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Euclidean distance degrees of real algebraic groups

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\textbf{A B S T R A C T}

We study the problem of finding, in a real algebraic matrix group, the matrix nearest to a given data matrix. We do so from the algebro-geometric perspective of Euclidean distance degrees. We recover several classical results, and among the new results that we prove is a formula for the Euclidean distance degree of special linear groups.

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\section{1. The distance to a matrix group}

In many applications across mathematics and engineering, one encounters the problem of finding the point on a real algebraic variety that minimises the distance to a given data point outside the variety, and of counting the critical points on the variety of the distance function to the data point. In this paper we discuss the special case of these problems where the variety is a real algebraic group realised as a matrix group. We
review known results for orthogonal and unitary groups, prove a general upper bound for compact tori, and discuss the problem of finding the nearest determinant-one matrix and the nearest symplectic matrix.

Let $V$ be an $n$-dimensional real vector space equipped with a positive definite inner product $(.,.)$, and write $\text{End}(V)$ for the space of linear maps $V \to V$. The inner product gives rise to a linear map $\text{End}(V) \to \text{End}(V)$, $a \mapsto a^t$ called transposition and determined by the property that $(av|w) = (v|a^t w)$ for all $v, w \in V$, and also to a positive definite inner product $\langle ., . \rangle$ on $\text{End}(V)$ defined by $\langle a, b \rangle := \text{tr}(a^t b)$. This inner product enjoys properties such as $\langle a, bc \rangle = \langle b^t a, c \rangle$. The associated norm $\| . \|$ on $\text{End}(V)$ is called the Frobenius norm. If we choose an orthonormal basis of $V$ and denote the entries of the matrix of $a \in \text{End}(V)$ relative to this basis by $a_{ij}$, then $\|a\|^2 = \sum_{ij} a_{ij}^2$. We will use the words matrix and linear maps interchangeably, but we work without choosing coordinates because it allows for a more elegant statement of some of the results. For $a, b \in \text{End}(V)$ and $v, w \in V$ we write $a \perp b$ for $\langle a, b \rangle = 0$, and $v \perp w$ for $\langle v|w \rangle = 0$.

Let $G$ be a Zariski-closed subgroup of the real algebraic group $\text{GL}(V) \subseteq \text{End}(V)$ of invertible linear maps. In other words, $G$ is a subgroup of $\text{GL}(V)$ characterised by polynomial equations in the matrix entries. Then $G$ is a real algebraic group and in particular a smooth manifold. The problem motivating this note is the following.

**Problem 1.1.** Given a general $u \in \text{End}(V)$, determine $x \in G$ that minimises the squared-distance function $d_u(x) := \| u - x \|^2$.

Here, and in the rest of this note, *general* means that whenever convenient, we may assume that $u$ lies outside some proper, Zariski-closed subset of $\text{End}(V)$. Instances of this problem appear naturally in applications. For instance, the *nearest orthogonal matrix* plays a role in computer vision [6], and we revisit its solution in Section 3. More or less equivalent to this is the solution to the orthogonal Procrustes problem [11]. For these and other matrix nearness problems we refer to [5,8]. More recent applications include structured low-rank approximation, for which algebraic techniques are developed in [10].

The bulk of this note is devoted to counting the number of critical points on $G$ of the function $d_u$, in the general framework of the *Euclidean distance degree* (ED degree) [2]. In Section 2 we specialise this framework to matrix groups. In Section 3 we discuss matrix groups preserving the inner product. In particular, we derive a conjecturally sharp upper bound on the ED degree of a compact torus preserving the inner product, revisit the classical cases of orthogonal and unitary groups, and express the ED degree as the algebraic degree of a certain matrix multiplication map. Then in Section 4 we discuss two classes of groups not preserving the inner product: the special linear groups, consisting of all determinant-one matrices, and the symplectic groups. For the former we determine the ED degree explicitly. We conclude the note with a conjecture for the latter.
2. The ED degree and critical equations

As is common in the framework of ED degree computations, we aim to count the critical points of the function $d_u$ over the complex numbers, as follows. A point $x \in G$ is critical for $d_u$ if and only if $(u - x) \perp a$ for all $a$ in the tangent space $T_x G \subseteq \text{End}(V)$. As $G$ is an algebraic group, we have $T_x G = x \cdot T_I G = xg$, where $T_I G = g$ is the tangent space of $G$ at the identity element $I$, i.e., the Lie algebra of $G$. So criticality means that

$$0 = \langle u - x, xb \rangle = \langle x^t(u - x), b \rangle$$

for all $b \in g$. Hence, given $u$, we look for the solutions of the critical equations

$$x^t(u - x) \perp g \text{ subject to } x \in G. \tag{1}$$

The number of solutions $x \in G$ to (1) can vary with $u \in \text{End}(V)$. But if we set $V_C := \mathbb{C} \otimes \mathbb{R} V$, let $G_C \subseteq \text{GL}_C(V_C) \subseteq \text{End}_C(V_C)$ be the set of complex points of the algebraic group $G$, and extend $\langle \cdot, \cdot \rangle$ to a symmetric $\mathbb{C}$-linear form on $\text{End}_C(V_C)$ (and not to a Hermitian form!), then the number of solutions to (1) will not depend on $u$, provided that $u$ is sufficiently general. Following [2], we call this number the Euclidean distance degree (ED degree for short) of $G$. This number gives an algebraic measure for the complexity of writing down the solution to the minimisation Problem 1.1. We now distinguish two classes of groups: those that preserve the inner product $(\cdot | \cdot)$ and those that do not.

