - \frac{\partial F_1(\chi_1)}{\partial u_2} \left( \frac{3}{3u_1} F_1(\chi_1) \right) \right] - \\
&- \left[ \frac{\partial F_1(\chi_2)}{\partial u_2} \left( \frac{3}{3u_2} F_1(\chi_2) \right) - \frac{\partial F_1(\chi_1)}{\partial u_2} \left( \frac{3}{3u_1} F_1(\chi_1) \right) \right] + \\
&- \left[ \frac{\partial F_1(\chi_2)}{\partial u_1} \left( \frac{3}{3u_2} F_1(\chi_2) \right) - \frac{\partial F_1(\chi_1)}{\partial u_1} \left( \frac{3}{3u_1} F_1(\chi_1) \right) \right] - \\
&- \left[ \frac{\partial F_1(\chi_2)}{\partial u_2} \left( \frac{3}{3u_2} F_1(\chi_2) \right) - \frac{\partial F_1(\chi_1)}{\partial u_2} \left( \frac{3}{3u_1} F_1(\chi_1) \right) \right] - \\
&- \left[ \frac{\partial F_1(\chi_2)}{\partial u_1} \left( \frac{3}{3u_2} F_1(\chi_2) \right) - \frac{\partial F_1(\chi_1)}{\partial u_1} \left( \frac{3}{3u_1} F_1(\chi_1) \right) \right] - \\
&- \left[ \frac{\partial F_1(\chi_2)}{\partial u_2} \left( \frac{3}{3u_2} F_1(\chi_2) \right) - \frac{\partial F_1(\chi_1)}{\partial u_2} \left( \frac{3}{3u_1} F_1(\chi_1) \right) \right]. \tag{50}
\end{align*}
$$
where

\[ u_i = \frac{k \cdot \chi_i}{k}, \quad i = 1, 2, \]

\[ \frac{2}{p} = \omega \frac{k - 2}{k}, \quad \omega = \frac{4 \pi m e^2}{m}, \]

\[ \tilde{F}_1(u) = \int d^3v \ F_1(v) \ \delta\left(\frac{k}{k} \cdot \chi - u\right), \]

and \( Z^\pm(u) = 1 + 2\pi i u \frac{\partial}{\partial u} \frac{\partial F^\pm}{\partial u}(u) \)

The reader is reminded of the general definitions

\[ \psi^+(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(u')}{u' - u} \, du', \quad \psi^-(u) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(u')}{u' - u} \, du'. \]

For small impact parameter \( (\chi_{12} \to 0) \) the relative difference between the Landau correlation \( \Delta_A^{(0)} \) and \( \Delta_B \) tends to zero, as shown in Appendix C, i.e.,

\[ \Delta_B(\chi; \chi_1, \chi_2) - \Delta_A^{(0)}(\chi; \chi_1, \chi_2) \]

\[ \Delta_A^{(0)}(\chi; \chi_1, \chi_2) = 0(\chi_1), \quad \chi_1 \to 0. \]

where \( \chi_1 = \chi - \frac{\chi \cdot \chi_{12}}{\chi_{12}^2} \chi_{12} \) as before.
In this chapter we shall derive an explicit expression for the asymptotic binary correlation, as part of a spatially uniformly valid solution of the B.B.C.K.Y. hierarchy equations [Eq. (4)]. To this end we distinguish three regions which partly overlap as discussed in Ch. I.

### III.1. The region $R_2(1,2)$ of large impact parameters.

Due to screening the asymptotic binary and ternary correlations $g_{2A}$ and $g_{3A}$ depart from their zero order approximations $g_{2A}^{(0)}$ and $g_{3A}^{(0)}$ as soon as the relevant impact parameters become comparable with the Debye-length.

In order to calculate this departure we define correction factors $g_{2A}^{(1)}$ and $g_{3A}^{(1)}$ by

\[ g_{2A}(x; x_1, x_2) = g_{2A}^{(0)}(x; x_1, x_2) + g_{2A}^{(1)}(x; x_1, x_2), \]

\[ g_{3A}(x_1, x_2; x_3) = g_{3A}^{(0)}(x_1, x_2; x_3) + g_{3A}^{(1)}(x_1, x_2; x_3), \]

and notice that the series expansions [Eqs. (34) and (39)] for $g_{2A}^{(0)}$ and $g_{3A}^{(0)}$ obtained by successive straight path approximations are valid within $R_2(1,2)$ and the complement of the union of $R_1(1,3)$ and $R_1(2,3)$, respectively. In view of this limited validity of the straight path approximation, the operators $L_1^{(1)}$, $L_1^{(2)}$, and $L_2$ [Eqs. (8) and (48)] are decomposed into two terms.
\[ L_1^{(j)} = L_1^{(j)*} + L_1^{(j)^*}, \]  
\[ L_2 = L_2^* + L_2^{(j)*}, \]  
\[ (56) \]

such that the \(^*\)-operators involve integration over the regions where the straight path approximation yields a convergent series representation, so that substitution of series (34) and (39) is valid in the integral terms with asterisk, but not in those with dagger index.

Since the Debye length is a typical unit of length within \( R_2^{(1,2)} \) the ordering of terms in the second hierarchy equation is given by Eq. (12).

Making use of Eq. (12), series expansions (34) and (39), scaling laws (35) and (40) and of decomposition (56), we obtain the following equation for the correction \( g_{2A}^{(1)} \) to the asymptotic binary correlation function:

\[ \chi_{12} \cdot \frac{\partial}{\partial \chi_{12}} \left[ g_{2A}^{(1)} \left( \chi_{12}, \chi_{21}, \chi_{22} \right) \right] = \sum_{n=0}^{1+n} \Delta_{A}^{(n)} \left( \chi_{12}^{*}, \chi_{12}^{n}, \chi_{21}, \chi_{22} \right) \]

\[ = \epsilon^{2} I_{2}^{(1)} \left( \chi_{12}, \chi_{21}, \chi_{22} \right) \left( F_{1}^{(1)} \chi_{12} \right) + \sum_{n=0}^{1+n} \Delta_{A}^{(n)} \left( \chi_{12}^{*}, \chi_{12}^{n}, \chi_{21}, \chi_{22} \right) \]

\[ g_{2A}^{(1)} \left( \chi_{12}^{*}, \chi_{12}^{n}, \chi_{21}, \chi_{22} \right) \times \sum_{n=0}^{1+n} \Delta_{A}^{(n)} \left( \chi_{12}^{*}, \chi_{12}^{n}, \chi_{21}, \chi_{22} \right) \]

\[ \epsilon^{2} I_{2}^{(1)} \left( \chi_{12}, \chi_{21}, \chi_{22} \right) \left( F_{1}^{(1)} \chi_{12} \right) + (1 \rightarrow 2) + \]

\[ \epsilon^{1+n} \Delta_{A}^{(1)} \left( \chi_{12}^{*}, \chi_{21}, \chi_{22}, \chi_{31}, \chi_{32}, \chi_{33} \right) \left( F_{1}^{(1)} \chi_{12} \right) + (1 \rightarrow 2) + \]

\[ \epsilon^{2+n} \Delta_{A}^{(1)} \left( \chi_{12}^{*}, \chi_{21}, \chi_{22}, \chi_{31}, \chi_{32}, \chi_{33} \right) \left( F_{1}^{(1)} \chi_{12} \right) + (1 \rightarrow 2) + \]
where the symbol \((1 \leftrightarrow 2)\) denotes the preceding term with particle indices 1 and 2 interchanged.

Next we investigate to which extent truncation of the B.B.G.K.Y. hierarchy is possible by neglect of the terms in Eq. (57) involving the ternary correlation function.

We first consider the expression

\[
+ c L^+_2 \mathfrak{g}^{(0)}_3 A (\xi_{13}/\epsilon, \xi_{23}/\epsilon; \chi_1, \chi_2, \chi_3) \mathfrak{g}^{(1)}_3 A (\xi_{13}/\epsilon, \xi_{23}/\epsilon, \epsilon; \chi_1, \chi_2, \chi_3)
\]

(57)

which has a leading term

\[
L = \epsilon^3 \int d^3 \nu_3 \int d^3 \xi \{ \nabla \phi (\zeta_{12} + \xi) \cdot \nabla + \nabla \phi (\xi) \cdot \nabla \} \mathfrak{D}^{(0)}_3 A (\xi_{12} + \xi, \xi; \chi_1, \chi_2, \chi_3)
\]

\[
\mathfrak{g}^{(1)}_3 A (\xi_{12} + \xi, \xi; \epsilon; \chi_1, \chi_2, \chi_3)
\]

(58)

where the prime in \( \int d^3 \xi \) indicates that integration is restricted to

\[
\{ \xi; \hat{\zeta}_1 \sim |1-R|; |\hat{\zeta}_1 + \hat{\zeta}_2| > |1-R| \}.
\]

Making use of a simple transformation of integration variables, we get

\[
|L| \sim \epsilon^{2+R} \int d^3 \nu_3 \int d^3 \xi \{ \nabla \phi (\zeta + \xi) \cdot \nabla + \nabla \phi (\xi) \cdot \nabla \} \mathfrak{D}^{(0)}_3 A (\zeta + \xi, \xi; \chi_1, \chi_2, \chi_3)
\]

\[
\mathfrak{g}^{(1)}_3 A (\zeta + \xi, \xi; \epsilon; \chi_1, \chi_2, \chi_3)
\]

(59)
where now the prime denotes restriction of the integration to the interval \( \{\xi_\perp > 1 \cap \xi_\perp + \xi_\perp > 1\} \), i.e. to an interval where \( D_\perp^{(0)} \) is at most of order unity. As can be trivially checked, the orders of magnitude of the \( v_1 \) - and \( v_2 \) -terms in Eq. (60) are identical.

Let us consider the \( v_1 \) -term. Since

\[
D_\perp^{(0)}(\xi_\perp, \xi_\perp; \chi_1, \chi_2, \chi_3) = 0(\xi_\perp^{-1}) \text{ as } \zeta_\perp \to \infty \text{ (cf. Appendix A)},
\]

we arrive at the estimate:

\[
|L| \leq 2^{2 R_1} \int_0^\infty d\xi \zeta_\perp \varphi(\xi_\perp) \left| g_3^{(1)} \right| \left/ \zeta_\perp \right| \leq 2^{2 R_1} \int_0^\infty d\xi \int_\rho(\rho^2 + z^2)^{-1} d\rho \left| g_3^{(1)} \right| .
\]

The factor \( g_3^{(1)} \) represents the decrease of \( g_3^{(1)} \) due to screening.

Therefore it is reasonable to assume that \( g_3^{(1)} \to 0 \text{ as } \rho \to \infty \), so that

\[
\int_\rho^\infty d\rho \left| g_3^{(1)} \right| \text{ converges and is at most of order } \ln \rho.
\]

Hence:

\[
|L| \leq 2^{2 R_1} \ln \rho^{-1},
\]

uniformly over \( R_2(1,2) \).

Since the source term \( \rho^{2 \gamma_2} \rho \left( \chi_1 \right) \right. F_1(\chi_1) F_1(\chi_2) - \epsilon \right) \left( \chi_1 \right) \right. F_1(\chi_2) \left( \chi_3 \right) \right. F_1(\chi_3) \right. \) in Eq. (57) is at least of order \( \epsilon^2 \) for interparticle distances not exceeding the Debye length in order of magnitude (all considerations in this report have been restricted to such lengths anyway), it can therefore be concluded that the term \( \epsilon L_2^* \) in Eq. (57) can be neglected, the relative error being of the order \( \epsilon^R \ln \rho \).

Next we consider the term

\[
T = \epsilon L_2^* (g_3^{(0)}(\xi_\perp / \epsilon, \xi_\perp / \epsilon; \chi_1, \chi_2, \chi_3) g_3^{(1)}(\xi_\perp / \epsilon, \xi_\perp / \epsilon; \chi_1, \chi_2, \chi_3)_
\]

(63)
in Eq. (57) and put $g_3^{(i)} = 1$ as an overestimate. First we notice that at least one of the spatial arguments of $g_3^{(0)}$ is very large:

$$\xi_{12} \approx \frac{-1}{\varepsilon R_2} \gg 1$$  \quad (64)

For the part \((\xi_3; \xi_2 < \varepsilon^{-1-R})\) of the integration interval in \(L^+_2\), the term \(I_2(\xi_{13}/\varepsilon; \chi_1, \chi_2, \chi_3)g_3^{(0)}(\xi_3; \chi_2, \chi_3)\) in Eq. (36) is, therefore, much larger than \(I_3(\xi_{13}/\varepsilon; \chi_1, \chi_2, \chi_3)\)

$$g_3^{(0)}(\xi_{13}/\varepsilon; \xi_3; \chi_1, \chi_2, \chi_3),$$

whereas for the part \((\xi_3; \chi_1 < \varepsilon^{-1-R})\) the term \(I_2(\xi_{13}/\varepsilon; \chi_1, \chi_2, \chi_3)g_3^{(0)}(\xi_3; \chi_1, \chi_3)\) dominates \(I_3g_3^{(0)}\). We may conclude that in Eq. (36) the term \(I_3g_3^{(0)}\) can be neglected when evaluating \(g_3^{(0)}(\xi_{13}/\varepsilon; \xi_3; \chi_1, \chi_2, \chi_3)\) within the domain of integration contained in \(L^+_2\):

\[
\psi_{13}^3 \cdot (\xi_{13} = \xi_3; \chi_1, \chi_2, \chi_3) = e^{i}I_2^{(0)}(\xi_{13}; \chi_1, \chi_3)\cdot (\chi_{13} = \chi_3; \psi_{13}^3; \chi_1, \chi_2, \chi_3)\]

\[
\psi_{13}^3 = \psi_{13}^0(\xi_{13}; \chi_1, \chi_2, \chi_3) + e^{i}I_2^{(0)}(\xi_{13}; \psi_{13}^3; \chi_1, \chi_2, \chi_3)\],
\]

(65)

provided \(\chi_{12} > e^{-R_2}, \chi_{13} < e^{-R_1} \cup \chi_{23} < e^{-R_1}\) (with \(r_L\) as unit).

In view of the fact that \(g_3^{(0)}\) is uniformly bounded [Eq. (30)] it follows from Eq. (65) that \(g_3^{(0)}\) behaves approximately like a homogeneous function of degree -1, i.e.:

\[
|g_3^{(0)}(\xi_{13}/\varepsilon; \xi_3; \chi_1, \chi_2, \chi_3)| \approx e^{-R_1} \cup \chi_{23} < \varepsilon^{-1-R_1}.
\]

(66)

Making use of Eq. (66) we get the following estimate for \(T\):

\[
|T| \leq e^{2}d^3\psi_3 |f^{(0)}d^3\xi_3(\psi_{13})\cdot \psi_{13} + \psi_{23})|g_3^{(0)}(\xi_{13}, \xi_3; \chi_1, \chi_2, \chi_3)\cdot \psi_{13}.
\]

(67)
the prime denoting restriction of the integration over $d^3 \xi_3$ to $\{\xi_3 : \xi_{13} < \epsilon^{1-R} \cup \xi_{23} < \epsilon^{1-R}\}$. Clearly, to the right hand side of Eq. (67) the regions $\{\xi_3 : \xi_{13} < \epsilon^{1-R}\}$ and $\{\xi_3 : \xi_{23} < \epsilon^{1-R}\}$ contribute terms of the same order of magnitude.

Let us consider the contribution $T_1$ to the right hand side of Eq. (67) coming from $\{\xi_3 : \xi_{13} < \epsilon^{1-R}\}$. Putting $\eta \equiv \xi_{13}$ we obtain:

$$\left| T_1 \right| \leq 2 \epsilon^2 \int d^2 v_3 f \int d^3 \eta \left| \{ \varphi ( \eta_1 ), \varphi ( - \xi_{12} + \eta_1 ), \varphi ( \xi_{12} + \eta_2 ) \} g_{3A}^{(0)} ( \xi_{12}, \xi_1, \xi_2, \xi_3 ) \right|$$

where the prime denotes that integration over $d^3 \eta$ is restricted to $\eta < \epsilon^{1-R}$. Provided the velocity gradients of the ternary correlation $g_{3A}^{(0)}$ are bounded in configuration space, there exists some function $M$ such that:

$$\left| \nabla \varphi g_{3A}^{(0)} ( \xi_{12}, \eta; \chi_1, \chi_2, \chi_3 ) \right| \leq M ( \chi_1, \chi_2, \chi_3 ), \quad i = 1, 2. \quad (69)$$

Even for situations outside equilibrium it is reasonable to assume all velocity integrations to be convergent; in particular:

$$\int d^3 v_3 M ( \chi_1, \chi_2, \chi_3 ) < \infty. \quad (70)$$

We now have the following estimate for the term $T_1$:

$$\left| T_1 \right| \leq 2 \epsilon^2 \left( \int d^3 \eta | \eta^{\perp} |^{-2} + \int d^3 \eta | \eta^{\perp} |^{2} \right), \quad (71)$$

where the prime denotes integration to be restricted to $\eta < \epsilon^{1-R}$. In order to determine the order of magnitude of
we perform the integrations over $d\eta_\perp$ and $d\eta_\parallel$, using cylindrical coordinates:

$$\int \sim \int_0^{2\pi} d\theta \int_0^{1-R_1} \eta_\perp d\eta_\parallel \int_{-\infty}^{\infty} d\eta_\parallel \left( (\eta_\parallel + \xi_\parallel)^2 + (\eta_\perp + \xi_\perp)^2 \right)^{-1} = \int_0^{2\pi} d\theta \int_0^{1-R_1} \eta_\perp d\eta_\parallel \int_{-\infty}^{\infty} d\eta_\parallel \left( \eta_\parallel^2 + \xi_\parallel^2 + \eta_\perp^2 + 2 \eta_\perp \xi_\perp \cos \psi \right)^{-1},$$

(73)

where $\psi$ is the angle between $\xi_{\perp\parallel}$ and $\eta_\perp$.

Integration over $d\phi$ yields

$$\int \sim \int_0^{2\pi} d\theta \int_0^{1-R_1} \eta_\perp d\eta_\parallel \int_{-\infty}^{\infty} d\eta_\parallel \left( \eta_\parallel^2 + \xi_\parallel^2 + 2 \eta_\perp \xi_\perp \cos \psi \right)^{-1} = \int_0^{2\pi} d\theta \int_0^{1-R_1} \eta_\perp d\eta_\parallel \int_{-\infty}^{\infty} d\eta_\parallel \left( \eta_\perp + \xi_\perp \right)^{-1},$$

(74)

where the integration interval is the interior of a circle, around the origin, with radius $\epsilon^{1-R_1}$.

Hence, $\gamma \lesssim \epsilon^{1-R_1}$, implying [Eqs. (71), (72)]

$$T \lesssim \epsilon^{2-R_1}.$$  

(75)

Since estimate (75) applies uniformly over $R_2(1,2)$ we conclude that for interparticle distances not exceeding the Debye length in order of magnitude:

$$T/\epsilon^2 \lesssim \epsilon^{1-R_1},$$

uniformly over $R_2(1,2)$. Combining this result with (62), we conclude that in Eq. (57) truncation can be made at the expense of a relative error of order maximum

$$\{ \epsilon R_1 \text{ inc}, \epsilon^{1-R_1} \}$$

uniformly over $R_2(1,2)$. The resulting closed equation for $g_{2\Lambda}(1)$ can be simplified to yield

$$\frac{\epsilon^2}{2} \int_{R_1} \int_{R_1} \frac{F_1(\chi_1)F_1(\chi_2)}{\epsilon^{1-R_1}} \lesssim \epsilon^{1-R_1},$$

uniformly over $R_2(1,2)$. Combining this result with (62), we conclude that in Eq. (57) truncation can be made at the expense of a relative error of order maximum

$$\{ \epsilon R_1 \text{ inc}, \epsilon^{1-R_1} \}$$

uniformly over $R_2(1,2)$. The resulting closed equation for $g_{2\Lambda}$ can be simplified to yield
\[
\begin{align*}
\epsilon^2 & x_{12} \cdot \frac{\partial}{\partial \xi_{12}} \Delta_A^{(0)}(\xi_{12}; \chi_1, \chi_2) \, g_{2A}^{(1)}(\xi_{12}/\epsilon, \epsilon; \chi_1, \chi_2) = \\
& \epsilon^2 I_2(\xi_{12}; \chi_1, \chi_2) F_1(\chi_1) F_1(\chi_2) + \epsilon^2 L_1^{(1)} \Delta_A^{(0)}(\xi_{23}; \chi_2, \chi_3) \\
& g_{2A}^{(1)}(\xi_{23}/\epsilon, \epsilon; \chi_2, \chi_3) F_1(\chi_1) + (1 \leftrightarrow 2) + \epsilon L_1^{(1)} \{ g_{2A}^{(0)}(\xi_{23}/\epsilon; \chi_2, \chi_3) \\
& g_{2A}^{(1)}(\xi_{23}/\epsilon, \epsilon; \chi_2, \chi_3) F_1(\chi_1) \} + (1 \leftrightarrow 2),
\end{align*}
\]

(76)

without further loss of accuracy.

The right hand side can be easily rearranged to yield:

\[
\begin{align*}
& \epsilon^2 x_{12} \cdot \frac{\partial}{\partial \xi_{12}} \Delta_A^{(0)}(\xi_{12}; \chi_1, \chi_2) \, g_{2A}^{(1)}(\xi_{12}/\epsilon, \epsilon; \chi_1, \chi_2) = \\
& \epsilon^2 I_2(\xi_{12}; \chi_1, \chi_2) F_1(\chi_1) F_1(\chi_2) + \epsilon^2 L_1^{(1)} \Delta_A^{(0)}(\xi_{23}; \chi_2, \chi_3) \\
& g_{2A}^{(1)}(\xi_{23}/\epsilon, \epsilon; \chi_2, \chi_3) F_1(\chi_1) + (1 \leftrightarrow 2) + \\
& \epsilon L_1^{(1)} \{ g_{2A}^{(0)}(\xi_{23}/\epsilon; \chi_2, \chi_3) - \Delta_A^{(0)}(\xi_{23}/\epsilon; \chi_2, \chi_3) \\
& g_{2A}^{(1)}(\xi_{23}/\epsilon, \epsilon; \chi_2, \chi_3) F(\chi_1) \} + (1 \leftrightarrow 2),
\end{align*}
\]

(77)

If the last two terms, i.e. those involving the \( L_1^{(1)} \)-operators, are neglected then we obtain for \( g_{2A}^{(1)} \) the following equation:

\[
\begin{align*}
\epsilon^2 & x_{12} \cdot \frac{\partial}{\partial \xi_{12}} \Delta_A^{(0)}(\xi_{12}; \chi_1, \chi_2) \, g_{2A}^{(1)}(\xi_{12}/\epsilon, \epsilon; \chi_1, \chi_2) = \\
& \epsilon^2 L_1^{(1)} \Delta_A^{(0)}(\xi_{23}; \chi_2, \chi_3) \\
& g_{2A}^{(1)}(\xi_{23}/\epsilon, \epsilon; \chi_2, \chi_3) F_1(\chi_1) \} + (1 \leftrightarrow 2).
\end{align*}
\]

(78)
As shown in appendix B, this neglect is consistent in the sense that substitution of the solution of Eq. (78) into the terms $\epsilon L_1^{(1,3)}$ and $\epsilon L_1^{(2,3)}$ in Eq. (77) and comparison with the source term in Eq. (77) yields:

\[
\left[ \epsilon L_1^{(1,3)} + \left( [g_2^A (\xi_{j3}/\epsilon; \chi_j, \chi_3) - \Delta_1^{(0)} (\xi_{j3}/\epsilon; \chi_j, \chi_3)] \right) \right] \\
\times g_2^A (\xi_{j3}/\epsilon, \epsilon; \chi_j, \chi_3) F_1 (\chi_i) / \epsilon^2 \int \left( \frac{\xi_{12}}{\epsilon} \right) F_1 (\chi_1) F_1 (\chi_2) \epsilon^{1-R} \right)
\]

where again considerations are restricted to interparticle distances not exceeding the Debye length in order of magnitude.

The solution of Eq. (78), which is exactly Bogolyubov's equation [Eq. (47)] for the binary correlation $\Delta_A^{(0)} (\xi_{12}, \chi_1, \chi_2)$ yields:

\[
\xi_{2A} (\xi_{12}, \epsilon; \chi_1, \chi_2) = g_2^A (\xi_{12}, \chi_1, \chi_2) \cdot \frac{\Delta_A^{(0)} (\xi_{12}, \chi_1, \chi_2)}{\Delta_A^{(0)} (\epsilon \xi_{12}, \chi_1, \chi_2)} + O(\epsilon^\gamma), \epsilon \to 0, \tag{80}
\]

with $\gamma \equiv \max \{ R, 1 \} | \ln \kappa |, \kappa \epsilon^{1-K_3}$.

In the next sections it will be shown that the solution (80), obtained in region $R_2(1,2)$, is also a good approximation for the asymptotic binary correlation in the rest of $\xi_{12}$-space, i.e. in regions $R_1(1,2)$ and $R_3(1,2)$ [cf. section 2 of Chapter I].
III.2. The region $R_3(1,2)$ of small interparticle distances.

Next we try to obtain a good approximation for $g_{2A}$ in the region
$R_3(1,2) = \{ x_{12} : x_{12} < \epsilon^{-R_2} r_L \}$ with $0 < R_1 \leq R_2 < 1$. To this end
we define a correction factor $g_{2A}^{(1)}$ for the deviation from the
Bogolyubov correlation $g_B$ such that $g_{2A}^{(1)}$ is of order unity for
distances of the order of the Debye length, i.e.,

$$g_{2A}(\xi_{12}, \epsilon; \chi_1, \chi_2) = \epsilon g_B(\xi_{12}; \chi_1, \chi_2) g_{2A}^{(1)}(\xi_{12}, \epsilon; \chi_1, \chi_2).$$

(81)

Scaling the second hierarchy equation by means of the units
($\epsilon^{-R_2} r_L, \nu_{th}$) we obtain

$$\frac{D^{(2)}}{D\epsilon} G_{2A}(1,2) = \epsilon^{2R_2} L_2(1,2) F_1(\chi_1 F_1(\chi_2) + \epsilon^{3R_2} L_2(1,2) G_{2A}(1,2) +$$

$$+ \epsilon^{2R_2} L_1(1) [G_{2A}(2,3) F_1(\chi_1)] + (1 \rightarrow 2) + \epsilon^{2R_2} L_2 G_{3A}(1,2,3),$$

(82)

where we defined

$$G_{2A} \equiv \epsilon^{-R_2} g_{2A},$$

$$G_{3A} \equiv \epsilon^{-2R_2} g_{3A},$$

$$\theta \equiv \epsilon^{-R_2} t,$$

(83)

where $t$ is measured in units $\frac{r_L}{\nu_{th}}$. The orders of magnitude of the
correlations have been estimated to equal those of the zero order
approximations $g_{2A}^{(0)}$ and $g_{3A}^{(0)}$ for impact parameters of order $\epsilon^{-R_2} r_L$, so that $G_{2A}$ and $G_{3A}$ are of order unity.
Since the importance of the integral terms decreases with decreasing interparticle distance, we can neglect these integral terms for distances smaller than $\varepsilon^{-R^2} r_L$, up to a relative accuracy of order $\varepsilon^{-R^2}$. But since the interaction operator $I_2$ is inversely proportional to the square of the interparticle distance we can not neglect the term $\varepsilon^{-3R^2} I_2(1,2) G_{2A}(1,2)$ in Eq. (82) uniformly over $R_3(1,2)$. Thereby we obtain the following equation for $\xi^{(1)}$: 

$$
\begin{align*}
\frac{d}{dt}[\xi_B(\varepsilon \xi_{12}; \xi_1, \xi_2) & \xi^{(1)}_2(\varepsilon; \xi_1, \xi_2)] = \\
& L_2(\varepsilon \xi_{12}; \xi_1, \xi_2) F_1(\xi_1) F_1(\xi_2) + L_2(\varepsilon \xi_{12}; \xi_1, \xi_2) \left[ \xi_B(\varepsilon \xi_{12}; \xi_1, \xi_2) \\
& - \xi^{(1)}_2(\varepsilon; \xi_1, \xi_2) \right].
\end{align*}
$$

(84)

Since over $R_3(1,2): |\varepsilon \xi_{12}| < \varepsilon^{-1-R^2}$, Eq. (53) implies that up to an accuracy of order $\varepsilon^{-1-R^2}$

$$
\xi_B(\varepsilon \xi_{12}; \xi_1, \xi_2) = \Delta^{(0)}_A (\varepsilon \xi_{12}; \xi_1, \xi_2)
$$

(85)

can be allowed as a uniform approximation within $R_3(1,2)$, so that we arrive at:

$$
\begin{align*}
\frac{d}{dt}[\xi^{(0)}_A (\varepsilon \xi_{12}; \xi_1, \xi_2) & \xi^{(1)}_2(\varepsilon; \xi_1, \xi_2)] = \\
& - L_2(\varepsilon \xi_{12}; \xi_1, \xi_2) [F_1(\xi_1) F_1(\xi_2) + \xi^{(0)}_A (\varepsilon \xi_{12}; \xi_1, \xi_2) \\
& - \xi^{(1)}_2(\varepsilon; \xi_1, \xi_2)].
\end{align*}
$$

(86)

up to a relative accuracy of order $\varepsilon^\sigma$, with $\sigma = \min\{R_2, 1-R^2\}$. 

Eq. (86) is identical with the equation for the binary correlation according to the Boltzmann theory [Eq. (16)]. Hence,

\[ g_{2A}^{(1)}(x_{12}, \varepsilon; x_1, x_2) = g_{2A}^{(0)}(x_{12}; x_1, x_2) + 0(\varepsilon^0), \quad \text{(87)} \]

Combination of results (80) and (87) leads to the conclusion that:

\[ g_{2A}(x_{12}, \varepsilon; x_1, x_2) = \frac{g_{2A}^{(0)}(x_{12}; x_1, x_2)}{\Delta_A^{(0)}(x_{12}; x_1, x_2)} g_{B}(x_{12}; x_1, x_2) + 0[\max \{\varepsilon^0, (\ln \varepsilon^{-1}) \varepsilon^{R_1}, \varepsilon^{1-R} \}], \quad \text{(88)} \]

uniformly over the union of \( R_2(1,2) \) and \( R_3(1,2) \).

III.3. The region \( R_1(1,2) \) of small impact parameters.

Finally, consider region \( R_1(1,2) = \{x_{12} : |x_{12}^\perp| < \varepsilon^{-1} r_L \} \) where

\[ x_{12}^\perp = x_{12} - \frac{x_{12}^2}{v_{12}^2} \quad \text{and} \quad 0 < R_1 < 1, \]

as introduced in Chapter I.

For a typical point in \( R_1(1,2) \)

\[ |x_{12}^\perp| \sim \varepsilon^{-R_1} r_L, \quad |x_{12}^\parallel| \sim r_D. \quad \text{(89)} \]

Since \( R_1 > 0 \), expansion (34) for the Boltzmann correlation converges for a typical point in \( R_1(1,2) \), and we find [Eq. (35)]

\[ |g_{2A}^{(0)}| \sim \varepsilon^{R_1} F_1(x_1)F_1(x_2), \quad \text{(90)} \]
for the order of magnitude of the Boltzmann correlation in a
typical point in \( R_1(1,2) \). The correction factor \( g_{2A}^{(1)} \) due to
screening tends to decrease the correlation, but can be taken
of order unity in the region where correlations are nonvanishing.
Hence, within \( R_1(1,2) \)

\[
|g_{2A}| \lesssim e^{R_1} F_1(x_1) F_1(x_2), \tag{91}
\]

whereas in the rest of space, where typically \( |x_{12}| \sim r_D \) isotropically,

\[
|g_{2A}| \sim \varepsilon F_1(x_1) F_1(x_2), \tag{92}
\]

as follows from Eqs. (34) and (35). Furthermore, the ternary correlation
\( g_{3A} \) can be assumed not to exceed \( F_1(x_1) F_1(x_2) F_1(x_3) \) in order of
magnitude, i.e.,

\[
|g_{3A}| \lesssim F_1(x_1) F_1(x_2) F_1(x_3) , \tag{93}
\]

within \( R_1(1,2) , R_1(1,3) \) and \( R_1(2,3) \), whereas in the rest of space
[Eqs. (39), (40)], where typically \( x_{ij} \sim r_D \),

\[
|g_{3A}| \sim \varepsilon^2 F_1(x_1) F_1(x_2) F_1(x_3) . \tag{94}
\]

From (89) it follows that

\[
|\nabla \phi| \sim \varepsilon^2 \frac{e^2}{r_L} , \tag{95}
\]

since the Coulomb potential is an isotropic function.

Finally, the integration elements \( d^3x_3 \) present in all integral terms
of the second hierarchy equation [Eq. (4)] must be ordered according to the scale dependence of the integrand, yielding

\[ d^3x_2 \sim d^3x_3 \varepsilon^{-1-2R_1}, \]  

(96)

where \( d^3x_3 \) is the integration element measured in units \( r_L \), if integration is to be performed over a region \( R_1 \), and

\[ \int d^3x_3 \sim \varepsilon^{-3} \int d^3x_3 \text{ otherwise.} \]  

(97)

Adopting \( \varepsilon^{-R_1} r_L \) as unit for \( x_{12} \), \( r_D \) as unit for \( |x_{12}| \), and making use of estimates (90) - (97), we obtain the following form for the nondimensional hierarchy equation relevant to region \( R_1(1,2) \):

\[
\varepsilon^{1+R_1} x_{12}^{-\frac{3}{2}} x_{12}^{-\frac{3}{2}} g_{2A}(1,2) = \varepsilon^2 I_2(1,2) \left[ F_1(x_1)F_1(x_2) +
+ \varepsilon^{R_1} g_{2A}(1,2) + \varepsilon^2 L_1^{(1)} \left[ g_{2A}(2,3)F_1^{(1)} \right] + (1 \leftrightarrow 2) +
+ \varepsilon^{3-2R_1} L_2 g_{3A}(1,2,3), \right.
\]

(98)

from which it follows that

\[
\chi_{12}^{-\frac{3}{2}} \chi_{12}^{-\frac{3}{2}} g_{2A}(1,2) = 0_{\max, \{ \varepsilon^{2-3R_1}, \varepsilon^{1-R_1} \}}. \]  

(99)

Hence, within a good accuracy the binary correlation function does not vary along the \( \chi_{12} \)-axis within region \( R_1(1,2) \), and can therefore be equated to its form within region \( R_3(1,2) \), implying [Eqs. (80), (88), (99)]
\[ \varepsilon_{2A}(\chi_{12}; \chi_1, \chi_2) = \frac{g^{(0)}_{2A}(\varepsilon \chi_{12}; \chi_1, \chi_2) s_B(\varepsilon \chi_{12}; \chi_1, \chi_2)}{\Delta_{A}^{(0)}(\varepsilon \chi_{12}; \chi_1, \chi_2)} + + O[\max \{\varepsilon, (\ln \varepsilon)^{-1}, \varepsilon^{-1}, \varepsilon^{-2}, \varepsilon^{-3} \}], \quad (100) \]

uniformly over \( R_1(1,2) \cup R_2(1,2) \cup R_3(1,2) \).

An accuracy of order \( \varepsilon^{1/2} \ln \varepsilon \) is now achieved by putting \( R_1=R_2=R \), i.e.,

\[ \varepsilon_{2A}(\chi_{12}; \chi_1, \chi_2) = \frac{g^{(0)}_{2A}(\varepsilon \chi_{12}; \chi_1, \chi_2) s_B(\varepsilon \chi_{12}; \chi_1, \chi_2)}{\Delta_{A}^{(0)}(\varepsilon \chi_{12}; \chi_1, \chi_2)} + + O(\varepsilon^{1/2} \ln \varepsilon), \quad (101) \]

uniformly in \( \chi_{12} \).

In Eq. (101), \( g^{(0)}_{2A} \) is explicitly given by Eq. (30), \( \Delta_{A}^{(0)} \) by Eqs. (44) and (46), while the Fourier-transform of \( s_B \) is given by Eq. (50). Through Eqs. (101) and the first hierarchy equation [Eq. (4), with \( s = 1 \)] the collision integral is determined, i.e.,

\[ \frac{\delta F}{\delta t} = \frac{n}{m_e} \frac{\partial}{\partial \chi_1} \int d^3x \int d^3v_2 \frac{g^{(0)}_{2A} s_B}{\Delta_{A}^{(0)}}. \quad (102) \]

Note that the dependence on the plasma parameter \( \varepsilon \) only occurs through the space coordinates in the Bogolyubov and Landau correlations, \( s_B \) and \( \Delta_{A}^{(0)} \), respectively. Since \( \frac{g^{(0)}_{2A}}{\Delta_{A}^{(0)}} \to 1 \) if \( \chi_{12} \to 0 \) [Eq. (A.21)] and \( \frac{s_B}{\Delta_{A}^{(0)}} \to 1 \) if \( \chi_{12} \to \infty \) [App. C] our collision integral evidently converges.
CHAPTER IV: THERMODYNAMIC EQUILIBRIUM

In this chapter the free energy density in thermodynamic equilibrium is derived from our formula [Eq. (101)] for the binary correlation function.

As is well known the free energy density in equilibrium is given by

$$F = F_0 + \frac{n^2}{2} \int_0^1 d\lambda/d^3 x \frac{e^{2}}{x} G(\lambda; x),$$

(103)

where $F_0$ is a constant and where $G(\lambda; x)$ is the space-dependent factor in the expression for the equilibrium binary correlation function, i.e.,

$$G(\lambda; x) = -\exp\left(\frac{-x}{r_D}\right) \left[1 - \exp\left(-\frac{x_L}{x}\right)\right]$$

(104)

in our case [Eq. (101)].

Defining

$$I(\lambda; r_L, r_D) = \int_0^\infty d\lambda r \exp\left(-\frac{\lambda^2 r^2}{r_D}\right) \left[1 - \exp\left(-\frac{\lambda r_L}{r}\right)\right]$$

(105)

and making use of Eq. (104) we obtain from Eq. (103)

$$F = F_0 - 2\pi n^2 e^2 \int_0^1 d\lambda I(\lambda; r_L, r_D).$$

(106)

In view of

$$\int_0^\infty dp \exp(-e^{\epsilon r_0}) \exp(-\lambda r_0) = 2 \frac{\lambda^{\frac{1}{2}}}{\epsilon} K_2(2\lambda^{\frac{1}{2}}\epsilon^{\frac{1}{2}})$$

(107)
and

\[ k_2(2x^2) = \frac{1}{2x} - \frac{1}{2} - \sum_{n=0}^{\infty} \frac{x^{1+n}}{n! (2+n)!} \left[ \ln x^2 - \frac{1}{2} \psi(n+1) - \frac{1}{2} \psi(n+3) \right], \tag{108} \]

we readily obtain the following approximation for the free energy density [Eq. (106)]

\[ F = F_0 - nT \left[ \frac{\epsilon}{3} + \frac{\epsilon^2 \ln \epsilon}{12} + O(\epsilon^2) \right], \tag{109} \]

where \( \epsilon = r_L/r_D \). This expression is in agreement with Guernsey's result \(^{16}\) and differs from the divergent expression obtained by Frieman and Book.\(^{13,17}\) It should be noted that the terms \( O(\epsilon^2) \) cannot be trusted in view of the relative error \( O(\epsilon^2) \) made in deriving our formula for the binary correlation function [Eq. (101)].
If the deviation from thermodynamic equilibrium is small, then the binary correlation only slightly differs from its equilibrium form and the collision integral can be written, as in dimensional form, as

$$\frac{\delta \mathcal{F}}{\delta \mathcal{T}} = A + B + C,$$  \hspace{1cm} (110)

where

$$A = \frac{n}{m} \frac{\delta}{\delta \mathcal{V}_1} \int d^3x \alpha(x) \nabla \phi(x) \int d^3\mathcal{V}_2 \, g_{_{2A}}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2),$$  \hspace{1cm} (111)

with

$$\alpha(x) = \left[ \frac{\mathcal{G}_{_{2A}}^{(0)}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2)}{\Delta_A^{(0)}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2)} \right] \quad \text{(eq.)}$$  \hspace{1cm} (112)

$$B = \frac{n}{m} \frac{\delta}{\delta \mathcal{V}_1} \int d^3x \beta(x) \nabla \phi(x) \int d^3\mathcal{V}_2 \, g_{_{2A}}^{(0)}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2),$$  \hspace{1cm} (113)

with

$$\beta(x) = \left[ \frac{\mathcal{G}_B(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2)}{\Delta_A^{(0)}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2)} \right] \quad \text{(eq.)}$$

$$C = \frac{n}{m} \frac{\delta}{\delta \mathcal{V}_1} \int d^3x \gamma(x) \nabla \phi(x) \int d^3\mathcal{V}_2 \, \Delta_A^{(0)}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2),$$  \hspace{1cm} (115)
with
\[ \gamma(x) = - \frac{\mathcal{E}_{2A}^{(0)}(\chi_1, \chi_2) \phi_B^{(0)}(\chi_1, \chi_2)}{\Delta_A^{(0)}(\chi_1, \chi_2)} \] (eq.) \hspace{1cm} \text{(116)}

In Eqs. (112), (114) and (116) the symbol \([\ldots]\) \(\text{eq.}\) denotes that the equilibrium form of the quantity between the square brackets should be taken; hence, \(\alpha, \beta\) and \(\gamma\) only depend on \(x\). In Eqs. (111), (113) and (115) the correlations \(g_B, g_{2A}^{(0)}\) and \(\Delta_A^{(0)}\), respectively, can be further linearized around thermodynamic equilibrium.

