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Published: 01/01/2015

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Link to publication

Citation for published version (APA):
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Effects of communication efficiency and exit capacity on fundamental diagrams for pedestrian motion in an obscure tunnel – a particle system approach

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July 15, 2015

Abstract

Fundamental diagrams describing the relation between pedestrians speed and density are key points in understanding pedestrian dynamics. Experimental data show complex behaviors in which the velocity decreases with the density and different logistic regimes are identified. This paper addresses the issue of pedestrians transport and of fundamental diagrams for a scenario involving the motion of pedestrians escaping from an obscure tunnel. We capture the effects of the communication efficiency and the exit capacity by means of two thresholds controlling the rate at which particles (walkers, pedestrians) move on the lattice. Using a particle system model, we show that in absence of limitation in communication among pedestrians we reproduce with good accuracy the standard fundamental diagrams, whose basic behaviors can be interpreted in terms of the exit capacity limitation. When the effect of a limited communication ability is considered, then interesting non–intuitive phenomena occur. Particularly, we shed light on the loss of monotonicity of the typical speed–density curves, revealing the existence of a pedestrians density optimizing the escape.

We study both the discrete particle dynamics as well as the corresponding hydrodynamic limit (a porous medium equation and a transport (continuity) equation). We also point out the dependence of the effective transport coefficients on the two thresholds – the essence of the microstructure information.

Keywords: Pedestrians transport in dark, lattice model, hydrodynamic limits, porous media equation, continuity equation, fundamental diagrams, evacuation scenario

MSC2010: 91D10; 82C22.
1 Introduction

*Fundamental diagrams* representing the dependence of the pedestrian speed on the local density are one of the basic methods in studying pedestrians dynamics. They contain macroscopic information to identify the key effects affecting the general behaviour of pedestrian flows and to test the validity of pedestrian models. In [1, 2], e.g., their main properties are discussed and experimental test are performed. In particular it is seen that many different effects, such as passing manoeuvres, space reduction, and internal friction, have to be taken into account to explain the main features of the diagrams.

In this paper, we use *Zero Range Processes* (ZRP), originally proposed by Spitzer [3], to recover the same behaviors of the fundamental diagrams, excepting perhaps the existence of an upper density above which the pedestrian velocity drops to zero. Our attention focuses on pedestrians moving in dark corridors, where the lack of visibility hinders them to find the exit. This research line follows a similar path as in [4–7], where the authors used a kinetic formulation to investigate the role of the leaders to control crowds evacuation when visibility is reduced, and extends our previous works on this topic; compare e.g. [8–10] (group formation and cooperation in the dark).

For the current framework, we assume that more particles can occupy the same site of a one–dimensional array of discrete positions (modeling a long dark corridor) and no interaction among the individuals takes place. The dynamics of the system is determined only by the escape rate, namely, the frequency at which a site releases the individuals. The key idea in our model is to assume that the escape rate is proportional to the number of individuals on the site up to a saturation threshold above which such a rate stays constant. The second ingredient we use is that escape is maintained low until a certain activation threshold is reached.

The rationale behind our modeling ideas fits the following *Gedanken experiment*. Imagine a flow of pedestrians on a lane and consider a partition of this lane in squared (or rectangular) cells; cf. Fig. 1. The rate at which a walker leaves one cell is proportional to the number of pedestrian occupying the cell up to a limit which is reached when the “forward row” of the cell is full, cf. the right panel of Fig. 1. In this case, indeed, the pedestrians on the back are prevented from exiting the cell due to the presence of an obstacle. Thus, the escape rate from a cell increases proportionally to the number of pedestrians within the cell until this number reaches the total number of walkers that can be fit into the first row. On the other hand, the escape rate from a cell increases proportionally to the number of individuals provided that an efficient communication network (allowing the individuals to exchange informations about the location of the exit) can established inside the cell. Now, assuming that the interaction range, cf. the left panel of Fig. 1, between any pair of individuals is finite and much less than the size of the cell, the onset of an efficient communication network requires the number of individuals to exceed a minimal value which allows a proper interaction within the cell.

These effects are captured by using ZRP with, respectively, a saturation and an activation threshold. In essence, our modelling is rather simple: no interaction between pedestrians on different cells is taken into account. This choice is deliberate – we want to keep the level of modelling as low as possible to show that, even in such cases, it is possible to recover the qualitative behavior of the fundamental diagrams.