3. Groups preserving the inner product

Assume that $(xv|zw) = (v|w)$ for all $x \in G$, so that $G$ is a subgroup of the orthogonal group of $(\cdot | \cdot)$. Then all elements $x \in G$ satisfy $x^t x = I$ and hence $\|x\|^2 = n$, that is, $G$ is contained in the sphere in $\text{End}(V)$ of radius $\sqrt{n}$. As a consequence, $g$ is contained in the tangent space at $I$ to that sphere, which equals $I^\perp$. Hence the critical equations simplify to

$$x^t u \perp g \text{ subject to } x \in G. \tag{2}$$

In other words, given the data matrix $u$ we seek to find all $x \in G$ that satisfy a system of linear homogeneous equations. Alternatively, we can write the critical equations as $u \in x \cdot g^\perp$. This proves the following proposition.

Proposition 3.1. If $G$ preserves the inner product $(\cdot | \cdot)$, then the ED degree of $G$ equals the degree of the multiplication map $G_C \times g_C^\perp \rightarrow \text{End}_C(V_C)$, $(x, s) \mapsto x \cdot s$. For general real $u$, among the real pairs $(x, s)$ satisfying $xs = u$, the one with the largest value of $\text{tr}(s)$ is the one that minimises $d_u(x)$. 
In other words, the ED degree counts the number of ways in which a general matrix $u$ can be decomposed as a product of a matrix in $G_C$ and a matrix in $g_C^\perp$. The last statement follows from
\[ d_u(x) = \|u - x\|^2 = \text{tr}(u^t u) - 2\text{tr}(u^t x) + n = \text{tr}(u^t u) - 2\text{tr}(s) + n, \]
in which only the second term is not constant.

### 3.1. Orthogonal groups

If $G$ is the full orthogonal group of $(\cdot,\cdot)$, then $g$ is the space of skew-symmetric matrices. Hence the decomposition of Proposition 3.1 boils down to the classical polar decomposition, where one writes a general matrix $u$ as $u = xs$ with $x$ orthogonal and $s$ symmetric. If $(x, s)$ is a solution, then
\[ s^2 = s^t s = (x^{-1} u)^t (x^{-1} u) = u^t u, \]
a quadratic equation for $s$ that has $2^n$ real solutions for general real $u$. Indeed, write
\[ u^t u = y \text{diag}(\lambda_1, \ldots, \lambda_n) y^t \]
where $y$ is orthogonal and the $\lambda_i$ are the eigenvalues of $u^t u$, which are positive and distinct for general $u$. Then the solutions $s$ of the quadratic equation are precisely the matrices of the form
\[ s = y \text{diag}(\pm \sqrt{\lambda_1}, \ldots, \pm \sqrt{\lambda_n}) y^t, \]
and for each of these the matrix $x = us^{-1}$ is orthogonal, since
\[ x^t x = s^{-1} (u^t u) s^{-1} = I. \]

We summarise our findings in the following, well-known theorem (see, e.g., [8]).

**Theorem 3.2.** The ED degree of the orthogonal group of the $n$-dimensional inner product space $V$ is $2^n$. Moreover, for general real $u$, all $2^n$ critical points of the squared distance function $d_u$ are real. The critical point that minimises $d_u$ is
\[ x = us^{-1} \text{ with } s = y \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) y^t, \]
where $y$ is an orthogonal matrix of eigenvectors of $u^t u$, and the $\lambda_i$ are the corresponding eigenvalues.

The last statement follows since $\text{tr}(s)$ is maximised by taking the positive square roots of the $\lambda_i$. 
Remark 3.3. The nearest orthogonal matrix to a general real $u$ has determinant $1$ if $\det(u) > 0$ and determinant $-1$ if $\det(u) < 0$. Half of the critical points of $d_u$ on the orthogonal group have determinant $1$, and half of the points have determinant $-1$, that is, the ED degree of the special orthogonal group is $2^{n-1}$. To find the special orthogonal matrix nearest to a matrix $u$ with $\det(u) < 0$, one replaces the smallest $\sqrt{\lambda_i}$ in the construction above by $-\sqrt{\lambda_i}$.