Conservation of number density and of particle momentum density are obviously checked from Eqs. (110), (111), (113), (115). Conservation of kinetic energy density is checked for terms \(A, B\) and \(C\) in Eq. (110) separately, as follows:

\____ Ad. \(A\) ______

Introduce the potential \(\psi(x)\) defined by
\[ \psi(x) = - \int_{-\infty}^{\infty} \alpha(\xi) \frac{d\phi(\xi)}{d\xi} \, d\xi \] \hspace{1cm} \text{(117)}
in terms of which \([\text{Eq. (111)}]\)
\[ A = \frac{n}{m} \frac{\partial}{\partial \chi_1} \int d^3 x \, \psi(x) \int d^3 v_2 \, g_B(\xi_1, \chi_2) \] \hspace{1cm} \text{(118)}

Making use of Parseval's theorem we then obtain
\[ A = \frac{n(2\pi)^3}{m} \frac{\partial}{\partial \chi_1} \int d^3 k \, k \hat{\psi}(k) \text{Im} \int d^3 v_2 \, \hat{g}_B(k; \chi_1, \chi_2) \] \hspace{1cm} \text{(119)}

where \(\psi\) and \(\hat{g}_B\) are the Fourier-transforms of \(\psi\) and \(g_B\) \([\text{Eq. (49)}]\),
and where \( \text{Im}[~] \) denotes the imaginary part of the quantity between brackets. It is straightforward to show\(^9,20\) that the right hand side can be written as

\[
A = -\frac{3e_2^2}{m^2 \delta y_1}, \int d^3v_2 \frac{\partial^2}{\delta y_1} \mathcal{Q}(y_1, y_2) \cdot \frac{\partial F_1(y_1)}{\partial y_1} F_1(y_2) - \frac{\partial F_1(y_2)}{\partial y_2} F_1(y_1),
\]

where

\[
\mathcal{Q}(y_1, y_2) = -\int d^3k \frac{\delta^{(3)}(k \cdot y_1 / k) \delta^{(3)}(k \cdot y_2 / k)}{\left| \Delta^+(i k \cdot y_1) \right|^2},
\]

where the dielectric function is defined by

\[
\Delta^+(k, p) = 1 - \frac{\omega^2}{k^2} \int d^3v \frac{\partial f_0}{\partial \varphi} \frac{\partial f_0}{\partial \varphi},
\]

Therefore

\[
\int \text{Im} \ v_1^2 A \ d^3v_1 = \frac{1}{2} \frac{d^3v_2}{\Delta^+(i k \cdot y_1 / k)} \mathcal{Q}(y_1, y_2) \cdot \frac{\partial F_1(y_1)}{\partial y_1} F_1(y_2) - \frac{\partial F_1(y_2)}{\partial y_2} F_1(y_1) = 0
\]

since \( y_1 \cdot \mathcal{Q} = 0 \).

Ad R:

Introduce a potential \( \xi(x) \) defined by

\[
\xi(x) = - \int \frac{d\beta(\xi)}{d\xi} \ d\xi,
\]

on

\[
\xi(x) = - \int \frac{d\beta(\xi)}{d\xi} \ d\xi,
\]

and

\[
\xi(x) = - \int \frac{d\beta(\xi)}{d\xi} \ d\xi,
\]

resulting

\[
\xi(x) = - \int \frac{d\beta(\xi)}{d\xi} \ d\xi,
\]

and

\[
\xi(x) = - \int \frac{d\beta(\xi)}{d\xi} \ d\xi,
\]

finally

\[
\xi(x) = - \int \frac{d\beta(\xi)}{d\xi} \ d\xi,
\]
in terms of which

\[ \frac{1}{m} \int d^3v_1 v_1^2 \mathcal{B} = \frac{1}{m} \int d^3v_1 \frac{\partial}{\partial \chi_1} \frac{\partial}{\partial v_1} \int d^3x \nabla \xi(x) \int d^3v_2 g_{(0)}^{(2)}(\chi; \chi_1, \chi_2) \]

(126)

Integrating by parts with respect to both \( \chi \) and \( \chi_1 \) and using a symmetry argument we readily obtain

\[ \frac{1}{m} \int d^3v_1 v_1^2 \mathcal{B} = \frac{1}{m} \int d^3v_1 \int d^3v_2 \int d^3x \xi(x) \chi_{12} \]

\[ - \frac{\partial}{\partial \xi} g_{(0)}^{(2)}(\chi; \chi_1, \chi_2) \]

(127)

Since [Eq. (16)]

\[ \chi_{12} \frac{\partial}{\partial \xi} g_{(0)}^{(2)}(\chi; \chi_1, \chi_2) = \frac{1}{m} \frac{\partial}{\partial v_1} \left( \frac{\partial}{\partial \chi_1} - \frac{\partial}{\partial \chi_2} \right) \left[ F_1(\chi_1) F_1(\chi_2) \right] + \]

\[ + g_{(0)}^{(2)}(\chi; \chi_1, \chi_2) \]

the right hand side of Eq. (127) is seen to be equal to zero using Gauss' theorem.

Ad C:

Quite similarly we find

\[ \int \frac{1}{m} v_1^2 \mathcal{C} d^3v_1 = \frac{1}{m} \int d^3v_1 \int d^3v_2 \int d^3x \eta(x) \nabla \cdot \left( \frac{\partial}{\partial \chi_1} - \frac{\partial}{\partial \chi_2} \right) F_1(\chi_1) F_1(\chi_2), \]

(128)

where the potential \( \eta \) is defined by
\[ \eta(x) = - \int_{-\infty}^{\infty} \gamma(\xi) \frac{d\phi(\xi)}{d\xi} \, d\xi \]  

(129)

and where the right hand side of Eq. (128) vanishes in view of Gauss' theorem.

The proof of kinetic energy conservation is thereby completed.
In this chapter we consider an unmagnetized and homogeneous plasma under the influence of a spatially independent but time-varying electric field, $\mathbf{E} \sim \exp(-i\omega t)$, and we will restrict considerations to the case when

$$v_{ei} < \frac{e|\mathbf{E}|}{m_v e} < |\omega| < \omega_p/e,$$  \hspace{1cm} (130)

where $v_{ei}$ is the relaxation frequency of electrons due to collisions with ions, $v_t$, and where $\omega_p = (e^2 n/e m_e)^{1/2}$ is the electron plasma frequency. The unperturbed electron and ion distributions are assumed to be Maxwellians with constant and equal temperatures,

$$f_e^0 = \frac{m_e v}{2\pi T_0}^{3/2} \exp\left(-\frac{m_e v^2}{2T_0}\right),$$

$$f_i^0 = \frac{m_i v}{2\pi T_0}^{3/2} \exp\left(-\frac{m_i v^2}{2T_0}\right).$$  \hspace{1cm} (131)

As a result of the strong inequality in (130) the plasma can be assumed quasi-neutral. The other two inequalities imply a possibility to analytically solve the kinetic equations for both electrons and ions by expanding in the small parameter

$$\varepsilon \equiv \frac{v_{ei} m_v e}{e|\mathbf{E}|} \sim \frac{e|\mathbf{E}|}{m_v e|\omega|}. \hspace{1cm} (132)$$

Since the relaxation frequency of ions is much lower than $v_{ei}$,
the ions can be described by the (collisionless) Vlasov-equation,

\[
\frac{\partial f_i}{\partial t} + \frac{e}{m_i} \mathbf{E} \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0 .
\]  

(133)

On the other hand, the collision terms in the electron kinetic equation are finite, though small [Eq. (132)],

\[
\frac{\partial f_e}{\partial t} - \frac{e}{m_e} \mathbf{E} \cdot \frac{\partial f_e}{\partial \mathbf{v}} = \frac{n}{m_e} \left( \frac{e}{m_e} \int d^3 \mathbf{x} \nabla \phi(\mathbf{x}) \int d^3 \mathbf{v}_2 \, g_{ee}(\mathbf{v}; \mathbf{v}_1, \mathbf{v}_2) - \right.
\]

\[
- \frac{n}{m_e} \left( \frac{e}{m_e} \int d^3 \mathbf{x} \nabla \phi(\mathbf{x}) \int d^3 \mathbf{v}_2 \, g_{ee}(\mathbf{v}; \mathbf{v}_1, \mathbf{v}_2) \right),
\]  

(134)

where \( g_{ee} = g \) as given by Eq. (101), and \( g_{ei} \) denotes the analog of \( g \) corresponding to electron-ion correlation.

Since \( m_e \ll m_i \) the current is predominantly carried by the electrons,

\[
\mathbf{j} = \int d^3 \mathbf{v} \, \mathbf{v} \, f_e .
\]  

(135)

The kinetic equations (133) and (134) can be used to determine the perturbed electron distribution \( \delta f_e \), explicitly in terms of the applied field \( \mathbf{E} \). Substitution into the expression for the current density [Eq. (135)] then yields an expression for the conductivity, in each order in \( \varepsilon \).

In first order in \( \varepsilon \), Eqs. (133) and (134) yield

\[
\delta f_e^{(1)} = \frac{e \mathbf{E} \cdot \mathbf{v}}{i \omega T_0} \, \delta f_e^0 , \quad \delta f_i^{(1)} = - \frac{e \mathbf{E} \cdot \mathbf{v}}{i \omega T_0} \, \delta f_i^0
\]  

(136)

for the first order perturbations in \( f_e \) and \( f_i \). The resulting conductivity bears no collisional effects, and is entirely due to electron inertia.
In second order in $\ell$,

$$-i\omega \delta f_e^{(2)} = C_{ei},$$

where $C_{ei}$ is the (linearized) electron-ion collision term, for which, in analogy with the linearized electron-electron collision term, can be written [Eqs. (110)-(116)]

$$C_{ei} = C_{ei}^{(A)} + C_{ei}^{(B)} + C_{ei}^{(C)},$$

where [cf. App. E]

$$C_{ei}^{(A)} = -\frac{n}{m_e} \frac{3}{\partial \chi_1} \int d^3x \frac{\partial}{\partial \chi_1} \left[ \psi(x) \int d^3v_2 \, g_{eiB}(\chi_1, \chi_2) \right],$$

with

$$\psi(x) = T_0 \int_x^\infty \frac{d \xi}{\xi} \left[ 1 - \exp\left( -\frac{r_L}{\xi} \right) \right];$$

$$C_{ei}^{(B)} = -\frac{n}{m_e} \frac{3}{\partial \chi_1} \int d^3x \, \exp\left(-\frac{x}{r_D}\right) \psi(x) \int d^3v_2 \, g_{eiA}(\chi_1, \chi_2);$$

$$C_{ei}^{(C)} = +\frac{n}{m_e} \frac{3}{\partial \chi_1} \int d^3x \, \eta(x) \int d^3v_2 \, \Delta_{eiA}^{(0)}(\chi_1, \chi_2),$$

with

$$\eta(x) = T_0 \int_x^\infty \frac{d \xi}{\xi} \left[ 1 - \exp\left(-\frac{r_L}{\xi} \right) \exp\left(-\xi r_D^{-1}\right) \right].$$
In Eqs. (139-143), $g_{eIB}$, $g_{eIA}$ and $\Lambda^{(0)}_{eiA}$ are the electron-ion analogies of $g_B$, $g_A$ and $\Lambda^{(0)}_A$ [Eqs. (50), (30), (45)] and

$$r_L = e^2 T_0^{-1}, \quad r_D = \left(\frac{\rho}{\omega_{ne}^2}\right)^{1/2}.$$

In deriving Eq. (137), note that the linearized electron-electron collision term can be discarded, since, due to conservation of linear momentum, its contribution proportional to $\delta f_e^{(1)}$ vanishes. Also it is noticed that a term $-\frac{e}{m_e} \frac{\partial}{\partial \rho} \delta f_e^{(1)}$ on the left hand side of Eq. (134) has been discarded, since, as this term is symmetric in $\chi$, it does not contribute to the current density [Eq. (135)].

Next the three collision terms in Eq. (137) are evaluated.

First consider $C_{ei}^{(A)}$ as defined by Eqs. (139) and (140). Making use of Parseval's theorem, we obtain

$$C_{ei}^{(A)} = \frac{n}{m_e} \frac{3}{\partial \chi_1} \int d^3k \, \frac{i \chi}{\delta \chi_1} \int d^3v \, \frac{\partial}{\partial v_2} g_{eIB}(k, \chi_1, \chi_2), \quad (144)$$

where the Fourier-transform $\phi = (2\pi)^{-3} \int d^3x \, \psi(x) \exp(-ik \cdot \chi)$ of $\psi$ equals

$$\phi(k) = \frac{T_0}{2\pi^2 k} \int_0^\infty dp \, \rho \sin(k \rho) \int^\infty_0 \left[1 - \exp\left(-\frac{L}{\chi}\right)\right] \frac{d\chi}{\rho}, \quad (145)$$

and where the expression for $\int d^3v_2 \, \frac{\partial}{\partial v_2} g_{eIB}(k, \chi_1, \chi_2) = (2\pi)^{-3} \int d^3v_2 \int d^3x \, g_{eIB}(\chi_1, \chi_2) \exp(-ik \cdot \chi)$ as given by e.g. fm(7.45) of Reference (20) can be substituted without any further, yielding

$$C_{ei}^{(A)} = -\frac{2}{\partial \chi_1} \cdot \chi_{eiB}(\chi_1), \quad (146)$$

with
\[ \mathcal{J}^{\text{e1}}(v_1) \equiv m_e n \int d^3 v_2 \delta(v_1 - v_2) \cdot \left[ \frac{f_e(v_2)}{m_e} \frac{\partial f_e}{\partial v_1} - \frac{f_i(v_1)}{m_i} \frac{\partial f_i}{\partial v_2} \right], \]

where

\[ \delta(v_1 - v_2) \equiv -\frac{4\pi^2 e^2}{m_e^2} \int d^3 k \frac{k \cdot v_1}{k^2} k^{-3} \varepsilon(k) \delta \left( \frac{k \cdot v_1}{k} \right) |\Delta^+(k, i k \cdot v_1)|^{-2}, \]

where

\[ 1 - \Delta^+(k, p) \equiv \frac{\omega_{\text{pe}}^2 i k}{k^2} \int d^3 v \frac{\partial f_e}{\partial v_1} + \frac{\omega_{\text{pi}}^2 i k}{k^2} \int d^3 v \frac{\partial f_i}{\partial v_2}. \]

In linearizing \( C^{(A)} \), notice that

\[ [f_e^0(v_2)] \frac{1}{m_e} \frac{\partial f_e^0}{\partial v_1} - f_i^0(v_1) \frac{1}{m_i} \frac{\partial f_i^0}{\partial v_2}. \quad \mathcal{O} = 0. \]

Therefore, the equilibrium distributions \( f_e^0 \) and \( f_i^0 \) may be substituted into \( \Delta^+ \), i.e.,

\[ \Delta^+ = \Delta^+(\text{eq.}) = 1 + (k \cdot p)^{-2} \left[ 2 + \frac{k \cdot v_1}{k v_e} \frac{k \cdot v_1}{k v_e} + \frac{k \cdot v_1}{k v_i} \frac{k \cdot v_1}{k v_i} \right]. \]

where \( Z \) is defined by

\[ Z(x) \equiv \frac{1}{V_n} \int_{-\infty}^{\infty} d\xi \exp(-\xi^2). \]

Since \( v_1 \gg v_e \) and \( v_2 \gg v_i \ll v_e \) we can approximate \( \Delta^+ \) [Eq.(151)] by
yielding for \( Q \) [Eq. (148)]

\[
Q \approx - \frac{4 \pi^2 e^2}{m_e} f d^3 k \frac{k}{k_e} k\delta \left( \frac{k \cdot v}{k} \right) \delta \left( \frac{k \cdot v}{k} - \frac{k \cdot v}{k} \right) \\
\times \left[ 1 + \left( k r_D \right)^{-2} \left[ 1 + \frac{k \cdot v}{k} z \left( \frac{k \cdot v}{k} \right) \right]^2 \right]^{-1} .
\]

(154)

Sinc \( v_1 \ll v_e \), \( k \cdot v_1 = 0 \) implies \( \left| \frac{k}{k_e} \cdot v \right| \ll v_e \), so that

\[
\frac{k}{k_e} \cdot v_1 \approx \frac{k}{k_e} \cdot \frac{v_1}{v} \approx \frac{v_1}{v} \approx 0 ,
\]

(155)

and hereby Eq. (154) reduces to

\[
Q(\psi_1, \psi_2) \approx - \frac{4 \pi^2 e^2}{m_e} f d^3 k \frac{k}{k_e} k \delta \left( \frac{k \cdot v}{k} \right) \delta \left( \frac{k \cdot v}{k} - \frac{k \cdot v}{k} \right) \left( 1 + k^{-2} r_D^{-2} \right)^{-2} .
\]

(156)

Making use of \( v_1 \ll v_e \), and substituting Eqs. (136) and (156) into Eq. (146), we obtain

\[
\mathcal{C}_{ei}(A) \approx - \frac{\omega^2 e^3}{m_e^2 (-i\omega + 0)} \frac{1}{\delta \psi_1} f d^3 k \frac{k}{k_e} \frac{k}{k} \frac{\psi(k)}{\delta} \frac{\delta \left( \frac{k \cdot v}{k} \right)}{\delta \left( \frac{k \cdot v}{k} \right)} \frac{\delta \left( \frac{k \cdot v}{k} \right)}{\delta \left( \frac{k \cdot v}{k} \right)} .
\]

(157)

In order to evaluate \( \mathcal{C}_{ei}^{(2)} \) [Eq. (141)], notice that the expression for the electron-electron Boltzmann correlation function [Eq. (30)] was derived by solving the characteristic differential equations
describing the particle trajectories, after reduction to those corresponding with a center of force problem. Therefore, the corresponding electron-ion formula is obtained by substituting appropriate values for the reduced mass, interaction strength and center of mass velocity, yielding

\[
g_{\text{el}}^{(0)}(\chi; \chi_1, \chi_2) = f_e\left[\chi - \frac{m_i}{m_e + m_i} \left(\frac{2e}{m_e} - \frac{2e^2}{m_e^2}\right) \frac{1}{e} \delta(\chi; \chi_1, \chi_2)\right]
\]

\[
f_i\left[\chi - \frac{m_i}{m_e + m_i} \left(\frac{2e}{m_e} - \frac{2e^2}{m_e^2}\right) \frac{1}{e} \delta(\chi; \chi_1, \chi_2)\right] - f_e(\chi_1)f_i(\chi_2),
\] (158)

where \(\chi = \frac{m_e \nu_1 + m_i \nu_2}{m_e + m_i}\).

Also notice that the influence of electron capture is excluded in a classical, i.e., nonquantal, theory of a fully ionized plasma. We shall account for this by writing \(\int d^3v_2 U(\nu_1^2 \frac{2e}{m_e^2})\) instead of \(\int d^3v_2\) in the expression for \(C^{(B)}_{\text{el}}\) \([\text{Eq. (140)}]\). Here \(U\) is the unit step function. Substitution of the expansion around equilibrium \([\text{Eq. (136)}]\) of the distribution functions \(f_e\) and \(f_i\) into \(\text{Eq. (158)}\) yields

\[
g_{\text{el}}^{(0)}(\chi; \chi_1, \chi_2) = -1 + \exp(e^2/T_0) - \exp\left(\frac{e^2}{T_0}\right) (\nu_1^2 - \frac{2e^2}{m_e^2}) \frac{1}{e} \delta(\chi; \chi_1, \chi_2) f_e(\chi_1)f_i(\chi_2).
\] (159)

Thereby we obtain \([\text{Eq. (141)}]\)

\[
C^{(B)}_{\text{el}} = \frac{e\nu_1}{m_e (-1 + 0)} \frac{3}{\delta\chi_1} \int d^3x \psi(x) \exp\left(-\frac{x}{T_0}\right) d^3v_2 U(\nu_1^2 - \frac{2e^2}{m_e^2}) (\nu_1^2 - \frac{2e^2}{m_e^2}) \frac{1}{e} \delta(\chi; \chi_1, \chi_2) \exp\left(\frac{e^2}{T_0}\right) f_e(\chi_1)f_i(\chi_2).
\] (160)
Finally, we evaluate $C^{(C)}_{ei}$ [Eq. (142)]. This is now most easily done by noting that $\int d^3v_2 \Delta^{(0)}_{eiA}$ can be obtained from $\int d^3v_2 \varphi_B(x; x_1, x_2)$ by taking the limit $r_D \to \infty$. The result is

$$C^{(C)}_{ei} = \frac{4\pi^2 n_e^3}{T_0 m_e^2 (-i\omega + 0)} \int d^3k \frac{k^3 k'}{k^2} \cdot \frac{\Phi(k)}{k} \frac{k \cdot x_1}{k} \exp(-\frac{k \cdot x_1}{k}) \exp(-\frac{r_D}{\lambda}) f_e^0(v_1),$$

where $\Phi(k) = \frac{1}{(2\pi)^3} \int d^3x \eta(x) \exp(-ik \cdot x)$ is equal to [Eq. (143)]

$$\Phi(k) = \frac{T_0}{2\pi^2 k} \int_0^\infty dp \sin(kp) \int_0^{\infty} \frac{d\xi}{\xi} [1 - \exp(-\frac{r_L}{\xi})] \exp(-\frac{r_D}{\xi}).$$

(Eq. 162)

Equations (157), (160) and (161) together determine the total electron-ion collision operator $C_{ei}$ on the right hand side of Eq. (137), thereby determining $\delta \varphi^{(2)}$ in terms of the electric field $\Phi$. Substitution of this expression into Eq. (135) for the current density then yields the contribution to second order in the electrical conductivity. Thereby we find the following formula for the electrical conductivity:

$$\sigma = -\frac{\omega}{4\pi \omega} \gamma + \sigma^{(A)} + \sigma^{(B)} + \sigma^{(C)},$$

where

$$\sigma^{(A)} = \frac{4\pi^2 n_e^2}{T_0 m_e^2 \omega} \int d^3v_1 \int d^3k \frac{k^3 k'}{k^2} \frac{\Phi(k)}{k} \frac{k \cdot x_1}{k} \exp(-\frac{k \cdot x_1}{k}) \exp(-\frac{r_D}{\lambda}) f_e^0(v_1)(1 + k - r_D)^{-2},$$

(Eq. 164)

$$\sigma^{(B)} = -\frac{\omega^2}{m_e^2 T_0} \int d^3v_1 \varphi(x) \exp(-\frac{x}{r_D}) \int d^3v_2 U(v^2 - \frac{2e^2}{m_e^2}).$$
\[
(v_{12} - \frac{2e^2}{m_e x})^{\frac{1}{2}} \exp\left(\frac{r_L}{x}\right) \tilde{g}(x; \chi_1, \chi_2) f_e^{0}(v_1) f_e^{0}(v_2),
\]

where \( \tilde{g} \) and \( \tilde{f} \) are defined by Eqs. (145) and (162), respectively, and \( \tilde{j} \) is the unit dyadic. In Eq. (163), the first term on the right is the first order (in \( \varepsilon \)) contribution and all other terms are collisional contributions, found in second order in \( \varepsilon \).

The integrals in Eqs. (164), (165) and (166) were performed by R.L. Guernsey [cf. Appendix D] and the result is

\[
\tilde{g} = -\frac{\omega}{4\pi i \omega} + \frac{2}{3} \frac{\omega}{2} - \frac{1}{2} \frac{3}{2} \left(-\ln \varepsilon - 2\gamma - \frac{1}{2} + 2 \ln 2\right),
\]

where \( \varepsilon = \frac{r_L}{x_0} \) is the plasma parameter and \( \gamma \) is Euler's constant, \( \gamma = 0.577... \)

The above expression agrees with R.L. Guernsey's result, if his result is corrected for the fact that R.L. Guernsey originally considered only two out of three contributions to the conductivity, whereas all three turn out to be of the same order a posteriori.
APPENDIX A:

SOME PROPERTIES OF THE CORRELATIONS ACCORDING TO THE BOLTZMANN THEORY.

In this Appendix it is shown that

1') In Eq. (30) for the asymptotic binary correlation \( g_{2A}(0) \) the unit vector \( \vec{e} \) is given by Eq. (31);

2') The Landau correlation \( \Delta_A^{(0)} \) can be obtained by Taylor-expansion of \( g_{2A}(0) \) around infinite \( |\vec{r}_{12}| \);

3') If \( \vec{e} \neq 0 \) , then

\[
D_A^{(0)}(\vec{e} + \xi, \xi; \chi_1, \chi_2, \chi_3) = O(\xi^{-1}), \quad \xi_1 + \omega
\]

where \( \xi_1 = \vec{e} - \vec{\xi} = \frac{\chi_{23} \bar{\chi}_{23}}{\chi_{23}^2} \).

\[\text{(A.1)}\]

**Ad. 1')**

Determination of \( \vec{e} \equiv \frac{\chi_{12}(\omega)}{|\chi_{12}(\omega)|} \).

Lemma: The vector \( \vec{e} \equiv \chi_{12} \times \vec{L} + e^2 \frac{\bar{\chi}_{12}}{|\chi_{12}|} \), \( \vec{L} \) being the angular momentum, is conserved.

For a proof: Landau & Lifschitz, "Mechanics", ch. III, last section.

Consequently:

\[
\chi_{12} \times \frac{m}{2} (\chi_{12} \times \bar{\chi}_{12}) + e^2 \frac{\bar{\chi}_{12}(\omega)}{|\chi_{12}(\omega)|} = \chi_{12}(\omega) \times \frac{m}{2} [\chi_{12}(\omega) \times \bar{\chi}_{12}(\omega)] + e^2 \frac{\bar{\chi}_{12}}{|\chi_{12}|},
\]

\[\text{(A.2)}\]
where \( \frac{\chi_{120}}{|\chi_{120}|} \) is the initial direction of the relative position of the particles, being opposite to the initial direction of the relative velocity.

\[
\frac{\chi_{120}(\omega)}{|\chi_{120}(\omega)|} = - \frac{\chi_{120}(\omega)}{|\chi_{120}(\omega)|} = - \hat{\xi}
\]  

(A.3)

Using (A.3), we can write (A.2) in the form:

\[
- \chi_{12} \times \hat{L} - e^2 \hat{\xi} = - |\chi_{12}(\omega)| \hat{\xi} \times \hat{L} + e^2 \frac{\chi_{12}}{\chi_{12}}.
\]  

(A.4)

Now, take the inner product with \( \hat{\xi} \):

\[
\hat{\xi} \cdot (\chi_{12} \times \hat{L}) + e^2 = - e^2 \hat{\xi} \cdot \frac{\chi_{12}}{\chi_{12}}.
\]  

(A.5)

Hence:

\[
\hat{\xi} \cdot \hat{L} = - e^2.
\]  

(A.6)

This equation gives the component of \( \hat{\xi} \) in the direction of \( \hat{L} \).

Since the motion takes place in a plane perpendicular to the angular momentum \( \hat{L} \),

\[
\hat{\xi} \cdot \hat{L} = 0.
\]  

(A.7)

A third constant vector of the motion is obtained by taking the outer product of (A.4) with respect to \( \hat{L} \) and inserting Eq. (A.2),

\[
\chi_{12} L^2 + e^2 \hat{L} \times \frac{\chi_{12}}{\chi_{12}} = v_{12(\omega)} \hat{\xi} L^2 - e^2 \hat{\xi} \times \hat{L}.
\]  

(A.8)
Now, take the inner product of Eq. (A.8) with respect to \( \xi \) to obtain

\[
\xi \cdot \left( v_{12} L^2 + e^2 \frac{L}{c} \times \frac{x_{12}}{x_{12}} \right) = \xi \cdot \psi = v_{12}^0(\xi)L^2, \tag{A.9}
\]

where we defined

\[
\psi = L \times \xi. \tag{A.10}
\]

Since \( \{\xi, L, \psi\} \) is an orthogonal basis Eqs. (A.6), (A.7) and (A.9) completely determine \( \xi \), i.e.,

\[
\xi = -e^2 \frac{c}{c^2} + v_{12}^0(\psi)L^2 \frac{\psi}{\psi^2}. \tag{A.11}
\]

Combination of Eqs. (28), (A.10) and (A.11) yields the desired result [Eq. (31)] for \( \xi \).

---

Expansion of \( g_{2A}^{(0)} \) around \( x_{12} = \infty \).

According to formula Eqs. (30) and (31) we have:

\[
g_{2A}^{(0)}(\xi, x_1, x_2) = F \left( x_1 + \frac{e^2}{4a^2_{mx}} \right) + F \left( x_2 + \frac{e^2}{4a^2_{mx}} \right)
\]

\[
(\psi \times L) + e^2 \frac{\hat{g}_x}{\sqrt{x}} - 2 \hat{g}_x + \left( \psi \times L \right) + e^2 \frac{\hat{g}_x}{\sqrt{x}} - 2 \hat{g}_x
\]

\[
\frac{1}{2} \left( u^2 + \frac{4a^2}{mx} \right) L^2 \left( \psi \times L \right) + e^2 \frac{\hat{g}_x}{\sqrt{x}} |\psi|L^2 +
\]

\[
e^2 \frac{\hat{g}_x}{\sqrt{x}} - 2 \hat{g}_x |\psi|L^2 - F(\psi)F(\psi), \tag{A.12}
\]
where we dropped indices 1, 2 in the first term and wrote \( \nu \) for the center of mass velocity, \( \nu^r \) for the relative velocity \( \nu_{12}^r \), 
\( \xi = \xi_{12} \) and \( L = \frac{m}{2} \xi \times \nu \) for the angular momentum \( \xi \times \nu / x \).

For large \( L \) we can approximate the second term in the argument of the distribution function \( F_j \) as follows:

\[
\pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( -\eta^2 \right) \left( \nu \times \nu^r \right)^2 = \pm \frac{1}{2} \left( -\eta^2 \right) u \frac{\nu \times \nu^r}{\left| \nu \times \nu^r \right|^2} + O(L^{-2}), \quad L \to \infty
\]

\[
= \pm \frac{\eta^2}{2L} \left( \xi_L \times \xi_u \right) + O(L^{-2}), \quad L \to \infty
\]

(A.13)

where \( \xi_L = \frac{\xi}{L} \).

And for the third term we can write:

\[
\pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( \nu \times \nu^r \right)^2 = \pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( \nu \times \nu^r \right)^2 = \pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( \nu \times \nu^r \right)^2
\]

\[
= \pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( \nu \times \nu^r \right)^2 = \pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( \nu \times \nu^r \right)^2
\]

\[
= \pm \frac{1}{2} \left( u^2 + \frac{4e^2}{m u} \right) \left( \nu \times \nu^r \right)^2
\]

(A.14)
Hence:

\[ \pm 4 \left( u^2 + \frac{4 \epsilon^2}{\mu \nu} \right) L^2 (\mu L^2 + \epsilon^2 \frac{\partial}{\partial \mu} \times \hat{e}_x) \left| \mu L^2 + \epsilon^2 \frac{\partial}{\partial \mu} \times \hat{e}_x \right|^{-2} = \]

\[ = \pm i \gamma \pm \frac{\epsilon^2}{2L} \frac{\partial}{\partial L} \times \hat{e}_x \times \hat{e}_x + \mathcal{O}(L^{-2}), \quad L \to \infty. \quad (A.15) \]

From (A.13) and (A.15) it follows, that (A.12) can be written as:

\[ g_{2A}^{(0)} (\xi, \nu + \frac{i}{2} \mu, \nu - \frac{i}{2} \mu) = F_1 \{ \nu + \frac{i}{2} \mu + \hat{\psi} (\xi, \nu) \} \]

\[ F_1 \{ \nu - \frac{i}{2} \mu - \hat{\psi} (\xi, \nu) \} - F_1 \{ \nu + \frac{i}{2} \mu \} \quad F_1 \{ \nu - \frac{i}{2} \mu \}, \quad (A.16) \]

where:

\[ \hat{\psi} (\xi, \nu) = \frac{\epsilon^2}{2L} \{ \hat{e}_x \times \hat{e}_u + \hat{e}_u \times \hat{e}_x \} + \mathcal{O}(L^{-2}), \quad L \to \infty \]

\[ = \hat{\psi} (\xi, \nu) + \mathcal{O}(L^{-2}), \quad L \to \infty, \quad (A.17) \]

the last equality defining \( \hat{\psi} \).

Provided \( F_1 \) is analytic in \( \nu + \frac{i}{2} \mu \), simple Taylor expansion of (A.16) leads to

\[ g_{2A}^{(0)} (\xi, \nu + \frac{i}{2} \mu, \nu - \frac{i}{2} \mu) = \hat{\psi} (\xi, \nu) \cdot \left( \frac{\partial}{\partial \chi_1} - \frac{\partial}{\partial \chi_2} \right) F_1 (\chi_1) F_1 (\chi_2) + \]

\[ + \mathcal{O}(L^{-2}), \quad L \to \infty \quad (A.18) \]
It remains to be shown that \( \chi' = \kappa' \), \( \kappa' \) being given by Eq. (46).

In view of the vector identity
\[
\mathbf{u} \cdot (\mathbf{x} \times \mathbf{u}) \times \mathbf{v} = \{\mathbf{u} \times (\mathbf{x} \cdot \mathbf{u})\} \mathbf{v} + (\mathbf{x} \times \mathbf{u}) \cdot (\mathbf{v} \times (\mathbf{x} \times \mathbf{u})),
\]
(A.19)

it is verified that
\[
\frac{e^2}{2L} \mathbf{e}_L \times \mathbf{e}_X + \frac{e^2}{2L} \mathbf{e}_L \times \mathbf{e}_X = \frac{e^2 \mu}{m\upsilon^2} + \left(1 + \frac{x_1 \upsilon}{x_2}\right) \frac{e^2}{2L} \mathbf{e}_L \times \mathbf{e}_U.
\]
(A.20)

It is readily seen that \( \kappa' \) as defined by Eq. (46) can be cast in the form of the right hand side of (A.20), so that we have found that
\[
\varepsilon_{2A}^{(0)}(\chi_{12}; \chi_1, \chi_2) = \Delta_A^{(0)}(\chi_{12}; \chi_1, \chi_2) + O(\chi_{12}^{-2}),
\]
(A.21)

(A.21') Proof of Eq. (A.1).

To this end we integrate Eq. (37) along its characteristics. This yields
\[
\Delta_A^{(0)}(\chi_{12} + \lambda \chi_{23}; \chi_1, \chi_2, \chi_3) = \int_0^\infty ds \lambda \chi_{12} \chi_{23} (s; \chi_1, \chi_2)
\]
\[
[F_1(2) \Delta_A^{(0)}(\lambda \chi_{23} - \chi_{13} s; \chi_1, \chi_3; s) +
\]
\[
+ F_1(1) \Delta_A^{(0)}(\lambda \chi_{23} - \chi_2 s; \chi_2, \chi_3; s)] +
\]
\[ + \int_0^\infty ds \, I_2(\chi_{12} + \lambda \chi_{23} - \chi_{13} s) [F_1(1)\Delta^{(0)}(\lambda \chi_{23} - \chi_{23} s; \chi_2, \chi_3; s) + F_1(3)\Delta^{(0)}(\chi_{12} - \chi_{12} s; \chi_1, \chi_2; s)] \]

\[ + \int_0^\infty \, ds \, I_2(\lambda \chi_{23} - \chi_{23} s) [F_1(2)\Delta^{(0)}(\chi_{12} + \lambda \chi_{23}; \chi_1, \chi_3; s) + F_1(3)\Delta^{(0)}(\chi_{12} - \chi_{12} s; \chi_1, \chi_2; s)]. \quad (A.22) \]

In view of the presence of \( \lambda \) within the \( I_2 \)-operators of the last two terms in (A.22), and its absence in \( I_2(\chi_{12} - \chi_{12} s; \chi_1, \chi_2) \), the first term in (A.22) dominates for large \( \lambda \), i.e.,

\[ D^{(0)}_A(\chi_{12} + \lambda \chi_{23}, \lambda \chi_{23}; \chi_1, \chi_2, \chi_3) \sim \int_0^\infty ds \, I_2(\chi_{12} - \chi_{12} s; \chi_1, \chi_2, \chi_3; s) \]

\[ + F_1(1)\Delta^{(0)}(\lambda \chi_{23} - \chi_{23} s; \chi_2, \chi_3; s), \lambda \to \infty. \quad (A.23) \]

From (Eq. 32), again by integration along characteristics,

\[ \Delta^{(0)}(\chi_{ij} - \chi_{ij}; s; \chi_i, \chi_j; s) = \int_0^s ds \, I_2(\chi_{ij} - \chi_{ij}; s; \chi_i, \chi_j; s) \]

\[ F_1(\chi_i)F_1(\chi_j) = \int_0^{2s} dn \, I_2(\chi_{ij} - \chi_{ij}\eta) \]

\[ F_1(\chi_i)F_1(\chi_j) = \Delta^{(0)}(\chi_{ij}; \chi_i, \chi_j; 2s) - \Delta^{(0)}(\chi_{ij}; \chi_i, \chi_j; s). \quad (A.24) \]
However, again from Eq. (32),

\[
\Delta^{(0)}(\xi_1, \chi_{1j}; t) = \int_0^t ds \Psi_0(\chi_{1j} - \chi_{1j} s).
\]

\[
\frac{3}{2} \frac{\partial}{\partial \chi_{1j}} F_1(\chi_{1j}) F_1(\chi_{2j}) = \frac{2}{m} \int_0^t ds \left( \frac{-\chi_{1j} + \chi_{12}}{[(\chi_{1j} - \chi_{12})^2 + \chi_{1j}^2]^{3/2}} \right) \chi_{12} - \chi_{12} \int_0^t ds \left( \frac{3}{2 \chi_{2j}} \right) F_1(\chi_{1j}) F_1(\chi_{2j}),
\]

where \( \chi_{1j} = \frac{(\chi_{1j} - \chi_{12})}{\chi_{12}} \) \( \chi_{12} = \chi - \chi_{1j} \).

Integration (cf. Gradshteyn & Ryzhik, p. 83, fm. 5) gives

\[
\Delta^{(0)}(\chi_1, \chi_{1j}, \chi_{2j}; t) = \frac{-1}{\chi_{12}} \left[ \frac{1}{\chi_{1j}} \right] - \frac{1}{[\chi_{1j}^2 + (t - \chi_{1j})^2]^{1/2}} \chi_{1j}, \chi_{1j} \rightarrow \infty.
\]

Using (A.27) and (A.23) we obtain

\[
|\Delta^{(0)}(\chi_{1j} \quad \chi_{1j}; s) | \leq A \frac{1}{\chi_{1j}}, \chi_{1j} \rightarrow \infty,
\]
\[ D_A^0(x_{12} + \lambda x_{23}; x_1, x_2, x_3) \mid z \mid \int_0^{\infty} ds \frac{1}{|x_{12} - x_{12}^s|^2} \]

\[ \{B \frac{1}{|x_{12} + \lambda x_{23}|_{(1,3)}^i} + C \frac{1}{|\lambda x_{23}|_{(2,3)}^i} \}, \quad (A.28) \]

where \( B \) and \( C \) are positive numbers independent of \( \lambda \) and \( s \), and

where "\( x \)" denotes the projection perpendicular to \( y_{ij} \), of \( x \).

Provided \( x_{1,1,2} \neq 0 \), the integral

\[ \int_0^{\infty} ds \frac{1}{|x_{12} - x_{12}^s|^2} \]

converges, so that \( [\text{Eq. (A.27)}] \)

\[ D_A^0(x_{12} + \lambda x_{23}; x_1, x_2, x_3) \mid z \mid \frac{D}{\lambda}, \lambda \to \infty, \]

where \( D \) is some positive constant.

This completes the proof of (A.1).
In this Appendix the relative error made in the iterative approach to Eq. (77) is estimated.