In the particular ZRP we introduce in this paper the two thresholds allow us to switch from an independent motion of the particles to a motion that can be mapped to a simple exclusion
process. When considering the hydrodynamic limits of our model [(i) reversible dynamics, (ii) dynamics with a drift], the resulting macroscopic dynamics exhibit a non–trivial dependence on the thresholds, which is, to our knowledge, yet unexplored.

The motivation for this study stems from our interest in the motion of pedestrian flows in dark or in heavily obscured corridors, where the internal dynamics of pedestrians can change depending on the willingness to cooperate (here: to adhere to large groups) or to be selfish (here: to perform independent random walks); see [8–10] for more details in this direction. To be able to understand the behavioural change leading individuals from cooperation to selfishness and eventually backwards, we consider the presence of the threshold in the particle system. From the evacuation point of view, the central question is:

Which values of the thresholds yield higher evacuation fluxes (currents), or, in other words, allow for lower (average) residence times?

It is worth noting that this particular traffic scenario is intimately related to the dynamics of molecular motors seen from the perspective of processivity (cf., e.g., [11]). For transporting at molecular scales, one distinguishes between processive and non–processive motors. The processive ones perform best when working in small groups (porters), while the non–processive motors work best in large groups (rowers). Their joint collective dynamics has been investigated in [12]. If the motors suddenly change their own processivity from porters to rowers (for instance, due to particular environmental conditions, or due to a command control from a hierarchical structure), then our approach based on zero range processes with threshold approximates conceptually well the changing–in–processivity dynamics.

Threshold effects are not new in microscopic dynamics. They are usually introduced to model dynamics undergoing sudden changes when some dynamical observable exceeds an a priori prescribed value. A natural application of this point of view appears in the context of infections propagation models, where an individual gets infected if the number of infected neighbours is large enough. A very well–studied situation is the Bootstrap Percolation problem [13] in which, for instance, on a square lattice, a site becomes infected as soon as the number of its neighbouring infected sites is larger than a fixed threshold value. In this context, the most interesting and surprising situation is the one in which the threshold is precisely half of the total neighbouring

Figure 1: Sketch of pedestrians moving through a cell of the obscure tunnel driven by a two-threshold biased dynamics. Left panel: the smaller black-filled circles represent the individuals located inside a cell, the bigger circles comprising the smaller black ones represent the interaction range of each individual. For an efficient communication network to be settled, a certain overlap among the bigger circles is needed, which is hence guaranteed by requiring the number of individuals in the cell to exceed the activation threshold $A$. Right panel: as soon as the front row of the cell is full, the number of individuals occupying that front row, corresponding to the saturation threshold $S$, fixes an upper bound to the escape rate from the cell.
sites. In such a case, new scaling laws have been discovered in the infinite volume limit [14, 15].

In the next sections we will focus on the hydrodynamic limit of our zero range process built on thresholds, subjected to periodic boundary conditions and equipped with either symmetric or asymmetric jump probabilities. The asymmetry in the jump probabilities will break detailed balance and will give rise to a net particle current across the system. We will explicitly highlight the effect of the threshold (microscopic information) on the macroscopic transport equations and discuss, in particular, the dependence of the structure of the effective diffusion coefficient and of the effective current on both the thresholds and the local pedestrian density. Our analysis allows one to recover some known results available for the independent particle model and for the simple exclusion process, and sets also the stage for a deeper understanding of the hydrodynamic limit of zero range processes with a fixed number of thresholds.

2 The model

We consider a positive integer \( L \) and define a ZRP [16, 17] on the finite torus (periodic boundary conditions) \( \Lambda_L := \{1, \ldots, L\} \subset \mathbb{Z} \).

We fix \( N \in \mathbb{Z}_+ \) and consider the finite state or configuration space \( \Omega_{N,L} := \{0, \ldots, N\}^\Lambda \).

We fix \( A,S \in \{1, \ldots, N\} \) with \( S \geq A \), the activation and saturation thresholds, respectively. We define the intensity function

\[
g(k) = \begin{cases} 
0 & \text{if } k = 0 \\
1 & \text{if } 1 \leq k \leq A \\
k - A + 1 & \text{if } A < k \leq S \\
S - A + 1 & \text{if } k > S 
\end{cases}
\]  

for each \( k \in \mathbb{Z}_+ \). The intensity function, and all the quantities that we shall define below, do depend on the two thresholds \( A \) and \( S \), but we skip them from the notation for simplicity.