3.2. Unitary groups

Assume that $n = 2m$ and let $V$ be an $m$-dimensional complex vector space, regarded as an $n$-dimensional real vector space. Let $h$ be a non-degenerate, positive definite, Hermitian form on $V$, where we follow the convention that $h(cv, w) = ch(v, w) = h(v, cw)$. Define $(v|w) := \text{Re } h(v, w)$. Then $(.,.)$ is a positive definite inner product on $V$ regarded as a real vector space, and the norm on $V$ coming from $(.,.)$ is the same as that coming from $h$. Let $G$ be the unitary group of $h$, which consists of all $x : V \rightarrow V$ that are not only $\mathbb{R}$-linear but in fact $\mathbb{C}$-linear and that moreover preserve $h$. Such maps $x$ also preserve $(.,.)$, so we are in the situation of this section. The converse is also true: if $x$ is $\mathbb{C}$-linear and preserves $(.,.)$, then

$$\text{Im } h(v, w) = -\text{Re } h(iv, w) = -(iv|w) = -(x(iv)|x(w)) = -(ix(v)|x(w))$$

$$= -\text{Re } h(ix(v), x(w)) = \text{Im } h(x(v), x(w)),$$

so that $x$ preserves both $\text{Re } h$ and $\text{Im } h$ and hence $h$.

Note that $\text{End}_\mathbb{R}(V)$ has dimension $n^2 = 4m^2$, but $G$ is contained in the real subspace $\text{End}_\mathbb{C}(V) \subseteq \text{End}_\mathbb{R}(V)$, which has real dimension $2m^2$. For a general data matrix $u \in \text{End}_\mathbb{R}(V)$, the critical points of $d_u$ on $G$ will be the same as the critical points of $d_{u'}$ where $u'$ is the orthogonal projection of $u$ in $\text{End}_\mathbb{C}(V)$. Hence in what follows we may assume that $u$ already lies in $\text{End}_\mathbb{C}(V)$, and we focus our attention entirely on the space $\text{End}_\mathbb{C}(V)$. For a linear map $u$ in the latter space, we write $u^*$ for the $\mathbb{C}$-linear map determined by $h(v, uw) = h(u^*v, w)$ for all $v, w \in V$. This map will also have the property that $(v|uw) = (u^*v|w)$, i.e., $u^*$ coincides with our transpose $u^t$ relative to $(.,.)$. In what follows we follow the convention to write $u^*$.

The Lie algebra $\mathfrak{g}$ consists of all skew-Hermitian linear maps in $\text{End}_\mathbb{C}(V)$, and its orthogonal complement $\mathfrak{g}^\perp$ inside $\text{End}_\mathbb{C}(V)$ therefore consists of all Hermitian $\mathbb{C}$-linear maps. Again, the decomposition of Proposition 3.1 boils down to the polar decomposition. Here is the result (see, e.g., [8]).

Theorem 3.4. The ED degree of the unitary group of a non-degenerate Hermitian form $(.,.)$ on an $m$-dimensional complex vector space $V$ equals $2^m$. For a general data point $u \in \text{End}_\mathbb{C}(V)$ the critical points are computed as follows. First write

$$u^*u = y \text{diag}(\lambda_1, \ldots, \lambda_m)y^*,$$
where $y$ is a unitary map and the $\lambda_i \in \mathbb{R}_{\geq 0}$ are the eigenvalues of $u^*u$, then pick any of the $2^m$ square roots

$$s = y \operatorname{diag}(\pm \sqrt{\lambda_1}, \ldots, \pm \sqrt{\lambda_m}) y^*$$

of $u^*u$, and finally set $x := us^{-1}$. Choosing all square roots positive leads to the nearest unitary matrix to $u$.

There is a slight subtlety in the last statement: to find the nearest matrix, we have to maximise $\operatorname{tr}_{\mathbb{R}}(s)$, where we see $s$ as an element of $\operatorname{End}_{\mathbb{R}}(V)$, while the sum of the $m$ eigenvalues $\pm \sqrt{\lambda_i}$ equals $\operatorname{tr}_{\mathbb{C}}(s)$. But in fact, for any $z \in \operatorname{End}_{\mathbb{C}}(V)$, we have $\operatorname{tr}_{\mathbb{R}}(z) = \operatorname{tr}_{\mathbb{C}}(z) + \operatorname{tr}_{\mathbb{C}}(z)$, so that $\operatorname{tr}_{\mathbb{R}}(s) = 2\operatorname{tr}_{\mathbb{C}}(s)$.

### 3.3. Compact tori

Assume that the real algebraic group $G$ is a compact torus, i.e., that it is an abelian compact Lie group and abstractly isomorphic to a power $(S^1)^m$ of circle groups. We continue to assume that $G$ preserves the inner product $(.,.)$. We will bound the ED degree of $G$, and unlike in the previous two examples we will make extensive use of the complexification $G_{\mathbb{C}}$ of $G$.

Indeed, $G_{\mathbb{C}}$ is now isomorphic to an algebraic torus $T := (\mathbb{C}^*)^m$. Let $X(T)$ be the set of all characters of $T$, i.e., of all algebraic group homomorphisms $\chi : T \to \mathbb{C}^*$. These are all of the form $\chi : (t_1, \ldots, t_m) \mapsto t_1^{a_1} \cdots t_m^{a_m}$, where $a_1, \ldots, a_m \in \mathbb{Z}$; and this gives an isomorphism $X(T) \cong \mathbb{Z}^m$ of finitely generated Abelian groups: $X(T)$ with respect to multiplication and $\mathbb{Z}^m$ with respect to addition. We will identify these groups, and accordingly write $t^x$ instead of $\chi(t)$ and write $+\,$ for the operation in $X(T)$, so $t^{x+y} = t^x \cdot t^y$ and $t^x = t_1^{a_1} \cdots t_m^{a_m}$ if $\chi = (a_1, \ldots, a_m)$.