When we substitute the solution of Eq. (78) into the $L_1^+$-terms of Eq. (77), we obtain, e.g. for the $L_1^{(1)+}$-term, a term the order of magnitude of which does not exceed

$$\varepsilon L_1^{(1)+} |(\varepsilon_{2A} (\xi_{23}; \chi_2, \chi_3) - \Delta_A (0) (\xi_{23}; \chi_2, \chi_3))|$$

$$g_B (\xi_{23}; \chi_2, \chi_3) F_1 (\chi_1)$$

$$\Delta_A (0) (\xi_{23}; \chi_2, \chi_3)$$

(B.1)

Since over the integration interval, $R_1(2,3)$, of the $L_1^{(1)+}$-operator the Landau correlation $\Delta_A (0)$ dominates in the factor between curly brackets, the order of magnitude of the expression in (B.1) does not exceed

$$\varepsilon^2 L_1^{(1)+} |g_B (\xi_{23}; \chi_2, \chi_3) F_1 (\chi_1)|$$

(B.2)

or, writing $L_1^{(1)+}$ explicitly:

$$\varepsilon^2 d^3 \nu_3 \int d^3 \eta \phi (\xi_{12} + \eta) g_B (\eta; \chi_2, \chi_3) F_1 (\chi_1)$$

(B.3)

where the prime denotes integration is restricted to $\eta \lesssim \varepsilon^{1-R_1}$.

From Eq. (53) it is clear that $g_B (2,3)$ is behaving singularly along the axis parallel to the relative velocity $\chi_{23}$, like $1/|\eta_\perp|$, where $\eta_\perp$ is the projection of $\eta$ in the plane perpendicular to $\chi_{23}$. For an order of magnitude estimate, $g_B (\eta; \chi_2, \chi_3)$ may
therefore be replaced by \( \frac{x(y_2, y_3)}{n_{1}} \), where \( x \) is some integrable function; through a trivial triangular inequality we now obtain the following estimate for the absolute value \( E \) of the \( L_{1}^{(1)} \) term in Eq. (78):

\[
E \leq \epsilon^{2}\int d^{3}v_{3} \int_{0}^{2\pi} d\psi \int dz \int_{-\infty}^{\infty} d\phi |x(y_2, y_3)| \frac{1}{(\xi_{1}^{2} + \rho^{2} + z^{2})^2 - 4\xi_{1}^{2}\rho^{2}}
\]

(B.4)

where we used cylindrical coordinates \( \xi = (z, \rho, \psi) \);

\( \xi = (\xi_{\parallel}, \xi_{\perp}, \theta) \).

Since

\[
\int_{0}^{2\pi} \frac{d\psi}{a + b \cos \psi} = \frac{\pi}{(a^{2} - b^{2})^{\frac{1}{2}}},
\]

(B.5)

if \( a > b \), we arrive at the following estimate:

\[
E \leq \pi \epsilon^{2}\int d^{3}v_{3} |x(y_2, y_3)| \int_{-\infty}^{\infty} dz \int_{-\infty}^{\epsilon^{1-R_{1}}} d\phi \frac{1}{(\xi_{1}^{2} + \rho^{2} + z^{2})^2 - 4\xi_{1}^{2}\rho^{2}} =
\]

\[
= 2\pi \epsilon^{2}\int d^{3}v_{3} \int_{0}^{\infty} d\phi \int_{-\infty}^{\epsilon^{1-R_{1}}} dz \frac{1}{\{[z^{2} + (\rho + \xi_{1})^{2}][z^{2} + (\rho - \xi_{1})^{2}]\}^{\frac{1}{2}}}
\]

(B.6)

Since

\[
\int_{0}^{\infty} \frac{dz}{[(z^{2} + a^{2})(z^{2} + b^{2})]^{\frac{1}{2}}} = \frac{1}{a} \log \frac{(a^{2} - b^{2})^{\frac{1}{2}}}{a},
\]

(B.7)
where $K$ is the complete elliptic integral of the first kind (cf., e.g., Gradshteyn & Ryzhik), estimate (B.6) can be simplified to yield

$$E \leq 2\pi \varepsilon^2 \int d^3 \nu_3 \chi(\nu_2, \nu_3) \frac{\int_0^{1-R_1} d\rho}{\rho + \xi_1} K \left( \frac{2\sqrt{\rho \xi_1}}{\rho + \xi_1} \right), \quad (B.8)$$

Since

$$\frac{2\sqrt{\rho \xi_1}}{\rho + \xi_1} = \sqrt{\frac{4 \rho \xi_1}{\rho^2 + \xi_1^2 + 2\rho \xi_1}} \leq 1 < \frac{\pi}{2} \quad (B.9)$$

for all $\rho$ and all $\xi_1$, it follows that

$$0 < K \left( \frac{2\sqrt{\rho \xi_1}}{\rho + \xi_1} \right) = K(1) = 1, \quad (B.10)$$

(cf., e.g. Abramowitz & Stegun p. 592, for all $\rho$ and all $\xi_1$.

Therefore estimate (B.9) can still be simplified to yield

$$E \leq 2\pi K(1) \varepsilon^2 \int d^3 \nu_3 |\chi(\nu_2, \nu_3)| \int_0^{1-R_1} \frac{d\rho}{\rho + \xi_1} \xi_1$$

$$\leq K(1) \varepsilon^2 2\pi \int d^3 \nu_3 \chi(\nu_23, \nu_2)| \ln \left(1 + \frac{\frac{1-R_1}{\xi_1}}{\xi_1} \right), \quad (B.11)$$

where $\xi_1 = \xi \sin \alpha$, $\alpha$ being the angle between $\nu_23$ and $\xi$.

Taking spherical coordinates $\nu_23 = \chi = (v, \alpha, \phi)$ with $\xi$ along the z-axis we can write

$$E \leq \varepsilon^2 \int_0^\infty dv \int_0^\infty d\phi \int_0^{\alpha(\sin)} \frac{\sin \alpha d\alpha |\chi(v, \alpha, \nu_3)| \ln \left(1 + \frac{\frac{1-R_1}{\xi \sin \alpha}}{\xi \sin \alpha} \right). \quad (B.12)$$
Except if \( \sin \alpha \approx \frac{e^{1-R_1}}{\xi} \) we may approximate the logarithm by
\[
\frac{e^{1-R_1}}{\xi \sin \alpha}.
\]
Since \( \chi \) is bounded we then obtain the following estimate:
\[
\mathcal{E} \approx \varepsilon^2 \int_0^1 \frac{\sin \alpha}{\xi \sin \alpha} + \int_0^1 \varepsilon \alpha \ln \alpha^{-1} \cdot \frac{2(1-R_1)}{\xi^2},
\]
(B.13)

which is clearly of order

\[
\text{Max. } \{ \varepsilon^2 \frac{e^{1-R_1}}{\xi}, \varepsilon^2 \frac{2(1-R_1)}{\xi^2} \},
\]
(B.14)

The relative error made in the iterative approach to Eq. (77)
can therefore be estimated by
\[
\left| \varepsilon^2 \left( \xi_1 \chi_1 + \chi_2 \right) \right| \frac{\xi}{\xi^2},
\]
(B.15)

provided we restrict considerations to \( |\xi| < 1 \), which we did from the outset.
In this Appendix it is shown that

\[ g_B(x_1, x_2) - \Delta_A^{(0)}(x_1, x_2) = 0(x_{12}), \quad x_{12} = 0, \]

where \( x_{12} = x - x \cdot \frac{x_{12}}{x_{12}} \), as before.

Define:

\[ \delta g_A(x_1, x_2) = g_B(x_1, x_2) - \Delta_A^{(0)}(x_1, x_2). \quad \text{(C.1)} \]

Since

\[ \Delta_A^{(0)}(x_1, x_2) = \frac{u_p}{(u_1 - u_2 - i \omega) 8 \pi n} \left[ \frac{\partial F(x_1)}{\partial u_1} - F(x_1), \frac{\partial F(x_2)}{\partial u_2} \right] \]

and in view of Eq. (50), it is clear that

\[ \delta \tilde{g}_A(x_1, x_2) = \frac{u_2}{(u_1 - u_2 - i \omega) 8 \pi n} \left[ x(x_1, x_2) + x(x_2, x_1) \right], \quad \text{(C.3)} \]

with

\[ x(x_1, x_2) = \eta(x_1, x_2) + \xi(x_1, x_2), \quad \text{(C.4)} \]

where
\[
\eta(k; \chi_1, \chi_2) = \frac{\partial F(\chi_1)}{\partial u_1} \frac{1}{\partial^2 F(u_2)} \left( -\hat{F}(u_2) + \frac{\partial^2 \hat{F}(u_2)}{\partial u_2} \right) \left( \frac{(\partial^2 \ln \hat{F}(u_2))^{-1}}{Z^+(u_2)} \right)
\]
and
\[
\xi(k; \chi_1, \chi_2) = \left[ \hat{F}(u_2) \frac{\partial}{\partial u_2} F(\chi_2) - F(\chi_2) \frac{\partial}{\partial u_2} \hat{F}(u_2) \right] \times \frac{2 \pi i u^2}{\partial^2 F(u_2)} \left[ 1 + \frac{2 \pi i u^2}{\partial^2 F(u_2)} \right]^{-1}
\]
(For convenience we omitted the index 1 in \(F_1\)).

Consequently, in evaluating \(\delta g_A(\lambda \chi, \chi_1, \chi_2)\) for small \(\lambda\), -the cylindrical coordinates being taken such that the axis is parallel to \(\chi_12\), we only have to distinguish two types of contributions, namely:

\[
\delta g_A^{(1)} = \int d^2 k_\perp \int d\lambda \exp(ik_\perp \cdot \chi_1) \frac{\exp(\i \lambda k_\perp \cdot \chi_1) u^2}{(u_1 - u_2 - \i \alpha)} \eta(k_\perp, k_\parallel; \chi_1, \chi_2)
\]

(C.7)

and

\[
\delta g_A^{(2)} = \int d^2 k_\perp \int d\lambda \exp(ik_\perp \cdot \chi_1) \frac{\exp(\i \lambda k_\perp \cdot \chi_1) u^2}{(u_1 - u_2 - \i \alpha)} \xi(k_\perp, k_\parallel; \chi_1, \chi_2).
\]

(C.8)

The function \(\hat{F}\) only depends on \(k_\parallel, k_\perp\) through \(u = \frac{k_\parallel \cdot \chi_1}{k}\).
We readily obtain:

\[ \delta_{g_{A}}^{(1)} = \lambda^{-1} \frac{1}{d\omega/dk_{\perp}} k_{\perp} k_{\parallel} \exp(i k_{\perp} x_{\perp} \cos \psi + i k_{\parallel} x_{\parallel} \lambda^{-1}) \left( \frac{\omega^{2}}{k^{2}} \right) \]

\[ \frac{2 \pi i \delta^{-} \left[ \frac{k_{\parallel} v_{12}}{(k_{\parallel}^{2} + k_{\perp}^{2})^{1/2}} \right]}{\partial \nu \left( u_{2} \right)} \frac{1}{8 \pi \hbar} \left[ \partial \nu \left( \nu^{-1} \right) \right] \frac{k_{\parallel} k_{\perp}}{(k_{\parallel}^{2} + k_{\perp}^{2})} \cdot \]

\[ \nu_{2} \left( \nu^{-1} \right) \left\{ -\left( \frac{2}{\nu} \left( u_{2} \right) + \frac{1}{\nu_{2}} \right) \frac{1}{2 \pi i} \int \frac{d\nu}{\nu - u_{2}} \left( 1 + \frac{2 \pi i \omega \lambda_{2}^{2}}{k_{\perp}^{2} + k_{\parallel}^{2}} \right) \right\}^{-1} \frac{\partial \nu \left( \nu^{-1} \right)}{\mu - u_{2}} \]

\[ \left( \frac{\partial \nu \left( \nu^{-1} \right)}{\mu - u_{2}} \right) \]

\[ \left( \frac{\partial \nu \left( \nu^{-1} \right)}{\mu - u_{2}} \right) \]

The functions \[ 1 + \frac{2 \pi i \omega \lambda_{2}^{2}}{k_{\perp}^{2} + k_{\parallel}^{2}} \frac{\partial \nu^{\pm}}{\partial \nu} \left( \nu \right) \] do not have zeros in the lower and upper half planes respectively, at least for a stable plasma.

The \( \int \) - integral can be calculated by closing the contour in the lower half plane.

Under the rather mild restrictions, the arc does not contribute in the limit \( R \to \infty \).

Therefore:
\[ \frac{1}{2\pi i} \oint \frac{d\mu}{\mu - u_2} = \left( \frac{\partial \ln \Phi(\mu)}{\partial \mu} \right)^{-1} \left[ 2\pi i \omega^2 \lambda^2 \frac{\partial^2 \Phi(\mu)}{\partial \mu^2} \right] \]

\[ \left[ 1 + \frac{p}{k_\parallel + k_\perp} \frac{1}{\partial \mu} \right] \]

\[ = \sum_j R_j(u_2) \]

\[ \frac{1}{k_\parallel^2 + k_\perp^2} \frac{\partial \Phi(\mu)}{\partial \mu} \]

where \( R_j \) is defined as the residue of \( \left( \frac{\partial \ln \Phi(\mu)}{\partial \mu} \right)^{-1} \frac{1}{\mu - u_2} \) at a point \( \mu_j \) of singularity in the lower half plane, i.e. \( \text{Im} \mu_j < 0 \).

Similarly, the \( \oint \) integral can be evaluated by closing the contour in the upper half plane.

Again the arc does not contribute in the limit \( R \to \infty \), and we find:

\[ -\frac{1}{2\pi i} \oint \frac{d\mu}{\mu - u_2} = \left( \frac{\partial \ln \Phi(\mu)}{\partial \mu} \right)^{-1} \left[ 2\pi i \omega^2 \lambda^2 \frac{\partial^2 \Phi(\mu)}{\partial \mu^2} \right] \]

\[ \left[ 1 - \frac{p}{k_\parallel + k_\perp} \frac{1}{\partial \mu} \right] \]

\[ = \sum_j \rho_j(u_2) \]

\[ \frac{1}{k_\parallel^2 + k_\perp^2} \frac{\partial \Phi(\mu)}{\partial \mu} \]

The summation now being over all residues \( \rho_j \) of \( \frac{1}{\mu - u_2} \left( \partial \ln \Phi(\mu) \right)^{-1} \) at \( \mu \) within \( \text{Im} \mu_j \geq 0 \).

Since for \( \lambda = 0 \) the sum of the left hand sides of (C.10) and
(C.11) is equal to \([\beta u_2 \ln \tilde{F}(u_2)]^{-1}\), we have the following expression for the factor between the doubly curled brackets of (C.9):

\[
-\mathbb{F}’(u_2) + \frac{\beta \mathbb{F}(u_2)}{\beta u_2} \left(\frac{1}{2\pi i} \oint \frac{d\mu}{\mu - u_2} \left[\beta \frac{\ln \tilde{F}(\mu)}{\mu} \right]^{-1} \frac{2\pi i \omega^2 \lambda^2 \beta \mathbb{F}(\mu)}{k_\parallel^2 + k_\perp^2 \beta \mu} \right)
\]

\[
- \frac{1}{2\pi i} \oint \frac{d\mu}{\omega - u_2} \left[\beta \frac{\ln \tilde{F}(\mu)}{\mu} \right]^{-1} \frac{2\pi i \omega^2 \lambda^2 \beta \mathbb{F}(\mu)}{k_\parallel^2 + k_\perp^2 \beta \mu}
\]

\[
\frac{\partial \mathbb{F}}{\partial u_2} \sum_j R_j \frac{2\pi i \omega^2 \lambda^2 (k_\parallel^2 + k_\perp^2)^{-1} \beta \mathbb{F}(\mu_j)}{[1 + \frac{2\pi i \omega^2 \lambda^2 \beta \mathbb{F}(\mu_j)}{k_\parallel^2 + k_\perp^2 \beta \mu_j}]} \beta \mu_j
\]

\[
- \frac{\partial \mathbb{F}}{\partial u_2} \sum_j \frac{2\pi i \omega^2 \lambda^2 (k_\parallel^2 + k_\perp^2)^{-1} \beta \mathbb{F}(\mu_j)}{[1 + \frac{2\pi i \omega^2 \lambda^2 \beta \mathbb{F}(\mu_j)}{k_\parallel^2 + k_\perp^2 \beta \mu_j}]} \beta \mu_j
\]  

(C.12)

N.B.: The \(R_j\) and \(p_j\) depend on \(u_j\) and on \(u_2\), hence on \(\psi, k_\parallel\) and \(k_\perp\).

The factor \(\delta \mathbb{F}(\chi_j), \kappa \kappa \delta \mathbb{F}(\chi_2)\) in (C.9) can be written as:

\[
\delta \mathbb{F}(\chi_j), \kappa \kappa \delta \mathbb{F}(\chi_2) = A \kappa_\parallel^2 + B k_\parallel k_\perp + C \kappa_\perp^2 ,
\]  

(C.13)

where \(A, B\) and \(C\) depend on \(\psi, \chi_1, \chi_2\) only.

Substitution of (C.12), (C.13) into (C.9) yields:

\[
\delta \mathbb{A}^{(1)}(\lambda \omega) = \frac{\lambda \omega}{8\pi \frac{n}{3}} \left\{ \int \frac{d\psi}{2\pi} \int \frac{dk_\parallel}{k_\parallel} \int \frac{dk_\perp}{k_\perp} \exp(ik_\perp \cos \psi + ik_\parallel \lambda^{-1}) \right\}
\]

\[
\left(\kappa_\parallel^2 + \kappa_\perp^2 \right)^{-2} \delta \left[ \frac{k \chi_2}{(k_\parallel^2 + k_\perp^2)^2} \right] (A \kappa_\parallel^2 + B k_\parallel k_\perp + C \kappa_\perp^2)
\]
In view of the equality:

$$\delta \left[ \frac{x}{(x^2 + a^2)^{1/4}} \right] = |a| \delta(x),$$

(c. 15)

Eq. (c. 14) can be written as:

$$\delta g_A^{(1)} = \frac{\lambda \omega^4}{2\pi} f_0^2 \psi \int_0^\infty dk_{\perp} k_{\perp} \int_{-\infty}^{\infty} dk_{||} \exp(ik_{\perp} x_{\perp} \cos \psi + ik_{||} x_{||} \lambda^{-1})$$

$$- \left( k_{||}^2 + k_{\perp}^2 \right)^{-2} \frac{1}{2\pi v_{12}} \delta(k_{||}) + \frac{1}{2\pi v_{12}} (k_{||}^2 + k_{\perp}^2)^{1/2} p(k_{||}^{-1})$$

$$\left( A k_{||}^2 + B k_{||} k_{\perp} + C k_{\perp}^2 \right) \left\{ \sum_{j} \left[ k_{||}^2 + k_{\perp}^2 - 2\pi i \omega^2 \frac{\partial F^j_{\perp}}{\partial u_j} \right] \right\}$$

$$R_j(u_2) \frac{\partial F^j(u_j)}{\partial u_j}$$

First consider the term in (c. 16) corresponding with the $\delta$-function.

As can be easily checked, only the subterm $\delta g_A^{(1)}$ corresponding with $C k_{\perp}^2$ remains:

$$\delta g_A^{(1)} = \frac{\lambda \omega^4}{4\pi v_{12}} \int_0^2 \psi \int_0^\infty dk_{||} \exp(ik_{\perp} x_{\perp} \cos \psi) \left\{ \sum_{j} (-) R_j(u_2) \right\}$$
It should be noted that, through $u_2$, $R_j$ and $C_j$ depend on $k_\parallel$, $k_\perp$, $\Psi$, $\chi_2$; however, since $u_2 = \frac{k_\parallel \chi_2}{k}$ the $R_j$ and $C_j$ are uniformly bounded in $k_\parallel$, $k_\perp$. In other words: there are functions $\hat{R}_j(\Psi, \chi_2)$ and $\hat{C}_j(\Psi, \chi_2)$ such that:

$$|R_j(u_2)| \leq \hat{R}_j(\Psi, \chi_2) \text{ for all } k_\parallel, k_\perp;$$

$$|C_j(u_2)| \leq \hat{C}_j(\Psi, \chi_2) \text{ for all } k_\parallel, k_\perp. \quad (C.18)$$

All terms in (C.17) can therefore be estimated to be, in absolute value, less than

$$\int_0^{\infty} \frac{dk}{k} \frac{1}{|k_\perp + 2\pi \omega_2 \frac{2}{p} \frac{\partial \hat{F}}{\partial \mu_j}|} = \int_0^{\infty} \frac{dk}{k} \frac{1}{|k_\perp + 2\pi \omega_2 \frac{2}{p} \frac{\partial \hat{F}}{\partial \mu_j}|} = 0(1), \lambda \to 0 \quad (C.19)$$

in order of magnitude. Hence:

$$\delta g_{II}^{(1)} = 0(1), \lambda \to 0. \quad (C.20)$$

Next consider the subterm $\delta g_{II}^{(1)}$ with $P(\frac{1}{k_\parallel})$ in eq. (C.16)
Write the $R_j$ and $\rho_j$ as a sum of a symmetric and an antisymmetric term with respect to $k_\parallel$:

\[
R_j = \frac{1}{2} [R_j (k_\parallel) + R_j (-k_\parallel)] + \frac{1}{2} [R_j (k_\parallel) - R_j (-k_\parallel)] = \\
\equiv R_j^{S} (k_\parallel) + R_j^{A} (k_\parallel),
\]

\[\text{(C.22)}\]

Similarly for $\rho_j$. Consider a term with $R_j$. Apart from the factor $\exp (ik_\parallel x_\parallel \lambda^{-1})$, the integrand is decomposed in a symmetric and an anti-symmetric part with respect to $k_\parallel$. The symmetric part gives rise to a cosine transform, the anti-symmetric part to a sine transform:

\[
\delta g_{\text{II}}^{(1)} (R_j) = \frac{2 \omega^4}{4 \pi^2 \nu \nu_1^2} \int_0^{2\pi} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{dk}{2\pi} \frac{dk}{2\pi} \frac{dk}{2\pi} \frac{dk}{2\pi} \exp (ik_\parallel x_\parallel \lambda^{-1})
\]

\[\text{(C.21)}\]
\[
\exp(ik_\perp x_\perp \cos \psi)(k_\parallel^2 + k_\perp^2)^{-3/2} \left[ R_{jS}(k_\parallel)(\frac{\partial F}{\partial \mu_j}) \cdot \frac{R_{jA}(k_\parallel)}{(k_\parallel^2 + k_\perp^2 + 2\pi i \lambda^2 \omega \frac{\partial F}{\partial \mu_j})} \right] + (Ak_\parallel + C \frac{k_\perp^2}{k_\parallel}) \frac{R_{jA}(k_\parallel)}{(k_\parallel^2 + k_\perp^2 + 2\pi i \lambda^2 \omega \frac{\partial F}{\partial \mu_j})} - \frac{\pi i \lambda \omega \frac{\partial F}{\partial \mu_j}}{2\pi^3 n v_{12}} \int_0^{2\pi} d\psi \int dk_\perp k_\perp \int dk_\parallel \sin(k_\parallel x_\parallel)^{-1}
\]

\[
\exp(ik_\perp x_\perp \cos \psi)(k_\parallel^2 + k_\perp^2)^{-3/2} \left[ R_{jS}(k_\parallel)(\frac{\partial F}{\partial \mu_j}) \cdot \frac{R_{jA}(k_\parallel)}{(k_\parallel^2 + k_\perp^2 + 2\pi i \lambda^2 \omega \frac{\partial F}{\partial \mu_j})} \right] + (Ak_\parallel + C \frac{k_\perp^2}{k_\parallel}) R_{jA}(k_\parallel)
\]

(C.23)

N.B. It should be kept in mind that \( R_j \) also depends on \( \psi, \mu_j, x_2 \) and \( k_\perp \). But \( R_j \) is uniformly bounded in \( k_\parallel, k_\perp \).

The cosine transform \( CT \) in (C.23) can be estimated to have an absolute value less than:

\[
|CT| \leq \frac{\lambda \omega^4}{2\pi n v_{12}} \left| \frac{\partial^2 F}{\partial \mu_j} \right| R_{jS \max} \int |k_\parallel|^2 \int dk_\perp \int dk_\parallel (k_\parallel^2 + k_\perp^2)^{-3/2}
\]

\[
\frac{\pi \lambda \omega^2 \omega^2 \frac{\partial F}{\partial \mu_j}}{2\pi^3 n v_{12}} \int_0^{2\pi} d\psi \int dk_\perp k_\perp dk_\parallel (k_\parallel^2 + k_\perp^2)^{-3/2}
\]

\[
|A| k_\parallel^2 R_{jS} + |C| k_\parallel^2 R_{jA} \cdot \frac{\lambda \omega^2 \omega^2 \frac{\partial F}{\partial \mu_j}}{2\pi^3 n v_{12}} \int dk_\perp k_\perp \int dk_\parallel (k_\parallel^2 + k_\perp^2)^{-3/2}
\]

(C.24)
where $k_{j\text{max.}} = \max k_j, \text{over } \psi : 0 + 2\pi$.

Through transformation of the integration variables $k_\parallel = \lambda k_\parallel, k_\perp = \lambda k_\perp$, one obtains the result that:

$$|C_l| \leq \frac{\omega_p^4 R_{\text{max}}^2 |B|}{\pi n v_{12}} \int_0^\infty dk_\perp k_\perp^2 \int_0^\infty dk_\parallel (k_\parallel^2 + k_\perp^2)^{-3/2}$$

$$\frac{|\nabla^2|}{\partial \mu_j^2} \left| \begin{array}{c} k_\parallel^2 + k_\perp^2 + 2\pi\omega_p^2 \frac{\partial F}{\partial \mu_j} \\ \pi n v_{12} \end{array} \right| + \frac{\omega_p^4}{\pi n v_{12}} \int_0^\infty dk_\perp \int_0^\infty dk_\parallel (k_\parallel^2 + k_\perp^2)^{-3/2}$$

$$|A| k_\parallel R_j + \left| \begin{array}{c} k_\perp^2 \\ k_\parallel \\ R_j \end{array} \right| \frac{|\nabla^2|}{\partial \mu_j^2} \left| \begin{array}{c} k_\parallel^2 + k_\perp^2 + 2\pi\omega_p^2 \frac{\partial F}{\partial \mu_j} \\ \pi n v_{12} \end{array} \right| \frac{\lambda^2}{(k_\parallel^2 + k_\perp^2 + 2\pi\omega_p^2 \frac{\partial F}{\partial \mu_j})} \frac{\lambda^2}{(k_\parallel^2 + k_\perp^2 + 2\pi\omega_p^2 \frac{\partial F}{\partial \mu_j})}.$$  (C.25)

Since $2^\gamma(u)$ has no zeros in the lower half plane, the factor

$$\frac{1}{|k_\parallel^2 + k_\perp^2 + 2\pi\omega_p^2 \frac{\partial F}{\partial \mu_j}|}$$

is regular.

Next we will investigate the convergence of terms in (C.25).

In a neighbourhood of $k_\parallel = 0 = k_\perp$, the integrand of the first term in (C.25) behaves like $-k_\perp^2 (k_\parallel^2 + k_\perp^2)^{-3/2}$, apart from the trivial case when $\frac{\partial F}{\partial \mu_j} = 0$; so its integral converges at $k_\parallel = 0 = k_\perp$. In a neighbourhood of $k_\parallel = \infty = k_\perp$, the integrand of the first term behaves like $k_\perp^2 (k_\parallel^2 + k_\perp^2)^{-3/2}$, so its integral converges at $k_\parallel = \infty = k_\perp$. Therefore the first term converges.

(In fact it can be proved to be less than $\frac{1}{\gamma_j} \int_0^\infty dk \ln$

$$2 |\gamma_j|^2 + 2 (\text{Re} \gamma_j) k^2 + 2 |\gamma_j| \sqrt{(k^2 + \text{Re} \gamma_j)^2 + (\text{Im} \gamma_j)^2}$$

$$\left(2 (\text{Re} \gamma_j) + 2 |\gamma_j| \right) k^2$$

$2\pi\omega_p^2 \frac{\partial F}{\partial \mu_j} = \gamma_j$)
The integrand of the second term in (C.25) behaves like
\[ k_\perp (k_\perp^2 + k_\parallel^2)^{-3/2} \cdot R_\perp \cdot R_\parallel \cdot (k_\parallel) \]
in a neighbourhood of zero.

Since \( R_\parallel \cdot (k_\parallel) \) is anti-symmetric in \( k_\parallel \), it vanishes for \( k_\parallel = 0 \), and we have: \( \lim_{k_\parallel \to 0} R_\perp \cdot (k_\parallel) = \text{finite} \). Now, it simply follows that \( \lim_{k_\parallel \to 0} R_\perp \cdot (k_\parallel) \).

In a neighbourhood of infinity the second term's integrand behaves like \( k_\perp R_\parallel (k_\parallel^2 + k_\perp^2)^{-3/2} (k_\parallel^2 + k_\perp^2)^{-1/2} \), so its integral converges at infinity. Thus we may conclude that the cosine-transform term in (C.23) is of order unity for \( \lambda \to 0 \):

\[ \text{CT} = 0(1), \lambda \to 0. \quad \text{(C.26)} \]

The sine-transform term in (C.23) can be simplified using the formula:

\[ \lim_{\lambda \to 0} k_\perp^{-1} \sin\left( \frac{k_\parallel x_\parallel}{\lambda} \right) = \pi \delta(k_\parallel), x \neq 0. \quad \text{(C.27)} \]

Thereby we obtain:

\[ \text{ST} = \frac{-2 \pi i^2 \omega}{8 \pi^3 n v_{12}} \int_{0}^{2 \pi} \int_{0}^{\infty} \exp(ik_\perp x_\perp \cos \psi) \frac{d^2 \psi}{3 \mu_j} \]

\[ \frac{1}{(k_\perp^2 + 2 \pi \omega \frac{\delta F}{3 \mu_j})} \cdot CR_\parallel (0). \quad \text{(C.28)} \]

Since its integrand behaves like \( k_\perp^{-2} \exp(ik_\perp x_\perp \cos \psi) \) for large \( k_\perp \), and like \( \exp(ik_\perp x_\perp \cos \psi) \) for small \( k_\perp \), the right hand side of (C.28) completely converges. Therefore, the sine-transform term in (C.23) is also of order unity for small values of \( \lambda \),
\[ ST = 0(1), \lambda \to 0. \]

(C.29)

From (C.26) and (C.29) it follows that [cf. (C.23)]:

\[ \delta g_{A_{II}}^{(1)} (\lambda \chi_1, \chi_2; \chi_1, \chi_2) = 0(1), \lambda \to 0. \]  

(C.30)

Combined with (C.20), (C.30) yields the result that:

\[ \delta g_{A}^{(1)} (\lambda \chi_1, \chi_2; \chi_1, \chi_2) = 0(1), \lambda \to 0. \]  

(C.31)

Next we consider the remaining term \( \delta g_{A}^{(2)} \) [cf. (C.8)],

\[ \delta g_{A}^{(2)} (\lambda \chi_1, \chi_2; \chi_1, \chi_2) = \frac{1}{8\pi^3} \int \frac{2\pi}{0} \int \frac{dk_{\perp}}{0} \int \frac{dk_{\parallel}}{\infty} \exp(i k_{\perp} \chi_1 \cos \psi) \]

\[ \exp(i \frac{u_{\parallel}}{1-u_{2}^{2}} \chi_{\parallel}) \xi(\kappa_{\parallel}, \kappa_{\parallel}; \chi_1, \chi_2). \]  

(C.32)

Substitution of (C.6) into (C.32) and transformation: \( k_{\parallel} \to \lambda k_{\parallel}; \)

\( k_{\perp} \to \lambda k_{\perp}, \) yields

\[ \delta g_{A}^{(2)} (\lambda \chi_1, \chi_2; \chi_1, \chi_2) = \frac{\lambda \omega^4 (2\pi i)^2}{8\pi^2 m v_{12}} \int \frac{2\pi}{0} \int \frac{dk_{\perp}}{0} \int \frac{dk_{\parallel}}{\infty} \exp(i k_{\perp} \chi_1 \cos \psi) \exp(i \frac{u_{\parallel}}{1-u_{2}^{2}} \chi_{\parallel}) \]

\( (k_{\parallel}^2 + k_{\perp}^2)^{-2} \)
\[ \delta g_A^{(2)} = \frac{1}{2\pi i} \left( k_\perp^2 + k_\parallel^2 \right)^\frac{1}{2} F \left( \frac{1}{k_\perp} \right) \]

\[ \left[ \frac{3}{\partial u_2} F(u_2) \left( \frac{\partial^2 F(u_2)}{\partial u_2} \right) - F(u_2) \frac{\partial^2 F(u_2)}{\partial u_2} \right] \]

\[ \frac{2\pi i \omega^2 k_\perp^2}{\left( k_\parallel^2 + k_\perp^2 \right)^2} \frac{\partial^2 F(u_2)}{\partial u_2} - 1 \]

(C.33)

Hence, the expression for \( \delta g_A^{(2)} \) is quite similar to that for

\[ \delta g_A^{(1)} \] [cf. (C.16)], the only important difference being that instead

of \( \frac{\partial F}{\partial u_2} \) we have: \( \frac{\partial^2 F}{\partial u_2} \), with \( u_2 = \frac{k_\perp}{k_\parallel} \) depending on \( k_\parallel \), \( k_\perp \), a.o..

However, \( \frac{\partial F(u_2)}{\partial u_2} \), as well as the other factor \( \frac{\partial F(u_2)}{\partial u_2} \)

are uniformly bounded in \( k_\parallel \), \( k_\perp \). (And independent of \( |k| \).

(C.33')

The term with the \( \delta \)-function contribution can be written as:

\[ \frac{\alpha(\omega k_\perp)}{2\pi^2 n v_{12}} \int_0^\infty \int_0^\infty \exp \left( i k_\perp x_\perp \cos \psi \right) \frac{\partial F(u_2)}{\partial u_2} \frac{\partial F(u_2)}{\partial u_2} \]

(C.34)

with

\[
\alpha = \lim_{k_\parallel \to 0^+} \left[ \frac{\partial F(u_2)}{\partial u_2} \right] \frac{\partial F(u_2)}{\partial u_2} \]

(C.35)

and

\[
\gamma = \lim_{k_\parallel \to 0^+} \frac{\partial F(u_2)}{\partial u_2} \frac{\partial F(u_2)}{\partial u_2}
\]

(C.36)

From (C.33), it follows that \( \alpha \) and \( \gamma \) are uniformly bounded in \( k_\perp \).

Hence the integrand of (C.34) behaves like \( \exp(ik_\perp x_\perp \cos \psi) \), for small \( k_\perp \), and like \( k_\perp^{-2} \exp(ik_\perp x_\perp \cos \psi) \), for large \( k_\perp \).
Therefore, the $\delta$-term (C.34) in (C.33) is equal to a $\lambda$-independent completely convergent integral, hence $O(1), \lambda \to 0.$ (C.37)

Finally we consider the $P$-term in (C.33).

Decompose the factor

$$F = \left[ \frac{3}{\delta u_2} F(\chi_2) \int F(u_2) - F(\chi_2) \frac{\partial^2 F(u_2)}{\partial u_2^2} \right] \frac{\partial^2 F(u_2)}{\partial u_2^2} [1 +$$

$$+ \frac{\lambda^2 2\pi i}{p} \frac{\partial^2 F}{\partial u_2^2}]^{-1}$$

into a symmetric and an antisymmetric part with respect to $k_2$:

$$F(\lambda) = F_A(\lambda) + F_A(\lambda)$$

Since $\frac{\partial^2 F(\chi_2)}{\partial u_2^2}$, $\frac{\partial^2 F}{\partial u_2^2}$ and $\frac{\partial^2 F}{\partial u_2^2}$ are completely regular in $k_2 = 0$, we know that:

$$\lim_{k_2 \to 0} k_2^{-1} F_A(\lambda)(k_2 = 0) = \text{finite.}$$ (C.40)

Therefore, the $P(\frac{1}{k_2})$-term in (C.33), can be decomposed into the sum of a sine-transform and a cosine-transform:

$$\delta_x(2)(\lambda, \chi_2; \chi_2; \chi_2) = \frac{\lambda^2 \pi i}{2\pi^3 n_{12}} \int_0^\infty dk_1' \int_0^\infty dk_1' \int_0^\infty$$

$$\exp(ik_1' \cos \psi) \cos(k_1' \lambda) \left( k_2^2 + k_1^2 \right)^{-3/2} \frac{1}{2\pi k_1'}$$

$$k_2^{-1} F_A(\lambda) (k_2, k_1') + \frac{\lambda^4 \pi i}{2\pi^3 n_{12}} \int_0^\infty dk_1' \int_0^\infty$$
\[
\exp(ik_1 x_1 \cos \psi) = \frac{\sin \frac{k_1 x_1 \lambda}{k_\parallel}}{k_\parallel} (k_\perp^2 + k_\parallel^2)^{-3/2} \text{f}_s(\lambda)(k_\parallel, k_\perp). \tag{C.41}
\]

The first term "C\!T" in (C.41) can be written, by transforming back to \( k_\parallel \rightarrow k_\parallel \lambda^{-1}, k_\perp \rightarrow k_\perp \lambda^{-1}:

\[
\text{C\!T} = \frac{\omega^3}{4\pi^3 m v_{12}} \frac{2\pi}{0} d\psi \frac{1}{dk_\perp} \frac{1}{dk_\parallel} \exp(ik_1 x_1 \cos \psi) \int_0^\infty dk_\parallel.
\]

\[
\cos(k_\parallel x_1)(k_\parallel^2 + k_\perp^2)^{-3/2} k_\parallel^{-1} \text{f}_A. \tag{1}(k_\parallel, k_\perp),
\]

which can be estimated by

\[
|\text{C\!T}| \lesssim \frac{\omega^3}{4\pi^3 m v_{12}} \frac{2\pi}{0} dk_\perp k_\parallel \int_0^\infty (k_\parallel^2 + k_\perp^2)^{-3/2}.
\]

\[
|k_\parallel^{-1} \text{f}_A. \tag{C.43}(k_\parallel, k_\perp)| dk_\parallel.
\]

In view of (C.39, C.40), the integrand of (C.43) is behaving like \( k_\perp (k_\parallel^2 + k_\perp^2)^{-3/2} k_\perp^2 \) in a neighbourhood of \( k_\perp = 0, k_\parallel = 0 \).

Therefore its integral converges at \( k_\parallel = 0 = k_\perp \).

Since

\[
\lim_{k_\parallel \rightarrow 0, k_\perp \rightarrow 0} [1 + \frac{2\pi i \omega^2}{\frac{\partial F}{\partial u_2}}]^{-1} = 1, \tag{C.44}
\]

it follows from (C.40) that the integrand of (C.43) is behaving like \( \frac{k_\perp}{k_\parallel} (k_\perp^2 + k_\parallel^2)^{-3/2} \), hence its integral also converges for \( k_\parallel \rightarrow \infty, k_\perp \rightarrow \infty \).
Consequently, the cosine-transform term (C.42) in (C.41) can be estimated by a convergent \( \lambda \)-independent integral:

\[
CT = O(1), \; \lambda \to 0 . \tag{C.45}
\]

Finally, for the sine-transform term \( ST \) in (C.41), we can write, again invoking the formula (for \( x_\parallel \neq 0 \))

\[
\sin(x_\parallel k_\parallel \lambda^{-1}) \omega \delta(k_\parallel) , \quad \lambda \to 0 ,
\]

\[
ST = \frac{\lambda \omega}{2 \pi n v_\perp 12} \int_0^{2\pi} d\psi \int_0^{\infty} dk_\perp k_\perp \exp(i k_\perp x_\perp \cos\psi) k_\perp^{-3} f(\lambda) .
\]

\( (k_\parallel = 0; k_\perp) , \tag{C.46} \)

or by transforming back to \( k_\parallel + k_\parallel \lambda^{-1} ; k_\perp + k_\perp \lambda^{-1} :\)

\[
ST = \frac{4 \omega}{2 \pi n v_\perp 12} \int_0^{2\pi} d\psi \int_0^{\infty} dk_\perp k_\perp \exp(i k_\perp x_\perp \cos\psi) k_\perp^{-3} f(1) .
\]

\( (k_\parallel = 0; k_\perp) , \tag{C.47} \)

which can be estimated by

\[
|ST| \leq \frac{2}{n v_\perp 12} \int_0^{\infty} dk_\perp k_\perp^{-2} f(1) (k_\parallel = 0; k_\perp) . \tag{C.48}
\]

Since from (C.39) it follows that

\[
F_g^{(1)} (k_\parallel = 0; k_\perp) \sim k_\perp, \; k_\perp \sim 0 ,
\]

\[
F_g^{(1)} (k_\parallel = 0; k_\perp) \sim 1, \; k_\perp \to \infty ,
\]

it is clear that the integral on the right hand side of (C.48) completely converges. Since it is also independent of \( \lambda \) we may
conclude that:

$$\Pi_T = 0(1), \lambda \to 0.$$  \hfill (C.49)

Combining (C.45) and (C.49), it follows that the $\mathcal{P}(\frac{1}{|\lambda|})$-term in the expression (C.33) for $\delta g_{a}^{(2)}$ is of order unity for $\lambda \to 0$.