The ZRP we consider in this paper is the Markov process \( \omega_t \in \Omega_{N,L} \), with \( t \geq 0 \), such that each site \( x \in \Lambda \) is updated with intensity \( g(\omega_x(t)) \) and, once such a site \( x \) is chosen, a particle jumps with probability \( p \in [0, 1] \) to the neighboring right site \( x + 1 \) or with probability \( 1 - p \) to the neighboring left site \( x - 1 \) (recall periodic boundary conditions are imposed). For more details we refer the reader to [17, 18]. In our model, the intensity function is related to the (time dependent, in general) hop rates

\[
r^{(x,x-1)}(\omega_x(t)) = (1 - p)g(\omega_x(t)) \quad \text{and} \quad r^{(x,x+1)}(\omega_x(t)) = pg(\omega_x(t))
\]

and coincides, hence, with the escape rate \( r^{(x,x-1)}(\omega_x(t)) + r^{(x,x+1)}(\omega_x(t)) = g(\omega_x(t)) \) at which a particle leaves the site \( x \).

Thus, the effect of the thresholds is to control the escape rate. More precisely, the activation threshold \( A \) keeps the escape rate low and fixed to unity for all sites for which \( \omega_x(t) \leq A \), regardless the number of particles on \( x \). The saturation threshold \( S \), instead, holds the escape rate fixed to a maximum value for all sites for which \( \omega_x(t) \geq S \), regardless, again, the number of particles on \( x \). In the intermediate case, \( A < \omega_x(t) < S \) the escape rate increases proportionally to the actual number of particles on \( x \), see (1).
It is worth noticing that in the limiting case \( A = 1 \) and \( S = \infty \), the intensity function becomes \( g(k) = k \), for \( k > 0 \), and thus the well known independent particle model is recovered. A different limiting situation is the one in which the intensity function is set equal to 1 for any \( k \geq 1 \) and equal to zero for \( k = 0 \). In this case we find a ZRP whose configurations can be mapped to the simple exclusion model states, see [16]. We shall refer to the latter case as to the simple exclusion–like model. Such a model is found, in our set–up, when \( A = S \). We stress that one of the interesting features of our model is the fact that it is able to tune between two very different dynamics, namely, the independent particle and simple exclusion–like behaviors [16]; this tuning can be realized in two ways, i.e., by keeping \( S = \infty \) and varying \( A \) or by keeping \( A = 1 \) and varying \( S \).

We are interested in studying the hydrodynamic limit of this model, i.e. as \( N \to \infty \) and \( L \to \infty \). In particular we shall exploit the fact that the intensity function is not decreasing to use well established theories and derive in our set–up the limiting (effective) diffusion coefficient as well as the limiting (effective) current in presence of the two thresholds. As we shall discuss later, the behavior of such macroscopic quantities with the local density will exhibit very peculiar features inherited from the microscopic properties of the dynamics. In particular, it will be possible to give a nice interpretation of the diagrams in terms of pedestrian motion, and the related fundamental diagrams will be explained in the framework of our very simple model.

We let the Gibbs measure with fugacity \( z \in \mathbb{R} \) of the ZRP introduced above be the product measure on \( \mathbb{N}^L \)

\[
\prod_{x=1}^{L} \nu_z(\eta_x) \quad \text{for any} \quad \eta = (\eta_1, \ldots, \eta_L) \in \mathbb{N}^L
\]  

with

\[
\nu_z(0) = C_z \quad \text{and} \quad \nu_z(k) = C_z \frac{z^k}{g(1) \cdots g(k)} \quad \text{for} \quad k \geq 1,
\]  

where \( C_z \) is a normalizing factor depending, in general, on \( z \), \( A \), and \( S \), namely,

\[
C_z = \left[ 1 + \sum_{k=1}^{\infty} \frac{z^k}{g(1) \cdots g(k)} \right]^{-1}
\]

It is of interest to compute the mean value (against the Gibbs measure) of the intensity function \( g \). By using (3), we get

\[
u_z[g(\omega_x)] = \sum_{k=0}^{\infty} \nu_z(k) g(k) = \sum_{k=1}^{\infty} \nu_z(k) g(k) = C_z z + C_z z \sum_{k=2}^{\infty} \frac{z^{k-1}}{g(1) \cdots g(k-1)} = z,
\]  

where we have used that \( g(0) = 0 \) and, in the last step, we recalled (4).