The isomorphism $T \ni G_{\mathbb{C}} \subseteq \operatorname{GL}_{\mathbb{C}}(V_{\mathbb{C}})$ gives $V_{\mathbb{C}}$ the structure of a $T$-representation. As such, it splits as a direct sum of one-dimensional $T$-representations:

$$V_{\mathbb{C}} = \bigoplus_{\chi \in X(T)} V_\chi,$$

where, for any $\chi \in X(T)$, we let $V_\chi$ be the corresponding eigenspace (or weight space), defined by

$$V_\chi := \{ v \in V_{\mathbb{C}} \mid \forall t \in T : tv = t^x v \}.$$

Here $tv$ is the vector obtained from $v$ by acting with $t$, and $t^x v$ equals $v$ stretched by the scalar $t^x$. Only finitely many weight spaces are non-zero, and their dimensions add up to $n$. Let $X_V \subseteq X(T) = \mathbb{Z}^m$ denote the set of characters $\chi$ for which $V_\chi$ is non-zero. The fact that the map $T \to G$ is an isomorphism implies that the lattice in $\mathbb{Z}^m$ generated by $X_V$ has full rank. We will prove the following result.
Theorem 3.5. The ED degree of a compact torus $G \cong (S^1)^m$ that preserves the inner product on $V$ depends only on $X_V$, and is independent of the dimensions of the weight spaces $\dim V_{\chi}, \chi \in X_V$. Moreover, it is bounded from above by the normalised volume of the convex hull $\Delta$ of $X_V \subseteq \mathbb{Z}^m$. Here the normalisation is such that the simplex spanned by 0 and the standard basis vectors has volume one.

Proof. Observe that the (complexified) bilinear form $(\cdot|\cdot)$ on $V_\mathbb{C}$ is preserved by $T$. For $v \in V_{\chi}$ and $w \in V_{\chi}$ and $t \in T$ we therefore have

$$t^{\chi + \lambda} \cdot (v|w) = (t^\chi v|t^\lambda w) = (tv|tw) = (v|w).$$

Hence, if $v$ and $w$ are not perpendicular, then $\chi + \lambda$ is the trivial character sending all of $T$ to 1 (so $\chi + \lambda = 0 \in \mathbb{Z}^m$). In other words, $(\cdot|\cdot)$ must pair each $V_{\chi}$ non-degenerately with the corresponding space $V_{-\chi}$, and is zero on all pairs $V_{\chi} \times V_{\chi}$ with $\lambda \neq -\chi$. In particular, we have $\dim V_{\chi} = \dim V_{-\chi}$ for all $\chi \in X(T)$, and $X_V$ is centrally symmetric.

We can now choose a basis $v_1, \ldots, v_n$ of $V_\mathbb{C}$ consisting of $T$-eigenvectors such that $(v_i|v_j) = \delta_{j,n+1-i}$. Let $\chi_i \in X_V$ be the character of $v_i$, i.e., we have $tv_i = t^{\chi_i} v_i$ for all $t \in T$. There will be repetitions among the $\chi_i$ if some of the weight spaces have dimensions greater than 1; and by the above we have $\chi_i = -\chi_{n+1-i}$. Relative to this basis we have

$$G_\mathbb{C} = \{ \text{diag}(t^{\chi_1}, \ldots, t^{\chi_n}) \mid t \in (\mathbb{C}^*)^m \} \quad \text{and} \quad g_\mathbb{C} = \{ \text{diag}(w \cdot \chi_1, \ldots, w \cdot \chi_n) \mid w \in \mathbb{C}^m \}$$

where $\chi_i \cdot w$ is the ordinary dot product of $w \in \mathbb{C}^m$ with $\chi_i \in \mathbb{Z}^m$. The bilinear form $\langle \cdot, \cdot \rangle$ on matrices takes the form

$$\langle E_{ij}, E_{kl} \rangle = \delta_{k,n+1-i} \delta_{l,n+1-j},$$

where $E_{ij}$ is the map whose matrix relative to the basis $v_1, \ldots, v_n$ has a 1 at position $(i, j)$ and zeroes elsewhere. Correspondingly, transposition of matrices is reflection in the anti-diagonal. Hence, for a data matrix $u \in \text{End}(V)$, the critical equations (2) translate into the following equations for the pre-image $t \in T$ of $x \in G_\mathbb{C}$:

$$(t^{\chi_n} u_{11}) (w \cdot \chi_n) + \cdots + (t^{\chi_1} u_{nn}) (w \cdot \chi_1) = 0 \quad \text{for all } w \in \mathbb{C}^m. \quad (3)$$

The ED degree of $G$ is the number of solutions $t \in (\mathbb{C}^*)^m$ to this system of $m$ Laurent-polynomial equations for general values of the $u_{ij}$. Grouping the indices $i$ for which the $\chi_i$ are equal and adding up the corresponding $u_{n+1-i,n+1-i}$ we find that the cardinality of the solution set is, indeed, independent of the dimensions $\dim V_{\chi_i}$, the first statement in the theorem.