Combined with (C.37), we reach the conclusion that

$$\delta g_{a}^{(2)}(\lambda \xi_{1}, \xi_{2} ; \xi_{1}, \xi_{2}) = 0(1), \lambda \to 0.$$  \hfill (C.50)

From (C.31) and (C.50), it follows that [cf. Eqs. (C.7, C.8)]

$$\delta g_{a}(\lambda \xi_{1}, \xi_{2} ; \xi_{1}, \xi_{2}) = 0(1), \lambda \to 0.$$  \hfill (C.51)

Equivalently: $g_{B}(\xi ; \xi_{1}, \xi_{2}) = \Delta_{a}^{(0)}(\xi ; \xi_{1}, \xi_{2}) + 0(1), \xi \to 0$.

I.e.: The difference of the correlation $g_{B}$ and the Landau correlation $\Delta_{a}^{(0)}$ is finite along the axis of relative velocity. This implies that the correlation $g_{B}$ can be approximated by the Landau correlation in the neighbourhood of this axis, screening effects apparently being small along the direction of $\xi_{12}$.

This result supports the result derived by means of proper scaling of the BBGKY equations, that the Boltzmann correlation is a good approximation for $g_{2a}$ along the $\xi_{12}$-axis. It also confirms that the formula (101) for $g_{2a}$ as derived for $R_{2}(1,2)$ and $R_{3}(1,2)$, is also valid in the region along $\xi_{12}$, even so for interparticle distances of the order of the Debye length.
APPENDIX D

In this Appendix, a presentation is given of the evaluation of the integrals in our expression [Eqs. (163), (164), (165) and (166)] for the high-frequency conductivity in a spatially homogeneous plasma, by R.L. Guernsey.²⁶

First consider \( g_0^{(A)} \) as given by Eq. (164). Partial integration yields [Eq. (145)]

\[
\int d^3k \frac{k \cdot k}{k^3} \hat{f}(k) \delta\left(\frac{k \cdot r}{k}\right)(1 + k^{-2} \tau_D^{-2})^{-2}
\]

\[
= \frac{1}{\sqrt{\nu}} \frac{1}{2\pi} \int_0^\infty dk \ k \hat{f}(k) (1 + k^{-2} \tau_D^{-2})^{-2} \frac{I}{\hat{m}}
\]

\[
= \frac{2\pi}{\sqrt{\nu}} \frac{T_0}{2\pi^2} \int_0^\infty \frac{d\rho}{\rho} \left[1 - \exp\left(-\frac{r_L}{\rho}\right)\right] \int_0^\infty \frac{dk}{k^2} [\sin k\rho - k\rho \cos k\rho]
\]

\[(1 + k^{-2} \tau_D^{-2})^{-2} \frac{I}{\hat{m}}
\]

where \( I \) is the unit dyadic.

Consider the integral involving \( \sin(k\rho) \), on the right hand side of (D.1). The integral over \( k \) can be splitted into two parts, one from 0 to \( \tau_D^{-1}\omega^{-1} \), and one from \( \tau_D^{-1}\omega^{-1} \) to \( \omega \), where \( \omega \) is the plasma parameter \( \omega \equiv \frac{r_L}{r_D} \). The first part can be written
\[ r_D^{-1} = \int_0^{k_0} \frac{dk}{(k^2 + r_D^2)^2} = \lim_{k_0 \to r_D^{-1}} (1 + \kappa^2 \beta) \int_0^{k_0} \frac{dk}{k^2 + \kappa^2} \]

\[ = \lim_{k_0 \to r_D^{-1}} (1 + \kappa^2 \beta) \left( \frac{1}{2i\kappa} \right) \left( \frac{1}{2i} \right) \beta \alpha \]

\[ \exp(-\alpha \beta \kappa) \left( \begin{array}{c} k_0 - \alpha \kappa \\ -\alpha \kappa \end{array} \right) \]

where the summation over \( \alpha \) and \( \beta \) is over +1 and -1, and where \( \kappa \equiv r_D^{-1} \).

The right hand side can be rearranged to yield

\[ \varepsilon^{-1} \kappa \left( \begin{array}{c} \int_0^{k_0} \frac{dk}{(k^2 + \kappa^2)^2} \sin(k\rho) \\ \int_0^{k_0} \frac{dk}{(k^2 + \kappa^2)^2} \cos(k\rho) \end{array} \right) = \frac{-1}{2\kappa} \left( \begin{array}{c} \varepsilon \alpha \beta \\kappa \end{array} \right) \lim_{k_0 \to \varepsilon} \left( \exp(-\alpha \varepsilon)(1 - \alpha \varepsilon) \right) \]

The second part, i.e. the integral from \( \varepsilon^{-1} \kappa \) to \( \infty \) is equal to

\[ \int_{\varepsilon^{-1} \kappa}^{\infty} \frac{dk}{k^2} \cos(k\rho) = \frac{\varepsilon \sin(k\rho \varepsilon)}{k} + \int_{\varepsilon^{-1} \kappa}^{\infty} \frac{dk}{k^2} \cos(k\rho) \]

Similarly, the integral involving \( \cos(k\rho) \) on the right hand side of Eq. (D.1) can be evaluated, and we obtain
Although individual pieces of the factor between the outer curly bracket give rise to divergent integrals, the whole expression on the right hand side of (D.5) is convergent. Putting $\xi = \rho \frac{\pi}{L}$, the integrals are divided into two parts, one involving integration from $\xi = 0$ to $\xi = \varepsilon^{-\frac{1}{2}}$, the other involving integration from $\xi = \varepsilon^{-\frac{1}{2}}$ to infinity. In the first part, $\exp(-\alpha \varepsilon \xi)$ can be expanded, in the second part $\exp(-\xi^{-1})$ can be expanded.

As a result one finds

$$\frac{-\pi \sin(\pi \rho \xi^{-1})}{2 \varepsilon \kappa^2 (1 + \varepsilon^{-2})} = \frac{\rho}{4} \left[ \frac{\pi}{\beta} \sum_{\beta} \frac{1 - \frac{\alpha \rho}{2} \exp(-\alpha \rho)}{1 - \frac{\alpha \rho}{2}} \right]$$

(D.5)

Equations (164), (D.1) and (D.6) together completely determine $\theta(\xi)$. 

Next, consider $\theta_B^E$ [Eq. (165)]. From symmetry arguments it follows that only the odd part of $v_{12}^I = \left( v_{12}^I - \frac{2 \xi^2}{\mu \varepsilon} \right) \frac{1}{\xi}$ as a function of $\xi$ is needed. This odd part can be written [Eq. (31)] as
It is clear, that the denominator in the right hand side of (D,7),
containing \( L^2 \), is of the form \( A - B \cos^2 \Theta \), as a function of the angle \( \Theta \)
between \( x \) and \( v_{12} \), where

\[
A \equiv e^4 + \mu^2 x^2 v_{12}^2 (v_{12}^2 - \frac{2e^2}{\mu x}),
\]

\[
B \equiv \mu^2 x^2 (v_{12}^2 - \frac{2e^2}{\mu x}).
\]

In spherical coordinates with \( v_{12} \) along the \( z \)-axis,

\[
\frac{x}{v_{12}} = \begin{pmatrix} \sin \Theta \cos \Phi \\ \sin \Theta \sin \Phi \\ \cos \Theta \end{pmatrix}
\]

The integrals over \( \sin \Theta \, d\Theta \) \( = dy \) and \( d\phi \), i.e., over the spatial
angles, in Eq. (165) can now be performed, yielding

\[
\frac{2\pi}{\mu x^3} \int_0^\pi \int_0^{2\pi} \frac{x}{v_{12}^2} v_{12}^2 \sin \Theta \, d\Theta \, d\Phi = 2\pi x^{-2} (v_{12}^2 - \frac{2e^2}{\mu x})^{\frac{1}{2}}
\]

\[
\frac{e^4 v_{12}^2 v_{12}^2}{v_{12}^2 B} \left[ \left( \frac{A}{B} \right)^{\frac{1}{2}} \ln \left( \frac{A^{\frac{1}{4}} + B^{\frac{1}{4}}}{A^{\frac{1}{4}} - B^{\frac{1}{4}}} \right) - 2 \right] - \frac{ue^2 x}{\mu x^3}
\]

\[
(v_{12}^2 - \frac{2e^2}{\mu x}) (1 - \frac{v_{12}^2 v_{12}^2}{v_{12}^2}) \left[ A^{-\frac{1}{4}} B^{-\frac{1}{4}} (1 - \frac{A}{B}) \ln \left( \frac{A^{\frac{1}{4}} + B^{\frac{1}{4}}}{A^{\frac{1}{4}} - B^{\frac{1}{4}}} \right) + \frac{2}{B} \right].
\]
In view of the symmetry* of the total integrand in Eq. (165), we may replace $\frac{\nu_{12}\nu_{12}}{\nu_{12}^2}$ by $\frac{1}{3} I$, $I$ being the unit dyadic.

If $\mu\nu_2\nu_{12}(\nu_{12}^2 - \frac{2e_x^2}{\mu x}) \ll e^2$, then the right hand side of Eq. (D.10) can be simplified further to yield

$$\int_0^{2\pi} \int_0^\pi \frac{4\pi}{x^3} \nu_{12}^2 \int_0^\infty \frac{4\pi}{x^2} I x^2 (\nu_{12}^2 - \frac{2e_x^2}{\mu x})$$

If, on the other hand, $\mu\nu_2\nu_{12}(\nu_{12}^2 - \frac{2e_x^2}{\mu x}) \gg e^2$, then the right hand side of (D.10) can be reduced to give

$$\int_0^{2\pi} \int_0^\pi \frac{4\pi}{x^3} \nu_{12}^2 \int_0^\infty \frac{4\pi}{x^2} I x^2 (\nu_{12}^2 - \frac{2e_x^2}{\mu x})$$

$$\frac{e^4}{u^2 x v_2 (v_{12}^2 - \frac{2e_x^2}{\mu x})} \ln \left[ \frac{2xv_2 (v_{12}^2 - \frac{2e_x^2}{\mu x})}{\frac{2e_x^2}{\mu x}} \right]$$

$$\frac{e^2}{\mu x (v_{12}^2 - \frac{2e_x^2}{\mu x})}$$

Formula (D.11) and (D.12) can be combined to the general formula

$$\int_0^{2\pi} \int_0^\pi \frac{4\pi}{x^3} \nu_{12}^2 \int_0^\infty \frac{4\pi}{x^2} I (\frac{e^4}{v_{12}^2 (v_{12}^2 - \frac{2e_x^2}{\mu x})})$$

$$\frac{e^2}{\mu x (v_{12}^2 - \frac{2e_x^2}{\mu x})}$$

*Footnote: Despite $\mu_v \neq \mu_x$, the mixed term in the exponent of $e^2(v_{12}^2 - \frac{2e_x^2}{\mu x})$ vanishes.
Now, the following transformation of integration variables
will be considered, in order to perform the velocity integrations:

\[
\phi = \frac{e^2}{x},
\]

\[\frac{1}{2} v_{12}^2 - \phi = \varepsilon, \quad (D)\]

(Its Jacobian equals \( \frac{-e^2}{\phi^2} \) \[2\mu(\varepsilon + \phi)]^{-\frac{1}{2}} \neq 0\).

By means of this transformation and of Eq. (D.13) we obtain

\[
\int d^3x \int d^3v_1 \int d^3v_2 \frac{2\phi}{\partial v_1} \frac{2\phi}{\partial v_2} \exp(-x) \frac{\varepsilon(v_1) \varepsilon(v_2)}{\Sigma_D} =
\]

\[
I(\frac{\mu}{2\gamma T_0})^{\frac{3}{2}} \frac{8\pi^2}{3} e^{\mu} \int_0^\infty ds \exp(-cT_0 s) \frac{\varepsilon}{\Sigma_D} \int_0^\infty \phi \phi^{-2}
\]

\[
\exp(-cT_0 \phi^{-1}) G(\frac{\phi}{c}), \quad (D)
\]

where \( G \) is defined by

\[
G(y) = \frac{1}{2} y^2 \ln \left( \frac{2 + y + 2\sqrt{1+y}}{2 + y - 2\sqrt{1+y}} \right) - 2y\sqrt{1+y}. \quad (D)
\]
The integral

\[ I(\varepsilon) = \int_0^\infty \int_0^\infty \text{dn} \exp(-\eta) \, dy \, \exp\left(-\frac{\varepsilon}{\eta y}\right) \]

\[ \left\{ \frac{1}{2} \ln \left( \frac{2 + y + 2(1 + y)^{1/2}}{2 + y - 2(1 + y)^{1/2}} \right) - \frac{2(1 + y)^{1/2}}{y} \right\} \]  \hspace{1cm} (D.17)

appears on the right hand side of (D.15), after partial integration.

Three regions are distinguished within the two-dimensional integration interval:

Region (a): \( \eta : 0 \to \infty, y : 0 \to \varepsilon^{1/4} \);

Region (b): \( \eta : 0 \to \varepsilon^{1/4}, y : \varepsilon^{1/4} \to \infty \);

Region (c): \( \eta : \varepsilon^{1/4} \to \infty, y : \varepsilon^{1/4} \to \infty \).

Correspondingly, \( I(\varepsilon) \) can be written

\[ I(\varepsilon) \equiv -2 \int_0^\infty \int_0^\infty \text{dn} \exp(-\eta) \, dy \, \exp\left(-\frac{\varepsilon}{\eta y}\right) \]

\[ \varepsilon^{1/4} \int_0^\infty \text{dn}(1 - \eta + \frac{1}{2} \ln^2 + \ldots) \int_0^\infty \ln \left( \frac{2 + y + 2\sqrt{1+y}}{2 + y - 2\sqrt{1+y}} \right) \]

\[ - \frac{2\sqrt{1+y}}{y} \exp\left(-\frac{\varepsilon}{\eta y}\right) + \int_0^\infty \text{dn} \exp(-\eta) \, dy \]

\[ \varepsilon^{1/4} \varepsilon^{1/4} \]

\[ \frac{1}{2} \ln \left( \frac{2 + y + 2\sqrt{1+y}}{2 + y - 2\sqrt{1+y}} \right) - \frac{2\sqrt{1+y}}{y} \]  \hspace{1cm} (D.18)

where we neglected terms of relative order \( \varepsilon^{1/4} \).
The third term in (D.18), corresponding to region (c), is readily evaluated,

\[ \int_0^\infty dn \exp(-n) \int_\frac{1}{4}^\infty dy \left[ \frac{1}{4} \ln \left( \frac{2 + \sqrt{y + 2} + y}{2 + y - 2\sqrt{1+y}} \right) - \frac{2}{y} \sqrt{1+y} \right] \]

\[ = -2 \exp\left( -\frac{1}{n} \right) \left[ \ln \left( \frac{4}{n^4} \right) - 1 \right] \gamma \int \ln \varepsilon - 4\ln 2 + 2 \]  

(D)

Putting \( u = \frac{\varepsilon}{ny} \), the second term in Eq. (D.18) can be written

\[ \frac{\varepsilon}{n} \int_0^\infty \frac{dn}{n} \int_\frac{1}{n}^\infty du \left[ \frac{1}{4} \ln \left( \frac{2nu + \varepsilon + 2\sqrt{nu + \varepsilon}}{2nu + \varepsilon - 2\sqrt{nu + \varepsilon}} \right) \right] \]

\[ - \frac{2\sqrt{1+y}}{y} \exp\left( -\frac{1}{ny} \right) \]

\[ = \frac{2\sqrt{1+n}}{n} \exp\left( -\frac{1}{n} \right) \exp(-u) \]

\[ = \varepsilon \left[ \frac{1}{4} \ln \varepsilon \int_\frac{1}{\varepsilon}^\infty \frac{du}{u} \exp(-u) \left\{ \frac{1}{4} \ln \left[ \frac{2ue^{1/2} + \varepsilon + 2\varepsilon^{1/2}u^{1/2}(\varepsilon + \varepsilon^{1/2}u^{1/4})}{2ue^{1/2} + \varepsilon - 2\varepsilon^{1/2}u^{1/2}(\varepsilon + \varepsilon^{1/2}u^{1/4})} \right] \right\} \right] \]

\[ - \frac{2}{\varepsilon} \left( \varepsilon + \varepsilon^{1/2}u^{1/4} \right)^{1/4} \right] - \varepsilon^{-\frac{1}{2}} \left( \ln \varepsilon^{1/4} - 1 \right) \left[ \ln \left( \frac{16}{\varepsilon^4} \right) \right] \]

\[ - \frac{2}{\varepsilon^{1/2}} \int_0^\infty \frac{du}{u} \exp(-u) \int_\frac{1}{\varepsilon}^\infty \frac{du}{u} \exp\left( -\frac{2\nu}{\varepsilon} \left( \nu + \varepsilon \right)^{1/2} \right) \]
which can be shown to be \( \varepsilon \ln \varepsilon \), term by term.

Finally, the first term on the right-hand side of Eq. (D.18) can be written as

\[
-2 \int_0^\infty \frac{d\eta}{\eta} \exp(-\eta) \int_0^{\eta^{-1/2}} \frac{du}{u} \exp(-u) = \\
-2 \int_0^{\eta^{-1/2}} \frac{du}{u} \exp(-u) \sum_{n=0}^{\infty} \frac{e^{-\varepsilon\eta}}{n!} n^m \\
= \int_0^{\infty} \frac{d\eta}{\eta} \exp(-\eta) \left( 1 - \frac{e^{-\varepsilon\eta}}{\eta} \right) - 2 \int_0^{\infty} \frac{d\eta}{\eta} \exp\left( -\frac{\varepsilon}{\eta} \right) \left( 1 - \eta + \frac{1}{2} \eta^2 + \ldots \right) + \\
\int_0^{\infty} \frac{d\eta}{\eta} \exp(-\eta) \left( 1 - \frac{e^{-\varepsilon\eta}}{\eta} \right) \left( \ln \varepsilon + \gamma \right), \\
\tag{D.21}
\]

having neglected terms of order \( o(1) \), \( \varepsilon \to 0 \).

Combination of (D.18), (D.19), (D.20) and (D.21) yields

\[
I(\varepsilon) = 2 \ln \varepsilon - 4 \ln 2 + 2(1 + 2\gamma) + o(1), \quad \varepsilon \to 0 \\
\tag{D.22}
\]

Eqs. (165), (D.15), (D.17) and (D.22) together determine \( \sigma_{(B)}^{(A)} \), for which we find

\[
\sigma_{(B)}^{(A)} = \frac{2}{\varepsilon} \frac{\omega^3}{\omega^2 (4\pi)^2} \left[ 2 \ln 2 - 1 - 2\gamma - \ln \varepsilon \right]. \\
\tag{D.23}
\]

The calculation of \( \sigma_{(C)}^{(A)} \) is very similar to that of \( \sigma_{(A)}^{(A)} \). Finally, one finds for the total conductivity \( \sigma \), as given by Eqs. (163-166) the result [Eq. (167)] mentioned in Chapter VI.
In this Appendix it is shown that formulae (140) and (143) may be used for $\Psi$ and $\eta$, respectively. Consider the sum of $C^{(A)}_{ei}$ and $C^{(C)}_{ei}$,

$$C^{(A)}_{ei} + C^{(C)}_{ei} = -\frac{n}{m} \frac{\partial}{\partial \nu_{i1}} \cdot \int d^3x \int d^3\nu \left( e_{eiB} \nu \cdot \frac{\partial^{(o)}}{\partial \nu_{i1}} - \Delta^{(o)} \nabla \eta \right),$$

where

$$\nabla \Psi = \left[ \frac{\partial^{(o)}}{\partial \nu_{iA}} \right] \quad \nabla \phi, \quad \nabla \eta = \left[ \frac{\partial^{(o)}}{\partial \nu_{iA}} \right] \quad \nabla \phi,$$

and where we suppressed particle variables.

We consider two regions:

Region (a): $|x| < \varepsilon^2 r_D$;

Region (b): $|x| > \varepsilon^2 r_D$.

Ad (a).

Here,

$$\frac{e_{eiB}}{\Delta^{(o)}_{eiA}} - 1 = O(\varepsilon^2),$$

as follows from the final result of Appendix C.

Consequently [Eq. E.2]

$$\nabla^{\nu} e_{eiB} - \nabla \eta \Delta^{(o)}_{eiA} = O(\varepsilon^2).$$

Therefore, region (a) does not contribute to the conductivity through $C^{(A)}_{ei} + C^{(C)}_{ei}$ (it does contribute through $C^{(B)}_{ei}$).
Ad (b).

In region (b),

\[
1 - \exp\left(-\frac{r_L}{x}\right) = \exp\left(-\frac{r_L}{x}\right) - 1 + 0(\epsilon).
\]

Consequently, within region (b) we may replace the expression for the equilibrium Boltzmann correlation, \(\exp\left(-\frac{r_L}{x}\right) - 1\) \(\mathcal{F}_m(v_1) \mathcal{F}_m(v_2)\) by \(-r_L\left[1 - \exp\left(-\frac{r_L}{x}\right)\right] \mathcal{F}_m(v_1) \mathcal{F}_m(v_2)\), at least in the sum of the Bogolyubov and Landau collision terms, \(C_{ei}^{(A)}\) and \(C_{ei}^{(C)}\). This replacement has been used in deriving Eqs. (140) and (143).

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List of References of Part I

PART II

HYBRID-KINETIC STABILITY ANALYSIS

OF HIGH-BETA PLASMAS
ABSTRACT

Low-frequency (|ω|<ω_{ci}≡|e|B/m_{i}c) stability properties of high-beta plasmas are investigated within the framework of a hybrid-kinetic model (drift-kinetic electrons and fully Vlasov ions). An eigenvalue equation is derived for perturbations around a static, axisymmetric, linear screw-pinch equilibrium, characterized by an isotropic ion energy distribution. Apart from the equilibrium pressure constraint, the density and magnetic field profiles are allowed to be arbitrary. This equation clearly exhibits the importance of finite electron temperature and parallel electron kinetic effects on stability behavior, and can be used numerically to obtain the dispersion relation.

In the limit of zero electron temperature, the Vlasov-fluid eigenvalue equation of Freidberg is recovered. Analytically, and within the region of validity of Turner's additional Finite-Larmor-Radius ordering, explicit expressions for the growth rates are obtained for the case of low-frequency (|ω|<ω_{ci}) perturbations about a sharp-boundary, near-theta-pinch (B_{θ}^{2}<B_{z}^{2}) equilibrium.

Furthermore, an eigenvalue equation is derived for electromagnetic perturbations around a similar equilibrium, but characterized by an anisotropic ion distribution, restricting consideration to a pure theta-pinch configuration (B_{θ}^{0}=0). This equation can be used numerically to generalize the description of the electromagnetic ion cyclotron instability, driven by ion energy anisotropy, with finite radial systems dimensions and arbitrary density profiles. Analytically, and within the region of validity of Freidberg's trial function method, explicit expressions are derived for the growth rates of perturbations with |ω|<ω_{ci}, assuming a sharp-boundary equilibrium configuration, with B_{θ}^{0}=0 inside the plasma column. It is found that the influence of
ion energy anisotropy on the growth rate and on marginal stability is similar to the effect of anisotropic pressure on the shear Alfvén wave as found from double-adiabatic theory.

Within the context of Turner's FLR ordering as well as Freidberg's trial function method, the effect of finite electron temperature occurs exclusively through the equilibrium pressure balance equation.

Finally, some critical remarks are made concerning the hybrid-kinetic model and Turner's FLR calculation in connection with the importance of finite resistivity effects on the stability of a sharp-boundary, near-theta-pinch ($B_0 \ll B_z$) equilibrium, assuming long-wavelength perturbations. An eigenvalue equation for these modes is derived, clearly exhibiting the importance of resistivity on stability behavior.
1. INTRODUCTION

In high beta plasmas, e.g., in high-density pinch experiments,\textsuperscript{1-10} the thermal ion Larmor radius $r_{Li}$ is often comparable with the length scale $L_\parallel$ for spatial variations. Consequently, neither ideal magnetohydrodynamics\textsuperscript{11-13} nor guiding center theory\textsuperscript{13-30} are able to give a complete description of the stability behavior. On the other hand, a fully Vlasov description of both ions and electrons is generally difficult\textsuperscript{31,32} and unnecessary since the thermal electron Larmor radius $r_{Le}$ is typically much smaller than $L_\parallel$. In his Vlasov-fluid model, Freidberg assumed the electrons to form a massless, pressureless fluid, while retaining a full Vlasov description for the ions. He was able to derive an eigenvalue equation for perturbations around a static equilibrium, characterized by an isotropic ion energy distribution. This equation has been successfully used\textsuperscript{1,33} to demonstrate finite-ion-Larmor radius stabilization of low-frequency ($|\omega|<\omega_{ci}$) instabilities in a sharp-boundary near-theta pinch, for azimuthal mode numbers $m>1$. Freidberg's eigenvalue equation has been extended to include finite electron pressure,\textsuperscript{34} adopting an adiabatic electron fluid description, assuming the electron-ion collision frequency $\nu_{ei}$ to be much greater than the perturbation frequency $\omega$.

However, if $|\omega|>\nu_{ei}$, a kinetic description of the electron is necessary. Even for relatively low frequency ($|\omega|<\omega_{ci}$), electron kinetic effects parallel to the magnetic field direction may be important for the stability behavior of long-wavelength modes. A natural extension of the Vlasov-fluid model to include a collisionless ($|\omega|>\nu_{ei}$), kinetic description of the electron species is provided by the hybrid-kinetic model of D'Ippolito and Davidson.\textsuperscript{35,36} In this model the electrons are described by means of a drift-kinetic
equation, obtained by expansion of the electron Vlasov equation in the small parameter $c=r_L/e/L_e$. Similarly, the electromagnetic fields are evaluated to the same accuracy in every order in $c$. The ions, however, are described by the unexpanded Vlasov equation. Since only low frequencies are considered ($|\omega| < \omega_{ci}$), lowest order hybrid-kinetic theory $^{35,36}$ may be adopted as the starting point of the stability analysis.

Furthermore, during the implosion phase of a pinch experiment, microinstabilities driven by cross-field currents can be expected $^{37-42}$ to heat the plasma, primarily in the direction perpendicular to the magnetic field (in addition, the ions are heated in the perpendicular direction as a consequence of bouncing off the magnetic piston). Therefore, there results a considerable anisotropy in the particle energy distribution. The electrons are believed to be isotropized $^{40,41,43}$ on the very fast time scale $\nu(\frac{1}{2} \beta_{e1})^{-1/2} |\omega_{ce}|^{-1}$, where $\beta_{e1} = \frac{8\pi nT_e/e}{B^2}$ and $\omega_{ce} = eB/m_e c$. Calculations for the case of a spatially homogeneous plasma $^{43}$ indicate that the electromagnetic ion cyclotron instability provides a mechanism for ion energy isotropization. However, the time scale for this mechanism is slow, i.e., of the order of $\omega_{ci}^{-1}$. Moreover, even after many ion cyclotron periods there still remains a considerable ion energy anisotropy, $^{43}$ (e.g., $(T_{ii} - T_{i\parallel})T_{i\parallel}^{-1} < 0.6$ in the case of an infinite homogeneous plasma with $B^0 = B_{0\parallel}$. Consequently, it is important to study the effect of ion energy anisotropy on micro- and macrostability properties during the immediate post-implosion phases of a pinch experiment.

After a discussion of the assumptions and basic equations [Sec. 2], the subsequent analysis can be divided into three parts. In Secs. 3, 4 and 5, the hybrid-kinetic stability properties of a linear screw-
pinch is studied for the case of low frequency electromagnetic perturbations (\(|\omega|<\omega_{ci}\)) about a static equilibrium with isotropic ions.

After a description of equilibrium properties [Sec. 3], the general eigenvalue equation for these perturbations is derived in Sec. 4.

In Sec. 5, this equation is applied to derive explicit expressions for the growth rate in the case of a sharp-boundary, near theta-pinch plasma with small but finite ion Larmor radius, adopting Turner's FLR ordering. The effect of finite electron temperature on the growth rate is determined and discussed in detail.

In Secs. 6 and 7, hybrid-kinetic stability properties are examined for perturbations about a static equilibrium with anisotropic ions for the case when there is no helical field component inside the plasma.

In Sec. 6, the eigenvalue equation is derived. In Sec. 7, Freidberg's trial function method is applied to this equation, in order to obtain an explicit expression for the growth rate of low-frequency (\(|\omega|<\omega_{ci}\)) modes in a sharp-boundary near-theta pinch plasma, with \(B_0^0=0\) inside the plasma column, and with \(r_{Li}<R_p\), where \(R_p\) is the radius of the plasma column. The effect on the growth rate and on the threshold for instability of both finite electron temperature and ion temperature anisotropy is discussed.

Finally, in Sec. 8, the hybrid-kinetic ordering as well as Turner's FLR ordering are critically examined in connection with the description of long-wavelength perturbations in a sharp-boundary, near-theta pinch plasma. It is shown that in some parameter regimes finite resistivity effects (both electron inertial and collisional) should be taken into account, and the corresponding eigenvalue equation is derived.
2. ASSUMPTIONS AND BASIC EQUATIONS

We shall investigate the stability properties of collisionless, linear pinch configurations [Fig. 1] for complex eigenfrequencies \( \omega = \omega_r + i \gamma \) and characteristic inhomogeneity lengths
\[ L = \left| \frac{\partial \phi}{\partial x} \right|^{-1} \]
satisfying
\[ \frac{|\omega|}{\omega_{ci}} \ll 1, \quad \frac{r_{Li}}{L} \ll 1. \] (1)

Here, \( \omega_{ci} = eB/m_i c \) is the ion cyclotron frequency, and \( r_{Li} = v_i/\omega_{ci} \) is the thermal ion gyroradius. The present analysis makes use of the recently developed hybrid-kinetic model \(^3\) for high-beta plasmas. Like the Vlasov-fluid model, the hybrid-kinetic model makes no assumption that the ion gyroradius is small. Therefore, the ions are described by the exact (unexpanded) Vlasov equation
\[
\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right\} f_i (\mathbf{x}, \mathbf{v}, t) = 0 ,
\] (2)
where \( f_i (\mathbf{x}, \mathbf{v}, t) \) is the density of ions in the phase space \((\mathbf{x}, \mathbf{v})\), \( \mathbf{E}(\mathbf{x}, t) \) is the electric field, \( \mathbf{B}(\mathbf{x}, t) \) is the magnetic field, \( +e \) and \( m_i \) are the ion charge and mass, respectively, and \( c \) is the speed of light in vacuo.

Unlike the Vlasov-fluid model, however, the hybrid-kinetic model retains the effects of finite electron temperature, as well as the influence of electron kinetic effects parallel to \( \mathbf{B} \). The electron Vlasov equation is expanded within the context of the inequalities
\[ \frac{|\omega|}{\omega_{ce}} \ll 1, \quad \frac{r_{Le}}{L} \ll 1 , \] (3)
where \( \omega_{ce} = -eB/m_e c \) is the electron cyclotron frequency, and \( r_{Le} \) is the thermal electron gyroradius. To lowest order, this leads to the drift-kinetic equation,
\[
\left\{ \frac{\partial}{\partial t} + (\mathbf{v} \cdot \mathbf{v}_E) \right\} \mathbf{v}_E + \left( \frac{\partial}{\partial \chi} + \left( \mathbf{v}_E \cdot \frac{\partial}{\partial \chi} - \frac{1}{2} \mathbf{v}_E^2 \mathbf{v}_n \mathbf{B} - \frac{e}{m_e} \mathbf{E}_E \right) \frac{\partial}{\partial \mathbf{v}_E} + [\mathbf{v}_E \cdot \nabla \mathbf{B} - (\nabla \cdot \mathbf{v}_E) \mathbf{v}_E] \mathbf{v}_E^2 \right\} \bar{F}_e (\chi, \mathbf{v}^2, \mathbf{v}_E, t) = 0.
\]

where \( \bar{F}_e (\chi, \mathbf{v}^2, \mathbf{v}_E, t) \) is the (gyrophase-independent) distribution of electron guiding centers in a frame of reference moving with velocity \( \mathbf{v}_E \).

\[
\mathbf{v}_E = \frac{c \mathbf{E} \times \mathbf{B}}{B}.
\]

In Eq. (4), \( \mathbf{\hat{e}} = \mathbf{v}_e / |\mathbf{v}_e| \) is a unit vector along the magnetic field direction, \( e \) and \( m_e \) are the electron charge and mass, respectively, \( I \) is the unit dyadic, \( \mathbf{v}_\perp \) is the speed perpendicular to \( \mathbf{B} \) (in a frame moving with velocity \( \mathbf{V} \)), \( \mathbf{v} \mathbf{v}_E \) is the parallel velocity, \( \mathbf{E}_E = \mathbf{E} \mathbf{\hat{e}} \) is the parallel electric field, and the superdot (\( \cdot \)) denotes

\[
\mathbf{\hat{v}} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \mathbf{v}_E) \mathbf{v}_E + \frac{\partial}{\partial \chi}.
\]

In the lowest-order hybrid-kinetic model, Eqs. (2) and (4) are supplemented by the following equations, which determine the self-consistent evolution of the electromagnetic fields \( \mathbf{E}_E \) and \( \mathbf{B}_E \):

\[
\left( \frac{\partial}{\partial t} + \mathbf{v}_E \cdot \frac{\partial}{\partial \chi} \right) \mathbf{v}_E = - (\nabla \cdot \mathbf{v}_E) \mathbf{v}_E - \left( \frac{c \mathbf{E}_E}{B} \mathbf{v}_E - \mathbf{A}_E \right) - \left( \frac{c \mathbf{E}_E}{B} \mathbf{v}_E - \mathbf{A}_E \right) \times \mathbf{v}_E,
\]

\[
\omega_{ve}^2 \mathbf{E}_E = \mathbf{\hat{e}} \cdot \left\{ \mathbf{E} - \left( \frac{4 \pi e}{m_i} \mathbf{v}_E - \frac{4 \pi e}{m_e} \mathbf{v}_E \right) \right\},
\]

\[
(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \frac{\partial}{\partial \chi}) \mathbf{B}_E = - (\nabla \cdot \mathbf{v}_E) \mathbf{v}_E - \left( \frac{c \mathbf{E}_E}{B} \mathbf{v}_E - \mathbf{A}_E \right) + \left( \frac{c \mathbf{E}_E}{B} \mathbf{v}_E - \mathbf{A}_E \right) \times \mathbf{v}_E.
\]
\[
\mathbf{p} \times \frac{\partial}{\partial t} \left( \mathbf{v}_i N_1 + N_m \mathbf{v}_E + m_e \mathbf{v}_e \right) - \frac{e}{4\pi} \mathbf{B} \times \left( \frac{\mathbf{B}}{\mathbf{n}_e} + \frac{\mathbf{B}}{\mathbf{n}_e} - \mathbf{T} \right) = 0 .
\]

In Eqs. (9) and (10), \( N_1 = N = N_{\text{i}} \) is the number density, \( \omega_{pe} = (4\pi n e^2/m_e)^{1/2} \) is the electron plasma frequency, \( \mathbf{T}(x,t) \) is the electromagnetic stress tensor,

\[
\mathbf{T} = \frac{1}{4\pi} \left( \mathbf{E} \mathbf{E} - \frac{1}{2} \mathbf{B}^2 \mathbf{I} + \mathbf{B} \mathbf{E} - \frac{1}{2} \mathbf{E}^2 \mathbf{I} \right) ,
\]

\( N_{\text{i}}(x,t) \) is the ion momentum density,

\[
N_{\text{i}} = m_1 \int d^3 \mathbf{v} \chi f_1 ,
\]

and \( \mathbf{n}_{\text{i}}(x,t) \) and \( \mathbf{n}_{\text{e}}(x,t) \) are the ion and electron stress tensors defined by

\[
\mathbf{n}_{\text{i}} = m_1 \int d^3 \mathbf{v} \chi \mathbf{v} f_1 ,
\]

\[
\mathbf{n}_{\text{e}} = P_{\text{el}} (1 - n_0) + P_{\text{el}} \mathbf{n}_{\text{e}} + N_m \mathbf{v}_E \mathbf{v}_E + m_e (n_\text{e} V_\text{e} V_\text{e}) \int d^3 \mathbf{v} \mathbf{v}_{\text{e}} \mathbf{v}_\text{e} .
\]

In Eq. (14), the perpendicular and parallel electron pressures are defined in terms of the guiding-center distribution \( \mathbf{f}_e(\chi, v_1^2, v_\parallel, t) \) by

\[
P_{\text{el}} = \frac{m_e}{2} \int d^3 \mathbf{v} v_1^2 \mathbf{f}_e ,
\]

\[
P_{\text{el}} = m_e \int d^3 \mathbf{v} v_\parallel^2 \mathbf{f}_e ,
\]

where \( \int d^3 \mathbf{v} = 2\pi \int_0^\infty dv_1 v_1 \int_{-\infty}^{\infty} dv_{\parallel} \). Equations (2) and (4), when supplemented by Eqs. (7) - (15), constitute a complete lowest-order hybrid-kinetic description of the system. \( ^{35} \) Note that the principal ingredients of the model are: (a) fully kinetic ions [Eqs. (1) and (2)], so that finite-ion gyroradius effects are retained in a fully consistent
manner, and (b) drift-kinetic electrons [Eqs. (3) and (4)], including the influence of finite electron temperature and electron kinetic effects parallel to \( \mathbf{B} \). Since higher-order electron drifts (e.g., polarization and VB drifts) are not included in Eq. (4), we emphasize that present analysis cannot be applied to high-frequency perturbations with \( |\omega| \lesssim \omega_{\text{ci}}^{1/2} \), and is necessarily restricted to relatively low frequencies with \( |\omega| \lesssim \omega_{\text{ci}} \) [Eq. (1)]. A higher-order version of the hybrid-kinetic model has been developed, and can be used to examine microstability behavior on the fast time scale \( \omega^{-1} \).

The equilibrium and stability studies in Secs. 3-5 are carried out within the context of Eqs. (2), (4), (7)-(15) and the following simplifying assumptions:

(a) Equilibrium properties are azimuthally symmetric \((\partial/\partial \phi=0)\) and independent of \( z \) \((\partial/\partial z=0)\). Referring to Fig. 1, the equilibrium \( \mathbf{E}_c \times \mathbf{B} \) velocity \( \mathbf{v}_c^0(\chi) \) can be expressed as

\[
\mathbf{v}_c^0(\chi) = -\frac{cE_\mathbf{r}^0(\chi)B_\mathbf{z}^0(\chi)\hat{\mathbf{e}}_\phi}{[B(\chi)]^2} + \frac{cE_\mathbf{r}^0(\chi)B_\mathbf{\phi}^0(\chi)\hat{\mathbf{e}}_z}{[B(\chi)]^2},
\]

where \( \hat{\mathbf{e}}_\phi \) is a unit vector in the azimuthal direction, \( E_\mathbf{r}^0(\chi)\hat{\mathbf{e}}_\phi = -(\partial \phi / \partial r)\hat{\mathbf{e}}_\phi \) is the equilibrium electric field, \( E_\mathbf{z}^0 = B_\mathbf{\phi}^0 + B_\mathbf{z}^0 \) is the equilibrium magnetic field, and \( r \) is the radial distance from the axis of symmetry.

(b) In the stability studies, all perturbation quantities \( \delta \phi(\chi,t) \) are assumed to vary according to

\[
\delta \phi(\chi,t) = \delta \phi(r) \exp \left[ i(m\phi + k_z z - \omega t) \right],
\]

where \( m \) is the azimuthal mode number, \( k_z \) is the axial wavenumber, and \( \omega = \omega_r + i \gamma \) is the complex eigenfrequency.
3. **EQUILIBRIUM PROPERTIES**

For a linear pinch, we introduce cylindrical polar coordinates \((r, \theta, z)\) [Fig. 1] and consider the equilibrium field configuration

\[
\hat{E}^0(r, \theta, z) = E^0_x(r) \hat{\epsilon}_x = -\frac{\partial \phi^0(r)}{\partial r} \hat{\epsilon}_r ,
\]

\[
\hat{B}^0(r, \theta, z) = B^0_z(r) \hat{\epsilon}_z + B^0_\theta(r) \hat{\epsilon}_\theta ,
\]

where \(\hat{\epsilon}_r\) and \(\hat{\epsilon}_z\) are unit vectors in the radial and axial directions.

