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Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions

by

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Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions

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Abstract. In this paper, we obtain the uniqueness of weak solutions for a two phase flow model in a porous medium. A particularity of the model is that the dynamic effects and hysteresis are included in the capillary pressure.

Keywords: Dynamic capillary pressure, two-phase flow, hysteresis, weak solution, uniqueness.

1 Introduction

We consider a mathematical model for two-phase flow in a porous medium. Two immiscible fluid phases are flowing through a porous medium occupying a bounded, connected domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$). Using $\bar{\Omega}$ and $\partial \Omega$ denote the closure and boundary of $\Omega$. Let $T > 0$ be a given time. The phase pressures are denoted by $p_w, p_n$. The non-wetting phase saturation is $s$. We assume the porous medium is saturated by the two phases. Then from the Darcy law and mass conservation for each fluid give the system (see [1, 16])

$$\partial_t s - \nabla \cdot (k_n(s) \nabla p_n) - \nabla \cdot (k_n(s) \vec{g}) = 0,$$

$$-\partial_t s - \nabla \cdot (k_w(s) \nabla p_w) - \nabla \cdot (k_w(s) \vec{g}) = 0.$$  \hfill (1)

Here $\vec{g} \in \mathbb{R}^d$ is the gravity vector in direction $-\vec{e}_d = (0, ..., 0, -1) \in \mathbb{R}^d$. $k_n(s), k_w(s)$ are the permeabilities - two nonlinear functions depending on $s$. The system is closed by the relation between the phase pressures and saturation. Standardly, equilibrium models assume $p_w - p_n = p_c(s), p_c$ - decreasing with respect to $s$ (see [12]). While experiments [3, 7, 17] have proved the limitation of this approach. Alternatively, models involving non-equilibrium effects are proposed in [2]:

$$p_n - p_w \in p_c(s) + \gamma(x) \text{sign}(\partial_t s) + \tau \partial_t s.$$  \hfill (3)
Here $\gamma \geq 0$, $\tau > 0$ are given and sign denotes the multi-valued function

$$\text{sign}(\xi) = \begin{cases} 
1 & \text{if } \xi > 0, \\
-1 & \text{if } \xi < 0, \\
[-1,1] & \text{if } \xi = 0.
\end{cases} \quad (4)$$

The second term on the right in (3) models a play-type hysteresis (see [2, 18]), while the last one accounts for dynamic effects in the phase pressure difference (see [15]). Following [11], for $\tau > 0$, the multi-valued function $\Phi : \xi \mapsto \tau \xi + \gamma \text{sign}(\xi)$ can be inverted. Its inverse $\Psi : \Phi^{-1} : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function satisfying

$$0 \leq \Psi'(\xi, x) \leq 1/\tau. \quad (5)$$

With this notation, (3) transforms into

$$\partial_t s = \Psi(p_n - p_w - p_c(s), x). \quad (6)$$

The model (1), (2), (6) is complemented by initial and boundary conditions

$$s(0, \cdot) = s_0, \quad (7)$$

$$p_n = p_w = 0 \text{ at } \partial \Omega, \text{ for all } t \geq 0. \quad (8)$$

**Remark 1.1:** Other boundary conditions are possible, but for clarity, we restrict the presentation to (8).

The following assumptions are made:

- **A1:** The functions $k_w, k_n : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. Further, $\delta, M_k > 0$ exist such that $\delta \leq k_w(s), k_n(s) \leq M_k < \infty$, for all $s \in \mathbb{R}$.

- **A2:** $p_c(\cdot) \in C^1(\mathbb{R})$ is increasing and Lipschitz continuous, there exist $m_p, M_p > 0$ such that $m_p \leq p_c'(s) \leq M_p$, for all $s \in \mathbb{R}$.

- **A3:** $\Omega$ is a $C^{1,\alpha}$ ($0 < \alpha \leq 1$) domain.

- **A4:** $\gamma(x) \in C^{0,1}(\bar{\Omega})$.

- **A5:** $s_0 \in C^{0,\alpha}(\bar{\Omega})$.