Letting $w$ run over a basis of $\mathbb{C}^m$, we obtain a system of $m$ Laurent polynomial equations for $t \in (\mathbb{C}^*)^m$ with fixed support $X_V$. The Bernstein–Kushnirenko–Khovanskii theorem [1, Theorems A, B] ensures that the number of isolated solutions to this system
is less than or equal to the normalised volume of $\Delta$. This proves that the ED degree does not exceed that bound, and hence the second part of the theorem. \hfill \square

**Example 3.6.** Let the group $SO_2 \cong S^1$ act on $V := (\mathbb{R}^2)^\otimes d$ via $g(v_1 \otimes \cdots \otimes v_d) = (gv_1) \otimes \cdots \otimes (gv_d)$, and let $(.,.)$ be the inner product on $V$ induced from that on $\mathbb{R}^2$. Let $G \subseteq GL(V)$ be the image of $SO_2$. Then the ED degree of $G$ equals $2d$ if $d$ is odd, and $d$ if $d$ is even. This is computed as follows. All elements of $SO_2$ are diagonalised over $\mathbb{C}$ by the choice of basis $f_1 := e_1 + ie_2$ and $f_{-1} := e_1 - ie_2$ of $\mathbb{C}^2$. The complexification of $SO_2$ is the image of the one-dimensional torus $T = \mathbb{C}^\times$ in its action on $\mathbb{C}^2$ via $(t, f_i) \mapsto t^i f_i$. The complexification $G_{\mathbb{C}}$ is the image of $T$ in its induced action on $\mathbb{C}^\otimes d$ via

$$(t, f_{i_1} \otimes \cdots \otimes f_{i_d}) \mapsto t^{\sum_i i_j} f_{i_1} \otimes \cdots \otimes f_{i_d} \quad \text{for all } i_1, \ldots, i_d \in \{\pm 1\}.$$ 

We have $X_V = \{-d, -d+2, \ldots, d\} \subseteq \mathbb{Z}^1$, where $\mathbb{Z}^1$ is the character lattice of $T$. If $d$ is odd, then the map $T \to G_{\mathbb{C}}$ is one-to-one, and $\Delta$ is a line segment of length $2d$. If $d$ is even, then that map is two-to-one (since all exponents $\sum_j i_j$ above are then even), and the character lattice of $G_{\mathbb{C}}$ is then $2\mathbb{Z}^1 \subseteq \mathbb{Z}^1$. In this case, the normalised volume of $\Delta$ is $d$. The theorem says that the ED degree is at most $2d$ for odd $d$ and at most $d$ for even $d$. In this case equality holds: the system (3) reduces to

$$u'_d t^{-d} + \cdots + u'_d t^d = 0,$$

where the $u'_j$ are sums of $u_{ii}$ corresponding to the same character; and for general values of the $u'_j$ this equation has exactly $2d$ solutions for $t$, and exactly $d$ solutions for $t^2$ when $d$ is even.

We do not know if the system (3), for a general choice of the $u_{ii}$, is always sufficiently general for the BKK-bound to hold with equality; but here is a class of examples where it is.

**Proposition 3.7.** If $\Delta$ is a simplicial polytope, i.e., if all codimension-one faces of $\Delta$ are simplices of dimension $m - 1$, then the ED degree of the torus $G$ equals the normalised volume of $\Delta$.

In particular, this implies that for all embeddings of a two-dimensional torus in some $GL(V)$ where it preserves the inner product, the ED degree is equal to the normalised surface of the corresponding lattice polygon.

**Proof.** It suffices to point out a single $u$ for which the system (3) has the expected number of (distinct, isolated) solutions for $t$. Let $I \subseteq [n]$ be such that the set $\{\chi_i \mid i \in I\}$ is exactly the set of extremal vertices of the polytope $\Delta$ without repetition. We take $u_{ii} = 1$ if $i \in S$ and $u_{ii} = 0$ otherwise.
Theorem B in [1] gives a sufficient condition for the number of roots of a system like (3) to have the expected number of distinct solutions in T. Translated to our setting, let \( \alpha \) be a non-zero vector in \( \mathbb{Q}^m \) and let \( \alpha \in \mathbb{Q} \) be the minimum of \( \alpha \cdot \chi_i \) over all \( i = 1, \ldots, n \) (or, equivalently, over all \( i \in I \)). Then the condition is that the system

\[
\sum_{i \in I : \alpha \cdot \chi_i = a} t^{x_i} (w \cdot \chi_i) = 0 \quad \text{for all } w \in \mathbb{C}^m
\]

does not have a solution \( t \in T \) (for any choice of \( \alpha \neq 0 \) with corresponding minimum \( a \)). Here the sum ranges over the vertices of the proper face of \( \Delta \) where the linear form \( \alpha \) is minimised, and by assumption this face is a simplex of some dimension \( d < m \). In particular, the \( \chi_i \) appearing in the sum are affinely independent. Moreover, the affine space that they span does not contain 0, since 0 is an interior point of \( \Delta \) (recall that \( \Delta \) is full-dimensional and \( \Delta = -\Delta \)). Hence for any fixed \( j \in I \) there exists a \( w_j \in \mathbb{C}^m \) such that \( w_j \cdot \chi_i = \delta_{ij} \) for all \( i \in I \). The equation corresponding to \( w_j \) says that \( t^{x_j} = 0 \), which indeed does not have a solution in \( T \). \( \square \)

**Problem 3.8.** Is the ED degree of a torus \((S^1)^m \cong G \subseteq GL(V)\) that preserves the inner product on \( V \) always equal to the normalised volume of the convex hull of the character set \( X_V \subseteq \mathbb{Z}^m \) appearing in \( V_{\mathbb{C}} \)?