All equilibrium quantities \((\partial/\partial t = 0)\) are denoted by a superscript zero.

It is assumed that the ions carry zero equilibrium current, i.e., the ions are electrostatically confined with \(\nu_i^0 = \left(\int dv_1 \nu_i^0 \right) / \left(\int dv_1 \nu_i^0 \right) = 0\). We therefore consider the class of self-consistent ion Vlasov equilibria of the form

\[
f_i^0 = f_i^0(H) ,
\]

where \(H\) is the ion energy defined by

\[
H = \frac{m_1}{2} \nu_i^2 + e\phi^0(x) .
\]

In Eq. (20), \(m_1\) is the ion mass, and \(+e\) is the ion charge.

It is readily shown that Eq. (19) satisfies the steady-state version of the ion Vlasov equation (2). Substituting Eq. (19) into Eq. (13) and making use of \(\nu_i^0 = 0\), we find that the equilibrium ion stress tensor can be expressed as

\[
\frac{\nu_i^0}{\nu_i^0} = p_i^0 \frac{I_i^0}{\nu_i^0} ,
\]
where

\[ p_i^0 = \frac{m_i}{3} \int d^3v \; v^2 \; f_i^0(H) \]  

(22)

Differentiating Eq. (22) with respect to \( r \) and making use of Eq. (20), we obtain

\[ N_i^0 eE^0_r = \frac{\partial}{\partial r} p_i^0, \]  

(23)

where \( E_r^0 = -\partial \phi^0 / \partial r \), and \( N_i^0(r) = \int d^3v \; f_i^0(H) \) is the ion density.

Equation (23) is valid for the class of ion equilibria described by Eq. (19). Note that Eq. (23) is simply a statement of equilibrium radial force balance on an ion fluid element with \( v_i^0 = 0 \).

In Sec. 4, the hybrid-kinetic stability formalism is developed for the general class of ion equilibria described by Eq. (19). In Sec. 5, however, we will sometimes especially consider the specific choice of ion equilibrium

\[ f_i^0(H) = N_i^0 \left( \frac{m_i}{2\pi T_i^0} \right)^{3/2} \exp \left\{ \frac{-H}{T_i^0} \right\} , \]  

(24)

where \( N_i^0 \) and \( T_i^0 \) are constants (independent of \( r \)). From Eq. (24), it is straightforward to show that

\[ p_i^0(r) = N_i^0(r) T_i^0 \]  

(25)
where the equilibrium ion density \( N_i^0(r) = \int d^3v N_i^0(N) \) can be expressed as

\[
N_i^0(r) = N_0 \exp \left\{ - \frac{e\Phi(r)}{T_i} \right\}.
\]

(26)

Note that

\[
N_i^0 eE^0 r = T_i \frac{3N_i^0}{\vartheta r}
\]

(27)

follows directly from Eq. (26), or from Eqs. (23) and (25).

For future reference, we define the ion thermal velocity by \( v_i = (2T_i/m_i)^{1/2} \), and the effective local ion Larmor radius by \( r_{Li} = v_i/(eE^0/m_i) \). Making use of \( V_i^0(r) = -cE(r)/E^0(r) \), where

\[
E^0 = \left[ E^0_x^2 + E^0_y^2 \right]^{1/2},
\]

Eq. (27) can be expressed in the equivalent form

\[
\frac{V_i^0}{v_i} = - \frac{r_{Li}}{2N_i^0} \frac{3}{\vartheta r} N_i^0.
\]

(28)

Introducing \( L_n = 3N_i^0/\vartheta r \), where \( L_n \) is the length scale for density variation, it follows from Eq. (28) that \( r_{Li}/L_n \) is equal to the equilibrium velocity measured in units of the ion thermal speed \( (V_i^0/v_i) \).

For the electron equilibrium, we first note from Eqs. (9), (13)-(16) and (18) that

\[
\begin{align*}
V_i^0 = 0, \\
\frac{E}{r} \cdot V_i^0 B^0 = 0, \\
\nabla \cdot V_i^0 = 0, \\
\nabla \times V_i^0 = \frac{1}{r} \frac{\vartheta}{\vartheta r} (\vartheta n_i^0) - \hat{e}_\vartheta \frac{3}{\vartheta r} n_i^0,
\end{align*}
\]

(29)

where \( n_i^0 = n_i^0(r)\hat{e}_0 + n_i^0(r)\hat{e}_z \). It then follows from Eq. (29) that any guiding-center distribution of the form

\[
\frac{e}{V_i^0} = \frac{e}{(v_i^2 + v_i^2, r)}
\]

(30)
is a solution to the steady-state version of the electron drift-kinetic equation (4). Note that the dependence on the radial variable \( r \) is arbitrary in Eq. (30). For simplicity, we restrict the stability analysis in Secs. 4 and 5 to the specific case where \( f_e^0(v_1^2, v_1, r) \) is an isothermal Maxwellian,

\[
\frac{\rho_e^0}{\rho_e} = \frac{\rho_e^0}{\rho_e} = N(r) \left( \frac{m_e}{2\pi kT_e} \right)^{3/2} \exp \left\{ -\frac{m_e}{2kT_e} v^2 \right\}
\]

where \( N(r) \) is an arbitrary (unspecified) function of \( r \), and \( T_e^0 \) is assumed constant (independent of \( r \)). Making use of Eqs. (15) and (31), the electron pressure is given by

\[
P_e^0(r) = \frac{\rho_e^0}{\rho_e} N(r) = N(r) T_e^0
\]

and \( N(r) = N_e^0(r) = \int d^3 v \frac{\rho_e^0}{\rho_e} \) is the electron density. For the particular choice of ion distribution function in Eq. (24), the density profile \( N(r) \) and electrostatic potential \( \phi^0(r) \) are readily related by the equilibrium quasineutrality constraint

\[
N(r) = N_e^0(r) = N_i^0(r)
\]

Combining Eqs. (26) and (33), we find

\[
N(r) = N_e^0 \exp \left\{ -\frac{\phi^0(r)}{T_i^0} \right\}
\]

Finally, substituting Eqs. (14), (16), (18), (21) and (32) into Eq. (10) with \( \partial^2/\partial t^2 = 0 \), we find

\[
\frac{3}{3r} \left\{ \frac{P_e^0(r) + N(r) T_e^0}{8} \right\} = -\frac{N_e^0 V_e^0}{r} - \frac{P_e^0(r)}{4\pi r}
\]
where use has been made of \( \int d^3 \nu \, v^0 = 0 \) [Eq. (31)], and the
electric components of the equilibrium electromagnetic stress tensor
\( \tau^0_\nu \) [Eq. (11)] have been neglected by virtue of \( N_e^0 = N_i^0 \). For the
applications of interest here, the characteristic magnitude of \( V_e^0(r) \)
is much less than the electron thermal speed \((2T_e/m_e)^{1/2}\). We therefore
neglect the electron centrifugal contribution on the right-hand side
of Eq. (35), and approximate Eq. (35) by the familiar pressure balance
equation
\[
\frac{3}{3} \left\{ P_\perp(r) + N(r)T_e + \frac{B^0_\theta(r) + B^0_z(r)}{8\pi} \right\} = -\frac{B^0_\theta(r)}{4\pi r} . \tag{36}
\]
Equation (36) is valid for arbitrary ion equilibrium distribution \( f_i^0(H) \). In the case where \( f_i^0(H) \) has the form prescribed
by Eq. (24), we make use of \( P^0_\perp(r) = N(r)T_e \), and Eq. (36) can be
expressed as
\[
\frac{3}{3} \left\{ N(r)(T_e + T_i) + \frac{B^0_\theta(r) + B^0_z(r)}{8\pi} \right\} = -\frac{B^0_\theta(r)}{4\pi r} . \tag{37}
\]
Since the functional forms of \( N(r) \) and one of the magnetic field
components, e.g., \( B^0_\theta(r) \), can be chosen arbitrarily, the flexibility
of describing general equilibrium profiles within the context of the
hybrid-kinetic model is evident. Once the forms of \( N(r) \) and \( B^0_\theta(r) \)
are specified, however, \( B^0_z(r) \), \( \phi^0(r) \), and \( V_e^0(r) \) are determined
self-consistently from Eqs. (37), (34) and (16), respectively. For
example, if \( N(r) \) is gaussian,
\[
N(r) = N_0 \exp(-r^2/k_0^2) , \tag{38}
\]
and \( B^0_\theta(r)=0 \), then \( B^0_\theta(r) \), \( \phi^0(r) \) and \( V_e^0(r) \) are given by
\[ E_z^0(r) = B_\infty [1-\hat{\beta}\exp(-r^2/R_0^2)]^{1/2}, \]

\[ \phi^0(r) = \frac{T_1}{e} \frac{r^2}{E_0}, \]

\[ \psi^0_E(r) = \frac{2cT_1}{eB_\infty} \frac{r^2}{E_0} [1-\hat{\beta}\exp(-r^2/R_0^2)]^{-1/2} \hat{e}_\theta, \]

where \( \hat{\beta} \) is the value of \( \hat{\beta} = 8\pi (P_1^{0} + P_0^{0})/B_\infty^2 \) on the cylinder axis and where \( B_\infty \) is the magnetic field strength in vacuum.
4. GENERAL STABILITY ANALYSIS

In this section, we make use of the hybrid-kinetic formalism discussed in Sec. 2 to describe stability properties of linear pinch equilibria. The analysis is carried out for perturbations about an equilibrium characterized by Maxwellian electrons [Eq. (31)] but arbitrary ion distribution function \( f^0_i(H) \) [Eq. (19)]. The linearized equations are discussed in Sec. 4.A. In Secs. 4.B and 4.C we evaluate the required perturbed electron and ion properties. In Sec. 4.D, the results are combined to give the desired eigenvalue equation, valid for general \( f^0_i(H) \). Stability properties for the sharp-boundary case are then discussed in Sec. 5.

4.A Linearized Equations

We now linearize Eqs. (2), (4) and (7)-(10) about the equilibrium described by Eqs. (16), (18), (19), (21), (29), (31) and (36). In this regard, we neglect electron inertial terms (proportional to \( m_e \)) in Eqs. (10) and (14), an approximation which is generally valid for \( |\omega||v_e|_{ci} \), provided the characteristic \( E \times B \) velocity is less than the electron thermal speed, i.e.,

\[
\frac{v_e^2}{c_e^2} \ll \frac{v_e^2}{c_e^2},
\tag{40}
\]

where \( v_e = \left( \frac{2T_e}{m_e} \right)^{1/2} \). Within the context of Eq. (40) and the quasi-neutrality assumption, \( N_i = N_e \), we also neglect the electric field components of the electromagnetic stress tensor \( T_\omega \) [Eq. (11)]. Finally, the term \( \nu_e (\partial / \partial t) (\mathbf{E} \times \mathbf{B}) / \epsilon_0 \) is neglected in comparison with \( \nu_e \mathbf{v}_e / \partial t \) in Eq. (10). It is straightforward to show that this is a
valid approximation provided $\nu^2/\zeta^2 = \omega^2/\omega_p^2 < 1$, a condition readily satisfied in high-density pinch experiments.

Combining the approximations enumerated in the previous paragraph, Eq. (10) can be expressed as

$$
\mathcal{R} \times \left\{ \frac{3}{2} \mathcal{N}_d + \nu \cdot \left[ \mathcal{N}_d + \mathcal{P}_{\theta e} + \frac{1}{4\pi} \left( \mathcal{P}_p - \frac{B^2}{2} \mathcal{N}_d \right) \right] \right\} = 0,
$$

(41)

where $\mathcal{N}_d$ and $\mathcal{P}_\theta$ are defined in Eqs. (12) and (13), and

$$
\mathcal{P}_{\theta e} = \mathcal{P}_{\theta e} (T - \mathcal{P}_{\theta e}) + \mathcal{P}_{\theta e} \mathcal{N}_d.
$$

(42)

We also neglect terms of order $(m_e/m_i)^{1/2}$ in Eq. (9), and approximate $\mathcal{E}_\|$, by

$$
\mathcal{E}_\| = -\frac{4\pi e}{m_e} N \cdot \left[ \mathcal{P}_{\theta e} \right],
$$

(43)

All quantities are expressed as an equilibrium value plus a perturbation, i.e.,

$$
\mathcal{E}_\| (\chi, t) = \delta \mathcal{E}_\| (\chi, t),
$$

$$
\nu_e (\chi, t) = \nu_e^0 (r) + \nu_e (r) + \delta \nu_e (\chi, t),
$$

$$
B(\chi, t) = B^0 (r) + \delta B (\chi, t),
$$

$$
\mathcal{P}_{\theta e} (\chi, t) = \mathcal{P}_{\theta e}^0 + \delta \mathcal{P}_{\theta e} (\chi, t),
$$

$$
\mathcal{P}_{\theta e} (\chi, \nu_\perp, \nu_\|, t) = \mathcal{P}_{\theta e}^0 (v_\perp, v_\|, r) + \delta \mathcal{P}_{\theta e} (\chi, v_\perp, v_\|, t),
$$

(44)

where $\mathcal{R} = \mathcal{R}^0 + \delta \mathcal{R}, \nu_e = \nu_e^0 + \delta \nu_e$, and all perturbation quantities are assumed to vary according to $\delta \phi (\chi, t) = \delta \phi (r) \exp(i(m_e + k_z)z - \omega t)$, where $1 \omega > 0$.

Making use of Eqs. (29) and (44), we linearize Eqs. (7), (8), (41) and (43) for small-amplitude perturbations, which gives
\[-i(ω - \omega_0 - k_z v_z^0) \frac{\delta E}{\delta B} + \frac{\delta v_{\gamma E}}{\delta B} \cdot \nabla \times B = -v_{\gamma E} \cdot \delta E_{\gamma E} \]

\[+ \delta \rho \cdot (\delta \rho \cdot v_{\gamma E}^0) + \rho_n \cdot (\delta \rho \cdot v_{\gamma E}^0) - \frac{c}{\delta B} \rho_n \cdot (\nabla \times v_{\gamma E}^0), \]

\[-i(ω - \omega_0 - k_z v_z^0) \delta B + \omega_E \delta n \cdot \hat{e}\theta \]

\[+ \delta v_{\gamma E} \frac{\partial \rho_n}{\partial r} = (\hat{\phi}_z - \hat{\rho} \cdot n_0) \frac{n_0}{r} \delta v_{\gamma E} \]

\[+ i \left( k_z n_z^0 + \frac{m_0}{r} \right) \delta v_{\gamma E} + \frac{\partial n_0}{\partial r} \frac{3 \delta \rho_n}{\delta r} \]

\[- \frac{c}{\delta B} \rho_n \cdot \nabla \times \delta E_{\gamma E} \]

\[\frac{\partial n_0}{\partial r} \left( \frac{\partial v_{\gamma E} \theta}{\partial r} + \frac{\partial v_{\gamma E} \phi}{\partial r} \right) + \frac{\partial n_0}{\delta n} \omega_E \delta n_r \]

\[+ \frac{c}{\delta B} \rho_n \cdot \nabla \times \delta E_{\gamma E} \]

\[\frac{\partial n_0}{\delta n} \cdot \nabla \times \delta E_{\gamma E} \]

\[\delta E_{\gamma E} = \begin{cases} \frac{T_{\gamma E}}{e} \delta \rho_n \cdot \nabla \times B - i \left( k_z n_z^0 + \frac{m_0}{r} \right) \frac{\delta \rho_n}{\delta r} \frac{3 \delta \rho_n}{\delta r} \end{cases} \]  \tag{46}

In obtaining Eqs. (45) – (48), we have defined

\[\omega_E(\tau) = \frac{v_{\gamma E}(\tau)}{r} \]

\[\delta v_{\gamma E} = \begin{cases} \left( \frac{\partial v_{\gamma E}^0}{\partial \tau} \right) \cdot \delta v_{\gamma E} & \frac{\partial v_{\gamma E}^0}{\partial \tau} \end{cases} \]

\[\delta v_{\gamma E} = \begin{cases} \left( \frac{\partial v_{\gamma E}^0}{\partial \tau} \right) \cdot \delta v_{\gamma E} & \frac{\partial v_{\gamma E}^0}{\partial \tau} \end{cases} \]  \tag{50}
and made use of $p^0_{\gamma e}=N(r)T_{\gamma e}$ [Eqs. (32) and (42)]. Moreover, $S_{\gamma i}$, $\delta_{\gamma i}$ and $\delta_{\gamma e}$ are defined by

$$\delta S_{\gamma i} = m_i \int d^3 v \chi \delta f_i,$$

$$\delta S_{\gamma e} = m_e \int d^3 v \chi \delta f_i,$$

$$\delta E_{\gamma e} = \delta E_{\gamma e}^0 (I-\delta R) + \delta E_{\gamma e}^0 \delta R,$$

where

$$\delta E_{\gamma e}^0 = m_e \int d^3 v v^2 \delta f_e,$$

$$\delta E_{\gamma e} = \frac{1}{2} m_e \int d^3 v v^2 \delta f_e.$$

For the ions, the linearized Vlasov equation for $f_i(\chi, \chi', t)$ can be expressed as

$$\left\{ \frac{\partial}{\partial t} + \chi' \cdot \frac{\partial}{\partial \chi} + \frac{e}{m_i} \left( \frac{\chi' \times \delta B}{c} \right), \frac{\partial}{\partial \chi} \right\} \delta f_i = 0$$

Neglecting initial values, the formal solution to Eq. (53) can be obtained using the method of characteristics. Integrating from $t'=\infty$ to $t'=t$ gives

$$\delta f_i(\chi, \chi', t) = -\frac{e}{m_i} \int_{-\infty}^{t} dt' \left[ \delta B(\chi', t') \cdot \frac{\chi' \times \delta B(\chi', t')}{c} \right]$$

where $\chi'(t')$ and $\chi'(t')$ are the particle trajectories in the equilibrium fields,
subject to the "initial" conditions \( \xi'(t'=t) = \xi \) and \( \eta'(t'=t) = \eta \). In Sec. 4.C, Eq. (54) is further simplified for the class of ion equilibria \( f_1^0(\xi, \eta) = f_1^0(\eta) \) \(^1\) [Eq. (19)]. The resulting expression for \( \delta f_1 \) is then used to evaluate the ion contribution \(-1\omega eM_1 + \nabla \cdot \delta \mathbf{P}_i\) in the macroscopic force equation (47).

For the electrons, Eq. (4) is linearized about the Maxwellian equilibrium in Eq. (31). Making use of Eqs. (16), (18), (29), (49) and (50), together with \( v_{\|}^{-1}\delta \phi_1^0/\partial \nu_{\|} \delta \phi_1^0/\partial \nu_{\bot} \), we readily obtain

\[
\delta \phi_1^0 = -i\left( \omega \nu_{\|} - k_\nu \nu_{\|}^0 - k_{\nu}^0 \nu_{\|} - m_\nu^0 \nu_{\|}/r \right)^{-1}.
\]

\[
\left\{ \begin{array}{l}
\delta V_{E_{\|}} \frac{\partial \nu_0}{\partial \nu_{\|}} + \nu_{\|} \left( \delta n_r \frac{\partial \nu_0}{\partial \nu_{\bot}} + \delta V_{E_{\|}} \frac{m_0}{e} \omega_{E_{\|}} \delta n_r \right) \\
- \frac{m_e}{e} \nu_0 \delta n_{E_{\|}} \cdot \frac{\partial \nu_0}{\partial \nu_{\bot}} - \delta V_{E_{\|}} + \frac{m_e}{e} \omega_{E_{\|}} \delta n_r
\end{array} \right. \quad (56)
\]

where \( \delta \mathbf{e}_M \) is defined in Eq. (31). The expression for \( \delta \phi_1^0 \) given in Eq. (56) is used in Sec. 4.B to evaluate the perturbed electron contribution to \( \delta \mathbf{e}_e \) [Eq. (51)] and \( \delta \mathbf{e}_{\|} \) [Eq. (48)].

In summary, Eqs. (45)-(48), (54) and (56), when supplemented by the definitions (49)-(52), constitute the final set of linearized equations that describe the stability properties of the system.
4.8 Perturbed Electron Properties

In this section we make use of the expression for $\delta \tilde{P}_e$ in Eq. (56) to evaluate the perturbed electron pressure $\delta P_{e_e}$ [Eq. (51)] and the electron contribution to $\delta E_{\parallel}$ [Eq. (48)], assuming perturbations about a Maxwellian equilibrium $f^0_e$ [Eq. (31)]. Substituting Eq. (56) into $\delta P_{e_{el}} = (m_e/2) \int d^3v \nabla^2 e \delta \tilde{P}_e$ [Eq. (52)], and integrating over velocity, it is straightforward to show that

$$
\delta P_{e_{el}} = \frac{iN_e}{(k_z^n + \omega n_\theta / \nu_e)} \left\{ I_1 \delta V_e \left( \frac{3}{\beta} \ln N \right) + v_e I_2 \left[ \delta n_r \frac{3 \ln N}{\nu_e} + \delta V_e \nu_e^2 \omega_e n_\theta \nu_e^2 - \nu_e^2 \omega_e n_\theta \nu_e \frac{3 \ln N}{\nu_e} \right] + \frac{e}{\nu_e} \delta E_{\parallel} - 2v_e^2 \omega_e n_\theta \nu_e \delta n_r \right) + 2I_3 \left( \frac{\delta n_r}{\nu_e} \delta V_e \right) + \frac{n_0^2}{\nu_e} \left( \frac{\delta n_r}{\nu_e} \delta V_e - \frac{\delta n_r}{\nu_e} \nu_e \frac{3 \ln N}{\nu_e} - \omega_e n_\theta \nu_e \frac{3 \ln N}{\nu_e} \right) \right\},
$$

where $I_1$, $I_2$, $I_3$ and $I_4$ are defined by

$$I_1 = - \frac{(k_z^n + \omega n_\theta / \nu_e)}{N(\nu_e)^2} \int d^3v \frac{\nu^2_v f^0_e}{v_{eM}} \left[ \omega - \mu \nu_e - k_v \nu_e^0 - (k_z^n + \omega n_\theta / \nu_e) \nu_e \right],$$

$$I_2 = - \frac{(k_z^n + \omega n_\theta / \nu_e)}{N(\nu_e)^2} \int d^3v \frac{\nu^2_v f^0_e}{v_{eM}} \left[ \omega - \mu \nu_e - k_v \nu_e^0 - (k_z^n + \omega n_\theta / \nu_e) \nu_e \right],$$

$$I_3 = 1 + n_\parallel(\eta),$$

$$I_4 = 1 + n_\parallel(\eta).$$
I_3 = - \frac{(k_n z^0 + mn_0^0/r)}{2N(r) v_e^3} \int \frac{d^3 v_v 4 \Gamma_0^0}{v \Gamma_0^0} = z(n),

(58)

I_4 = - \frac{(k_n z^0 + mn_0^0/r)}{N(r) v_e^3} \int \frac{d^3 v_v 2 \Gamma_0^2}{v \Gamma_0^2} = \eta[1+\eta Z(n)],

In Eq. (58), \( v_e = (2T_e/m_e)^{1/2} \) is the electron thermal speed, \( \eta = (\omega - m\omega_E - k_z V^0_E)/(k_n z^0 + mn_0^0/r) v_e \), and \( Z(n) \) is the plasma dispersion function

\[
Z(n) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x}} \exp\left(-\frac{x^2}{\eta^2}\right).
\]

Similarly, substituting the expression for \( \delta f_e^0 \) [Eq. (56)] into \( \delta P_{e\parallel} = m_e \int d^3 v v_\parallel^2 \delta f_e^0 \) [Eq. (52)], and integrating over velocity, we obtain

\[
\delta P_{e\parallel} = \frac{2iN e}{m_e} \left\{ \frac{\partial \ln N}{\partial r} K_1 \delta V_{Er} + v_e K_2 \right\}
\]

\[
\left( \delta n_r \frac{\partial n_N}{\partial r} + \delta V_{Er} 2 v_e \omega_E n_0 \delta r - 2v_e \omega_E n_0 \delta r \right) + K_3 \left\{ \vdots \right. \]

\[
+ \frac{2}{r} \left( \frac{n_0^0}{\delta r} \delta V_{Er} - \delta n_r \right) \left( \frac{n_0^0}{\delta r} \delta V_{Er} - \right) + K_4 \left( \frac{n_0^0}{\delta r} \delta V_{Er} - \right)
\]

where \( K_1, K_2, K_3, \) and \( K_4 \) are defined by

\[
K_1 = - \frac{(k_n z^0 + mn_0^0/r)}{N(r) v_e^3} \int \frac{d^3 v v_\parallel^2 \Gamma_0^0}{v \Gamma_0^0} = \eta[1+\eta Z(n)],
\]

(60)
It is instructive to examine the form of Eqs. (57) and (60) in the limit where \( k_z \to 0 \) and \( n_0 \to 0 \), i.e., pure flute perturbations in a theta-pinch.

Making use of

\[
Z(n) = -\frac{1}{n} - \frac{1}{2n^3} - \ldots , \text{ for } |n| \gg 1 ,
\]

we obtain

\[
\delta P_{e_1} = \frac{\text{INT}}{\omega - m_0 E} \left( \frac{\delta \ln N}{\delta Y} \delta V_{Er} + 2V - \delta V_{e_1} \right) ,
\]
\[
\delta P_{e_2} = \frac{\text{INT}}{\omega - m_0 E} \left( \frac{\delta \ln N}{\delta Y} \delta V_{Er} + V - \delta V_{e_1} \right) ,
\]

valid for \( k_z = 0 \), \( n_0 = 0 \) (i.e., \( \delta \) in the \( z \)-direction). Equations (63) and (64) are simply statements that the perturbed pressures behave adiabatically, with \( \gamma_\perp = 2 \) for the perpendicular pressure, and \( \gamma_\parallel = 1 \) for the parallel pressure. For the special case of "incompressible" perturbations with \( \mathbf{v} \cdot \delta \mathbf{V} = 0 \) and \( k_z = n_0 = 0 \) we note from Eqs. (63) and (64) that \( \delta P_{e_1} = \delta P_{e_2} \).
We now return to the case where \( k_z \) and \( n_0 \) are arbitrary. For future reference, it is useful to express \( \delta P_{e1} \) [Eq. (57)] and \( \delta P_{eII} \) [Eq. (60)] directly in terms of \( \delta n \). To this end, we first establish some identities that relate \( \delta V_{E1} \) and \( \delta V_{EII} \) to \( \delta P_{e1} \) and \( \delta n \). Taking the radial component of Eq. (46), and making use of Eq. (48) to eliminate \( \delta E_{II} \) gives

\[
\delta V_{E1} = \left( -\frac{imc}{erB_0} + \Gamma_1 \right) \delta P_{e1} + \delta F,
\]

(65)

where \( \delta F \) and \( \Gamma_1 \) are defined by

\[
\begin{align*}
\delta F &= \frac{1}{2} \left[ \omega_k - \frac{k \cdot v^0}{m} \right] - m_a - \left( n_0 - \frac{k \cdot r n_0}{r} \right) m_a \delta n_r,
\end{align*}
\]

(66)

\[
\Gamma_1 = \frac{imc}{erB_0} \left( 1 - a \right) + \frac{ik z n_0}{erB_0},
\]

(67)

where \( \omega = -(\epsilon T_e/erB_0) \delta nN/\delta r \) is the electron diamagnetic drift frequency. Moreover, to express \( \nabla \cdot \delta V_{EII} \) in terms of \( \delta P_{eII} \) and \( \delta n \), we multiply Eq. (46) by \( B^0 \) and take the divergence, i.e.,

\[
\nabla \cdot B^0 \left\{ -i(\omega - m a - k \cdot v^0) \delta n + \nabla \cdot \left( -n_0 v^0 \frac{\partial n}{\partial r} \right) \right\}
\]

\[
= -B^0 \left( k \frac{3n_0}{r} + m \frac{2a}{r} \right) \delta V_{E1} + \left( imc \frac{m_0}{r} \right) \delta V_{E1} + \left[ \frac{imc}{erB_0} \frac{m_0}{n_0} \right] \delta V_{EII}
\]

\[
+ iB^0 \left( k \frac{n_0}{r} + \frac{m_0}{r} \right) \nabla \cdot \delta V_{EII}
\]

(67)
Eliminating $\delta E_{||}$ and $\delta V_{||}$ by means of Eqs. (48) and (65), and making use of (66), we find from Eq. (67) that

$$\nabla \cdot \delta V_{||} = \left( \frac{imc}{e\varepsilon_0 n} \frac{\partial n_B^0}{\partial r} + \Gamma_2 \right) \delta P_{e||} + \delta G,$$

where $\Gamma_2$ and $\delta G(\delta n)$ are defined by

$$\Gamma_2 = -\frac{\partial n_B^0}{\partial r} \left( \frac{imc}{e\varepsilon_0 n} (1-n) + \frac{ikzcn_0}{\varepsilon B_0 N} \right)
+ \frac{ic}{e\varepsilon_0 n} \left( k z \frac{\partial n_0}{\partial r} - m \frac{n_0}{\varepsilon B_0 N} \right),$$

$$\delta G = -\frac{\partial n_B^0}{\partial r} \left( \frac{k z n_0}{r} \cdot \frac{n_0}{r} \right) \omega_de \delta n_r
+ \frac{(\omega_{m} - k z \frac{\partial n^0}{\partial r})}{(k z n + mn_0/r)} \nabla \cdot \delta n,$$

where $\omega_{de} = -(e T_e / e B_0) \partial n_B / \partial r.$

Substituting Eqs. (48), (65) and (68) into Eq. (57), $\delta P_{e||}$ can be expressed directly in terms of $\delta P_{e||}$ and $\delta n$. This gives

$$\delta P_{e||} = L \delta P_{e||} + \delta I_4$$

where $L$ and $\delta I_4(\delta n)$ are defined by
\[
L_2 \equiv \frac{iN}{(k_\perp n_\perp + imn_\perp /r) e} \left( -\frac{\text{Im} e^{-i\theta}}{\text{er} B} + i \right) \frac{\partial n_\perp}{\partial \theta} \frac{I_1}{r} \\
+ 2v_0 I_2 \left( \omega_E n_\perp - V_0 \cdot \frac{\partial n_\perp}{\partial \theta} \right) + 2I_4 \frac{n_\theta}{r} \\
- 2I_3 \frac{n_\theta}{r} \right) e^{-i\theta} - \frac{ev}{Te} I_2 \left( \frac{ik}{cN} + \Gamma_3 \right) \\
+ 2I_3 \left( \frac{\text{Im} e^{-i\theta}}{\text{er} B} + i \right) \frac{\partial n_\perp}{\partial \theta} + \Gamma_2 \right),
\]
\[
(71)
\]
\[
\delta I_4 \equiv \frac{iN}{(k_\perp n_\perp + imn_\perp /r) e} \left[ \delta F \left( \frac{\partial n_\perp}{\partial \theta} \right) I_1 + 2v_0 I_2 \\
\times \left( \omega_E n_\perp - V_0 \cdot \frac{\partial n_\perp}{\partial \theta} \right) + (2I_4 - 2I_3) \frac{n_\theta}{r} \right) \\
+ 2I_3 \delta G + R_4 \right),
\]
\[
(72)
\]
where \( \Gamma_3 \) and \( R_4(\delta n) \) are defined by
\[
\Gamma_3 = [ik_\perp (n_\perp^2 - 1) + imn_\perp /r] \frac{1}{eN},
\]
\[
R_4 = -v_0 I_2 \frac{m}{Te} r\omega_E^2 \delta n_r \\
- 2I_3 \left[ \delta n_r \cdot (\delta n_\perp^0 \cdot \mathbf{v}) \omega_0 \mathbf{v} + \delta n_\perp^0 (\delta n_r \cdot \mathbf{v}) \omega_0 \mathbf{v} \right] \\
- 2I_4 \left( \delta n_r \frac{\partial n_\perp^0}{\partial \theta} \cdot \frac{\partial n_\perp}{\partial \theta} + n_\theta^0 \omega_E \delta n_r \right)
\]

In Eqs. (71) and (72), \( \delta F \) and \( \delta G \) are defined in Eqs. (66) and (69) and \( I_1, I_2, I_3 \) and \( I_4 \) are defined in Eq. (58). Similarly, substituting Eqs. (48), (65) and (68) into Eq. (60), \( \delta P_{\parallel \parallel} \) can be expressed in a form similar to Eq. (70), i.e.,
\[
\delta P_{\parallel \parallel} = I_{\parallel \parallel} \delta P_{\parallel \parallel} + \delta I_{\parallel \parallel},
\]
\[
(73)
\]
where $L_{\parallel}$ and $\delta I_{\parallel}$ ($\delta q$) are defined by

$$L_{\parallel} = \frac{2i\Pi_{\parallel}}{(k_{z} n_{z} + \alpha_{n_{z}})} e \left\{ \left( - \frac{imc}{erB N} + \Gamma_{1} \right) \left( \frac{3\alpha_{nN}}{\bar{\alpha}} K_{1} + \frac{\alpha_{nN}}{\bar{\alpha}} K_{2} \right) \right. \right.$$ 

$$+ 2v_{e}^{-1} K_{2} \left( \omega_{p0}^{0} - \Phi_{E}^{0} \cdot \frac{\alpha_{nN}^{0}}{\bar{\alpha}} \right) + (2K_{4} - K_{3}) \frac{n_{\theta}^{02}}{r} \right\} ,$$

$$\delta I_{\parallel} = \frac{2i\Pi_{\parallel}}{(k_{z} n_{z} + \alpha_{n_{z}})} e \left\{ \delta F \left( \frac{3\alpha_{nN}}{\bar{\alpha}} K_{1} + 2v_{e}^{-1} K_{2} \right) \right. \right.$$ 

$$\times \left( \omega_{p0}^{0} - \Phi_{E}^{0} \cdot \frac{\alpha_{nN}^{0}}{\bar{\alpha}} \right) + (2K_{4} - K_{3}) \frac{n_{\theta}^{02}}{r} \right\} ,$$

$$+ K_{3} \delta G + R_{\parallel} \right\} , \quad (75)$$

where $R_{\parallel}$ ($\delta q$) is defined by

$$R_{\parallel} = -v_{e}^{-1} K_{2} \frac{m_{e}}{T_{e}} r_{E}^{2} \delta n_{r}$$

$$- K_{3} \left[ \delta n_{r} \cdot (\Phi_{00}^{0} \cdot \Phi_{E}^{0}) + \Phi_{00}^{0} \cdot (\delta q \cdot \Phi_{E}^{0}) \right]$$

$$- 2K_{4} \left( \delta n_{r} \cdot \Phi_{E}^{0} \cdot \frac{\alpha_{nN}^{0}}{\bar{\alpha}} + n_{0}^{0} \alpha_{E} \delta n_{r} \right) .$$

Combining Eqs. (70) and (73), the expressions for $\delta P_{e\perp}$ and $\delta P_{e\parallel}$ become

$$\delta P_{e\perp} = \delta I_{\perp} + \frac{L_{\perp}}{1 - L_{\parallel}} \delta I_{\parallel} , \quad (76)$$

$$\delta P_{e\parallel} = \frac{\delta I_{\parallel}}{1 - L_{\parallel}} , \quad (77)$$
where \( L_\perp, \delta L_\perp, L_\parallel \) and \( \delta L_\parallel \) are defined in Eqs. (71), (72), (74) and (75), respectively. Moreover, making use of Eqs. (48) and (77), \( \delta E_\parallel \) can be expressed as

\[
\delta E_\parallel = - \frac{T_e}{e} \delta n \frac{\partial N}{\partial y} - \frac{1}{eN} \left( \frac{k}{z} n^0 + \frac{m_0}{\gamma} \right) \frac{\delta Y_\parallel}{1 - L_\parallel} \tag{78}
\]

Since \( \delta I_\parallel \) and \( \delta I_\perp \) are defined in terms of \( \delta F \) and \( \delta G \), and hence in terms of \( \delta N \) [Eqs. (66) and (69)], Eqs. (76) - (78) constitute final expressions for \( \delta P_{\parallel}, \delta P_{\perp} \) and \( \delta E_\parallel \) directly in terms of \( \delta N \).

Note also that electron kinetic effects parallel to \( B_\parallel \) play an important role in Eqs. (76) - (78), as manifest by the various terms proportional to \( Z(\eta) \), occurring through \( I_1, I_2, I_3, I_4, K_1, K_2, K_3 \) and \( K_4 \) [Eqs. (58) and (61)].
4. C Perturbed Ion Properties

In this section, we evaluate $\delta f_1$ in terms of $\delta P_{e\parallel}$ and the perturbed electromagnetic fields. For $f_1^0(x, \lambda) = f_1^0(H)$, it follows from Eq. (20) that $\partial f_1^0 / \partial y = m \lambda^0 \partial f_1^0 / \partial \lambda$. Since $\partial f_1^0 / \partial \lambda$ is independent of $t'$, Eq. (54) can be expressed as

$$\delta f_1(x, \lambda, t) = -e \frac{\partial f_1^0}{\partial \lambda} \int_{-\infty}^{t} dt' \delta E(x', \lambda') \cdot \gamma'. \quad (79)$$

It is useful to introduce the electromagnetic field potentials $\delta \phi$ and $\delta A$, where

$$\delta E = -\nabla \delta \phi + \frac{i \omega}{c} \delta A, \quad (80)$$

$$\delta B = \nabla \times \delta A. \quad (81)$$

As indicated in Appendix A, we choose a gauge condition such that

$$\delta \pi - \delta \phi + \frac{\delta A \cdot \mu^0}{c} = 0, \quad (81)$$

and

$$\delta A \cdot \mu^0 = \delta A_{\parallel} = 0. \quad (82)$$

In Eq. (81), $\delta \pi$ and $\mu^0$ are defined by

$$\delta \pi = \frac{\delta P_{e\parallel}}{eN(r)}, \quad (83)$$

$$\mu^0 = \nu^0_{de}, \quad (84)$$

where $\nu^0_{de}$ and $\nu^0_{de}$ are the electron diamagnetic drift velocities, respectively, i.e.,
Within the context of Eq. (82) we define a displacement vector \( \delta A_i \) by

\[
\delta A_i = \varepsilon_{i\lambda} B^0_{\lambda}.
\]

The gauge condition in Eq. (81) can then be expressed as

\[
\delta \tau - \delta \phi + \frac{\varepsilon_{\tau\lambda}^0}{c} \cdot (B^0_{\lambda} \chi^0) = 0.
\]

Substituting Eq. (80) into Eq. (79), and making use of Eqs. (17) and (82), the perturbed ion distribution function can be expressed as

\[
\delta f_i(\chi', \chi, t) = e \left[ \frac{\partial f^0_i}{\partial \chi} \right] \int_{-\infty}^{t} dt' \left[ \frac{1}{c} \left( \varepsilon_{i\lambda} B^0_{\lambda} \chi^0 \right) \right] \chi' \cdot \chi' + \varepsilon_{\tau\lambda}^0 \frac{\partial A(\chi', t')}{\partial \chi'} \chi' \cdot \chi',
\]

where \( \chi' \) and \( \chi' \) are the particle trajectories in the equilibrium field configuration, \( E^0_{\chi \tau} \) and \( B^0_{\chi \tau} = B^0_{\chi \rho} + B^0_{\chi \theta} \) [Eq. (55)]. The identity

\[
\frac{d}{dt'} \delta \phi(\chi', t') = (-i \omega + \chi' \chi' \chi') \delta \phi(\chi', t')
\]

can be used to simplify the first term on the right-hand side of Eq. (87). Integrating by parts with respect to \( t' \), we obtain

\[
\delta f_i(\chi, \chi', t) = e \delta \phi(\chi', t) \left[ \frac{\partial f^0_i}{\partial \chi} \right] + e i \omega \left[ \frac{\partial f^0_i}{\partial \chi} \right] \int_{-\infty}^{t} dt' \left[ \frac{\partial \phi(\chi', t')}{\partial \chi} \chi' \delta A(\chi', t') \right].
\]
Equation (89) can be further simplified by noting that Eqs. (55) and (81)-(85) combine to give the identity

$$\delta \phi = -\frac{\chi' \cdot \delta \phi}{c} = \delta \pi + \xi_\perp \cdot \left( \frac{\mathbf{\rho}^0 \times \chi'}{c} - \frac{\mathbf{\rho}^0 \times \chi_\perp}{c} \right)$$

$$= \delta \pi + \xi_\perp \cdot \left( \frac{\mathbf{\rho}^0 \times \chi'}{c} - \left( \frac{\mathbf{\rho}^0}{c} - \frac{m_1}{e} \frac{d \chi'}{d \tau'} \right) \cdot \xi_\perp \right)$$

$$= \delta \pi + \xi_\perp \cdot \frac{\mathbf{\rho}^0 \times \chi_\perp}{c} + \frac{m_1}{e} \frac{d \chi'}{d \tau'} \cdot \xi_\perp$$

where all quantities in Eq. (90) are evaluated at \((x', t')\). Substituting Eq. (90) into Eq. (89), and making use of \((d \chi'/d \tau') \cdot \xi_\perp = (d/d \tau') (\chi'_\perp \cdot \xi_\perp) - \chi' \cdot \delta \xi_\perp / \delta \tau \), we obtain

$$\delta f_1 (x, y, t) = e \delta \phi (x, y) \frac{\partial f^0}{\partial \mathbf{H}} + i \omega \frac{m_1}{e} \frac{d \chi'}{d \tau'} \frac{\partial f^0}{\partial \mathbf{H}}$$

$$- i \omega \hat{S} \frac{\partial f^0}{\partial \mathbf{H}}$$

where \(\hat{S}\) is defined by

$$\hat{S} = S - S_v - S_\pi$$

with

$$S = m_1 \int_{-\infty}^{t} dt' \chi' \cdot \frac{d}{d \tau'} \xi_\perp (x', t')$$

$$S_v = \frac{e}{c} \int_{-\infty}^{t} dt' \xi_\perp (x', t') \cdot [\mathbf{\rho}^0 (x') \times \chi_\perp (x')]$$

$$= \int_{-\infty}^{t} dt' \xi_\perp (x', t') \frac{\partial}{\partial \tau'} N(x')$$

$$S_\pi = \int_{-\infty}^{t} dt' \frac{\delta \mathbf{p}_{\perp} (x')}{\partial \mathbf{H} N(x')}$$

In the limit of cold electrons with \(T_e = 0\) and \(\delta \mathbf{p}_{\perp} = 0\), Eq. (93) gives \(S_v = 0 = S_\pi\), and Eq. (91) reduces to the expression for \(\delta f_1\) obtained by Freidberg from the Vlasov-fluid model.
We now make use of Eq. (91) to evaluate the ion contribution to the linearized momentum transfer equation (47). Rather than calculate \( \rho_i^0 \times (-i\omega \delta M_i^0 + \nabla \cdot \delta P_i^0) \) directly, we operate on the linearized Vlasov equation (53) with \( \rho_i^0 \times m_i \int d^3v \ldots \) This readily gives

\[
\rho_i^0 \times (-i\omega \delta M_i^0 + \nabla \cdot \delta P_i^0)
\]

\[
= \rho_i^0 \times \left\{ e \int d^3v \left( \frac{\nabla \times B_i^0}{c} + \frac{\nabla \times \delta E_i}{c} \right) \delta f_i \right\} + e \int d^3v \left( \frac{\nabla \times \delta E_i}{c} \right) \delta f_i,
\]

(94)

where \( \rho_i^0 = \frac{\rho_i^0}{\mu} \), \( \rho_i^0 = \rho_i^0(r) \delta_i^0 + B_i^0(r) \delta_i^0 \), \( \nabla \times B_i^0 = -\frac{\delta_i^0}{\mu} \delta_i^0 / \partial r \) and \( \delta E_i = -\nabla \delta \phi + (\omega/c) \delta \phi \).