**Remark 1.2:** Commonly, the permeabilities encountered in the literatures ([4]) are

$$k_w(s) = (1 - s)^p, \quad k_n(s) = s^q, \quad \text{with } p, q > 1,$$

and

$$p_c(s) = (1 - s)^{-\frac{1}{\lambda}}, \quad \lambda > 1, \text{ for } s \in [0,1].$$

Then A1 is not satisfied when $s$ approaches to 0 or 1. We consider here a regularized approximation of these functions.
2 Uniqueness

Existence results for the model considered here are proved in [11]. In this section, we provide a rigorous proof of the uniqueness of weak solutions to (1), (2), (6). We use common notations for function spaces, namely $L^2$, $W^{1,2}$, $W^{1,2}_0$, and Bochner space $L^2(0,T;X)$. Further, by $C > 0$, we have a generic constant. We follow [11] and consider weak solutions solving

**Problem $P_e$:** Given $s_0$ satisfying A5, find $p_n \in L^2(0,T;W^{1,2}_0(\Omega))$, $p_w \in L^2(0,T;W^{1,2}_0(\Omega))$ and $s \in W^{1,2}(0,T;L^2(\Omega))$, such that $s(\cdot,0) = s_0$ in $\Omega$, and

\[
(\partial_t s, \phi) + (k_n(s) \nabla p_n, \nabla \phi) + (k_n(s) \overrightarrow{g}, \nabla \phi) = 0, \\
(-\partial_t s, \psi) + (k_w(s) \nabla p_w, \nabla \psi) + (k_w(s) \overrightarrow{g}, \nabla \psi) = 0, \\
(\partial_t s, \rho) = (\Psi(p_n - p_w - p_c(s), x), \rho),
\]

for any $\phi, \psi \in L^2(0,T;W^{1,2}_0(\Omega))$ and $\rho \in L^2(0,T;L^2(\Omega))$.

In [11], the hysteresis is modeled by considering (3) valid a.e.. This immediately implies that (6) holds a.e. and further (11). In this respect, the weak solution of Problem $P_e$ is also a solution in [11]. The existence of weak solutions for Problem $P_e$ has been proved in [11]. Here we show that weak solution is unique. Unique results for a similar model but without hysteresis are obtained in [5]. To this aim, some intermediate results are needed. We start with essential bounds for the gradients of $p_n$ and $p_w$.

**Theorem 2.1.** Let $(p_n, p_w, s)$ be a weak solution to Problem $P_e$. Then one has $\nabla p_n, \nabla p_w \in L^\infty((0,T] \times \Omega)$.

**Proof.** First we show that $\|\nabla p_n\|_{L^2(\Omega)} \in L^\infty(0,T)$ and $\|\nabla p_w\|_{L^2(\Omega)} \in L^\infty(0,T)$.

Taking $\phi = p_n$ in (9), $\psi = p_w$ in (10) and adding the resulting equations give

\[
(\partial_t s, p_n - p_w) + \|\sqrt{k_n(s)} \nabla p_n\|_{L^2(\Omega)}^2 + \|\sqrt{k_w(s)} \nabla p_w\|_{L^2(\Omega)}^2 + (k_n(s) \overrightarrow{g}, \nabla p_n) + (k_w(s) \overrightarrow{g}, \nabla p_w) = 0. \\
\]

For the first term of (12), we note that (3) holds almost everywhere. Then since $\text{sign}(\xi) \xi \geq 0$ for any $\xi \in \mathbb{R}$, one has

\[
\int_\Omega \partial_t s(p_n - p_w) \geq \int_\Omega \tau |\partial_t s|^2 dx + \int_\Omega p_c(s) \partial_t s dx \geq \frac{\tau}{2} \|\partial_t s\|_{L^2(\Omega)}^2 - \frac{1}{2\tau} \int_\Omega |p_c(s)|^2 dx.
\]

Further, since $s \in L^\infty(0,T;L^2(\Omega))$ (see [11, 14]), by using the Cauchy-Schwarz inequality, A1 and A2, (12) gives

\[
\|\nabla p_n\|_{L^2(\Omega)}^2 + \|\nabla p_w\|_{L^2(\Omega)}^2 \leq C, \quad \text{for almost every } t.
\]