### 3.4. Other reductive groups preserving the form

After this more or less satisfactory result for tori, it is tempting to hope that the ED degree of any group \( G \) preserving the bilinear form should be expressible in terms of the highest weights appearing in the complexification \( V_{\mathbb{C}} \) as a \( G_{\mathbb{C}} \)-module. After all, \( G_{\mathbb{C}} \) is then a reductive group and much is known about its representations. By Proposition 3.1 an upper bound to the ED degree is the degree of a general orbit of \( G_{\mathbb{C}} \) in its action on \( \text{End}_{\mathbb{C}}(V_{\mathbb{C}}) \) by left multiplication. A formula for this degree is known by [7]; see also [4, Theorem 8]. However, the space \( g^\perp \) is in general not sufficiently general for that upper bound to be tight.

To test said hope, we have experimented with \( G_{\mathbb{C}} \) equal to the image of \( \text{SL}_2(\mathbb{C}) \) in its irreducible representation \( V_{\mathbb{C}} \) of highest weight \( m \) with \( m \) even. Thus \( V_{\mathbb{C}} \) is the \( m \)-th symmetric power \( S^m \mathbb{C}^2 \) where \( \mathbb{C}^2 \) is the standard representation of \( G_{\mathbb{C}} \). Since \( m \) is even, the bilinear form on \( S^m \mathbb{C}^2 \) induced by the symplectic \( \text{SL}_2(\mathbb{C}) \)-invariant form on \( \mathbb{C}^2 \) is, indeed, an invariant symmetric bilinear form. In this case, the formula in [4, Theorem 8] evaluates to \( m^3 \). Below is a small table of ED degrees, in which we could not yet find a pattern.

<table>
<thead>
<tr>
<th>( m )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>ED-degree of ( \text{SL}_2 ) on ( S^m \mathbb{C}^2 )</td>
<td>1</td>
<td>4</td>
<td>40</td>
<td>156</td>
<td>400</td>
</tr>
<tr>
<td>( m^3 )</td>
<td>0</td>
<td>8</td>
<td>64</td>
<td>216</td>
<td>512</td>
</tr>
</tbody>
</table>
Note that the formula $m^3$ does not apply for $m = 0$ since the representation does not have a finite kernel. Note also that the 4 is consistent with Theorem 3.2 and the fact that the map $\text{SL}_2(\mathbb{C}) \to \text{SL}(S^2\mathbb{C}^2)$ has image $G_\mathbb{C}$ equal to $\text{SO}_3(\mathbb{C})$.

**Problem 3.9.** Determine the ED-degree of $\text{SL}_2(\mathbb{C})$ on $S^m\mathbb{C}^2$ with $m$ even. More generally, find a formula for that ED degree for any group $G_\mathbb{C}$ on a representation $V_\mathbb{C}$ with an invariant symmetric bilinear form.

4. Groups not preserving the inner product

If $G$ does not preserve the inner product $(\cdot, \cdot)$, then it is much harder to compute (or even estimate) the ED degree of $G$. The following two classes of groups illustrate this.

4.1. The special linear groups

Consider the group

$$G := \text{SL}^\pm(V) = \{ x \in \text{End}(V) \mid \det x = \pm 1 \}.$$ 

Its Lie algebra $\mathfrak{g}$ is the space of matrices with trace equal to zero. Given a real data matrix $u$, the critical equations for the nearest $x \in G$ become

$$\langle x^t(u - x), a \rangle = 0 \quad \forall a \in \text{End}(V), \quad \text{tr}(a) = 0 \quad \text{subject to } \det x = 1.$$ 

Since this equation must hold for all $a$ with $\text{tr}(a) = 0$ we see that $x^t(u - x)$ must be of the form $cI$ for some $c \in \mathbb{R}$, so that $u = cx^{-t} + x$. From this expression for $u$ we find

$$u^t u = (cx^{-1} + x^t)(cx^{-t} + x) = c^2 (x^t x)^{-1} + 2cI + x^t x.$$ 

Hence $s := x^t x$ must be a symmetric matrix of determinant 1 satisfying

$$u^t u = c^2 s^{-1} + 2cI + s = s^{-1} (cI + s)^2. \quad (4)$$

Conversely, if $s$ is a symmetric determinant-1 matrix satisfying this equation, then we can set $x := u^{-t}(cI + s)$. This matrix then satisfies

$$x^t x = (cI + s)u^{-1}u^{-t}(cI + s) = (cI + s)^2(u^t u)^{-1} = s,$$

where we have used that $s$ commutes with $u^t u$. This means that $\det x = \pm 1$. We also find $x^t(u - x) = (cI + s) - s = cI$. Thus to compute the ED degree of $G$ it suffices to count the symmetric matrices $s$ solving (4).

Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $u^t u$. Since $u$ is general, these are all distinct, and (4) forces $s$ to be simultaneously diagonalisable with $u^t u$. Thus we need only find the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $s$, where $\lambda_i$ corresponds to $\mu_i$. Eq. (4) translates into

$$\mu_i = c^2 \frac{1}{\lambda_i} + 2c + \lambda_i, \quad i = 1, \ldots, n.$$
Multiplying by $\lambda_i$ and adding the condition $\lambda_1 \cdots \lambda_n = 1$, the system to solve becomes

\[
\begin{cases}
  f_i := c^2 + (2c - \mu_i)\lambda_i + \lambda_i^2 = 0, & i = 1, \ldots, n \\
  \lambda_1 \cdots \lambda_n = 1.
\end{cases}
\]  

(5)

Substituting $\lambda_n := (\lambda_1 \cdots \lambda_{n-1})^{-1}$ into $f_n$, we find an ideal $I$ generated by $n$ equations $f_1, \ldots, f_n$ in the ring $\mathbb{R}[\lambda_1^{\pm 1}, \ldots, \lambda_{n-1}^{\pm 1}, \mu_1, \ldots, \mu_n, c]$. This ideal is prime, since the equations can be read as defining the graph of a map from the Cartesian product of an $(n - 1)$-dimensional torus with coordinates $\lambda_1, \ldots, \lambda_{n-1}$ with the affine line with coordinate $c$ to the affine space with coordinates $\mu_1, \ldots, \mu_n$. The ED degree is the degree of this map. To determine it, we determine the intersection $I \cap \mathbb{R}[\mu_1, \ldots, \mu_n, c]$. For this, we eliminate the $\lambda_i$ successively, as follows. For $i = 0, \ldots, n - 1$ define $\lambda_{(i)} := \lambda_1 \cdots \lambda_i$. Define

\[
R_n := c^2 \lambda_{(n-1)}^2 + (2c - \mu_n)\lambda_{(n-1)} + 1,
\]

which is just $f_n$ multiplied by $\lambda_{(n-1)}^2$. Now recursively define, for $i = n - 1, \ldots, 1$,

\[
R_i := \text{Res}_{\lambda_i}(R_{i+1}, f_i),
\]

where $\text{Res}$ is the resultant given by the determinant of a suitable Sylvester matrix. The first two are as follows:

\[
R_{n-1} = \det \begin{bmatrix}
  c^2 \lambda_{(n-2)}^2 & (2c - \mu_n)\lambda_{(n-2)} & 1 & 0 \\
  0 & c^2 \lambda_{(n-2)}^2 & (2c - \mu_n)\lambda_{(n-2)} & 1 \\
  1 & (2c - \mu_{n-1}) & c^2 & 0 \\
  0 & 1 & (2c - \mu_{n-1}) & c^2 \\
\end{bmatrix} = c^8 \lambda_{(n-2)}^4 + \cdots + 1,
\]

where the dots stand for terms of degrees strictly between 0 and 4 in $\lambda_{(n-2)}$; and similarly

\[
R_{n-2} = \det \begin{bmatrix}
  c^8 \lambda_{(n-3)}^4 & \cdot & \cdot & \cdot & 1 & 0 \\
  0 & c^8 \lambda_{(n-3)}^4 & \cdot & \cdot & \cdot & 1 \\
  1 & (2c - \mu_{n-2}) & c^2 & 0 & 0 & 0 \\
  0 & 1 & (2c - \mu_{n-2}) & c^2 & 0 & 0 \\
  0 & 0 & 1 & (2c - \mu_{n-2}) & c^2 & 0 \\
  0 & 0 & 0 & 1 & (2c - \mu_{n-2}) & c^2 \\
\end{bmatrix} = c^{24} \lambda_{(n-3)}^8 + \cdots + 1.
\]

By induction, we find $R_i = c^{m_i} \lambda_{(i-1)}^{2^{n-i+1}} + \cdots + 1$ where the remaining terms have $\lambda_{(i-1)}$-degree strictly between zero and $2^{n-i+1}$. The exponents $m_i$ satisfy the recursion
Remark

Theorem

by kernels

all eliminate

where

Theorem 4.1 as $x$ above, $1$ and $-1$.

The degree-$n$ equation $R_1$ in $c$ lifts to as many distinct solutions to the system (5). Thus we have proved the following theorem.

**Theorem 4.1.** The ED degree of $\text{SL}^\pm(V)$ equals $n2^n$, and the ED degree of $\text{SL}(V)$ equals $n2^{n-1}$.

The last statement follows from the fact that there exists an orthogonal transformation of $\text{End}(V)$ that takes the matrices with determinant $1$ into the matrices with determinant $-1$ and vice versa (e.g., in matrix terms, multiplying the first column by $-1$). Hence the two connected components of $\text{SL}^\pm(V)$ have the same ED degree.