Making use of \( \delta A_i = \delta_i^0 \times \rho_i^0 \) [Eq. (85)], and the assumption that the ions carry zero equilibrium current \([i.e., \delta_i^0 = \int d^3v \nabla \times \delta f_i = 0] \), Eq. (94) readily reduces to

\[
\rho_i^0 \times (-i\omega \delta M_i^0 + \nabla \cdot \delta P_i^0)
\]

\[
= \rho_i^0 \times \left\{ -i\omega e \int d^3v \left( \frac{\nabla \times B_i^0}{c} + \frac{\nabla \times \delta E_i}{c} \right) \frac{\delta f_i^0}{\delta \phi} \right\} + e \int d^3v \left( -\frac{\nabla \times \delta E_i}{c} \right) \frac{\delta f_i^0}{\delta \phi}
\]

\[
+ \frac{e i \omega}{c} \int d^3v \left( \frac{\nabla \times \delta E_i}{c} \delta f_i^0 + \omega_i \delta \phi \times \frac{\nabla \times B_i^0}{c} \frac{\delta f_i^0}{\delta \phi} \right)
\]

(95)

where \( \delta \) is defined in Eq. (92). In obtaining Eq. (95),
we have expressed \( \delta f_1 \) [Eq. (91)] as

\[
\delta f_1 = [e\delta \phi + i\omega n \xi_1 \cdot \gamma - i\omega \frac{3}{2}] \delta f_1^0 / \partial H.
\]

Making use of the fact that the velocity dependence of \( f_1^0 (H) \) occurs through the variable \( H = m_1^2 / 2e\phi(r) \), it is straightforward to show that the final two terms on the right-hand-side of Eq. (95) exactly cancel. Noting that

\[
\int d^3 v \, \kappa_1^0 \, e_i \phi \frac{\partial f_1^0}{\partial H} = -\delta \phi \left( \int d^3 v \, f_1^0 \right),
\]

Eq. (95) can now be expressed as

\[
\nabla^0 \times (-i\omega \delta \xi_1 + \nabla \times \delta \phi)
\]

\[
= \nabla^0 \times \left\{ -i\omega \left( \int d^3 v \left( \frac{\partial f_1^0}{\partial H} + \frac{\nabla \times \delta \phi}{\kappa_1^0} \right) \right) \delta \phi \right\}
\]

\[
= -\int d^3 v \, f_1^0 \cdot \nabla[N(r)]
\]

where

\[
\int d^3 v \, f_1^0 \cdot \nabla[N(r)] [\text{Eq. (33)}].
\]

The gauge condition (81) can be used to eliminate \( \delta \phi \) in favor of \( \delta P_{e\perp} \) and \( \delta \phi = \xi_1 \times \delta \phi \) in Eq. (97). We find

\[
-\nabla^0 \times \nabla[N(r) \delta \phi]
\]

\[
= -\nabla^0 \times \nabla \delta P_{e\perp} - \nabla^0 \times \nabla \left( \frac{e}{c} \nabla \cdot \delta P_{e\perp} \right)
\]

\[
= -\nabla^0 \times \nabla \delta P_{e\perp} - \nabla^0 \times \nabla [\xi_1 \cdot \nabla P_1(r)],
\]

where

\[
P(r) = N(r) T_e + P_1^0 (r)
\]

is the total unperturbed pressure, and use has been made of Eq. (23) and the definition of \( \nu_n^0 \) [Eq. (83)]. Substituting Eq. (98) into Eq. (97) gives
\[ n^0_c \times (-i \omega \delta M^0_c + \mathbf{v} \cdot \delta \mathbf{P}^0_c) = \mathbf{p}^0_c \times \left\{ -i \omega \int \frac{d^3 \mathbf{v}}{c} (E^0_c + \frac{\mathbf{v} \times \mathbf{B}^0_c}{c}) \frac{\delta f^0}{\delta \mathbf{H}} \right. \]
\[ \left. - \nabla \delta P_{\ell} - \nabla (\mathbf{e}_\ell \cdot \mathbf{v} \mathbf{P}) \right\}. \tag{100} \]

Equation (100) constitutes the final expression for the ion contribution to be used in the linearized momentum transfer equation (47).

4.D General Eigenvalue Equation

In this section, the results of Secs. 4.A-4.C are combined to give an eigenvalue equation for \( \ell \). In particular, we substitute the expressions for \( \mathbf{p}^0_c \times (-i \omega \delta M^0_c + \mathbf{v} \cdot \delta \mathbf{P}^0_c) [\text{Eq. (100)}] \) and \( \delta \mathbf{P}_{\ell} \) [Eqs. (51), (76) and (77)] into the linearized force equation (47). Operating on Eq. (47) with \( \mathbf{p}^0_c \times \) gives

\[ -i \omega \int \frac{d^3 \mathbf{v}}{c} (E^0_c + \frac{\mathbf{v} \times \mathbf{B}^0_c}{c}) \frac{\delta f^0}{\delta \mathbf{H}} + \nabla \mathbf{v} (\delta \mathbf{P}_{\ell} - \delta P_{\ell}^{\parallel}) \]
\[ + \frac{n^0_c}{\ell} (\delta P_{\ell}^{\perp} - \delta P_{\ell}^{\parallel}) \frac{\delta n^0}{\ell} = \nabla \mathbf{v} (\mathbf{e}_\ell \cdot \nabla \mathbf{P}) \tag{101} \]
\[ + \frac{1}{4\pi} \delta P^{\perp} \right)_{\ell} + \frac{1}{4\pi} \left( \mathbf{P}^{\parallel} \cdot \nabla \delta \mathbf{B} \right)_{\ell} - \frac{1}{4\pi} \nabla \mathbf{v} (\mathbf{P}^{\parallel} \cdot \delta \mathbf{B} \right), \]

where \( \nabla \mathbf{v} (I - \mathbf{P}^0 \cdot \mathbf{v}) \cdot \nabla \) denotes gradient perpendicular to \( \mathbf{P}^0 \) and \( A_{\ell} \equiv (I - \mathbf{P}^0 \cdot \mathbf{v}) \).

In deriving (101), use has been made of [Eq. (42)]

\[ \nabla \cdot \delta \mathbf{P}_{\ell} = \nabla \delta \mathbf{P}_{\ell} - n^0_c (\mathbf{e}_\ell \cdot \nabla) (\delta \mathbf{P}_{\ell}^{\perp} - \delta \mathbf{P}_{\ell}^{\parallel}) \]
\[ - (\delta \mathbf{P}_{\ell}^{\perp} - \delta \mathbf{P}_{\ell}^{\parallel}) \frac{\delta n^0}{\ell} \cdot \nabla \mathbf{v} \]. \tag{102} \]

Note that the right hand side of Eq. (101) can be identified with the
perpendicular component of the ideal, incompressible magnetohydrodynamic force defined by

\[
F_{\text{mhd}}^g(\xi_g) = \nabla(\xi_G \cdot \Omega) + \frac{1}{4 \pi} \left[ (\nabla \times \mathbf{B}) / r \right] \times \partial \mathbf{B} \\
+ (\nabla \times \mathbf{B}) \times \mathbf{R}^0, \tag{103}
\]

with \( \mathbf{B} = \nabla \times (\xi_G \times \mathbf{R}^0) \) [Eq. (85)]. Moreover, the component of \( F_{\text{mhd}}^g \) parallel to \( \mathbf{R}^0 \) is identically zero, i.e.,

\[
F_{\text{mhd}}^g(\xi_g) = 0 \quad \text{if} \quad \mathbf{R}^0 \parallel \mathbf{R}^0.
\]

Combining Eqs. (101)-(103) gives the eigenvalue equation

\[
F_{\text{mhd}}^g(\xi_g) = -i \omega S \int d^3 \nu \left( \frac{\nabla \cdot \mathbf{B}}{c} \right) \frac{\partial \mathbf{F}}{\partial \mathbf{H}}
+ \nabla \cdot (\delta \mathbf{P}_{\text{e}} - \delta \mathbf{P}_{\text{e}||}) + \frac{n_e}{\tau} (\delta \mathbf{P}_{\text{e}||} - \delta \mathbf{P}_{\text{e}||}) \mathbf{R}^0.
\tag{105}
\]

In Eq. (105), \( S \) is defined by [Eqs. (92) and (93)]

\[
\frac{\partial}{\partial \mathbf{r}} = \int_{-\infty}^{t} dt' \left\{ m_e \mathbf{v} \cdot \frac{d}{dt'} \xi_{e} \mathbf{r} \left( \mathbf{x}', t' \right) - T_e \xi_{e} \mathbf{r} \left( \mathbf{x}', t' \right) + \frac{\partial}{\partial \mathbf{r}} \right\} \left( \mathbf{F} \mathbf{n} \right) \left( \mathbf{r}' \right).
\tag{106}
\]

Moreover, \( \delta \mathbf{P}_{\text{e}||} \) and \( \delta \mathbf{P}_{\text{e}||} \) are expressed in terms of \( \mathbf{R}^0 \) by Eqs. (76) and (77). In this regard it is important to recognize that \( \delta \mathbf{n} \left( \mathbf{r} \mathbf{R}^0 \mathbf{B}^0 \right) \mathbf{B}^0 \) is related to \( \xi_g \) by
Equation (105) constitutes the final eigenvalue equation for $\xi$ obtained within the framework of the hybrid-kinetic model described in Sec. 2. We summarize here several properties of the general eigenvalue equation (105).

(a) Equation (105) is valid for arbitrary equilibrium profiles $N(r)$, $B^0(r)$ and $E^0_z(r)$ consistent with the equilibrium constraint in Eq. (36). Moreover, Eq. (105) includes the influence of finite electron temperature and finite ion gyroorbits on stability behavior. Also included are electron kinetic effects parallel to $\xi^0$, as manifest by the various contributions to $\delta P_{e\parallel}$ and $\delta P_{e\perp}$ [Eqs. (76) and (77)] proportional to $Z(\eta)$ [see also Eqs. (58) and (61)].

(b) The Vlasov-fluid eigenvalue equation\(^1\) is recovered from Eq. (105) by considering the limit where

$$T_e = 0 .$$

In this case it follows that $\delta P_{e\parallel} = \delta P_{e\perp} = 0$ [Eqs. (76) and (77)]. Equation (105) then reduces to

$$F_{vahd}(\xi) = -\lambda \int d^3v \left( \xi^0 + \frac{v \times E^0}{c} \right) \hat{s} \frac{\partial f^0}{\partial H} ,$$

where

$$\hat{s} = \int_{-\infty}^{t} dt' m_{\parallel} v' \cdot \frac{d}{dt'} \xi_{\parallel}(v', t') ,$$

which is identical to the eigenvalue equation obtained by Freidberg [compare Eq. (109) with Eq. (19) of Ref. 1].

(c) For future reference, we simplify the eigenvalue equation
(105) for the case when $T_e$ is arbitrary, but the poloidal magnetic field is identically zero [$B^0_\theta(r) = 0$]. Equation (105) then reduces to

$$F_{\text{ahd}}(\xi) = -i \omega_e \int d^3 \mathbf{v} \left( \frac{F^0_\phi}{v_e} + \frac{v_e \omega_e}{c} \right) \mathbf{S} \frac{\partial F^0_{\phi i}}{\partial \mathbf{H}}$$

$$+ \nabla \cdot (\delta P_{\phi i} - \delta P_{\phi i}) ,$$

(111)

where $\mathbf{S}$ is defined by Eq. (106). The perturbed electron pressures are given by Eqs. (76) and (77). In Eq. (76) the expressions for $L_\perp$ [Eq. (71)] and $\delta I_\perp$ [Eq. (72)] reduce to

$$L_\perp (B^0_\theta = 0) = \frac{m(-\omega_e + 2\omega_B)}{k_z v_e} Z(\eta) + 1 + nZ(\eta) ,$$

(112)

$$\delta I_\perp (B^0_\theta = 0) = \frac{\int T_e}{k_z v_e} \left\{ \frac{\partial F}{\partial \mathbf{r}} \frac{\partial n}{\partial \mathbf{r}} Z(\eta) + 2Z(\eta) \delta G ight\} - \frac{2^n}{v_e} \left[ 1 + nZ(\eta) \right] \delta n ,$$

(113)

where use has been made of Eq. (58). In Eq. (113), $\eta = (\omega - \omega_B)/k_z v_e$, $\omega = V_e \omega_0 / r$, $v_e = (2T_e / m_e)^{1/2}$, $\omega_B = (cT_e / eB_z)^{1/2} \delta n / \partial \mathbf{r}$, $\omega_{de} = -(cT_e / eB_z) \delta n / \partial r$, and $\delta F$ and $\delta G$ [Eqs. (66) and (69)] reduce to

$$\delta F (B^0_\theta = 0) = - \left( \frac{1}{k_z} \left[ \omega - \omega_{de} - 2\omega_B \right] \delta n \right) ,$$

(114)

$$\delta G (B^0_\theta = 0) = - \frac{\partial n}{\partial \mathbf{r}} \left[ \frac{\omega}{k_z} \delta n \right] - \frac{(\omega - \omega_B)}{k_z} \nabla \cdot \delta n .$$

(115)

Similarly, the expressions for $L_\parallel$ and $\delta I_\parallel$ [Eqs. (74) and (75)] simplify to give

$$L_\parallel (B^0_\theta = 0) = 2 \left\{ \frac{m(-\omega_e + 2\omega_B)}{k_z v_e} \eta \left[ 1 + nZ(\eta) \right] \right\} + \frac{1}{2} + n + n^3 Z(\eta) ,$$

(116)
\[ \delta I^e \left( B_{e0}^0 = 0 \right) = \frac{2iNT_e}{k z v_e} \left\{ \left( \delta F \frac{\partial nN}{\partial r} + \delta G \right) \eta \left[ 1 + n z(\eta) \right] \right\} \]
\[ - \frac{2\sigma_e^2}{v_e} \left[ \frac{1}{2} + n^2 + n^2 z(\eta) \eta \right] \right\} . \]  

(117)

When, in addition to \( B_{e0}^0 = 0 \), the perturbations are flute-like with \( k_z = 0 \),

(118)

then the expressions for \( \delta P_{e1} \) and \( \delta P_{eII} \) [Eqs. (76) and (77)] can be further simplified. For present purposes, however, we make use of Eqs. (63) and (64). It follows that

\[ \nabla \left( \delta P_{e1} - \delta P_{eII} \right) = \nabla \left( \frac{INT_e}{\omega - m_0} v \cdot \delta V_{e} \right) \]  

(119)

on the right-hand side of Eq. (111). Moreover, from Eqs. (64) and (65), the perturbed parallel electron pressure \( \delta P_{eII} \) that occurs in the orbit integral \( \delta \) [Eq. (105)] can be expressed as

\[ \delta P_{eII} = \frac{INT_e}{\omega - m_0} \left( \frac{\partial}{\partial r} \ln N \right) \delta F + v \cdot \delta V_{e} . \]  

(120)

In addition,

\[ \delta P = \delta \left[ - (\omega - m_0) \delta V_{e} \right] \]  

(121)

follows from Eqs. (66) and (107). As discussed in Sec. 4.B, when \( k_z = 0 \) the perturbed electron pressures behave adiabatically, with \( \gamma_1 = 2 \) for the perpendicular pressure, and \( \gamma_II = 1 \) for the parallel pressure.

In circumstances where the perturbed electron \( e^- \)-motion is incompressible, i.e., \( \nabla \cdot \delta V_{e} = 0 \), Eqs. (119) and (120) simplify considerably

\[ \nabla \left( \delta P_{e1} - \delta P_{eII} \right) = 0 , \]  

(122)
and
\[ \delta P_{el} = \frac{1}{2} \int_{\Omega} \frac{\varepsilon_{ij}}{\omega_{ij} + \varepsilon_{ij}} \left( \frac{\partial}{\partial \varepsilon_{ij}} \ln \gamma \right) \delta \varepsilon \ . \] (123)

It should be noted from Eq. (68) that the condition \( \nabla \cdot \delta \varepsilon_{el} = 0 \) does not generally correspond to "incompressible" displacements characterized by \( \nabla \cdot \varepsilon_{el} = 0 \).
5. EFFECT OF FINITE ELECTRON TEMPERATURE ON LOW FREQUENCY LONG WAVELENGTH MODES IN A NEAR-THETA-PINCH WITH SHARP BOUNDARY AND $r_{Li}/r_p < 1$

5.A Basic Ordering

As an application of the stability theory developed in Secs. 2-4, we consider perturbations about a sharp-boundary screw pinch equilibrium with $g_i = 8\pi T_1^0 / R_1^0 \omega_i$ [Fig. 2]. Consideration is restricted to relatively small ion gyroradii with

$$\sigma = \frac{r_{Li}}{r_p} < 1,$$

where $r_p$ is the radius of the plasma column [Fig. 2]. In addition, we consider low-frequency, long-wavelength perturbations satisfying

$$k^2 T_p \sim \sigma^2 \sim \frac{|\omega_i|}{\omega_{ci}} ,$$

$$|v_1| \sim \frac{1}{r_p} ,$$

$$\hat{e}_z \cdot (v_1 \times \hat{e}_z) = 0 .$$

Introducing

$$\eta = |\xi_1| / r_p$$

as a dimensionless parameter measuring the perturbation amplitude, we assume the following ordering in the subsequent analysis in Sec. 5

$$|v_1 \cdot \xi_1| \sim \eta \sigma^2 ,$$

$$|v_1^2 \xi_1| \sim \eta \sigma |k_z| .$$

Moreover, we restrict the analysis to near-theta-pinch configurations,
The ordering in Eqs. (124)-(128) has been adopted following Turner. Finally, we make no a priori ordering of \( \frac{p_0^0}{p_1^0} \), and therefore assume

\[
\tau = \frac{p_0^0}{p_1^0} \sim 1
\]

(129)

without loss of generality. However,

\[
\frac{V_1}{V_e} \ll \frac{c}{a}
\]

(130)

is assumed in the present analysis. Note that the present ordering does not exclude MHD instability, since \( \frac{B_0^2}{r^2} \) may exceed \( \frac{k^2 B_0^2}{2} \), in view of Eqs. (125) and (128).

### 5.8 Simplified Eigenvalue Equation

Consider the interior region of the equilibrium plasma configuration. In the sharp-boundary model [Fig. 2], the density can be expressed as

\[
N = NU(R_p - r)
\]

(131)

where \( \hat{N} \) is constant, and \( U \) is the unit step function. Hence

\[
0 = \psi_0 e = \frac{\partial \psi_0}{\partial r} e = \frac{\partial \psi_0}{\partial r} e = \frac{\partial \psi_0}{\partial r} = \frac{\partial \ln N}{\partial r} = \frac{\partial \ln B_0}{\partial r}
\]

(132)

inside the plasma column \( (0 < r < R_p) \). The equilibrium magnetic field is given by

\[
\mathbf{B}^0 = \begin{cases} \delta \mathbf{e}_z, & 0 < r < R_p \\ \delta \mathbf{e}_z, & R_p < r < R_c \end{cases}
\]

(133)
where $B_0$, $B_0$, and $B_z$ are constants. Substitution of Eq. (132) into Eqs. (112)-(117) yields [Eqs. (76) and (77)]

$$\delta P_{el} - \delta P_{el} = NT_{e} \frac{\omega_0}{k_z v_e} Z \left( \frac{\omega_0}{k_z v_e} \right) \nabla \cdot \xi_1 ,$$  

(134)

$$\delta P_{el} = -NT_{e} \nabla \cdot \xi_1 ,$$  

(135)

for $0 < r < R$. Substituting Eqs. (134) and (135) into the eigenvalue equation (111) yields

$$F_{\text{mhd}}(\xi_1) = -i \omega e \int d^3 \nu \left( \frac{\nu X R_0}{c} \right) \frac{\partial f_0}{\partial H} S + NT_{e} \frac{\omega_0}{k_z v_e} Z \left( \frac{\omega_0}{k_z v_e} \right) v_1 (\nu, \xi_1) ,$$  

(136)

where

$$S = m_1 \int_{-\infty}^{t} dt' \nu' \cdot \frac{d}{d t'} \xi_1 + T_{e} \int_{-\infty}^{t} dt' \nabla' \cdot \xi_1 ,$$  

(137)

follows from Eqs. (92), (93) and (135). In obtaining Eqs. (136) and (137), we have assumed that the class of ion orbits hitting the pinch boundary is negligibly small [Eq. (124)].

Substituting Eq. (137) into Eq. (136) and integrating by parts gives

$$F_{\text{mhd}}(\xi_1) = \frac{i \omega} {c} \frac{\nu X R_0}{c} \times \xi_1 + i \omega e \int d^3 \nu \left( \frac{\nu X R_0}{c} \right) \frac{\partial f_0}{\partial H} \int_{-\infty}^{t} dt' \left[ e^{\xi_1} \right]$$  

$$\left( \frac{\nu X R_0}{c} \right) - T_{e} \nu' \cdot \xi_1 + NT_{e} \frac{\omega_0}{k_z v_e} Z \left( \frac{\omega_0}{k_z v_e} \right) v_1 (\nu, \xi_1) ,$$  

(138)

Next, from Eqs. (132), (86), (91), (135), we obtain

$$\delta f_1 = -T_{e} \frac{\partial f_0}{\partial H} \nu \cdot \xi_1 + i \omega e \frac{\partial f_0}{\partial H} \int_{-\infty}^{t} dt' \left[ e^{\xi_1} \right] \cdot \left( \frac{\nu X R_0}{c} \right)$$  

$$\left( \frac{\nu X R_0}{c} \right) - T_{e} \nu' \cdot \xi_1 ,$$  

(139)

In view of the ordering in Eqs. (124)-(130), the first and last terms on the right-hand side in Eq. (138) stand in the ratio
In obtaining Eq. (140), use has been made of \( |\omega/k_x v_x| \) and \( |\omega/k_z v_z| \) [Eqs. (124) and (125)]. Furthermore, the terms inside the square brackets in Eq. (138) stand in the ratio

\[
\nu \frac{\omega}{\omega_{ci}} : \frac{\nu_1}{\nu_{ci}} : \frac{1}{2} \frac{\nu_1}{\omega_{ci}} \frac{\sigma^2}{c^2} ,
\]

Consequently, up to and including second order in \( \sigma \), the eigenvalue equation [Eq. (138)] can be approximated (inside the plasma) by

\[
F_{\text{mhd}}(\xi) = \frac{i\omega N}{c} \xi \times B_0^0 + i\omega^2 \int d^3 v \left( \frac{\xi x B_0^0}{c} \right) \frac{3f_1^0}{3H} \left( \frac{\xi \cdot \xi}{c} \right) ,
\]

which is identical in form to the result obtained for zero electron temperature. (We emphasize, however, that finite \( T_e \) effects are still manifest through the equilibrium constraint in Eq. (36).) Moreover, the estimate in Eq. (141) implies that Eq. (139) can be approximated by
\[
\delta f_1 = -T_e \frac{\partial f_0^0}{\partial H} \nabla \cdot \xi_L + i \omega m_1 \int_{-\infty}^{t} dt' \left\langle \xi_L \left( \frac{\nabla \times B_0}{c} \right) \right\rangle 
\]

(143)

up to and including second order in \( \sigma \). Since \(-T_e \frac{\partial f_0^0}{\partial H} \nabla \cdot \xi_L\) is even in \( \chi \), it follows from Eq. (143) that the eigenvalue equation (142) can be expressed in the equivalent form

\[
F_{\text{mhd}}(\xi_L) = \frac{i \omega m_1}{c} \xi_L \times B_0 + \frac{eN}{c} \delta U \times B_0 ,
\]

(144)

where \( \delta U = \int d^3v \chi \delta f_1 \).

From the evaluation\(^{33}\) of the orbit integral in Eq. (143), we conclude that the finite \( T_e \) correction to \( \delta f_1 \) is of order \( \sigma \), i.e.,

\[
\left| T_e \frac{\partial f_0^0}{\partial H} \nabla \cdot \xi_L \right| : \left| \omega m_1 \int_{-\infty}^{t} dt' \left\langle \xi_L \left( \frac{\nabla \times B_0}{c} \right) \right\rangle \right| ,
\]

\[
\sim T_e \frac{\partial f_0^0}{\partial H} \nabla \cdot \xi_L \sim \sigma^2 \sim \frac{1}{2} \sigma \sigma.
\]

Expanding \( \xi_L \) in powers of \( \sigma \) (\( \delta f_1 = \delta f_{10} + \sigma \delta f_{11} + \ldots \)), we therefore find

\[
\delta f_{10} = i \omega m_1 \left\langle \frac{\partial f_0^0}{\partial H} \right\rangle \nabla \cdot \xi_L,
\]

(146)

\[
\delta f_{11} = \left\langle \frac{1}{2} \omega m_1 \left( \frac{\partial f_0^0}{\partial H} \right) \frac{\nabla^2}{\omega^2} \right\rangle - \frac{T_e}{\frac{\partial f_0^0}{\partial H}} \nabla \cdot \xi_L
\]

\[
+ \frac{i \omega m_1}{\omega c_i} \left\langle \frac{\partial f_0^0}{\partial H} \right\rangle \left\{ \frac{2}{\nabla^2} \left( \frac{\partial \xi_x}{\partial x} - 3 \frac{\partial \xi_y}{\partial y} \right) - \frac{2}{\nabla^2} \left( \frac{\partial \xi_y}{\partial x} - 3 \frac{\partial \xi_x}{\partial y} \right) \right\}.
\]

(147)

Consequently, the expressions for \( \delta N \), \( \delta \xi_L \) and \( \delta U \) (up to leading order in \( \sigma \)), when modified to include the effects of finite \( T_e \), can be expressed as

\[
\delta N = -T_e \nabla \cdot \xi_L \int d^3v \frac{\partial f_0^0}{\partial H} - \frac{1}{2} \frac{1}{M_1(\omega)} \nabla \cdot \xi_L ,
\]

(148)

\[
\delta \xi_L = -1 \omega \xi_L ,
\]
\[
\delta P_{\text{irr}} = \left( \frac{p_0^0}{p_i^0} - \gamma(\omega) \right) p_i^0 \mathbf{v} \cdot \mathbf{\xi}_l + \frac{i\omega p_0^0}{\omega} \frac{\partial \mathbf{\xi}_l}{\partial r}, \quad (149)
\]
\[
\delta P_{\text{irr}} = \frac{i\omega p_0^0}{\omega} \frac{\partial \mathbf{\xi}_l}{\partial r}, \quad (150)
\]
\[
\delta P_{\text{irr}} = -\frac{i\omega p_0^0}{\omega} \frac{\partial \mathbf{\xi}_l}{\partial r}, \quad (151)
\]
\[
\delta P_{\text{izz}} = \rho_0 \frac{2}{\omega^2} \left( 1 - \frac{m_{\perp}(\omega)}{2N} \right) \mathbf{v} \cdot \mathbf{\xi}_l + p_0^0 \mathbf{v} \cdot \mathbf{\xi}_l, \quad (152)
\]

where use has been made of Eq. (32), and \(\gamma(\omega)\) and \(M^\perp(\omega)\) are defined by

\[
\gamma(\omega) = \frac{m_{\perp}(\omega)}{4p_0^0} M_2(\omega), \quad (153)
\]
\[
M^\perp(\omega) = m_{\perp} \omega \int \frac{d^3v}{k_{\perp}v_{\perp}} \frac{2n}{\omega} \frac{\partial \mathbf{\xi}_l}{\partial r}. 
\]

Evidently, up to and including \(O(\sigma^2)\), the effects of finite \(T_e\) do not modify \(\delta P_{\text{irr}}\) [Eq. (147)]. Therefore, the form of the final eigenvalue equation is identical to Turner's equation \(^{33}\) in leading nontrivial order [cf. Eq. (46) of Ref. 33], i.e.,

\[
\mathbf{\lambda}_{\text{mhd}}(\xi_l) = -p_{i0}^0 \mathbf{\gamma}(\omega) \mathbf{v} \cdot (\mathbf{\xi}_l) - \rho_0 \omega^2 \mathbf{\xi}_l, \quad (154)
\]

where \(\rho_0^0 = N_{e_i} m_i\) and \(\mathbf{\lambda}_{\text{mhd}}(\xi_l)\) is defined in Eq. (103).

5.C Solution to Eigenvalue Equation

Making use of Eqs. (103) and (154), the solution for \(\xi_{\perp}\) inside the plasma \((0 < r < R_p)\) can be expressed as

\[
\xi_{\perp}(r) = C_{m} \mathbf{I}_{m}(\omega), \quad (155)
\]
\[
\xi_{\parallel}(r) = C_{m} \mathbf{I}_{m}(\omega), \quad (156)
\]
where \( w \equiv |k_z| r \), \( I_m'(w) \equiv dI_m/dw \), and

\[
\delta = \left[ \frac{k_z^2 B_0^2}{k_z^2 [B_0^2 + 4 \pi \gamma (\omega) P_0^2]} \right]^{1/2}.
\]  

(156)

The perturbed vacuum magnetic field \((r_p < r < r_c)\), neglecting displacement current, can be expressed as

\[
\delta B = -B^0 \nabla \delta \psi (\xi, t)
\]

where \( \nabla^2 \delta \psi = 0 \), and \( B^0 = (B_\theta^2 + B_z^2)^{1/2} \) [Eq. (133)]. This gives

\[
\delta \psi (\xi, t) = \eta [a_{m, m} (u) + b_{m, m} (u)] \exp [i(-\omega t + m \theta + k_z z)]
\]

(157)

where \( u = k_z |r| \). Making use of \( \delta B = -B^0 \nabla \delta \psi \), we obtain

\[
\delta B = -B^0 \left[ \left| k_z \right| \left\{ \frac{\partial \delta \psi}{\partial u} \hat{\xi}_r + \frac{\partial \delta \psi}{\partial r} \hat{\xi}_\xi + \frac{\partial \delta \psi}{\partial \theta} \hat{\xi}_\theta + \frac{\partial \delta \psi}{\partial z} \hat{\xi}_z \right\} \right]
\]

(158)

in the region outside the plasma column.

We now address the question of boundary conditions at \( r = r_p \) and \( r = r_c \). In the drift-kinetic description, the electrons are described by a guiding-center model, and the plasma is characterized by local charge neutrality. Moreover, the magnetic field must be tangential to the plasma surface, i.e.,

\[
\hat{n} \cdot (B^0 + \delta B)_{\text{vac}} (\xi, t) = 0
\]

(159)

at any position \( \xi \) on the boundary. Here \( \hat{n} \) is the unit normal to the plasma boundary at \( \xi \). Furthermore, since \( \delta \psi_{\xi} = -i \omega \delta \xi \) to leading order in \( \sigma \), the displacement of the plasma boundary is equal to the magnetic field line displacement \( \hat{\xi}_\xi \). Therefore, the normal \( \hat{n} (\xi, t) \) is given by

\[
\hat{n} = \hat{\xi}_r - \frac{im}{r} \xi (\xi_p, \theta, z, t) \hat{\xi}_\theta - i k_z \xi (\xi_p, \theta, z, t) \hat{\xi}_z
\]

(160)

in cylindrical coordinates. Here \( r = \xi (\xi_p, \theta, z, t) = \xi_p + \xi (\xi_p, \theta, z, t) \) specifies
the radius of the perturbed boundary at given \((\theta, z, t)\). Making use of Eqs. (133), (157) and (158), the boundary condition in Eq. (159) yields

\[
a I'(u_0) + b K'(u_0) = -i \left( \frac{m B + k R B}{u_0 B_0} \right) \xi'(R_p),
\]

(161)

where \(u_0 = \left| k_z \right| P \) and \(\xi'(R_p) = \xi'(\delta | k_z | R_p) \) [Eq. (155)].

Furthermore, since \(B_{r0}\) must be tangential to the perfectly conducting wall at \(r = R_c\), it follows from Eq. (158) that

\[
a I'(U) + b K'(U) = 0,
\]

(162)

where \(U = \left| k_z \right| R_c\).

Apart from the electromagnetic boundary conditions, i.e., Eqs. (161) and (162), a third boundary condition can be obtained by integrating the force balance equation (41) across the perturbed boundary, at fixed values of \(\theta\) and \(z\). [Note that in the perpendicular direction Eq. (41) differs from Turner's equation (56) only in the presence of the electron pressure gradient term \(\nabla p_e\)]. Equation (41) can be expressed as

\[
(I - \mathbf{n}_n) \left\{ -i \omega_n \mathbf{n}_m U_{\mathbf{n}} + \mathbf{v} \cdot \mathbf{P} - \frac{1}{8\pi} \mathbf{V} \mathbf{B}^2 - \frac{1}{4\pi} (B \cdot \mathbf{V}) \mathbf{B} \right\} = 0,
\]

(163)

where \(\mathbf{P} = \mathbf{P}_e + \mathbf{P}_e\) is the total kinetic pressure, and \(\mathbf{n} = \mathbf{B} / |\mathbf{B}|\). Since \((I - \mathbf{n}) \cdot \mathbf{v} = \mathbf{v} e_{\mathbf{n}}\) and \(U^0 = 0\), integrating Eq. (163) across the perturbed boundary yields (after linearization)

\[
4\pi \delta \rho p \frac{R_{\mathbf{p}}}{R_{\mathbf{p}}} - \frac{2}{B_0} \left\{ \frac{1}{R} \frac{\partial}{\partial R} \left[ R \mathbf{v} p \right] + \frac{i m}{L} \xi'(R_p) \right\} + B_0^2 \frac{\xi'(R_p)}{R_p}
\]

\[
= -B_0 \left\{ \frac{i (m B + k R B)}{R_p} \right\} [a I_m (u_0) + b K_m (u_0)],
\]

(164)
In Eq. (165) $\delta P_{irr}$ is given by Eq. (149), and $\delta P_{err} = \delta P_{ei}$ is obtained from Eqs. (134) and (135), i.e.,

$$
\delta P_{err} = N_T e \left( \omega \frac{e}{k z} - Z \left( \frac{\omega}{k z} \right) - 1 \right) \nabla \cdot \varepsilon_i.
$$

Equation (164) is identical in form to Turner's boundary condition. However, in the present analysis, $\delta P_{rr}$ represents the total (electron plus ion) pressure perturbation. It should also be noted that the constants $B_0$ and $B_0$ occurring in Eqs. (164) and (133) are related by

$$
\frac{B_0^2 + B_0^2}{8\pi} = N_T e + P_i (r = 1) + \frac{B_0^2}{8\pi}.
$$

Equation (167) has been obtained by integrating Eq. (36) across the boundary at $r = R_p$.

Substituting Eqs. (149) and (166) into Eq. (165) gives

$$
\delta P_{rr} = -\gamma(\omega) P_1^0 \nabla \cdot \varepsilon_i + \frac{\omega}{k z} \left( \frac{\omega}{k z} \right) \nabla \cdot \varepsilon_i + \frac{i \omega}{\omega c_i} P_1^0 \frac{\partial \varepsilon_i}{\partial r}.
$$

Note that the terms proportional to $N_T e \nabla \cdot \varepsilon_i$ in the expressions for $\delta P_{irr}$ and $\delta P_{err}$ exactly cancel. Evidently, the correction to $\delta P_{rr}$ associated with finite $T_e$ is very small, i.e.,

$$
\gamma(\omega) P_1^0 \nabla \cdot \varepsilon_i = |N_T e \frac{\omega}{k z} \left( \frac{\omega}{k z} \right) \nabla \cdot \varepsilon_i |,
$$

$$
\sim 1 : \frac{\omega}{k z},
$$

$$
\sim 1 : \frac{V_i}{V_e},
$$

$$
\sim 1 : \tau \sigma.
$$

Therefore, to leading order, $\delta P_{rr}$ can be approximated by
\[
\delta P_{rr} = -\gamma(\omega)P_1^{(0)}v_r + \frac{\omega}{\omega_c} p_1^{(0)} \frac{\partial^2 \phi}{\partial \tau^2}, \tag{170}
\]

where terms of order \( \sigma \) have been neglected.

Combining Eqs. (161), (162) and (164) [with \( \delta P_{rr} \) specified by Eq. (170)] yields a dispersion relation that determines the complex eigenfrequency \( \omega \). Moreover, within the context of the ordering assumed in Sec. 5.A, the influence of finite \( T_e \) on stability properties occurs only through the equilibrium constraint in Eq. (167).