We find relevant to stress that the expression of such an expectation, as a function of the activity, does not depend on the particular choice of the intensity function. Note, also, that whereas the intensity is a site–dependent function, its expected value, with respect to the Gibbs measure given in (2), is not. This is due to the fact that the Gibbs measure is not site–dependent. This fact stems directly from the imposed periodic boundary conditions and from requiring the jump probabilities to be translationally invariant.
In discussing the hydrodynamic limit, a special role will be played by
\[ \bar{\rho}(z) = \sum_{k=0}^{\infty} k \nu_z(k) \] (6)

It is possible to prove a nice expression for the function \( \bar{\rho} \) independently on the particular choice of the intensity function. Indeed, recalling (3), equation (6) can be rewritten as
\[ \bar{\rho}(z) = C_z \sum_{k=1}^{\infty} \frac{z^k}{g(1) \cdots g(k)} = z \frac{d}{dz} \sum_{k=1}^{\infty} \frac{z^k}{g(1) \cdots g(k)} = z C_z \frac{1}{dz} \sum_{k=1}^{\infty} \nu_z(k) \]
which implies
\[ \bar{\rho}(z) = z C_z \frac{1}{dz} \sum_{k=1}^{\infty} \frac{z^k}{g(1) \cdots g(k)} = -\frac{z}{C_z} \frac{d}{dz} C_z = -z \frac{d}{dz} \log C_z . \] (7)

At the same level of generality, it is not difficult to prove that \( \bar{\rho}(z) \) is an increasing function of the fugacity. Indeed, after some straightforward algebra, one can prove that
\[ \frac{\partial}{\partial z} \bar{\rho}(z) = \frac{\partial}{\partial z} C_z \sum_{k=1}^{\infty} \frac{z^k}{g(1) \cdots g(k)} = \frac{1}{z} [\nu_z(\eta_1^2) - (\nu_z(\eta_1))^2] > 0 . \] (8)

We mention that the above result is strictly connected to the fact the \( -\log C_z \) is a convex function.

Finally, we observe that \( \bar{\rho} \) is defined for any positive \( z \) if \( A \) is finite and \( S = \infty \). On the other hand, it displays a singularity, that is to say it is defined for \( z \) small enough, if \( S \) is finite or when \( A = S = \infty \) (simple exclusion–like model); see Fig. 2.

3 Hydrodynamic limit for reversible dynamics

We consider the case \( p = 1/2 \) so that the dynamics becomes now reversible with respect to the invariant measure. The evolution of the distribution of the particles on the space \( \Lambda \) under the ZRP with thresholds \( A \) and \( S \) introduced above can be described in the diffusive hydrodynamic limit via the time evolution of the density function \( \rho(x,t) \) with the space variable \( x \) varying in the interval \([0,1]\) and \( t \geq 0 \). We again refer to [17] for a detailed and precise derivation of the hydrodynamic limit.

Here, we just recall the main results: it is proven that, in this symmetric case \( p = 1/2 \), the continuous space density \( \rho(x,t) \) is the solution of the partial differential equation
\[ \frac{\partial}{\partial t} \rho = -\frac{\partial}{\partial x} J(\rho) \] (9)
where the macroscopic flux \( J(\rho) \) is defined as
\[ J(\rho) = -\frac{1}{2} D(\rho) \frac{\partial}{\partial x} \rho \] (10)
with the diffusion coefficient \( D \) given by
\[ D(\rho) = \frac{\partial}{\partial \rho} \nu_z(\rho) [g(\omega_1)] . \] (11)
Figure 2: Left panel, top row: Graph of the function $\bar{\rho}(z)$ for $A = 1$ and for different values of the saturation threshold, i.e. $S = 1, 2, 5, \infty$. Right panel, top row: Graph of the function $\bar{\rho}(z)$ for $S = \infty$ and for different values of the activation threshold, i.e. $A = 1, 2, 5, \infty$. Left panel, bottom row: Graph of the function $\bar{\rho}(z)$ for $A = 3$ and for $S = 3, 4, 10, \infty$. Right panel, bottom row: Graph of the function $\bar{\rho}(z)$ for $S = 10$ and for $A = 1, 2, 5, 10$.

Note that the diffusion coefficient is here computed in terms of the mean of the intensity function evaluated against the single site Gibbs measure with fugacity corresponding to the local value of the density.

Recall that, even if it is not coded in the notation, the diffusion coefficient $D$ crucially depends on the value of the thresholds. One of the main multiscale aspects is precisely the link between the two thresholds $A$ and $S$ with the effective diffusion coefficient $D$.