Then substituting (6) into (1) and (2) respectively, one has

\[
- \nabla \cdot (k_n(s) \nabla p_n) = -\Psi(p_n - p_w - p_c(s), x) + \nabla \cdot (k_n(s) \overrightarrow{g}),
\]

(15)
\[-\nabla \cdot (k_w(s)\nabla p_w) = \Psi(p_n - p_w - p_c(s), x) + \nabla \cdot (k_w(s)\overline{\gamma}). \tag{16}\]

Using Theorem 14.1 in [13] gives for almost every \( t \),
\[
\|p_n\|_{C^{0,\alpha}(\bar{\Omega})} + \|p_w\|_{C^{0,\alpha}(\bar{\Omega})} \leq C. \tag{17}\]

Further, from (6), for almost every \( x, y \in \Omega \ (x \neq y) \) and \( t > 0 \), \( \zeta \) and \( \tilde{\zeta} \) depending on \( x, y, t \) exist, such that
\[
\partial_t \frac{s(t, x) - s(t, y)}{|x - y|^\alpha} = \frac{\Psi((p_n - p_w - p_c(s))(t, x), x) - \Psi((p_n - p_w - p_c(s))(t, y), y)}{|x - y|^\alpha} \]
\[
= \frac{\Psi(p_n(t, x) - p_w(t, x) - p_c(s(t, x)), x) - \Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), x)}{|x - y|^\alpha} \]
\[
+ \frac{\Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), x) - \Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), y)}{|x - y|^\alpha} \]
\[
= \Psi' (\zeta, x) \left( \frac{p_n(t, x) - p_n(t, y)}{|x - y|^\alpha} - \frac{p_w(t, x) - p_w(t, y)}{|x - y|^\alpha} - \frac{p_c(s(t, x) - p_c(s(t, y))}{|x - y|^\alpha} \right) \]
\[
+ \frac{\Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), x) - \Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), y)}{|x - y|^\alpha} \]
\[
= \Psi' (\zeta, x) \left( \frac{p_n(t, x) - p_n(t, y)}{|x - y|^\alpha} - \frac{p_w(t, x) - p_w(t, y)}{|x - y|^\alpha} - \frac{p_c(\tilde{\zeta}) \cdot s(t, x) - s(t, y)}{|x - y|^\alpha} \right) \]
\[
+ \frac{\Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), x) - \Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), y)}{|x - y|^\alpha}. \tag{18}\]

Define
\[
\Gamma(t, x, y) = \frac{\Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), x) - \Psi(p_n(t, y) - p_w(t, y) - p_c(s(t, y)), y)}{|x - y|^\alpha}. \tag{19}\]

By A2 - A4, and since \( p_n, p_w \in C^{0,\alpha}(\bar{\Omega}) \), for almost every \( t \), we have
\[
|\Gamma(t, x, y)| + \sup_{x, y \in \Omega, x \neq y} \frac{|p_n(t, x) - p_n(t, y)|}{|x - y|^\alpha} + \sup_{x, y \in \Omega, x \neq y} \frac{|p_w(t, x) - p_w(t, y)|}{|x - y|^\alpha} \leq C. \tag{20}\]

Defining \( w : (0, T] \times \Omega^2 \to \mathbb{R} \) as
\[
w = \frac{s(t, x) - s(t, y)}{|x - y|^\alpha}, \tag{21}\]
\( w \) satisfies
\[
\partial_t w = fw + g, \tag{22}\]
where \( f(t, x) = -\Psi'(\zeta, x) \cdot p'_c(\zeta) \) and \( g(t, x) = \Psi'(\zeta, x)(\frac{p_n(t, x) - p_n(t, y)}{|x-y|^\alpha} - \frac{p_w(t, x) - p_w(t, y)}{|x-y|^\alpha}) + \Gamma(t, x, y) \). Note that, (5), (20) and A2 give \( f, g \in L^\infty((0, T] \times \tilde{\Omega}) \).