5.D Discussion of Dispersion Relation

Making use of Eqs. (155), (156) and (170), and expanding Eqs. (161), (162) and (164) with \( k \approx \frac{a^2 - a^2}{\beta_0} \approx 2 \omega^2 \ll 1 \), we obtain the dispersion relation

\[
\omega^2 - \frac{1}{2} \left( \frac{\tau_{Li}}{R_p} \right)^2 m(m-1) \omega + \gamma_0 = 0, \tag{171}
\]

where

\[
\hat{\omega} = \frac{\omega}{\omega_c}, \quad \gamma_0 = -\frac{\tau_{Li}^2}{2 R_p} \left( \frac{k^2 - m^2}{\beta_i} \right), \tag{172}
\]

\[
\tau_{Li} = k R_p, \quad \gamma_0 \approx -2 \left( \frac{B_0^2 - \frac{1}{m} (m\beta_v + k R_p B)^2}{R_p} \right). \]

Normalizing the growth rate with respect to \( \omega_c = |e| B_0 / m \), we find

\[
\frac{\nu}{\gamma_c} = \left( \frac{\gamma_0}{\omega_c} - m^2 \right)^{1/2} \left( \frac{\tau_{Li}}{2 R_p} \right). \tag{173}
\]

From the equilibrium constraint [Eq. (167)] it follows that, for fixed ion pressure and vacuum magnetic field, the finite electron temperature effectively reduces the magnetic field strength \( B_0 \) within the plasma column, thereby increasing \( \tau_{Li} \). Indeed, the internal magnetic field strength \( B_0 \) is reduced by a factor \( \left[ 1 - \tau_{Li}^2 (1-\beta_0)^{-1} \right]^{1/2} \), where \( \beta_0 = \frac{8\pi p_1^{(0)} / B_0^2}{z} \) and \( \tau = T_e / T_i \), thereby increasing the ideal MHD
\[ \gamma_{mhd}^2 = \frac{1}{4\pi Nm_1} \left\{ -k_z^2 B_0^2 + \frac{m}{R_p^2} \left( B_e^2 - \frac{1}{m} (mB_e + k_x R_p B_z)^2 \right) \right\} \] (174)

and the thermal ion Larmor radius \( r_{Li} = v_i/\omega ci \). Maximizing the growth rate \( \gamma \) [Eq. (173)] with respect to \( k_z \), we find (at maximum growth)

\[ \gamma_M^2(m) = m^2 h^2 R_p^2 \nu_A^2 (2-\beta_0) \left( 1 + \frac{\beta_0^{\gamma}}{2-\beta_0} \right) \]

\[ + m(1-m)h^2 R_p^2 - \frac{m^2}{16} (m-1)^2 \frac{r_{Li}(0)}{R_p^2} \nu_A^2 \left( 1 - \frac{\beta_0^{\gamma}}{1-\beta_0} \right) \] (175)

where \( \nu_A = B_z^2 / 4\pi Nm_1 \), \( h = B_0 / B_z \), and \( r_{Li}(0) \) is the ion Larmor radius calculated from the \( T_e = 0 \) value of \( B_0 \). The hybrid-kinetic and ideal MHD maximum growth rates are plotted versus \( \tau = T_e / T_i \) in Fig. 3 for \( m=1 \) and \( m=2 \). Evidently, for small values of \( \tau \), there is a slight increase in the maximum growth rate. For sufficiently large values of \( \tau \), however, \( \gamma_M \) is a sharply decreasing function of \( \tau \) for \( m=2 \). For \( m=1 \), it is evident from Eqs. (173) and (175) that the growth rate is unchanged from its ideal MHD value.
6. INFLUENCE OF ION ENERGY ANISOTROPY ON THE THETA PINCH

STABILITY PROPERTIES

Thus far, the analysis has been restricted to equilibrium ion energy distributions of the form \( f_i^0 = f_i^0(H) \) [Eqs. (19) and (20)] which are isotropic in energy. In this section the effect of ion energy anisotropy on stability behavior is investigated. Allowing for this anisotropy is particularly important since the immediate post-implosion phase of a theta pinch is not generally characterized by isotropic ions, except on a sufficiently long time scale \((\frac{1}{\omega_{ci}})\) for collective instability mechanisms, such as the electromagnetic ion cyclotron instability, to isotropize the ion energy.

For simplicity, we assume a pure theta pinch equilibrium characterized by

\[ B_\theta^0 = 0 \] (176)

As before, the ions are assumed to be electrostatically confined with \( v_i^0 = (\frac{1}{3}v_i^3 f_i^0) / (\frac{1}{3}v_i^3 f_i^0) = 0 \). We therefore consider the class of self-consistent ion Vlasov equilibria of the form

\[ f_i^0 = f_i(H, v_z^2) \] (177)

where \( v_z \) is the axial velocity, and \( H_\perp \) is the perpendicular ion energy defined by

\[ H_\perp = \frac{m_i}{2} v_i^2 + e \phi_0(r) \] (178)

where \( v_i = (v_r^2 + v_\theta^2)^{1/2} \) is the ion speed perpendicular to \( B_0^0 B_z^0(r) \). It is readily shown that Eq. (177) satisfies the steady-state ion Vlasov equation with \( \partial / \partial t = 0 \). Substituting Eq. (177) into Eq. (13) and making use of \( \phi_0 = 0 \), we find that the equilibrium ion stress tensor
can be expressed as

\[
\Pi_{\text{el}}^0 = p_{\text{el}}^0 (e^{-P_{\text{el}}} + 1) = p_{\text{el}}^0 \quad (179)
\]

where \( p_{\text{el}}^0 \) and the pressures \( P_{\text{el}}^0 (r) \) and \( P_{\text{el}}^0 (r) \) are defined by

\[
P_{\text{el}}^0 (r) = \frac{m_1}{2} \int d^3v \frac{v_2}{v} f_{\text{el}}^0 (H_1, v_z^2)
\]

\[
P_{\text{el}}^0 (r) = m_1 \int d^3v \frac{v_2}{v} f_{\text{el}}^0 (H_1, v_z^2)
\]

Differentiating Eq. (180) with respect to \( r \) and making use of Eq. (178), we obtain

\[
\frac{\partial N_1^0}{\partial r} = \frac{3}{3} \frac{\partial E}{\partial r} = \frac{3}{3} \frac{\partial E}{\partial r} = \frac{3}{3} \frac{\partial E}{\partial r} = \frac{3}{3} \frac{\partial E}{\partial r} = \frac{3}{3} \frac{\partial E}{\partial r}
\]

where \( E = -\frac{3}{3} \frac{\partial E}{\partial r} \), and \( N_1^0 \) is the ion density.

In this section, the hybrid-kinetic stability formalism is developed for the general class of ion equilibria described by Eq. (177). In Sec. (7), however, we examine stability properties for perturbations about the specific choice of ion equilibrium

\[
f_{\text{el}}^0 (H_1, v_z^2) = N_0 \left( \frac{m_1}{2 \pi T_{\text{el}}^0} \right)^{1/2} \left( \frac{m_1}{2 \pi T_{\text{el}}^0} \right)^{1/2}
\]

____

(182)

where \( N_0, T_{\text{el}}^0 \) and \( T_{\text{el}}^0 \) are constants (independent of \( r \)). From Eq. (182) it is straightforward to show that

\[
P_{\text{el}}^0 (r) = N_1^0 (r) T_{\text{el}}^0
\]

\[
P_{\text{el}}^0 (r) = N_1^0 (r) T_{\text{el}}^0
\]

where the equilibrium ion density \( N_1^0 (r) = \int d^3v f_{\text{el}}^0 (H_1, v_z^2) \) can be expressed
Note that
\[ N_i^0(x) = N_0 \exp \left( -\frac{\mathcal{E}_i^0(x)}{T_{ii}} \right) \]  \hspace{1cm} (184)

follows directly from Eq. (184). For future reference, we define the ion thermal speed perpendicular to \( B_z^0(x) \) by
\[ v_{iL} = \left( \frac{2T_{ii}}{m_i} \right)^{1/2} \]
and the effective local ion Larmor radius by
\[ r_{L_i}(r) = \frac{v_{iL}}{v_{e}\sqrt{2\pi T_{e}}} \]
so that Eq. (185) can be expressed as
\[ \frac{v_{iL}^0}{v_{e}} = -\frac{r_{L_i}^0}{2N_i^0} \frac{\partial N_i^0}{\partial r} \]  \hspace{1cm} (186)
where \( v_{e}^0(r) = -eE_r^0(r)/B_z^0(r) \). Making use of Eqs. (14), (16), (18), (32), (40), (176) and (177), the steady-state momentum transfer equation can be expressed as
\[ P_{iL}^0(r) + N(r)T_{e} \frac{B_{02}^2(r)}{8\pi} = \frac{B_{\infty}^2}{8\pi} \]  \hspace{1cm} (187)
where the electric field components of the equilibrium electromagnetic stress tensor \( \tau_{ij} \) [Eq. (11)] have been neglected \( N_i^0 = N_1^0 = N(r) \). For the case of bi-Maxwellian ions [Eq. (182)], Eq. (187) becomes
\[ N(r)(T_{iL} + T_{e}) + \frac{B_{02}^2(r)}{8\pi} = \frac{B_{\infty}^2}{8\pi} \]  \hspace{1cm} (188)

The analysis of the perturbed electron properties (Sec. 4B) can be directly applied in the present analysis. Since \( B_z^0 = 0 \) [Eq. (124)], the perturbed electron pressures satisfy Eqs. (76) and (77), with \( L_i, \delta I_i, L_{ii} \) and \( \delta I_{ii} \) given by Eqs. (112), (113), (116) and (117) respectively.

We now evaluate \( \delta f_i \) in terms of \( \delta P_{eii} \) and the perturbed electromagnetic fields. For \( P_{iL}(x,y) = P_{iL}(H_x, v_z^2) \), it follows from Eq. (178) that
\[ P_{iL}^0(x,y) = P_{iL}^0(H_x, v_z^2) \]
Since $\delta f_i / \partial H_z$ and $\delta f_i / \partial v_z$ are independent of $t'$, Eq. (54) can be expressed as

$$
\delta f_i(x, y, z, t) = -e \left[ \frac{\delta f_i}{\partial H_z} \int_{-\infty}^{t} dt' \delta R(x', t') \cdot \chi' \right.
- \frac{e}{m_i} \left( \frac{\delta f_i}{\partial v_z} - \frac{\delta f_i}{\partial H_z} \right) \int_{-\infty}^{t} dt' \delta R(x', t')
+ \frac{\chi' \times \delta R(x', t')}{c} \bigg|_{z},
$$

where $[\delta H]_z$ denotes $A_z$. Note that if the ion equilibrium is isotropic with $f_i^0 = f_i^0(\mathbf{H}_i + \frac{1}{2} \mathbf{v}_i \times \mathbf{v}_i)$, then the final term on the right-hand side of Eq. (189) is identically zero. It is useful to introduce electromagnetic field potentials $\delta \phi$ and $\delta A_z$ [Eq. (80)]. Since the analysis in Appendix A is independent of the form of $f_i^0$, it follows that we can still choose a gauge condition such that [Eqs. (81), (82)]

$$
\delta \pi + \frac{\delta A_z}{c} = 0,
$$

and

$$
\delta A_z = 0,
$$

where $\delta \pi$ and $\mu_0^z$ are defined by

$$
\delta \pi = \frac{\delta \pi}{e N},
$$

$$
\mu_0^z = \left[ \omega_E(r) + \omega_{de}(r) \right] \mathbf{e}_z = \mu_0^z \mathbf{e}_z,
$$

and $\omega_E = -c e E / r \mathbf{e}_z$ and $\omega_{de} = \left( \pi / m_e \right) (\partial / \partial r) \ln N$. From Eq. (191) it follows that a magnetic field line displacement vector $\mathbf{e}_z$ can be defined by
where \( \mathcal{A}_0^0 = B_z(r) e_\perp \). The gauge condition in Eq. (190) can then be expressed as

\[
\delta \pi - \delta \phi + \frac{k_t}{c} \cdot (\mathcal{A}_0^0 \times \mathcal{A}_0^0) = 0.
\]  

Substituting Eq. (80) into Eq. (189), and making use of Eqs. (17) and (191), the perturbed ion distribution function can be expressed as

\[
\delta f_i(\chi', \chi, t) = -e \int_{-\infty}^t dt' \left[ -V' \delta \phi(\chi', t') + \frac{i \omega}{c} \delta \mathcal{A}(\chi', t') \right] \chi' 
\]

\[
- \frac{e}{m_1} \left( \frac{\partial f_0^0}{\partial z} - m_1 v \frac{\partial f_0^0}{\partial H_\perp} \right) \frac{i k_t}{c} \int_{-\infty}^t dt' \left[ \delta \phi(\chi', t') \right] 
\]

\[
+ \frac{V'}{c} \delta \mathcal{A}(\chi', t') \right)
\]  

where \( \chi' \) and \( \chi \) are the particle trajectories in the equilibrium field configuration.

Substituting Eq. (88) into Eq. (196) and integrating by parts with respect to \( t' \), we obtain

\[
\delta f_i(\chi, \chi', t) = e \delta \phi(\chi', t) \frac{\partial f_0^0}{\partial H_\perp} + e i \omega \frac{\partial f_0^0}{\partial H_\perp} \int_{-\infty}^t dt' \left[ \delta \phi(\chi', t') \right] 
\]

\[
- \frac{V'}{c} \delta \mathcal{A}(\chi, t') \right)
\]  

where \( \lambda \) is defined by

\[
\lambda = \frac{1}{\omega} \left( \frac{k_v}{m_1 v} \right) \left( \frac{1}{m_1 v} \frac{\partial f_0^0}{\partial H_\perp} \frac{1}{\partial f_0^0} - \frac{\partial f_0^0}{\partial H_\perp} \right).
\]  

Note that \( \lambda = 1 \) for isotropic ion equilibria with \( f_0^0 = f_0^0(H_\perp + m_1 v^2/2) \).

Equation (197) can be further simplified by noting that Eqs. (55) and (190)-(193) combine to give the identity
\[ \delta \phi = \frac{\gamma' - \delta A}{c} = \delta \pi + \delta \mu \cdot \frac{(\rho_0 \times \mathbf{x}_0')}{c} - \gamma' \cdot \frac{(\xi_0 \times \mathbf{y}_0')}{c} = \delta \pi + \delta \mu \cdot (\mathbf{B}_0 \times \gamma_0') \]
\[ - \left( \frac{m_i}{e} \frac{d \mathbf{v}_i'}{dt'} \right) \cdot \delta \pi = \delta \pi \frac{d}{dt'} \left( \frac{\mathbf{B}_0 \times \gamma_0'}{c} \right) \]
\[ + \frac{m_i}{e} \frac{d \mathbf{v}_i'}{dt'} \cdot \delta \mu \]

where \( \gamma_{de}(x_0') = \omega_{de}(r_0') \gamma_{de}' \), and all quantities are evaluated at \((x_0', t_0')\).

Substituting Eq. (199) into Eq. (197), and making use of \( (d \mathbf{v}_i'/dt') \cdot \delta \pi = (d/dt')(\gamma'_i \cdot \mathbf{p}_i') - \gamma'_i \cdot d \mathbf{p}_i'/dt' \), we obtain
\[ (200) \]
where \( \gamma_{de} \) is defined in Eqs. (92) and (93). For the case of isotropic ions with \( f_1^0(H_1', \mathbf{v}_z) = f_1^0(H_1 + m_1 v_z^2/2) \), it follows that \( (m_1 v_z)^2 \delta f_1^0 / \delta v_z = \delta f_1^0 / \delta H_1 \) and \( \lambda = 1 \), and hence Eq. (200) reduces to Eq. (91).

We now operate on the linearized Vlasov equation [Eq. (53)] with \( \mathbf{p}_0 \times m_1 \int d^3 \mathbf{v} \). This yields
\[ \mathbf{p}_0 \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]
\[ = \mathbf{p}_0 \times \left[ \int d^3 \mathbf{v} \left( \frac{\gamma \times \mathbf{B}}{c} \right) \delta f_1 \right] \]
\[ + \mathbf{p}_0 \times \left( \frac{\gamma \times \mathbf{B}}{c} \right) f_1^0 \]

Substituting Eq. (200) for \( \delta f_1 \) on the right-hand side of Eq. (201), and making use of \( \mathbf{p}_0 = \mathbf{p}_0 \), \( \mathbf{E} = \mathbf{E} \), \( \mathbf{B} = \mathbf{B} \), \( \delta \phi = \delta \phi \), \( \delta \phi = \delta \phi \), \( \delta \phi = \delta \phi + \frac{im}{c} \delta \mathbf{A} \) [Eqs. (80) and (176)] and the assumption that the ions carry zero equilibrium current [i.e., \( \gamma_0^0 = \int d^3 \mathbf{v} \gamma^0 f_1^0 = 0 \)], Eq. (201) readily yields
\[
\frac{\partial}{\partial t} \mathbf{\tilde{s}} = f_{1}^{0} \mathbf{\tilde{E}} + \frac{e_{0} \mathbf{\tilde{B}}}{c} \times \mathbf{\tilde{E}} + \frac{e_{0} \mathbf{\tilde{B}}}{c} \times \mathbf{\tilde{B}} + \frac{e_{0} \mathbf{\tilde{B}}}{c} \times \mathbf{\tilde{B}} + \frac{e_{0} \mathbf{\tilde{B}}}{c} \times \mathbf{\tilde{B}}.
\]

(202)

where

\[
\mathbf{\tilde{s}} = \mathbf{\tilde{s}}_{\perp} + (1 - \lambda) \mathbf{\tilde{s}}_{\parallel},
\]

(203)

and \( \lambda \) and \( \mathbf{\tilde{s}} \) are defined by Eqs. (198) and Eqs. (92) and (93). In obtaining Eq. (202), we have expressed \( f_{1}^{0} \) [Eq. (200)] as \( f_{1}^{0} = f_{1}^{0} \mathbf{\tilde{E}} + f_{1}^{0} \mathbf{\tilde{B}} \) and \( f_{1}^{0} \mathbf{\tilde{B}} \). Making use of the fact that the perpendicular velocity dependence of \( f_{1}^{0}(H_{\perp}, v_{\perp}) \) occurs only through the variable \( H_{\perp} = v_{\perp}^{2}/2 + e\phi_{0}(r) \), it is straightforward to show that the final two terms on the right-hand-side of Eq. (202) exactly cancel. Noting that \( \int d^{3}v f_{1}^{0} \mathbf{\tilde{E}} \cdot \mathbf{\tilde{E}} = e\phi_{0} \left( \int d^{3}v f_{1}^{0} \right) \), Eq. (202) can be expressed as

\[
\frac{\partial}{\partial t} f_{1}^{0} \mathbf{\tilde{E}} = \int d^{3}v f_{1}^{0} (\mathbf{\tilde{E}} \cdot \mathbf{\tilde{E}}) - e\phi_{0} \left( \int d^{3}v f_{1}^{0} \right).
\]

(204)

where \( \int d^{3}v f_{1}^{0} N(r) \) [Eq. (33)]. The gauge condition (195) can be used to eliminate \( \mathbf{\tilde{E}} \) in favor of \( \mathbf{\tilde{E}}_{\parallel} \) and \( \mathbf{\tilde{E}}_{\perp} \) in Eq. (204). This yields

\[
\mathbf{\tilde{E}}_{\perp} = \mathbf{\tilde{E}}_{\parallel} \mathbf{\tilde{B}}_{\perp} \mathbf{\tilde{B}}_{\parallel}.
\]

(205)
where

\[
P_{\perp}(r) = N(r) T_e + \frac{P_0}{\nu} (r)
\]  

is the total perpendicular pressure, and use has been made of Eq. (181) and the definition of \( \xi^0 \) [Eq. (193)]. Substitution of Eq. (205) into Eq. (204) gives

\[
-\omega \delta \frac{\partial \delta P_{\perp}}{\partial \nu} = \rho_0 \times \left[ -\frac{\partial \delta P_{\perp}}{\partial \nu} \right]
\]  

In order to obtain the desired eigenvalue equation for \( \xi_{\perp} \), we substitute the expressions for \( \rho_0 \times (\omega \delta \delta M_{\perp} + \nabla \cdot \delta P_{\perp}) \) [Eq. (207)] and \( \delta P_{\perp} \) [Eqs. (76) and (77)] into the linearized force equation. Operating on Eq. (47) with \( \rho_0 \times \) then gives

\[
-\omega \delta \frac{\partial \delta P_{\perp}}{\partial \nu} = \frac{1}{4\pi} \left( \nabla \cdot \delta P_{\perp} \right) + \frac{1}{4\pi} \nabla \times \left( \delta P_{\perp} \nabla \cdot \delta \right)
\]  

where \( \nu_{\perp} = (\frac{1}{\nu} - \frac{1}{\nu_0}) \nabla \cdot \delta \), \( \nu = \frac{1}{\nu} \frac{\partial \delta}{\partial r} \), and \( \delta \nu = \delta \beta \nabla \cdot \delta \). Note that the right-hand side of Eq. (208) can be identified with the perpendicular component of the ideal, incompressible magnetohydrodynamic force defined by

\[
F_{\text{mhd}}(\xi_{\perp}) = \nabla \cdot (\xi_{\perp} \nabla P_{\perp}) + \frac{1}{4\pi} \left( (\nabla \times \rho_0^0) \times \nabla \times (\xi_{\perp} \rho_0^0) \right)
\]  

(209)
\[ \mathbf{F}_{\text{mhd}}(\mathbf{\xi}_0) = -i\omega \mathbf{\xi}_0 + \mathbf{\xi}_0 \times \mathbf{B}_0, \]

\[ \mathbf{F}_{\text{mhd}}(\mathbf{\xi}_0) = -i\omega \mathbf{\xi}_0 + \mathbf{\xi}_0 \times \mathbf{B}_0. \]

In Eq. \((210)\), \(\mathbf{S}\) is defined by \([\text{Eqs. (92), (93) and (203)}]\)

\[ \mathbf{S} = \lambda \int_{-\infty}^{\infty} \mathbf{e}^{0} \cdot \mathbf{d}c \left[ \mathbf{e}^{0} \cdot \mathbf{d}c \right] + (1-\lambda) \mathbf{e}^{0} \cdot \mathbf{d}c \]

where \(\lambda\) is defined in Eq. \((198)\). Moreover, \(\delta P_{\parallel}\) and \(\delta P_{\perp}\) are expressed in terms of \(\delta \rho\) by \([\text{Eqs. (76) and (77)}]\). Note that for the case where \(B_0^z = 0\) \([\text{Eq. (176)}]\), \(\delta \rho_0 = (\mathbf{T}_0 \cdot \mathbf{B}_0^z) \cdot \mathbf{B}_0^z / B_0^z\) is simply related to \(\mathbf{\xi}_0\) by

\[ \delta \rho_0 = \mathbf{k}_z \cdot \mathbf{\xi}_0. \]

As shown in Appendix B, Eq. \((210)\) leads to the correct dispersion relation for low-frequency \((|\omega| < \omega_{ci})\) transverse electromagnetic perturbations about a spatially homogeneous equilibrium. As is well-known, these perturbations exhibit an instability driven by ion energy anisotropy when \(T_{\parallel} > T_{\perp}\), with a typical growth rate of order \(\omega_{ci}\). Moreover, the eigenvalue equation \((210)\) can be used to generalize the description of this ion energy anisotropy instability to include the effects of finite plasma size and equilibrium spatial gradients.

Finally, as discussed in Sec. 7, Eq. \((210)\) can also be used to investigate the effect of ion energy anisotropy on the growth rate of instabilities with \(|\omega| < \omega_{ci}\) in a sharp-boundary screw pinch, where \(B_0^z = 0\) inside the plasma.
7. **Effects of Finite Electron Temperature and Ion Energy Anisotropy on Low Frequency Long Wavelength Modes in a Near Theta Pinch with Sharp Boundary and $r_{Li}/R_p \approx 1$**

In this section, we consider the sharp-boundary equilibrium described in Fig. 2 and Eq. (133) for the case of a near theta pinch with

$$h = \frac{B_0}{B_{z}} << 1.$$  (213)

A trial function method first used by Freidberg in the Vlasov-fluid model shall be applied to obtain the dispersion relation for low-frequency, long-wavelength perturbations characterized by

$$\frac{\omega}{\omega_{ci}} << 1, \ k_{z}^2 R_p^2 << 1.$$  (214)

The present analysis generalizes Freidberg's work in two respects. First, the electron temperature is allowed to be finite. Second, ion energy anisotropy effects are taken into account by assuming that the unperturbed ion distribution $f_1^0$ is bi-Maxwellian, i.e.,

$$f_1^0 = N \frac{m_i}{2\pi T_{1,11}} \left( \frac{m_i}{2\pi T_{1,11}} \right)^{1/2} \exp\left\{ -\frac{m_i v_1^2}{2T_{1,11}} - \frac{m_i v_{1z}^2}{2T_{1,11}} \right\},$$  (215)

with temperatures $T_{1,11}$ and $T_{1,11}^0$ independent of $r$.

Within the framework of the trial-function method of Freidberg, only those ion orbits striking the plasma boundary contribute to the correction term in the dispersion relation. On the other hand, in Turner's calculation [see also Sec. 6] based on a FLR ordering with $r_{Li} << R_p$, only those ion orbits not striking the boundary were taken into account. In view of the discrepancies of the results of

*According to (133) the magnetic field inside the plasma is directed along the $z$-axis so that $v_z$ is indeed a constant of the motion.
these two calculations, it is likely that the two calculations apply to different parameter regimes in the sense that the contribution from inner orbits can be discarded only if these orbits form a minority, i.e., if

\[ r_{Li} > R_p. \]  

(216)

On the other hand, the present analysis assumes that \( r_{Li} \) is small enough to satisfy

\[ \frac{k^2 r_{Li}^2}{z^2} \ll 1. \]  

(217)

Finally, we restrict the present analysis to the case where

\[ \frac{k^2 r_{Li}^2}{z^2} \gg \frac{|\omega|^2}{\omega_{ci}^2} \frac{m_e T_e}{m_i T_i}. \]  

(218)

From Eq. (134), an order-of-magnitude estimate gives

\[ \left| \frac{\partial}{\partial t} \left( \delta P_{e1} - \delta P_{e2} \right) \right| \leq \frac{\omega}{k^2 v_e}. \]

From Eq. (218), the term \( \frac{\partial}{\partial t} \left( \delta P_{e1} - \delta P_{e2} \right) \) gives a negligibly small contribution to eigenvalue equation (210). A similar order-of-magnitude estimate yields

\[ \left| \frac{\partial}{\partial t} \left( \delta P_{e1} - \delta P_{e2} \right) \right| \leq \frac{\omega}{k^2 v_e}. \]

Therefore, making use of Eq. (218), the term \( \frac{\partial}{\partial t} \left( \delta P_{e1} - \delta P_{e2} \right) \) gives a negligibly small contribution to eigenvalue equation (210). A similar order-of-magnitude estimate yields

\[ \left| \frac{\partial}{\partial t} \left( \delta P_{e1} - \delta P_{e2} \right) \right| \leq \frac{\omega}{k^2 v_e}. \]

where \( \delta_i = 8 \pi P_i^0 / B_0^2 \). Evidently, the orbit-integral contribution to the eigenvalue equation is also small by virtue of (214) and (216).
We conclude that $\xi_{\perp}$ is approximately determined by

$$F_{\text{mhd}}(\xi_{\perp}) = 0,$$  \hspace{1cm} \quad (219)$$

which has the solution

$$\xi_{\perp} = A r^{m-1}(\xi_{\perp}^0 + i\xi_{\perp}^0) \exp(-i\omega t + i0 + ik_z z), \quad 0 < r < R_p$$ \hspace{1cm} \quad (220)$$

provided $k^2 \frac{R_p^2}{\lambda^2} << 1$ [Eq. (214)]. Following Freidberg, we now adopt Eq. (220) as a trial function-form for $\xi_{\perp}$ in the eigenvalue equation (210). In particular, we multiply Eq. (210) by $\xi_{\perp}^*$ and integrate over space to obtain the dispersion relation.

Substituting Eq. (220) into Eqs. (113) and (117) for $\delta I_{\perp}$ and $\delta I_{\parallel}$, and substituting into Eqs. (112), (113), (116) and (117) we readily obtain on the boundary ($r = R_p$)

$$L_\perp = 1 - \frac{\omega \delta e - \omega B}{\omega E},$$ \hspace{1cm} \quad (221)$$

$$\delta I_{\perp} / \left[ -N \text{e} \left( \frac{\omega \delta e - \omega B}{\omega E} \right) \left( \frac{\partial}{\partial r} \ln N \right) \xi_r \right] = 1,$$ \hspace{1cm} \quad (222)$$

$$L_{\parallel} = \frac{\omega \delta e - \omega B}{\omega E},$$ \hspace{1cm} \quad (223)$$

$$\delta I_{\parallel} / \left[ -N \text{e} \left( \frac{\omega \delta e - \omega B}{\omega E} \right) \left( \frac{\partial}{\partial r} \ln N \right) \xi_r \right] = 1,$$ \hspace{1cm} \quad (224)$$

$$\delta N = i \left( k^2_n z^0 + \frac{m n^0}{r} \right) \xi_{\perp} + O(h),$$ \hspace{1cm} \quad (225)$$

and inside the plasma ($0 < r < R_p$)

$$L_\perp - l \neq 0,$$ \hspace{1cm} \quad (226)$$

$$\delta I_{\perp} = 0,$$ \hspace{1cm} \quad (227)$$

$$L_{\parallel} - l \neq 0,$$ \hspace{1cm} \quad (228)$$

$$\delta I_{\parallel} = 0,$$ \hspace{1cm} \quad (229)$$

$$\delta N = i k_\perp \xi_{\perp}.$$ \hspace{1cm} \quad (230)$$
In obtaining Eqs. (225) and (230) use has been made of Eqs. (107), (213), (214) and (220), and Eqs. (107) and (133), respectively.

From (221)-(230), it readily follows that

\[
\delta P_{e\parallel} = -NT \left( \frac{\partial \ln N}{\partial r} \right) \delta_r, \quad 0 < r < R_p,
\]

\[
\delta P_{e\perp} = -\delta P_{e\parallel} = 0, \quad 0 < r < R_p,
\]

where terms of order \( h = \frac{B_0}{E_0} \) have been neglected, and \( \delta P_{e\perp} - \delta P_{e\parallel} \) is nonzero but finite at \( r = R_p \). From Eq. (231) it follows that the finite electron temperature corrections \( S_\gamma \) and \( S_\eta \) [Eqs. (92), (93)] to the orbit integral cancel, i.e.,

\[
\tilde{S} = S = \frac{m_e}{\hbar} \int_{-\infty}^{t} dt' \chi' \cdot \frac{d\xi_{e\perp}}{dr'}.
\]

From Eq. (232), it follows within the context of the trial function method that the term \( \nabla_\perp (\delta P_{e\perp} - \delta P_{e\parallel}) \) does not contribute to the dispersion relation, i.e.,

\[
\int d^3x \nabla_\perp \cdot (\delta P_{e\perp} - \delta P_{e\parallel}) = \int d^3x (\nabla_\perp \xi_{e\perp}^*(\delta P_{e\perp} - \delta P_{e\parallel})) = 0,
\]

where we made use of Eq. (220), i.e., \( \nabla \cdot \xi_{e\perp}^* = 0 \).

From Eqs. (233) and (234), one important conclusion can immediately be drawn. Namely, the finite electron temperature corrections only occur through a modification in the equilibrium parameters [Eq. (167)]. Substituting Eq. (233) into the eigenvalue equation (210), multiplying by \( E_0^0 + \frac{\chi_{e\perp}E_0^0}{c} \) and integrating over space we obtain

\[
\frac{2}{\omega_{mh}} = \frac{i \omega e}{k_{T_{\perp}}} \int d^3x d^3\nu \xi_{e\perp}^* \cdot \left( E_0^0 + \frac{\chi_{e\perp}E_0^0}{c} \right) f_{\perp}^0 \lambda S, \tag{235}
\]

where [from Eqs. (198) and (215)]

\[
\lambda = 1 + \frac{k_{T_{\parallel}}}{\omega} \left( \frac{T_{\parallel}}{T_{\perp}} - 1 \right), \tag{236}
\]
In Eq. (235), the ideal magnetohydrodynamic frequency $\omega_{\text{mhd}}$ is defined by

$$\omega_{\text{mhd}}^2 = -\frac{1}{2K_m} \int d^3x \, \xi_\perp \cdot F_{\text{mhd}}(\xi_\perp).$$

We introduce the notation

$$\frac{D}{Dt} = i\omega + \frac{d}{dr}, \quad \hat{\omega}_{\text{mhd}} = \left(\frac{D}{Dt}\right)_{k_z=0},$$

and make use of [Eq. (220)]

$$\omega_{\text{mhd}}, \xi_\perp = \hat{\omega}_{\text{mhd}}(\hat{\xi}_\perp).$$

Equation (233) can then be expressed as

$$S = -i \frac{m_i}{m} (\omega - k_z v_z) \chi \cdot \xi_\perp + \frac{1}{m} \int_{-\infty}^{t} dt' \chi' \cdot \hat{D}_{\text{mhd}}\xi_\perp$$

$$+ (\omega - k_z v_z)^2 \frac{m_i}{m} \int_{-\infty}^{t} dt' \chi' \cdot \xi_\perp,$$

where the final term on the right-hand side of Eq. (241) is of order $\text{Max}[(\omega R / \omega_{ci} L_i)^2 (k_m^2 R)^2]$ in comparison with the leading term.

Hence the dispersion relation (238) can be approximated by

$$\omega_{\text{mhd}}^2 = -\frac{1}{2K_m} \int d^3x \int d^3y \, \xi_\perp \cdot \left( E \cdot \frac{\nabla E}{c} \right) f_1 \lambda$$

$$\times \left[ -\frac{i}{m} (\omega - k_z v_z) \chi \cdot \xi_\perp + \int_{-\infty}^{t} dt' \chi' \cdot \hat{D}_{\text{mhd}} \xi_\perp \right].$$

Due to symmetry, the magnetic force term multiplying $(\omega - k_z v_z) \chi \cdot \xi_\perp$ on the right-hand side of Eq. (242) vanishes, whereas the electric force contribution (denote E) can be expressed as
Making use of $P_i^0 = \frac{T_{\perp \perp}}{e} \partial \ln n/\partial \theta$ [Eq. (185)] and the definition of $K_m$ [Eq. (237)], the term on the right-hand side of Eq. (243) proportional to $\omega^2$ can be evaluated by means of partial integration. Noting that 

$$m_j \int d^3v \frac{v_z^2}{2} f_i^0 = \frac{1}{4} \int d^3v f_i^0,$$

the remaining term on the right-hand side of (243) is readily evaluated. Making use of Eqs. (242) and (243), we obtain

$$\omega^2_{\text{nhd}} = \omega^2 - \frac{k^2}{m_j} \left( T_{\perp \perp} - T_{\parallel \parallel} \right) + \gamma,'$$

(244)

where

$$\gamma = \frac{-i \omega m_j}{2k v_z} \int d^3x \int d^3v \left( \frac{v_z^2}{2} f_i^0 \right) \left( \frac{x_j^*}{C} \right) \int_0^\infty dt' \left( \gamma' \cdot \gamma \cdot \gamma' \cdot \gamma \right).$$

(245)

Let us compare Eqs. (244) and (245) with the corresponding formulae in the Vlasov-fluid model. The only difference between our orbit integral $m_j \int \frac{1}{\gamma} dt' (\gamma' \cdot \gamma) \cdot \gamma'$ and Freidberg's integral $-\delta_1$ is the existence of the extra phase factor

$$\exp(ik_z t') = \exp(ik_z) \exp[i k_z v_z (t' - t)]$$

in Eq. (245). The factor $\exp(ik_z t')$ thus cancels the factor $\exp(-ik_z t)$ in $\delta_1$. Since $v_z$ is a constant of the motion, the factor $\exp(-ik_z v_z t)$ is independent of $t'$. The factor $\exp(-ik_z v_z t')$ effectively replaces $\omega$ by $\omega - k_z v_z$ as the argument of $\delta_1$. Since $\delta_1(\omega - k_z v_z)$ varies like $\exp(-i \omega t + ik_z v_z t)$, we conclude that

$$m_j \int \frac{1}{\gamma} dt' (\gamma' \cdot \gamma) \cdot \gamma' = \delta_1(\omega - k_z v_z).$$

(246)
In the MHD limit where \(|\omega| < \omega_{\text{ci}}|\) and \(k_z^2 r_{Li}^2 < 1\) [Eqs. (214) and (217)], we obtain

\[
\frac{m_i}{2} \int_{-\infty}^{t} dt' \chi' \cdot B' \left| \xi_\ell \right| \omega \to 0 \lim \xi_\ell \left( \omega \right).
\] (247)

The right-hand side of Eq. (247) has been evaluated by Freidberg for the \(m=1\) and \(m=2\) modes, and is independent of \(v_z\). Since \(\xi_\ell \)

\[
\left( \frac{R}{C} + \frac{v_x \beta_0}{c} \right) \xi_\ell
\]

is an even function of \(v_z\), it follows that the factor \(\lambda\) in Eq. (245) may be replaced by unity. Consequently, our expression for \(\lambda\) [Eq. (245)] reduces to Freidberg's expression, i.e.,

\[
\lambda = 0, \quad \text{for } m=1,
\] (248)

and

\[
\lambda = \frac{1}{2} \omega \frac{v_{\text{ii}}}{L} \frac{Z}{Z_R} \left( \frac{R_p}{r_{Li}} \right), \quad \text{for } m=2.
\] (249)

Here \(r_{Li}\) is the thermal ion Larmor radius inside the plasma, \(v_{\text{ii}}\) is the thermal ion speed perpendicular to \(\hat{\xi}_\ell\) and \(Z\) is the plasma dispersion function defined in Eq. (59).

In summary, the dispersion relation can be expressed as

\[
\omega^2 = \omega_{\text{mhd}}^2 + \frac{k_z^2}{m_i^2} \left( T_{\text{ii}} - T_{\text{ii}} \right), \quad \text{for } m=1
\] (250)

and

\[
\omega^2 = \omega_{\text{mhd}}^2 + \frac{k_z^2}{m_i^2} \left( T_{\text{ii}} - T_{\text{ii}} \right) + \frac{1}{2} \omega \frac{v_{\text{ii}}}{R_p} \frac{Z}{Z_R} \left( \frac{R_p}{r_{Li}} \right), \quad \text{for } m=2.
\] (251)

The effect of the ion energy anisotropy on stability behavior is analogous to its effect on the shear Alfvén wave as found from double-adiabatic theory.\(^{13,18}\) Note that the term \(k_z^2 \left( T_{\text{ii}} - T_{\text{ii}} \right) / m_i\) has a stabilizing effect (when \(T_{\text{ii}} > T_{\text{ii}}\)), and that the ion energy anisotropy correction may very well be comparable to \(\omega_{\text{mhd}}^2\).
From Eq. (251), the explicit solution for the growth rate \( \gamma \) for \( m=2 \) can be expressed as

\[
\gamma = -A \frac{1}{\sqrt{2}} \left( \frac{2^{1/2} \left( A^2 - \frac{1}{4} + \frac{2}{4} \right)^{1/2} - \left( A^2 - \frac{1}{4} + \frac{2}{4} \right)^{1/2}}{A^2 - \frac{1}{4} + \frac{2}{4}} \right)
\]

(252)

where

\[
A = A_r + i A_i \equiv \frac{v_{Li}}{4 R_p} \left( \frac{R_p}{r_{Li}} \right)
\]

(253)

and

\[
\tau^2 = \omega_{\text{mhd}}^2 + \frac{k^2}{2} \left( T_{ii} - T_{ij} \right).
\]

(254)

Evidently, the effect of finite electron temperature is qualitatively similar to the case of small ion Larmor radius [Section 5.D]. It does not shift the point of marginal stability, and it alters \( \gamma_{\text{mhd}} \) and the effective ion Larmor radius through a reduction of the magnetic field strength inside the plasma column. In fact, for typical parameters satisfying the restrictions in Eqs. (214) and (216), the expression for the growth rate simplifies considerably. This can be seen by noting, for \( r_{Li} > R_p \) and \( \omega_{ci} > \omega_{\text{mhd}} \), that Eq. (252) reduces to

\[
\gamma \propto 2^{\frac{1}{2}} \frac{\gamma_{\text{mhd}}}{(v_{Li}/R_p)^{1/2}} \ll \gamma_{\text{mhd}}, \text{ for } m=2.
\]

(255)

A plot of \( \gamma \) versus \( T_e/T_{ii} \), is shown in Fig. 4.

The ion temperature anisotropy not only affects the \( m=2 \) modes, but also modifies the kink mode (\( m=1 \)) by reducing its growth rate when \( T_{ji} > T_{ij} \), as is often the case in pinch experiments. [Fig. 5] Moreover, it shifts the point of marginal stability by reducing the magnitude of the critical wavenumber \( k_c \) for the onset of instability. For the kink
mode this ion energy anisotropy is the only stabilizing factor and the marginally stable mode has wavenumber \( k_c \) satisfying [Eqs. (172), (254)]

\[
\left( \frac{B_0^2 + B_z^2}{4\pi m_1 N} + \frac{T_{1\perp} - T_{1\parallel}}{m_1} \right) k_c^2 + \frac{B_z B_0 k_c}{2\pi m_1 N p} = 0.
\]

Consequently, modes are unstable if and only if

\[
0 > \frac{\gamma}{k} > \frac{-2h}{2 - \beta_0 + \frac{1}{2} \alpha \beta_0},
\]

where \( \beta_0 \equiv 1 - \frac{B_0^2}{B_z^2}, \ \alpha \equiv \frac{B_0 / B_z}{T_{1\perp} - T_{1\parallel} / T_{1\parallel}} \) and \( k \equiv k R_p \). A plot of \( k_c \) versus \( \alpha \) is shown in Fig. 6.
8. A CRITICAL COMMENT ON OHM'S LAW; FINITE RESISTIVITY EFFECTS ON THE
STABILITY PROPERTIES OF LONG-WAVELENGTH E.M. PERTURBATIONS IN A
NEAR-THETA PINCH WITH SHARP BOUNDARY.

