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by

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Analysis of an interior penalty discontinuous Galerkin scheme for two phase flow in porous media with dynamic capillarity effects

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Abstract We present an interior penalty discontinuous Galerkin scheme for two-phase flow with dynamic capillary pressure effects. The mass-conservation laws are approximated directly, without the introduction of a global pressure. We prove existence and convergence of the scheme and obtain error-estimates for sufficiently smooth data.

1 Introduction

Flow and transport processes in porous media are of high interest in many different fields of application, for example, modeling of CO₂-storage [30], solute transport [32], designing diapers [16], filters, etc. It is not always possible nor feasible to use experiments to understand the behaviour of these processes. To overcome this, modeling, mathematical, and numerical tools become necessary. The porous media models developed to describe these processes vary greatly in their scale and scope. To solve these models many different simulation and discretization techniques have been developed, for example, finite volume methods [20, 24], finite element methods [33, 11, 12], mixed finite element methods [31, 32, 17], discontinuous Galerkin methods [19, 2, 35, 5]. Since the early phase of development of the porous media models ground water flow and oil-reservoir modeling have been the most important areas of application. As a result, the models for large scale applications are well developed and rigorously studied [6, 24].

However over the last couple of decades more and more interest has been generated in small scale applications which has lead to new modeling and discretization approaches. Where, at larger scales, dynamic effects are negligible and the processes can be considered in a quasi-equilibrium state, at smaller scales, for example in the case of batteries or diapers, significant dynamic phenomena is observed which can not be described with the existing models [15]. To account for these effects efforts have been made to extend the constitutive relationships in different ways. One approach to

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this effect has been to extend the capillary pressure relationship with a dynamic term [23, 22], leading
to some new and interesting results, for example, the travelling waves [37], existence and uniqueness
results [33, 27, 28, 21, 9], and numerical treatment of the model [27, 25, 26].

In our work we explore the application of discontinuous Galerkin (DG) methods to such extended
capillary porous media models. DG methods have grown more popular during the last decades due
to their versatility and easy adaptation to include heterogeneities, parallelization, and hp-adaptivity.
The numerical analysis of the DG methods, well developed and widely used for classical two-
phase flow problems [18], has not, to the best of our knowledge, been implemented for two phase flow
with dynamic capillary effects. In this paper we present the numerical analysis of a primal interior
penalty discontinuous Galerkin scheme for an incompressible two phase flow in porous media with
extended capillary pressure effects.

For analysis it is more common to use a global pressure approach. The advantage of this is that
the a priori pressure-estimate can be separately obtained, which can then be used to estimate the
saturation, as done by Epshteyn and Riviere [18]. The downside is however that the global pressure
and the total velocity are difficult to interpret in a physical setting. In our approach we do not
ref ormulate the mass balance equations in terms of global pressure, but rather use the physical
variables. With this approach the a priori estimate for the pressure cannot be calculated directly due
to the stronger coupling of the mass balance equations. Instead one has to estimate both pressure
and saturation simultaneously, like in the work of Eymard et al. [20] and Schweizer et al. [27].

In [20] and [27] finite volume and finite element approaches have been used respectively. We extend
the analysis to an interior penalty discontinuous Galerkin approximation of the arising dynamic
capillary based two-phase flow model and prove convergence of the scheme.

In Section 2 we present the mathematical model and describe the physical properties as well as
the dynamic capillary effects. In Section 3 we present the discretization scheme as well as the basic
notations and assumptions for the interior penalty discontinuous Galerkin approximation. In Section
4 we give the main results of the paper, like existence of a discrete solution to our scheme as well as
the convergence. Finally in Section 5 we present a numerical example.

2 Mathematical model

Governing equations Given an open bounded polygonal subset Ω ⊂ \mathbb{R}^d with d = 2, 3 and a time
interval [0, T] with T ∈ \mathbb{R}, T > 0, denoting the end time. The mathematical model considered here
is based on the following assumptions:

• A macro-scale non-degenerate two phase fluid system is considered.

• All physical processes are isothermal.

• Fluids under consideration are assumed to be incompressible.