Multiplying (22) by \( w \) and integrating from 0 to \( t \) leads to

\[
\frac{1}{2} w^2(t) = \int_0^t f w^2(z) dz + \int_0^t g w(z) dz + \frac{1}{2} \left( \frac{s_0(x) - s_0(y)}{|x-y|^\alpha} \right)^2.
\]  

(23)

Since \( f, g \in L^\infty((0, T] \times \tilde{\Omega}) \) and \( s_0 \in C^{0,\alpha}(\Omega) \) from A5, we have

\[
w^2(t) \leq C(1 + \int_0^t w^2 dz), \quad \text{for every } t.
\]  

(24)

Using Gronwall’s inequality yields \( w \leq C \), implying that

\[
\frac{|s(t, x) - s(t, y)|}{|x-y|^\alpha} \leq C, \quad \text{for almost every } x, y \in \Omega, \text{ for every } t.
\]  

(25)

Let \( \Omega_c \) be the subset of \( \Omega \), where (25) holds everywhere. Clearly, \( \Omega \setminus \Omega_c \) is zero measured. For any \( x \in \Omega \setminus \Omega_c \), we consider a sequence \( \{x_n\}_{n \in \mathbb{N}} \in \Omega_c \) converging to \( x \), and define

\[
s(t, x) = \lim_{n \to \infty} s(t, x_n).
\]  

(26)

In the view of (25), \( s(x) \) does not depend on the choice of \( \{x_n\}_{n \in \mathbb{N}} \). With this choice, \( s \in C^{0,\alpha}(\Omega) \) (see [8]).

Finally, by Theorem 8.33 and Corollary 8.35 in [10] (see also [6]), we get

\[
|p_n|_{1,\alpha} \leq C(|p_n|_0 + |\Psi|_0 + |k_n(s)|_{0,\alpha}),
\]  

(27)

\[
|p_w|_{1,\alpha} \leq C(|p_w|_0 + |\Psi|_0 + |k_w(s)|_{0,\alpha}),
\]  

(28)

implying \( \nabla p_n, \nabla p_w \in L^\infty((0, T] \times \tilde{\Omega}) \).

\( \square \)

**Theorem 2.2.** Problem \( P_e \) has at most one solution.

**Proof.** Let \( (u, p^u_n, p^w_n) \) and \( (v, p^v_n, p^v_w) \) be the two solutions of Problem \( P_e \), then one has

\[
(\partial_t (u - v), \phi) + (k_n (v) \nabla (p^u_n - p^w_n), \nabla \phi) + ((k_n (u) - k_n (v)) \nabla p^u_n, \nabla \phi) + ((k_n (u) - k_n (v)) \nabla p^w_n, \nabla \phi) = 0,
\]  

(29)

\[
- (\partial_t (u - v), \psi) + (k_w (v) \nabla (p^u_n - p^w_n), \nabla \psi) + ((k_w (u) - k_w (v)) \nabla p^u_n, \nabla \psi) + ((k_w (u) - k_w (v)) \nabla p^w_n, \nabla \psi) = 0,
\]  

(30)

and

\[
(\partial_t (u - v), \rho) = (\Psi(p^u_n - p^w_n - p_c (u), x) - \Psi(p^v_n - p^w_n - p_c (v), x), \rho),
\]  