We shall first recall the very well known results which are valid in the limiting cases of the independent particles and simple exclusion–like dynamics.

**Remark 3.1.** Independent particle model: For $A = 1$ and $S = \infty$, one has $C_z = \exp\{-z\}$. Hence, by (7), it holds $\bar{\rho}(z) = z$. Thus, recalling (5) and the definition of $\bar{z}$ given below (8), one finds $\nu_{\bar{z}(\rho)}[g(\omega_1)] = \nu_{\rho}[g(\omega_1)] = \rho$. Thus, by using (11), the diffusion coefficient reads $D(\rho) = 1$.

**Remark 3.2.** Simple exclusion–like model: For $A = S$ (either finite or infinite), one has $g(k) = 1$ for any $k \geq 1$ and $g(0) = 0$. Hence, $C_z = 1 - z$, and it holds $\bar{\rho}(z) = z/(1 - z)$. Thus, proceeding as above, one finds the law $D(\rho) = 1/(1 + \rho)^2$, cf. [19].

Let us now illustrate the general strategy to compute the diffusion coefficient $D$ in presence of arbitrary values of the thresholds $A$ and $S$. To this aim, one uses, first, (4) and (7) to compute $\bar{\rho}(z)$, whose explicit expressions are deferred to the App. A. Then, the expression of the diffusion coefficient can be then obtained using the general recipe in equation (11) and
recalling (5). Indeed,

\[ D(\rho) = \frac{\partial}{\partial \rho} \nu_\varphi(\rho) [g(\omega_1)] = \frac{\partial}{\partial \rho} z(\rho) = \left( \frac{\partial}{\partial z} \tilde{\rho}(z) \right)^{-1} \bigg|_{z=\tilde{z}(\rho)} \]

(12)

We remark that the explicit expression of the quantity \( \partial \tilde{\rho}(z)/\partial z \) appearing in (12) is quite lengthy, hence will be omitted here.

Let us now discuss the above results in detail.

Fig. 3 shows the behavior of the diffusion coefficient as a function of the local density, and parameterized by the values of the thresholds. In particular, the upper left panel of Fig. 3 refers to the case \( A = 1 \) and for different values of \( S \): the simple exclusion–like model is recovered for \( S = 1 \), while the independent particle model is attained for \( S = \infty \). Similarly, the upper right panel illustrates the case with \( S = \infty \) and for different values of \( A \): here the independent particle model corresponds to \( A = 1 \) and the simple exclusion–like model is recovered for \( A = \infty \). As shown in both the upper panels of Fig. 3, in the independent particle case the diffusion coefficient is constant with respect to the local density, and is equal to unity.

A noteworthy feature of the diffusion coefficient, clearly visible in the upper right panel as well as in both the lower panels of Fig. 3, is the loss of monotonicity of the function \( D(\rho) \) occurring at values of \( \rho \) exceeding some critical value (depending, in general, from \( A \) and \( S \)). This observation can be interpreted as the effect, at the hydrodynamic level, of an activation threshold \( A > 1 \) and/or \( S < \infty \) acting at the more microscopic, dynamical, level: both conditions locally pulls the dynamics away from the independent particle behavior. Note, for instance, the behavior of \( D(\rho) \) displayed in the lower left panel of Fig. 3, referring to the case \( A = 3 \). Considering, in particular, the green curve, corresponding to \( S = 10 \), one observes the onset of a double loss of monotonicity of the function \( D(\rho) \): for small values of the density, \( D \) stays close to the simple exclusion–like behavior and decreases with \( \rho \), then, after one first critical value of the density, it starts rising up, until it eventually drops down again, when \( \rho \) exceeds an upper critical value. This reflects precisely the existence of a double threshold for the intensity function, described by (1).

The effect of the two thresholds on the diffusion coefficient is, thus, clear: the larger the activation threshold \( A \) and/or the smaller the saturation threshold \( S \), the smaller is, hence, is the resulting value of the diffusion coefficient. Moreover, in presence of reversible dynamics, the dependence of the diffusion coefficient with respect to density becomes non–monotonic.