In this section we adhere to a sharp boundary profile [Eq. (133)]
with $B_0^\Omega = 0$ inside the plasma. We only consider modes with

$$|\delta E_z| \ll |\delta P_\perp|$$  \hspace{1cm} (258)

and long wavelengths, i.e.,

$$k_z l_\perp \ll 1$$  \hspace{1cm} (259)

Hence $\mathcal{E}_z \cdot (\nabla \delta \rho)$ may very well exceed $\nabla \cdot \delta \rho$ in order of magnitude.

In this respect it is important to note that the quasi-neutrality
condition does not imply the smallness of $\mathcal{E}_z \cdot (\nabla \delta \rho)$ itself, but
only

$$\mathcal{E}_z \cdot (\nabla \delta \rho) \ll \frac{\omega}{\omega \omega_{ce}} \frac{|\delta \rho|}{L_\perp}$$  \hspace{1cm} (260)

so that Turner's ordering equation $0 = \mathcal{E}_z \cdot (\nabla \delta \rho)$ is an additional
restriction to the validity of his calculation.

*This contention has nothing to do with the controversy in the literature
about the treatment of the kinetic boundary condition that, in the
opinion of the author, is still unresolved. 33,47-50
In the derivation of (260) use has been made of the continuity equation relating \( \delta N \) to \( \delta \nu \) and of the circumstance that, even for high values of plasma-beta (\( B_0 \)) the displacements of fluid elements and of magnetic lines are of the same order of magnitude.

Inequality (260) is readily satisfied if e.g. \( \omega \approx 10^{-5} \omega_{pe} \), since typically: \( \omega_{ce} \approx \frac{\omega_{pe}}{5} \). Henceforth we consider perturbations with

\[
|E_z(\nabla \times \delta N)| \approx \frac{\delta N}{L_1}. \tag{261}
\]

Comparison of the lowest-order drift-kinetic expression for the pressure term in the parallel component of Ohm's law [Eqs. (48), (134), (135)] with the term \( \frac{\partial \rho}{\partial t} \) therein \( \rho \), shows that these terms have the following ratio:

\[
\frac{\rho}{\frac{\partial \rho}{\partial t} + \rho \cdot \nabla \cdot \nabla} \cdot (\nabla \cdot \left( \frac{e \rho}{m_i} \nabla v_i - \frac{e \rho}{m_e} \nabla v_e \right)) = \frac{\omega_{ce}^{2c^2}}{\omega_{pe}^2} \frac{\omega_{pe}}{v_e} \frac{1}{kL_1}.
\]

Since typically, \( \omega_{ce} \approx \frac{1}{5} \omega_{pe} \), \( \frac{2}{c^2} \approx 10^{-3} \), it follows that for \( kL_1 \approx 10^{-3} \) and \( \omega \approx 10^{-5} \omega_{pe} \) the term \( \frac{\partial \rho}{\partial t} \) may well exceed the pressure term, thereby becoming the leading term in the equation determining the parallel electric field. Although often the collision frequency of electrons with ions is less than the perturbation frequency \( \omega \), the ratio \( \frac{\nu_{ei}}{\omega} \) may still be large enough to make the collisional correction \( \left( \text{coll} \right) \) in Ohm's law comparable with the pressure term.

On the other hand, the collisional correction to the pressure term through collision terms in the drift-kinetic equation is typically of order \( \frac{\nu_{ei}}{|\omega|} \) times the pressure and hence can be neglected. This can also be seen from Eq. (60), where a collisional correction in the drift-kinetic
equation would yield a term \( \nu \frac{e_i}{|\omega|} \delta P_{ei} \) with the following ratio to the collisional correction via the parallel field \( E_\parallel \), as determined from Ohm's law:

\[
\nu \frac{e_i}{|\omega|} \delta P_{ei} : \nu \frac{e_i}{k_z v_e} \frac{N T_e}{T_e} \frac{e v_a \delta E}{r} = \nu \left| \frac{v}{\delta v} \right| : \frac{\epsilon}{c^2} = 1 ,
\]

(263)

and this is the reverse of ratio (262), i.e.: the collisional correction to the pressure via the collision terms in the drift-kinetic equation can be neglected to the extend in which \( \nu \delta v / \delta t \) is the leading term in the parallel component of Ohm's law.

In summary, we will assume

\[
\nu \frac{e_i}{|\omega|} < 1 ,
\]

(264)

\[
\alpha \equiv k_z L_\perp \left( \frac{\omega p_e}{\omega c_e} \right) \frac{\frac{\omega p_e}{\omega c_e}}{c^2} \frac{v_e^2}{c^2} < 1 ,
\]

and neglect terms of order \( \nu \frac{e_i}{|\omega|} \) or smaller compared with the leading term in Ohm's law parallel to \( B \). Determination of \( E_\parallel \) to this accuracy is needed, since \( E_\parallel \) is present in the dominant term in the drift-kinetic equation. This yields the following set of equations:

(a) The total momentum balance equation [Eq. (47)] as before, since no collisional effects on total momentum transfer are present in a fully ionized plasma, since momentum is conserved in collisions;
(b) A modified Ohm's law parallel to $\mathbf{B}$:

$$
\frac{\omega^2}{\rho_0} \mathbf{E}_\parallel = - \frac{4\pi e}{m_e} \mathbf{R} \cdot (\nabla \times \mathbf{B}) + \omega^2 \eta_\parallel \mathbf{j}_\parallel,
$$

(265)

where $\eta_\parallel$ is the sum of inertial and collisional resistivity;

(c) The drift-kinetic equation [Eq. (4)];

(d) The full Vlasov equation for the ions [Eq. (2)];

(e) The $\nabla \times \mathbf{E} = \mathbf{J}$ equation [Eqs. (7) and (8)].

Note that since $j_\parallel^0 = 0$, only the unperturbed resistivity enters in the linearized version of Eq. (265). In the calculation of $\eta_\parallel^0$, the perturbed electric field $E_\parallel$ can be taken spatially homogeneous along the $z$-axis provided that $k_2 \rho_{\text{m.f.p.}} \lesssim 1$, where $\rho_{\text{m.f.p.}}$ is the mean free path for electrons. In the present analysis, this inequality is assumed to hold. Then $\eta_\parallel^0$ only depends on the frequency $\omega$ and the plasma parameter.

Linear perturbation analysis will now be applied to the aforementioned set of equations, strictly adhering to the inner region of the sharp-boundary plasma [Eq. (133)]. In terms of the vector potential [Eq. (80)] the perturbed current density $\delta j_\parallel$ can be written, using Ampère's law, as

$$
\delta j_\parallel = \frac{c}{4\pi} \mathbf{R}^0 \cdot [\nabla (\nabla \times \mathbf{A}) - \mathbf{V}^2 \mathbf{A}] .
$$

(266)

Substituting Eqs. (80) and (266) into the linearized Ohm's law [Eq. (265)], we obtain
Define $\delta Q$ by

$$
\delta Q = ik_{\mathbf{z}} [-\delta \Phi + \frac{\delta \Phi_{\nu}}{e N} - \eta^o \frac{c}{4\pi} \nabla \cdot \delta A] + \frac{\gamma^o c}{4\pi} \frac{1}{r} \frac{3}{\partial r} (r \frac{3}{2\pi} \delta A)\ 
$$

and consequently the gauge condition $\delta Q = 0$ can be imposed by

$$
\delta A' = \delta A + \nabla \delta \lambda, \ 
$$

Choosing

$$
\delta \lambda = -[\frac{aw}{c} + \frac{ik_{\mathbf{z}}}{4\pi} \eta^o c \left( \frac{m^2}{r^2} + \frac{k^2}{z^2} \right)] \delta Q, \ 
$$

which is possible provided that

$$
1 \neq -i \frac{\eta^o \frac{c^2}{4\pi}}{w^2} \frac{m^2}{r^2} + \frac{k^2}{z^2}. \ 
$$

Since the leading term in $\eta^o$ is equal to $-\frac{4\pi a w}{\omega^2}$ [cf. Part I, Eq. (163), of this thesis], condition
(272) is generally satisfied, its right hand side being dominantly real and negative. Therefore, a gauge condition can be chosen such that

$$\delta Q = 0.$$  \hfill (273)

Combining Eqs. (257), (268) and (273) we conclude that a gauge condition can be imposed such that

$$\left[-\frac{\eta_\parallel^c}{4\pi} \left( \frac{m}{v^2} + k_2^2 \right) - i\omega \right] \delta A_z = 0,$$

or, in view of (272),

$$\delta A_z = 0. \quad (274)$$

Consequently, like in the collisionless case, a perturbation $\xi_\parallel$ can be defined by

$$\delta A = \xi_\parallel \times \mathbf{B}^0,$$

in terms of which our gauge condition reads

$$\delta \phi - \frac{\delta P_\parallel}{eN} + \frac{\eta_\parallel^c}{4\pi} \mathbf{E}_z \cdot (\nabla \times \xi_\parallel) = 0.$$  \hfill (276)

Because of Eq. (275), the linearized Ohm's law parallel to $\mathbf{B}$ reads

$$\delta E_\parallel = -ik_2^2 \frac{\delta P_\parallel}{eN} + \frac{ik_2^2 c}{4\pi} \eta_\parallel \mathbf{E}_z \cdot (\nabla \times \xi_\parallel). \quad (277)$$

The moments [Eqs. (57), (60)] of the drift-kinetic equation and Eq. (277) form a closed set of equations for $\delta E_\parallel$, $\delta P_\parallel$, and $\delta P_\perp$. In the case of the sharp-boundary profile
[Eq. (133)] these moment equations reduce to [Eqs. (57), (60)]

\[
\frac{\delta p}{\delta \rho} = \frac{iN T_e}{k_z v_e} \left( I_2 \frac{e v}{T_e} \delta E \| + 2 I_3 v \cdot \delta E \perp \right),
\tag{278}
\]

\[
\frac{\delta p}{\delta E} = \frac{2iN T_e}{k_z v_e} \left( K_2 \frac{e v}{T_e} \delta E \| + K_3 v \cdot \delta E \perp \right),
\tag{279}
\]

where [Eqs. (58), (61)]

\[
I_2 = 1 + n Z(n),
\]

\[
I_3 = Z(n),
\]

\[
K_2 = \frac{1}{2} + n^2 + \frac{3}{2} n Z(n),
\]

\[
K_3 = n[1 + nZ(n)],
\]

with \( n = \frac{\omega}{k_z v_e} \) and where \( Z \) is the plasma dispersion function [Eq. (59)]. In Eqs. (278) and (279), \( \nabla \delta \rho \rho \) is related to \( \xi_\perp \) by Eq. (68), i.e.

\[
\nabla \delta \rho \rho \| = -i \omega \nabla \xi_\|,
\tag{280}
\]

within the plasma.

By means of Eqs. (277), (279) and (280) we obtain

\[
\delta P_\| = -NT_e \nabla \rho \xi_\| + \frac{2K_2}{2K_2-1} (NT_e) \frac{\omega e n \| c^2}{2\pi v_e^2} \left[ \frac{\delta}{\delta z} \cdot (\nabla \times \xi_\|) \right].
\tag{281}
\]

Similarly, from Eqs. (277), (278) and (280), we have

\[
\delta P_\perp = \delta P_\perp = NT_e \frac{\omega}{k_z v_e} \frac{Z(\omega)}{k_z v_e} \nabla \rho \xi_\perp -
\]

\[
NT_e \left( I_2 + \frac{2K_2}{2K_2-1} \right) \frac{\omega e n \| c^2}{2\pi v_e^2} \left[ \frac{\delta}{\delta z} \cdot (\nabla \times \xi_\perp) \right].
\tag{282}
\]
Eqs. (281) and (282) constitute the generalizations of Eqs. (76) and (77) to finite resistivity (inertial and collisional). Next we consider the perturbed ion distribution, which is the solution

$$\delta f_i = -\frac{\partial f_i^0}{\partial H} \int_{-\infty}^t dt' \delta F, \chi'$$

(283)

of Vlasov's equation ($v_i < v_e < |\omega|$), [Eq. (79)]

In terms of electromagnetic potentials $\delta \phi$, $\delta \psi$, and by means of Ohm's law, we obtain

$$\delta f_i = i \omega \frac{\partial f_i^0}{\partial H} \int_{-\infty}^t dt' \left\{ \frac{\partial \rho_{||}}{\partial cN} (\chi', t') + \frac{\eta^0 \rho_{||} c B}{4\pi} \right\}$$

$$- \delta \phi (\chi', t') \left( \frac{\varepsilon_{||} \times \kappa_{||} (\chi', t')}{c} \right) \frac{\partial f_i^0}{\partial H}$$

(284)

where

$$\chi' \left( \frac{\varepsilon_{||} \times \kappa_{||}}{c} \right) = -\frac{1}{\varepsilon_{||}} \chi', \left( \frac{\varepsilon_{||} \times \kappa_{||}}{c} \right) = -\frac{m_i}{e} \frac{dv'}{dt'}$$

inside the plasma column. By partial integration, it now follows that

$$\delta f_i = i \omega \frac{\partial f_i^0}{\partial H} \left\{ \frac{\varepsilon_{||} \times \kappa_{||} \cdot \chi + e \delta \phi (\chi', t') \frac{\partial f_i^0}{\partial H}}{\partial H} \right\} - \text{Im} \gamma (c) \frac{\partial f_i^0}{\partial H}$$

(285)

where

$$\gamma (c) \equiv S - S_\pi - S_\eta$$

(286)

and $S$ and $S_\pi$ are defined by Eq. (93) whereas

$$S_\eta = \frac{e c B}{4\pi} \eta^0 \int_{-\infty}^t dt' \delta F, [\varepsilon \times \kappa_{||} (\chi', t')]$$

(287)
and where, following Turner, only inner orbits are considered. Operating on the linearized Vlasov equation with \( \mathbf{p}^0_1 \times \mathbf{m}_1 / d^3v \mathbf{y} \) and making use of Eq. (285) we readily obtain

\[
\mathbf{p}^0_1 \times (-\omega \mathbf{M}_1 + \nabla . \mathbf{P}_1) = \mathbf{p}^0_1 \times [ -\omega e \int d^3v \left( \frac{\mathbf{v} \times \mathbf{B}^0}{c} \right) \mathbf{g}(c) \frac{\partial f^0_1}{\partial \mathbf{H}} - e \nabla (\phi / d^3v f^0_1)]
\]

(288)

for the inner plasma region. Using Ohm's law [Eq. (276)] to eliminate \( \phi \) in favor of \( \mathbf{E}_1 \) and \( \mathbf{P}_1 \) (Eq. (281)), we obtain

\[
\mathbf{p}^0_1 \times (-\omega \mathbf{M}_1 + \nabla . \mathbf{P}_1) = \mathbf{p}^0_1 \times -\omega e \int d^3v \left( \frac{\mathbf{v} \times \mathbf{B}^0}{c} \right) \mathbf{g}(c) \frac{\partial f^0_1}{\partial \mathbf{H}} - e \mathbf{v} \mathbf{l}(\phi / d^3v f^0_1) \]

(289)

Substitution of Eq. (289) into the total momentum balance equation [Eq. (47)] then yields the eigenvalue equation

\[
\mathbf{F}_{\text{mhd}} \left( \xi_1 \right) = -\omega e \int d^3v \left( \frac{\mathbf{v} \times \mathbf{B}^0}{c} \right) \mathbf{g}(c) \frac{\partial f^0_1}{\partial \mathbf{H}} + \\
+ \mathbf{v}_1 \left[ \delta \mathbf{p}_1 - \mathbf{p}_1 \right] - \frac{e \mathbf{N} \mathbf{c}_2}{4\pi} \mathbf{H}_1 \mathbf{B}_0 \mathbf{E}_2 \cdot (\nabla \times \xi_1) \right)
\]

(290)

where \( \mathbf{F}_{\text{mhd}} \) is the ideal incompressible magnetohydrodynamic force [Eq. (103)]. Equation (290) is the generalization of Eq. (105) to include finite resistivity effects. In two respects, however, Eq. (290) is less general than Eq. (105): it is only valid for the case of a sharp-boundary profile (133), and only inner orbits are considered.
(i.e. $r_i/L_e$ is assumed to be small). Eq. (290) should be supplemented by Eqs. (285) and (287) defining $\dot{\gamma}(c)$ and by Eqs. (281) and (282) defining $\dot{y}_{el}$ and $\dot{y}_{el} - \dot{y}_{el}$ in terms of $\dot{y}_{el}$.

Together with the usual boundary conditions, the eigenvalue equation (290) can be applied to determine the dispersion relation for long-wavelength incompressible electromagnetic perturbations in a sharp-boundary plasma with the profile given by Eq. (133). This dispersion relation is only valid if $|\omega| \gg \omega_e$, and

$$\alpha = k_z L_e \left( \frac{\omega}{\omega_e} \right) \left( \frac{\omega}{\omega_e} \right) \left( \frac{v_e^2}{c^2} \right) < 1$$

to the extent that terms of order $\frac{v_e}{\omega_e}$ are neglected.
9. CONCLUSIONS

We have studied the stability of high-beta linear-pinch configurations, to low-frequency \(|\omega|<\omega_{ci}\), electromagnetic perturbations, using lowest-order hybrid-kinetic theory.\(^{35,36}\) Following a discussion of the assumptions and basic equations [Secs. 1 and 2], we derived an eigenvalue equation for perturbations about a steady-state axisymmetric linear screw-pinch equilibrium characterized by an isotropic ion energy distribution [Eq. (105), Sec. 4]. This equation clearly exhibits the importance of finite electron temperature and electron kinetic effects parallel to the magnetic field direction, and reduces to the Vlasov-fluid eigenvalue equation\(^{1}\) in the limit of zero electron temperature. This equation can be investigated numerically\(^{45,46}\) to study linear stability properties.

For the special case of very low frequency \(|\omega|<\omega_{ci}\) modes in a sharp-boundary near-theta-pinch configuration [Eq. (133)] with small but finite ion Larmor radius, we have obtained a dispersion relation analytically, adopting Turner's FLR ordering\(^ {33}\) [Sec. 5]. In this case, the effects of finite electron temperature modify stability behavior exclusively through the equilibrium pressure constraint [Eq. (167)], and parallel electron kinetic effects are found to be of higher order \(|\omega|<|k_z|v_e|\). The dependence of the growth rate on electron temperature is discussed in Sec. 5.D and illustrated in Fig. 3.

In Sec. 6, we derived the eigenvalue equation [Eq. (210)] for low-frequency \(|\omega|<\omega_{ci}\), electromagnetic perturbations about a steady-state axisymmetric theta-pinch equilibrium \((B_\theta^0=0)\) with an anisotropic ion energy distribution. Allowing for this anisotropy is especially important in view of the anomalous perpendicular heating\(^ {8,38}\) \((T_{\perp}>T_{\parallel})\) that is known to take place during the implosion.
phase of theta-pinch experiments. The eigenvalue equation (210) can be studied numerically\textsuperscript{45,46} to generalize the dispersion relation\textsuperscript{43} for transverse, electromagnetic perturbations ($k \parallel B_0$ and $|\omega| < \omega_{ci}$) parallel to the magnetic field about a homogeneous equilibrium to the case of perturbations about a spatially inhomogeneous plasma.

This analysis forms the basis for generalizing the discussion of the electromagnetic ion cyclotron instability driven by ion energy anisotropy.

As an analytical application of the eigenvalue equation (210), we have obtained the dispersion relation for low-frequency ($|\omega| << \omega_{ci}$), long-wavelength modes in a sharp-boundary, near theta-pinch plasma, with $B_0^z=0$ inside the plasma column. This analysis assumes that the thermal ion Larmor radius is comparable with or larger than the radius of the plasma column [Sec. 7, Eqs. (250), (251)]. It is concluded that Freidberg trial-function method\textsuperscript{1} is valid only in the limit $r_{Lz} >> R_p$. Application of this method yields a result which is qualitatively similar to the effect of pressure anisotropy found in double-adiabatic theory.\textsuperscript{13,18} However, in the present analysis, the correction term due to pressure anisotropy is not a priori small, in view of the large values of local beta allowed in the theory, and in view of the fact that considerable ion energy anisotropy\textsuperscript{43} can be maintained for several ion cyclotron periods. In addition to the effect of ion energy anisotropy, the influence of finite electron temperature is apparent from Eqs. (250) and (251), but only occurs through the equilibrium pressure constraint [Eq. (167), see also Sec. 5].

Finally, in Sec. 8, we have examined the validity of the hybrid-kinetic ordering\textsuperscript{35,36} for the case of long wavelength ($|k_z| L_z^{-1} \sim 10^{-3}$), electromagnetic perturbations in a sharp-boundary pinch with $B_0^z=0$ inside the plasma. It is argued that if $(\omega_{ce}/\omega_{pe})(2e^2/\nu_e^2)(\omega/\omega_{pe})(k_z L_z)^{-1}$
then finite resistivity effects can no longer be neglected in the analysis. An eigenvalue equation for this case has been derived [Eq. (290)]. It is also noted that in Turner's FLR ordering, the equation $\hat{e}_{\|} (\nabla \times \mathbf{E}) = 0$ is not a consequence of the assumption of quasineutrality but an additional restriction that can be relaxed. Since the finite resistivity correction terms in the eigenvalue equation (290) are proportional to $\hat{e}_{\|} (\nabla \times \mathbf{E})$, this implies that the effects of finite resistivity (electron inertial, but also collisional if $|\omega| \gg v_{ei}$) on stability behavior are nontrivial.
THE GAUGE CONDITION

The gauge condition for the electromagnetic potentials [Sec. 4.3] is chosen in such a manner that the parallel component of Ohm's law [Eq. (48)] adopts a convenient form. Since

$$\delta E_0 = \delta E_c \cdot \rho_0 + E_0^0 \cdot \delta n,$$

(A.1)

Eq. (48) can be expressed as:

$$\delta E \cdot \rho_0 + \delta n \cdot \left( E_0^0 + \frac{T_e}{e} \nabla \ln N \right) + \rho_0^0 \cdot \nabla \left( \frac{\delta E^0}{eN} \right) = 0.$$  \hspace{1cm} (A.2)

Making use of $\delta n = \delta E_0^0 - B_0^0 \delta B^0$, and $B_0^0 = 0$, Eq. (A.2) becomes

$$\rho_0^0 \cdot \delta E \cdot \frac{\delta B}{B} \cdot \left( E_0^0 + \frac{T_e}{e} \nabla \ln N \right) + \rho_0^0 \cdot \nabla \delta n = 0,$$ \hspace{1cm} (A.3)

where $\delta n = \delta E^0/eN$. Making use of $\nabla \times E_0^0 = 0$ and $\delta E = \nabla \times \delta A$, we find that

$$\left( \nabla \times \delta A \right) \cdot \left( E_0^0 + \frac{T_e}{e} \nabla \ln N \right) = \nabla \cdot \left\{ \delta \vec{A} \times \left( E_0^0 + \frac{T_e}{e} \nabla \ln N \right) \right\}$$  \hspace{1cm} (A.4)

Since $E_0^0 \cdot \vec{B} = 0 = (\nabla \ln N) \cdot \vec{B}$, we have

$$E_0^0 + \frac{T_e}{e} \nabla \ln N = \frac{1}{c} \frac{B_0^0 \times \vec{\rho}_0^0}{\mu_0},$$  \hspace{1cm} (A.5)

where $\vec{\rho}_0^0$ is defined by

$$\vec{\rho}_0^0 = \frac{c}{\mu_0} \left( E_0^0 + \frac{T_e}{e} \nabla \ln N \right) \times \frac{B_0^0}{\mu_0}.$$  \hspace{1cm} (A.6)

Since $\nabla \times \vec{\rho}_0^0 = 0$, it follows from (A.4), (A.5) and (A.6) that Ohm's law [Eq. (A.3)] can be expressed as

$$\rho_0^0 \cdot \nabla \left( \delta \vec{n} - \delta \Phi + \frac{\delta \vec{A} \times \vec{\rho}_0^0}{c} \right) = -\frac{1}{c} (\omega - k_0 u_0^0 - m u_0^0 / r) \delta \vec{A} \cdot \vec{\rho}_0^0$$  \hspace{1cm} (A.7)
The gauge condition is now chosen such that
\[ \delta \pi - \delta \phi + \frac{\delta A^0}{c} = 0. \]  
(A.8)

Since Im\(\omega\) \(\neq 0\) is assumed, this gauge can always be adopted. Moreover, it follows from Eq. (A.6) that
\[ \frac{\pi^0}{\mu} \cdot \delta A = 0, \]  
(A.9)

for the choice of gauge in Eq. (A.7).
Here we make use of Eq. (210) to derive the dispersion relation for low-frequency ($\omega \ll \omega_{ci}$) transverse electromagnetic perturbations propagating parallel to the magnetic field $B_0 = B_0 z$ in a high-beta, spatially homogeneous plasma. The ion distribution is assumed to be bi-Maxwellian [Eq. (182)]. In addition, we consider axial wavenumbers $k_z$ with

$$k_{z, \text{Le}}^2 \ll 1.$$ 

Since $\xi_B = 0$, the perturbations satisfy

$$\nabla \cdot \mathbf{\xi} = 0.$$ 

Adopting Cartesian geometry, the perturbation $\mathbf{\xi}_{\perp}$ can be expressed as

$$\mathbf{\xi}_{\perp} = (\xi_x e + \xi_y e_y) \exp(-i\omega t + i k_{z, \perp} z),$$

where $\xi_x$ and $\xi_y$ are constant complex amplitudes. Making use of Eq. (B.2) and the spatial homogeneity of the equilibrium state, we obtain [Eqs. (76), (77), (113) and (117)],

$$\delta P_{eL} = 0 = \delta P_{eH}.$$ 

The eigenvalue equation (210) then reduces to [Eqs. (236) and (B.4)]

$$\mathcal{E}_{\text{mdh}}(\lambda_{\perp}) = \frac{i \omega e}{T_{i, i}} \int d^3 v \left( \frac{\lambda \times \mathbf{E}_0}{c} \right) f_i^0 \left( 1 + \frac{k_z v_z}{\omega} \left( \frac{T_{i, i}}{m_i} - 1 \right) \right)$$

$$\times m_i \int_{-\infty}^{t} \left( \lambda \cdot \frac{d \mathbf{\xi}_{\perp}}{dt} \right) ' dt',$$
where the orbit integral can be written [Eq. (B.3)] as
\[
\int_{-\infty}^{t} \left( \chi' - \frac{d\chi}{dt} \right) \cdot dt' = i(\omega-k_z)v_z \int_{-\infty}^{t} \chi' \cdot \delta_\chi(\chi',t') dt',
\]
with
\[
\chi' \cdot \delta_\chi(\chi',t') = (\xi_{x0}' \xi_x + \xi_{y0}' \xi_y) \exp[-i\omega t' + ik_z z + ik_z v_z (t'-t)].
\]

Making use of the periodicity of the orbits, we obtain from Eqs. (B.6) and (B.7)
\[
\int_{-\infty}^{t} \left( \chi' - \frac{d\chi}{dt} \right) \cdot dt' = \frac{-i(\omega-k_z v_z) \exp(ik_z z - i\omega t)}{1 - \exp\left\{i(\omega-k_z v_z) \frac{2\pi}{\omega}\right\}}
\]
\[
\times \int_0^{2\pi/\omega} c_i d\tau \left( \xi_{x0}' \xi_x + \xi_{y0}' \xi_y \right) \exp[i(\omega-k_z v_z) \tau],
\]
where \(\xi_{x0}'\) and \(\xi_{y0}'\) are defined by
\[
\xi_{x0}' = v_x \cos(\phi + \omega_c \tau),
\]
\[
\xi_{y0}' = v_y \sin(\phi + \omega_c \tau).
\]

Because of the spatial homogeneity assumption, \(v_z\) is a constant of the motion, and evaluation of the orbit integral is straightforward. We obtain
\[
\int_{-\infty}^{t} \left( \chi' - \frac{d\chi}{dt} \right) \cdot dt' = \left( \omega-k_z v_z \right) \frac{v_z}{2} \left( \frac{\xi_{x0}' - i\xi_{y0}'}{\omega - \omega \omega_c} \right) \exp(i\phi + ik_z z - i\omega t)
\]
\[
+ \frac{(\xi_{x0}' + i\xi_{y0}')}{\omega - \omega \omega_c} \exp(-i\phi + ik_z z - i\omega t).
\]

Substituting Eq. (B.10) into the eigenvalue equation (B.5), and performing the integration over the gyrophase-angle \(\phi\), we readily obtain
\( \mathcal{F}_{\text{mhd}}(\xi_{\perp}) = \frac{e_{\text{um}} B_{0} z}{2 T_{i} c} \phi_{x} \int d^{3} v \int_{0}^{1} \left( 1 + \frac{k_{z} v_{z}}{\omega} \left( \frac{T_{i}}{T_{i} \mu} - 1 \right) \right) \)

\( \times (\omega - k_{z} v_{z}) \frac{v_{\perp}}{2} \left[ \frac{\xi_{x} + i \xi_{y}}{\omega - \omega_{ci} - k_{z} v_{z}} - \frac{\xi_{x} - i \xi_{y}}{\omega + \omega_{ci} - k_{z} v_{z}} \right] \)

\( - \frac{i e_{\text{um}} B_{0} z}{2 T_{i} c} \phi_{y} \int d^{3} v \int_{0}^{1} \left( 1 + \frac{k_{z} v_{z}}{\omega} \left( \frac{T_{i}}{T_{i} \mu} - 1 \right) \right) \)

\( \times (\omega - k_{z} v_{z}) \frac{v_{\perp}}{2} \left[ \frac{\xi_{x} + i \xi_{y}}{\omega - \omega_{ci} - k_{z} v_{z}} + \frac{\xi_{x} - i \xi_{y}}{\omega + \omega_{ci} - k_{z} v_{z}} \right], \)

where

\[ \mathcal{F}_{\text{mhd}} = - \frac{k_{0} B_{0} z}{4 \pi} \mathcal{F}_{\perp} \]

follows from equilibrium spatial homogeneity and \( B_{\nu} = B_{0} c_{\nu} \) [Eq. (103)].

Therefore, the eigenvalue equation (B.11) can be written in matrix form as

\[
\begin{pmatrix}
\frac{-k_{z}^{2} B_{0}^{2}}{4 \pi} + D_{+} - D_{-} \\
i(D_{+} + D_{-})
\end{pmatrix}
\begin{pmatrix}
\xi_{x} \\
\xi_{y}
\end{pmatrix}
= 0,
\]

where the functions \( D_{+} \) and \( D_{-} \) are defined by

\[
D_{\pm} = \frac{e_{\text{um}} B_{0} z}{4 T_{i} T_{i} c} \int d^{3} v v_{\perp}^{2} f_{1}^{0} \left( 1 + \frac{k_{z} v_{z}}{\omega} \left( \frac{T_{i}}{T_{i} \mu} - 1 \right) \right) \left( \frac{\omega - k_{z} v_{z}}{\omega + \omega_{ci} - k_{z} v_{z}} \right). \]

Making use of Eq. (B.13), we obtain the desired dispersion relation, i.e.

\[
\frac{k_{z}^{2} B_{0}^{2}}{4 \pi} = \pm 2 D_{\pm}.
\]

The functions \( D_{\pm} \) defined in Eq. (B.14) can be readily expressed in terms of the plasma dispersion function

\[ z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \exp \left( - \frac{x^{2}}{\xi} \right). \]
This gives
\[ D_\pm = \frac{\omega B^0 Z}{2c} \pm \frac{B^0_1}{4\pi} \frac{\omega}{2c} \frac{\omega^2}{p_i^2} \frac{Z(\xi_{i\pm})}{k_z v_{i\|}} \]
\[ \pm \frac{B^0_1}{4\pi} \frac{\omega}{p_i} \frac{N}{2c} \left( \frac{T_{i\|}}{T_{i\|}} - 1 \right) [1 + \xi_{i\mp} Z(\xi_{i\pm})], \]
\[ (B.16) \]

where \( \omega^2_{p_i} = 4\pi N e^2/m_i \) is the ion plasma frequency-squared and \( \xi_{i\pm} \) is defined by
\[ \xi_{i\pm} = \frac{\omega \xi_{i\mp}}{k_z v_{i\|}.} \]
\[ (B.17) \]

Noting that \( \omega_{p_i}^2 = \omega_{p_e}^2 \), and substituting (B.16) into Eq. (B.15), we obtain
\[ k_z^2 = \frac{\omega^2}{\omega_{p_e}^2} + \frac{\omega^2}{\omega_{p_i}^2} \frac{1}{k_z v_{i\||}} Z(\xi_{i\pm}) + \frac{\omega^2}{p_i^2} \left( \frac{T_{i\|}}{T_{i\|}} - 1 \right) [1 + \xi_{i\mp} Z(\xi_{i\pm})], \]
\[ (B.18) \]

which is the well-known dispersion relation for low frequency (\( |\omega| < \omega_{p_e} \)) transverse electromagnetic perturbations propagating parallel to \( E_0^0 \) in a spatially homogeneous plasma.

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I feel greatly indebted to Dr. R. C. Davidson for his suggestions, his guidance and his continuous and stimulating interest in my work. Also, useful interactions with Drs. J. Choe, W. A. D'Ippolito, J. Freidberg, R. L. Guernsey, H. R. Lewis and L. Turner are acknowledged. Special thanks are due to Mrs. Janet Wolfsheimer for her careful typing of the manuscript.
LIST OF REFERENCES OF PART II

8. R.C. Davidson and J.P. Freidberg, invited paper presented at the Third Topical Conference on Pulsed High Beta Plasmas, Abingdon,
Oxon, UK, 1975, paper A.1.15.


34. R. Gerwin, private communication (1976).
Figure Captions

Fig. 1: General Equilibrium Configuration.

Fig. 2: Density (...........), axial (―――――) and helical (――――――) magnetic field in sharp-boundary screw pinch equilibrium, for $E_0 B_z^{-1} = 0.25$ and $\beta = 0.8$.

Fig. 3: Hybrid-kinetic and ideal mhd growth rates for $m = 1$ and for $m = 2$, as a function of $\tau = T_e T_i^{-1}$, for $T_e = 2$ keV, $N = 4 \times 10^{16}$ cm$^{-3}$, $B_z = 70$ kg, $B_0 = 5$ kg, $\beta_i = 0.56$.

Fig. 4: Maximal growth rate as a function of $\tau = T_e T_i^{-1}$ for $\tau_e \gg \tau_p$ and extremely high beta ($\beta_o = 0.929$), $B_z = 50$ kg, isotropic ions, $N = 4 \times 10^{16}$ cm$^{-3}$, $R_p = 0.1$ cm.

Fig. 5: Ratio of kink mode (maximum) growth rate versus its value according to ideal mhd, as a function of ion energy anisotropy ($\alpha = 1 - T_e T_i^{-1}$) at $\beta = 0.9$ (exterior, total beta).

Fig. 6: Ratio of marginally stable wavenumber $K_m$ and marginally stable wavenumber as found from ideal mhd, $K_{m \text{mhd}}$, as a function of anisotropy ($\alpha$) for total exterior beta equal to 0.8.
CURRICULUM VITAE

Ik ben op 30 November 1948 te Amsterdam geboren.

Ruim een half jaar na mijn afstuderen ben ik in 1973 als wetenschappelijk medewerker in langdurig tijdelijke dienst getreden van de Technische Hogeschool Eindhoven (vakgroep transportfysica van de afdeling natuurkunde). Onder supervisie van Dr. P. P. J. M. Schram heb ik voornamelijk gewerkt aan het onderzoek dat zijn beslag gevonden heeft in deel I van dit proefschrift. In 1975 werd mij een Fulbright-beurs voor het verrichten van onderzoek aan een Amerikaanse Universiteit verleend. Hiervan heb ik gebruik gemaakt om mij als lid van de plasma fysica groep van de Universiteit van Maryland breder te oriënteren in mijn vakgebied. Bovendien heb ik aldaar onder
leiding van Prof. Dr. R. C. Davidson het onderzoek verricht dat deel II van het nu tot stand gekomen proefschrift is geworden.

Grote dank ben ik verschuldigd aan mijn beide promotoren, Dr. P. F. J. M. Schram en Prof. Dr. R. C. Davidson, voor hun supervisie, expertise en stimulerende interesse met betrekking tot mijn onderzoeken. Financieel-organisatorisch gaat mijn dank uit naar de Technische Hogeschool Eindhoven, de "Netherlands-America Commission for Educational Exchange" en de Universiteit van Maryland (USA).
STELLINGEN

I De berekening door Freidberg van de groeiensnelheid van de laagfrequente elektromagnetische verstoring met azimutale golfgetal $m=2$ in een pinch met scherpe rand, is onjuist, tenzij de gyrotietstraal van de thermische ionen tenminste even groot is als de straal van de plasma kolom.
(J.P. Freidberg, Phys. Fluids 15, 1102 (1972))

II De voorwaarde dat de rotatie van de verplaatsing van het magnetische veld ten opzichte van een evenwichtsconfiguratie met scherpe rand geen component langs het magnetische veld heeft, wordt niet dwingend afgelast door de eis van quasi-neutraliteit: deze voorwaarde is een extra beperking in Turner's theorie over correcties op magnetohydrodynamische groeiensheden ten gevolge van de eindigheid van de gyrotietstralen.
(L. Turner, Rapport LA-UR-76-397 van Los Alamos Scientific Laboratory of the University of California, Los Alamos, New Mexico, U.S.A.)

III De methode van Baus om sterke twee-deeltjes wisselwerkingen te verwaarlozen bij het afleiden van hogere orde correcties op de Balescu-Guernsey-Lenard theorie kan niet gecouveerd worden via een schatting van de omwisselkeurigheid die een dergelijke verwaarlozing met zich meebrengt.
(M. Baus, Physica 66, 421 (1973))

IV De theorie van golf-golf interacties in sterk turbulente plasmas heeft zich tot nog toe slechts ontkend op grond van modelvergelijkingen welke niet gebaseerd zijn op de fundamentele bewegingsvergelijkingen. Dit vindt zijn oorzaak in de afwezigheid van een kleine parameter in de basissetvergelijkingen. Er is momenteel sprake van een wenselijk dilemma, vanwege het feit dat voorspellingen op grond van genoemde modelltheorieën sterk afhankelijk zijn van een niet of vaak gemotiveerde keuze van de waarden der koppelingselementen.
V De golf-deeltjes en golf-golf interacties vergroten slechts reeds aanwezige correlaties, en wel op een manier analog aan de vergroting van een verstoring ten gevolge van een instabiliteit. Het verdient aanbeveling de in stelling IV genoemde impasse te doorbreken door middel van een schatting van initiële correlaties in termen van de kleine plasma parameter, op grond van de overweging dat uiteindelijk directe deeltjeswisselwerkingen de oorzaak van de multiple correlaties zijn.


VI In plaats van de mogelijkheid van bewustzijn als eigenschap van de levenlose materie open te laten, leert de quantummechanica ons veelal te streven naar een operationele definitie van dit begrip, indien toegepast op materie.

(P.P.J.M.Schram, proefschrift Rijksuniversiteit Utrecht, stelling 10, 1964)

VII De theoretische rechtvaardiging en de beperkingen in de toepasbaarheid van de continuum mechanica kunnen slechts worden bepaald vanuit de via de kinetische theorie verworven kennis en inzichten.

(R.Benach, proefschrift Technische Hogeschool Eindhoven, 1974)

VIII Het Iconoclastische aspekt van het karakter van beoefenaren van de wetenschap bevordert alleen dan de wetenschap als het gepaard gaat met voldoende creativiteit.

(S.Nagarajan, proefschrift Technische Hogeschool Eindhoven, stelling 12, 1975)
IX Een hoger onderwijs systeem zonder waterdichte selectie draagt niet alleen bij tot incompetentie, maar ook tot werkloosheid onder academici. In een later stadium draagt deze werkloosheid bij tot een ongewenst selectieproces, waarin rijkdom, in geld en relaties, het gebrek aan vooruitzichten kan compenseren bij de beslissing een academische studie te entameren.

X Het onderscheid tussen zuiver en toegepaste wetenschap is vrij kunstmatig en komt de vooruitgang vaak niet ten goede, aangesien het door sommigen wordt gebruikt om kortsichtige politici om de tuin te leiden, en door anderen om te verklaren waarom hun wetenschappelijk werk niet tot nuttige resultaten leidt.

XI De degradatie van het Nederlandse stelsel van wetenschappelijk onderwijs door een op loting gebaseerde "numerus fixus" vindt haar oorsprong één generatie terug, toen de verantwoordelijken tot de universiteiten werden toegezaten.

Eindhoven, 3 juni 1977

J.P. Mondt