(31)
Further, let \((\phi, \psi)\). Since \(\Psi\) is Lipschitz, for almost every \((x, t) \in \Omega_T\), a \(\xi\) exists, such that
\[
(\partial_t (u - v), \rho) = (\Psi'(\xi, x)((p_{n}^u - p_{n}^v) - (p_{w}^u - p_{w}^v) - (p_c(u) - p_c(v))), \rho).
\]
Further, let \((G_{u-v}, \tilde{G}_{u-v})\) be the weak solution pair of the elliptic system (see [5, 9]),
\[
-\nabla \cdot (k_n(v)\nabla G_{u-v}) + \Psi'(\xi, x)(G_{u-v} + G_{u-v}) = \Psi'(\xi, x)(u - v),
\]
\[
-\nabla \cdot (k_w(v)\nabla \tilde{G}_{u-v}) + \Psi'(\xi, x)(G_{u-v} + \tilde{G}_{u-v}) = \Psi'(\xi, x)(u - v),
\]
where
\[
G_{u-v}, \tilde{G}_{u-v} = 0, \quad \text{at} \partial \Omega.
\]
The existence and uniqueness follow the Lax-Milgram lemma. Further, one has
\[
(\Psi'(\xi, x)G_{u-v}, \lambda) + (\Psi'(\xi, x)G_{u-v}, \lambda) + (k_n(v)\nabla G_{u-v}, \nabla \lambda) = (\Psi'(\xi, x)(u - v), \lambda),
\]
for any \(\lambda, \tilde{\lambda} \in W^{1,2}_0(\Omega).
Using the properties of \(\Psi, k_w, k_n\), one immediately gets
\[
\|G_{u-v}\|^2_{W^{1,2}(\Omega)} \leq C\|u - v\|^2_{L^2(\Omega)}, \quad \text{and} \quad \|	ilde{G}_{u-v}\|^2_{W^{1,2}(\Omega)} \leq C\|u - v\|^2_{L^2(\Omega)}.
\]
Taking \(\phi = G_{u-v}\) in (29), and \(\psi = \tilde{G}_{u-v}\) in (30), one has
\[
(\partial_t (u - v), G_{u-v}) + (k_n(v)\nabla (p_{n}^u - p_{n}^v), \nabla G_{u-v})
+ ((k_n(u) - k_n(v))\nabla p_{n}^u, \nabla G_{u-v}) + ((k_n(u) - k_n(v))\nabla \tilde{g}, \nabla G_{u-v}) = 0,
\]
\[
-(\partial_t (u - v), \tilde{G}_{u-v}) + (k_w(v)\nabla (p_{w}^u - p_{w}^v), \nabla \tilde{G}_{u-v})
+ ((k_w(u) - k_w(v))\nabla p_{w}^u, \nabla \tilde{G}_{u-v}) + ((k_w(u) - k_w(v))\nabla \tilde{g}, \nabla \tilde{G}_{u-v}) = 0.
\]
Choosing \(\lambda = p_{n}^u - p_{n}^v\) in (36) and \(\tilde{\lambda} = p_{w}^u - p_{w}^v\) in (37) gives
\[
(k_n(v)\nabla G_{u-v}, \nabla (p_{n}^u - p_{n}^v)) = (\Psi'(\xi, x)(u - v), p_{n}^u - p_{n}^v) - (\Psi'(\xi, x)G_{u-v}, p_{n}^u - p_{n}^v)
- (\Psi'(\xi, x)G_{u-v}, p_{n}^u - p_{n}^v),
\]
Substitute (41) into (39) and (42) into (40), we find that

\[
(k_w(v)\nabla \tilde{G}_{u-v}, \nabla (p_w^u - p_w^v)) = (\Psi'(\xi, x)(u-v), p_w^u - p_w^v) - (\Psi'(\xi, x)G_{u-v}, p_w^u - p_w^v)
- (\Psi'(\xi, x)\tilde{G}_{u-v}, p_w^u - p_w^v).
\]

Substituting the above two equations into (46) leads to

\[
-(\partial_t(u-v), \tilde{G}_{u-v}) - (\Psi'(\xi, x)G_{u-v}, p_w^u - p_w^v) - (\Psi'(\xi, x)G_{u-v}, p_n^u - p_n^v)
- (\partial_t(u-v), G_{u-v}) - (\Psi'(\xi, x)G_{u-v}, p_n^u - p_n^v)
+ (\Psi'(\xi, x)(u-v), p_w^u - p_w^v) + ((k_n(u) - k_n(v))\nabla p_n^u, \nabla G_{u-v})
+ ((k_n(u) - k_n(v)) \tilde{g}, \nabla G_{u-v}) = 0,
\]

(43)

Taking \( \rho = u - v \) into (32) yields

\[
(\Psi'(\xi, x)(p_n^u - p_n^v), u - v) = (\partial_t(u-v), u-v) + (\Psi'(\xi, x)(p_w^u - p_w^v), u-v)
+ (\Psi'(\xi, x)(p_c(u) - p_c(v)), u-v).
\]

(45)