This observation can also be interpreted by recalling that the number \( \mathcal{N}(t) \) of particles departing from, or arriving at, the site \( x \in [0, 1] \) of the lattice, is described by a non–homogeneous Poisson process with time–dependent rate parameter \( g(\omega_x(t)) \). Thus, given a small \( \delta > 0 \), it holds

\[ P_t[\mathcal{N}(\delta) = 1] \simeq g(\omega_x(t))\delta \]

where \( P_t[\mathcal{N}(\delta) = 1] \) is the probability of exactly one change in \( \omega_x(t) \) within the time interval \( (t, t + \delta) \). Then, for values of the threshold \( A \) and \( S \) different, respectively, from 1 and \( \infty \) (independent particle model), \( g(\omega_x) \) – hence the escape rate from the site \( x \) – takes a lower value compared to that referring to the independent particle model, with a minimum (corresponding to \( g(\omega_x) = 1 \)) attained when \( A = S \) (i.e., simple exclusion–like model).

The effect of the threshold on the dynamics, in the hydrodynamic limit, is also visible in Fig. 4, showing the profiles, at different times, of the function \( \rho(x, t) \) solving (9)–(10), in the two limiting cases, i.e. independent particle and simple exclusion–like models.
Figure 3: Left panel, top row: Behavior of the diffusion coefficient $D(\rho)$ vs. $\rho$ for $A = 1$ and for different values of the saturation threshold, i.e., $S = 1, 2, 5, \infty$. Right panel, top row: Behavior of the diffusion coefficient $D(\rho)$ vs. $\rho$ for $S = \infty$ and for different values of the activation threshold, i.e., $A = 1, 2, 5, \infty$. Left panel, bottom row: Behavior of the diffusion coefficient $D(\rho)$ vs. $\rho$ for $A = 3$ and for $S = 3, 4, 10, \infty$. Right panel, bottom row: Behavior of the diffusion coefficient $D(\rho)$ vs. $\rho$ for $S = 10$ and for $A = 1, 2, 5, 10$.

Figure 4: Plot of $\rho(x, t)$ vs. $x$, with a given initial condition and with periodic boundary conditions, for the two limiting cases corresponding, respectively, to the independent particle model (red curve), and to the simple exclusion–like process (blue curve) at different times (from left to right): $t = 0$, $t = 0.01$, $t = 0.02$, $t = 0.05$. 

9
4 Hydrodynamic limit in presence of a drift

In section 3 we have discussed the effect of the thresholds on the diffusion equation describing the macroscopic behavior of the system in the hydrodynamic limit. In this section we investigate how the dynamics depends on the thresholds under the effect of an external field breaking the detailed balance condition and inducing a non–vanishing particle current across the system. Hence, we consider the model in Section 2 with \( p \neq \frac{1}{2} \).

The evolution of the distribution of the particles on the space \( \Lambda \) in a ZRP with thresholds under drift can be again described, in the hydrodynamic limit, via the time evolution of the density function \( \rho(x,t) \) with the space variable \( x \) varying in the interval \([0,1]\) and \( t \geq 0 \).

It can be then proven that the equation governing the evolution of the macroscopic local density \( \varrho \) is (9) with the macroscopic current \( J(\rho) \) defined as

\[
J(\rho) = (2p - 1) \bar{z}(\rho) \left[ g(\omega_1) \right]
\]

where, we recall, the intensity function in defined in (1) and the Gibbs measure is defined in (3), see [20, equation (1.3)].

In this out–of–equilibrium regime, the relevant quantity we look at is the velocity, defined as \( v(\rho) = J(\rho)/\rho \). In the setup of ZRP subjected to dynamical thresholds, it is therefore important to shed light on how the constitutive relation \( v \) vs. \( \rho \) is actually affected by both the activation and saturation thresholds. This point can also pave the way to a better understanding of the so–called fundamental diagrams typically invoked in the context of pedestrian flows investigations.

We can now use our results of Section 2 to compute the current. First, note that, for any value of the threshold, by (5), it holds

\[
J(\rho) = (2p - 1) \bar{z}(\rho).
\]

It is not possible to write such an expression explicitly, but for the independent particle and simple exclusion–like cases, in which cases it is straightforward to derive the well known results

\[
v(\rho) = 2p - 1 \quad \text{and} \quad v(\rho) = \frac{2p - 1}{1 + \rho},
\]

respectively, where we used the results in Remark 3.1 and Remark 3.2.

Fig. 5 shows the behavior of the velocity \( v \) as a function of the local density for different values of \( A \) and \( S \). An inspection of the upper left panel of Fig. 5 confirms that the velocity divided by the bias \( (2p - 1) \) is equal to unity for the independent particle model, and behaves as \( (1 + \rho)^{-1} \) in the simple exclusion–like case.