**Remark 4.2.** As a referee pointed out, by a computation similar to the one in the proof above, one can show that the ED degree of the hypersurface $H$ in $\mathbb{R}^n$ with equation $x_1 \cdots x_n = 1$, relative to the standard inner product, equals $n2^{n-1}$. This is related to Theorem 4.1 as follows: if the data matrix $u$ is a general diagonal matrix $\text{diag}(u_1, \ldots, u_n)$, then all critical points of $d_u$ on $\text{SL}(V_\mathbb{C})$ are also diagonal, and they correspond bijectively to the critical points of the distance function to $(u_1, \ldots, u_n)$ on $H_\mathbb{C}$. A similar statement holds for orthogonally diagonalisable, i.e., symmetric matrices $u$. But for general matrices $u$, the critical points $x$ are not simultaneously diagonalisable with $u$.

The proof gives rise to the following algorithm for finding the nearest matrix in $G$ to a given real data matrix $u$: first diagonalise $u^t u$ as

$$ u^t u = T \text{diag}(\mu_1, \ldots, \mu_n) T^t, $$

where $T$ is a real orthogonal transformation and the $\mu_i$ are positive. Then successively eliminate $\lambda_n, \ldots, \lambda_1$ as above, using Sylvester matrices for the resultants $R_i$. Compute all real roots $c$ of $R_1$. For each of these, compute the corresponding $\lambda_1, \ldots, \lambda_n$ from the kernels of the Sylvester matrices: since the data is sufficiently general, each of those kernels will be one-dimensional and spanned by a vector of powers of the relevant $\lambda_i$. Since all $\lambda_i$ are $\mathbb{Q}$-rational functions of $\mu_1, \ldots, \mu_n, c$, the $\lambda_i$ are, indeed, real. Then construct $s$ by

$$ s = T \text{diag}(\lambda_1, \ldots, \lambda_n) T^t. $$
Finally, construct $x$ by

$$ x = u^{-1}(cI + s). $$

We have already verified that $x$ satisfies $x^t x = s$, so that the $\lambda_i$ are necessarily positive, but this can also be seen directly from (5).

It would be useful to know in advance which real root $c$ corresponds to the nearest matrix $x$. Experiments with the algorithm above suggests that it may be the real root that is smallest in absolute value.

**Problem 4.3.** Is it true that the real root $c$ of $R_1$ of smallest absolute value gives rise to the matrix $x \in \text{SL}^\pm(V)$ that is nearest to $u$?

5. The symplectic groups

As a final case in our quest for ED degree of real algebraic groups we fix an even $n = 2m \in \mathbb{N}$ and study the symplectic group

$$ \text{Sp}_n := \{ x \in \mathbb{R}^{n \times n} \mid x^t J x = J \}, $$

where $J$ has the block structure

$$ J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}. $$

In the case of $\text{SL}(V)$ the ED degree did not depend on the choice of an inner product on $V$, because $\text{SL}(V)$ acts transitively on inner products (up to positive scalars). But for $\text{Sp}_n$ the ED degree may well depend on the relative position of the symplectic form given by $J$ and the inner product. A general study of orbits of pairs of a symplectic and a symmetric form is performed in [9,3], based on classical work by Kronecker. We choose the standard inner product. This choice is rather special in the sense that the complexified group $\text{Sp}_n(\mathbb{C})$ intersects the complexified group $O_n(\mathbb{C})$ in a large group, containing a copy of the group $\text{GL}_m(\mathbb{C})$. This is not immediately clear from the chosen coordinates, but relative to the basis

$$ v_1 := \frac{e_1 + ie_{m+1}}{\sqrt{2}}, \quad \ldots, \quad v_m := \frac{e_m + ie_{2m}}{\sqrt{2}}, $$

$$ v_{m+1} := \frac{ie_1 + e_{m+1}}{\sqrt{2}}, \quad \ldots, \quad v_{2m} := \frac{ie_m + e_m}{\sqrt{2}} $$

of $\mathbb{C}^n$ the symplectic form still has Gram matrix $J$, while the standard symmetric bilinear form on $\mathbb{C}^n$ has Gram matrix

$$ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. $$

(6)
Now all complex matrices that relative to the basis of the \( v_i \) have the block structure

\[
\begin{bmatrix}
g & 0 \\
0 & g^{-T}
\end{bmatrix}
\]

lie both in \( \text{Sp}_n(\mathbb{C}) \) and in \( \text{O}_n(\mathbb{C}) \). This shows that we could have chosen the symmetric form with Gram matrix (a scalar multiple of) that in (6), without changing the ED degree.

We have implemented Eqs. (1) and computed the ED degree for very small values of \( n \). The resulting table is as follows:

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>ED degree of ( \text{Sp}_n )</td>
<td>4</td>
<td>24</td>
<td>544</td>
</tr>
</tbody>
</table>

The pattern might be that the answer is \( 2^{m^2} + 2^{2m-1} \), but we do not know how to prove this.

**Problem 5.1.** Determine the ED degree of \( \text{Sp}_n \) for general even \( n \).

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**References**