Using this into (43) and subtracting (44), the resulting equation gives

\[
(\partial_t(u-v), G_{u-v}) + (\Psi'(\xi, x)G_{u-v}, p_w^u - p_w^v) - (\Psi'(\xi, x)G_{u-v}, p_n^u - p_n^v)
+ (\partial_t(u-v), \tilde{G}_{u-v}) + (\Psi'(\xi, x)\tilde{G}_{u-v}, p_w^u - p_w^v) - (\Psi'(\xi, x)\tilde{G}_{u-v}, p_n^u - p_n^v)
+ (\partial_t(u-v), u-v) + (\Psi'(\xi, x)(p_c(u) - p_c(v)), u-v)
+ ((k_n(u) - k_n(v))\nabla p_n^u, \nabla G_{u-v}) - ((k_n(u) - k_n(v))\nabla p_n^v, \nabla \tilde{G}_{u-v})
+ ((k_n(u) - k_n(v)) \tilde{g}, \nabla G_{u-v}) - ((k_n(u) - k_n(v)) \tilde{g}, \nabla \tilde{G}_{u-v}) = 0.
\]

(46)

Further, taking \( \rho = G_{u-v} \) and \( \rho = \tilde{G}_{u-v} \) in (32) respectively give

\[
(\partial_t(u-v), G_{u-v}) + (\Psi'(\xi, x)G_{u-v}, p_w^u - p_w^v) - (\Psi'(\xi, x)G_{u-v}, p_n^u - p_n^v)
= - (\Psi'(\xi, x)G_{u-v}, p_c(u) - p_c(v)),
\]

(47)

and

\[
(\partial_t(u-v), \tilde{G}_{u-v}) + (\Psi'(\xi, x)\tilde{G}_{u-v}, p_w^u - p_w^v) - (\Psi'(\xi, x)\tilde{G}_{u-v}, p_n^u - p_n^v)
= - (\Psi'(\xi, x)\tilde{G}_{u-v}, p_c(u) - p_c(v)).
\]

(48)

Substituting the above two equations into (46) leads to

\[
(\partial_t(u-v), u-v) - (\Psi'(\xi, x)G_{u-v}, p_c(u) - p_c(v)) - (\Psi'(\xi, x)\tilde{G}_{u-v}, p_c(u) - p_c(v))
+ (\Psi'(\xi, x)(u-v), p_c(u) - p_c(v)) + ((k_n(u) - k_n(v))\nabla p_n^u, \nabla G_{u-v})
- ((k_n(u) - k_n(v))\nabla p_n^v, \nabla \tilde{G}_{u-v}) + ((k_n(u) - k_n(v)) \tilde{g}, \nabla G_{u-v})
- ((k_n(u) - k_n(v)) \tilde{g}, \nabla \tilde{G}_{u-v}) = 0.
\]

(49)
Integrating (49) from 0 to \( \tilde{t} \), for any \( \tilde{t} \in (0, T] \). Since \( \nabla p_n, \nabla p_w \in L^\infty((0, T] \times \bar{\Omega}) \), by using (5), A1, A2 and (38), we obtain

\[
\|(u - v)(\cdot, \tilde{t})\|^2_{L^2(\Omega)} \leq C \int_0^{\tilde{t}} \|(u - v)(\cdot, t)\|^2_{L^2(\Omega)} dt.
\] (50)

By Gronwall’s inequality, \( \|(u - v)(\cdot, \tilde{t})\|^2_{L^2(\Omega)} = 0 \). Since \( \tilde{t} \) is arbitrary, this gives \( u = v \) a.e. in \( \Omega \) and for all \( t \in (0, T] \).

To show that \( p_n^u = p_n^v, p_w^u = p_w^v \), we use (29) and (30). Since \( u = v \), one has

\[
(k_n(u) \nabla(p_n^u - p_n^v), \nabla \phi) = 0,
\] (51)

\[
(k_w(u) \nabla(p_w^u - p_w^v), \nabla \psi) = 0,
\] (52)

for any \( \phi, \psi \in W_0^{1,2}(\Omega) \), for almost every \( t \).

The rest of the proof follows straightforwardly by taking \( \phi = p_n^u - p_n^v, \psi = p_w^u - p_w^v \), and recalling that \( p_n^u, p_n^v, p_w^u, p_w^v \) have equal traces on \( \partial \Omega \).

3 Conclusion

In this paper, we have proved the uniqueness of weak solutions to a non-degenerate system which models two-phase flow in porous media including hysteresis and dynamic effects in the capillary pressure. In doing so, we use arguments based on Green’s function.

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