Similarly to the case of the diffusion coefficient, we now notice the presence of a non–monotonic behavior of \( v \) as a function of \( \rho \), occurring if \( A > 1 \) and/or \( S \) is finite. Again, this effect can be ascribed to the peculiar properties of the microscopic dynamics, constrained by the two thresholds.

In particular, in the right top panel in figure 5 the case \( S = \infty \) is depicted. In absence of limitations due to the exit capacity, if no limitation on the communication occurs \( (A = 1) \) the typical speed is maximal and it does not depend on the local density. On the other hand, when \( A > 1 \), the speed decreases until the density exceeds a critical value (depending on the two thresholds), and, after that, it starts to increase until it attains the ideal maximal value at
Figure 5: Left panel, top row: Behavior of the velocity $v/(2p - 1)$ vs. $\rho$ for $A = 1$ and for different values of the saturation threshold, i.e., $S = 1, 2, 5, \infty$. Right panel, top row: Behavior of the velocity $v/(2p - 1)$ vs. $\rho$ for $S = \infty$ and for different values of the activation threshold, i.e., $A = 1, 2, 5, \infty$. Left panel, bottom row: Behavior of the velocity $v/(2p - 1)$ vs. $\rho$ for $A = 3$ and for $S = 3, 4, 10, \infty$. Right panel: Behavior of the velocity $v/(2p - 1)$ vs. $\rho$ for $S = 10$ and for $A = 1, 2, 5, 10$. 
large $\rho$. In the extreme case $A = \infty$, no communication is possible however large is the density, hence the typical velocity is a monotonic decreasing function of $\rho$.

In the right bottom panel the case $S = 10$ is portrayed: the graphs show that as a result of the constraints imposed by the two dynamical thresholds, there exists a local value of the density optimizing the typical speed. Such a density has to be large enough so that communication is efficient but, also, small enough so that the limitation on the escape capacity do not cause an abrupt drop of the typical velocity.

5 Possible interpretations of the two thresholds

It is worth mentioning that working with two thresholds leads to rich descriptions in terms of modeling. In particular, a double-threshold dynamics is amenable to be interpreted in multiple fashions, viz.

(i) **Porous media interpretation**: Essentially, the bulk porosity estimates how many particles can be accommodated in a cell. This connects to the activation threshold. The saturation threshold is essentially proportional to the surface porosity, since it is a measure of the exit capacity. We refer to [21] for building a possible closer look on the porous media interpretation.

(ii) **Mechanical interpretation**: Imagine, for a moment, that the tunnels are equipped with valve-like doors whose opening results from the balance between the pressure inside the cell and an outer pressure exerted by a spring. A minimal – structural – opening of the door, with the spring maintained at rest, corresponds to the presence of an activation threshold. Any further opening of the door is hence achieved by compensating the external pressure of the spring, which is considered to increase proportionally to the displacement of the door, as dictated by the Hooke’s law of mechanics. Finally, the maximal opening of the door, in presence of the minimum elongation of the spring, corresponds to the saturation threshold. See, e.g., [22] for a scenario describing how pressure/temperature-controlled shape-memory alloys facilitate the functioning of the Japanese rice cooking machine.

(iii) **Psychologico–geometrical interpretation**: The activation threshold is a measure of the domain of communication between the individuals and the level this communication is processed towards a decision on the motion (either on orientation in the dark, or on the chosen speed). Essentially, we imagine that this activation threshold is inversely proportional to the level of trust (see our interpretations proposed in [23]). The saturation threshold is then directly proportional to the capacity of the exit(s).

(iv) **A phase transitions perspective**: The assumption here is that pedestrians evacuating the obscure tunnel undergo a first transition of first kind (like the ice-water transition, cf. Landau’s classification): from being trapped in the dark tunnel and being free to go in corridors where they can choose their own desired velocity. The parallel can be made a bit more precise by applying the Clapeyron equation in this context to translate difference in temperatures into difference in pressures. The two thresholds can now be seen as the direct counterparts of the accumulated heat content (amount of phonons) needed to melt the ice (the activation threshold) and the amount of accumulated heat content needed to
evaporate water (the saturation threshold). Essentially, we mean here that the dynamics is “frozen” for densities below the activation threshold and people “evaporate” from the tunnel for densities of the order of magnitude of the saturation threshold. Remotely related connections to phase transitions supposed to happen in social systems are reported, for instance, in [24, 25].

6 Discussion

We considered a one–dimensional ZRP equipped with periodic boundary conditions and with symmetric and asymmetric jump probabilities. The novelty of our approach stems from the introduction in the zero range process of the two thresholds $A$ and $S$ together with their interpretations in terms of communication efficiency and exit capacity.

From the mathematics viewpoint, the thresholds can be tuned to control the magnitude of the intensity function, thus allowing one to span a broad variety of zero range dynamics, ranging from the independent particle models to the simple exclusion–like processes.

We then investigated the hydrodynamic limit of the considered ZRP for different values of the thresholds, and discussed the effect of such dynamical constraints on some macroscopic quantities, e.g. the effective diffusion coefficient, the particle density and the effective outgoing current. We recover known results in the limiting scenarios, and also provide explicit formulae for arbitrary thresholds, provided the activation and saturation thresholds coincide. Our investigation thus provides a noteworthy bridge between the features of the underlying microscopic dynamics and macroscopic quantities relevant in the hydrodynamic limit of the model, which are also experimentally accessible. Further investigations are needed, next, to extend our results to the even more challenging scenario characterized by the use of non–periodic boundary conditions in the zero range dynamics.

From the pedestrians evacuation viewpoint, we explored the effects of communication on the effective transport properties of the crowd of pedestrians. More precisely, we were able to emphasize the effect of two thresholds on the structure of the effective nonlinear diffusion coefficient. One threshold models pedestrians’s communication efficiency in the dark, while the other one describes the tunnel capacity. Essentially, we observe that if the evacuees show a maximum trust (leading to a fast communication), they tend to quickly find the exit and hence the collective action tends to prevent the occurrence of disasters. In our context, “a high activation threshold increases the diffusion coefficient” means that “higher trust among pedestrians improves communication in the dark” and therefore the exits can be found easier.

The exit capacity is given by the size of the saturation threshold. Consequently, a higher saturation threshold leads to an improved capacity of the exists (e.g. larger doors, or more exits [26]) and, hence, the evacuation rate is correspondingly higher.

Similarly, in presence of a drift, the fundamental diagrams become non–monotonic with respect to the local pedestrian density. We were able to point out that the fundamental diagrams become independent on the local density as soon as the exit capacity is unbounded. Interestingly, we were able to detect situations (see, for instance, Fig. 5) in which there are particular pedestrian densities optimizing the speed. It appears that such an optimizing density must be large enough so that communication is efficient but, also, small enough so that the limitation on the escape capacity do not cause an abrupt drop of the typical flow velocity.
Acknowledgements

The authors wish to thank Errico Presutti (Gran Sasso Science Institute, L’Aquila, Italy), Anna De Masi (University of L’Aquila, Italy), and Claudio Landim (IMPA, Rio de Janeiro, Brazil) for useful discussions. ENMC thanks ICMS (TU/e, Eindhoven, The Netherlands) for the very kind hospitality and for financial support.

A Some explicit formulas for arbitrary values of the thresholds

We provide, here, the general form of the normalization constant $C_z$ and the function $\bar{\rho}(z)$ for arbitrary values of the thresholds $A$ and $S$. It holds

$$C_z = z \left[ \frac{z - z^A}{1 - z} + \frac{z^{2+S}}{\Gamma(2+S-A)} \left( \frac{1}{1 - z + S - A} + e^z E_{(-1-S+A)}(z) \right) \right]^{-1},$$

where

$$\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds$$

is the Gamma function and

$$E_n(x) = \int_1^\infty \frac{e^{-sx}}{s^n}ds$$

denotes the generalized exponential integral function [27]. Moreover,

$$\bar{\rho}(z) = C_z z \left[ \frac{z^2 + z^A(1-A(1-z) - 2z)}{z(1-z)^2} + \frac{z^{1+S}}{\Gamma(2+S-A)} \times \right.$$  

$$\left. \times \left( \frac{z + (A + z)(1 - z + S - A)}{(1 - z + S - A)^2} + e^z(-1 + A + z)E_{(-1-S+A)}(z) \right) \right]. \quad (16)$$

One can then verify that, by taking $A = 1$ and $S = \infty$, one recovers the expressions for $C_z$ and $\bar{\rho}(z)$ corresponding to the independent particle model (see Remark 3.1), whereas, for $A = S$, one obtains the results pertaining to the simple exclusion–like model (see Remark 3.2).

References


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