• Gravitational forces are neglected.

• Flow velocities lie well within the Darcy regime.

• Porous matrix is rigid and homogeneous.

Following those, the two phase flow through a rigid porous medium for incompressible fluids like
oil and water can be described mathematically with the following mass conservation laws defined
phase-wise, i.e., for each phase \( \alpha = n, w \),

\[ \partial_t (S_\alpha \rho_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha) = q_\alpha. \] (1)
The indices $\alpha = n, w$ stand for non-wetting and wetting phases respectively. $\phi$ is the porosity of the medium, $\rho_\alpha$ the fluid phase densities, $u_\alpha$ the fluid phase velocities, and $q_\alpha$ the volumetric sources or sinks.

The momentum conservation can be reduced to Darcy’s law

$$u_\alpha = -\lambda_\alpha (S_w) K \nabla p_\alpha.$$  \hspace{1cm} (2)

$p_\alpha$ are the phase pressures, $K$ the intrinsic permeability tensor, and $\lambda_\alpha = \frac{k_{r,\alpha}}{\mu_\alpha}$ the fluid mobility functions with relative permeabilities $k_{r,\alpha}$ and dynamic viscosities $\mu_\alpha$.

To close the system, we relate the phase saturations through the summation relationship as,

$$S_w + S_n = 1$$  \hspace{1cm} (3)

and the phase pressures through the capillary pressure, as,

$$p_n - p_w = p_c(S_w, \partial_t S_w)$$  \hspace{1cm} (4)

We chose the non-wetting phase pressure $p_n$, the capillary pressure $p_c$ and the wetting phase saturation $S_w$ as our primary variables, which, using the assumptions, gives us the following system:

$$-\partial_t S_w \phi - \nabla \cdot (\lambda_n (S_w) K \nabla p_n) = q_n$$  \hspace{1cm} (5)

$$\partial_t S_w \phi - \nabla \cdot (\lambda_w (S_w) K (\nabla p_n - \nabla p_c)) = q_w$$  \hspace{1cm} (5)

Note for readability we still use $p_w = p_n - p_c$, although it is a secondary variable.

**Capillary pressure** The pressure difference across the wetting and non-wetting phase interface is called the capillary pressure. This pressure difference arises due to balancing of cohesive forces between the fluids and the adhesive forces between the fluid-matrix interfaces. On a pore scale, the capillary pressure is inversely related to the radius of the pore-throat. On a macro scale however, the capillary pressure is defined as an average pressure depending on the pore-size distribution, and $S_w$. Several parameterizations exist which relate $p_c$ and $S_w$ using soil specific parameters, for instance, the Brooks-Corey model [7], or van Genuchten model [38], [29], etc. These models however, assume equilibrium or steady state conditions across the fluid-fluid interface. This assumption holds well when the permeability of the porous medium is sufficiently large. For media with smaller permeabilities however, the capillary forces require a finite time to balance and thus, dynamic conditions are expected to prevail. To account for this dynamic effects are proposed in [23] or [22], where

$$p_c = p_{c,eq}(S_w) - \tau \partial_t S_w.$$  \hspace{1cm} (6)

$p_{c,eq}$ is the capillary pressure at equilibrium, and $\tau$ is a material parameter, often called capillary damping coefficient. In our model, we assume $\tau$ to be a positive constant.

**Relative permeabilities** The relative permeability factors for both phases are evaluated using the standard models like Brooks-Corey or van Genuchten models in conjunction with the Burdine theorem [7, 38, 29, 8].
Summary  Using the constitutive relationships for capillary pressure and the relative permeability as well as the before made assumptions, (5) is reformulated to the following set of equations:

\[- \partial_t S_w \phi - \nabla \cdot (\lambda_n(S_w)K(\nabla p_n)) = q_n,\]
\[\partial_t S_w \phi - \nabla \cdot (\lambda_w(S_w)K(\nabla p_n - \nabla p_c)) = q_w,\]  \(7\)
\[p_c = p_{c,eq}(S_w) - \tau \partial_t S_w.\]

To close the system, we assume the following initial and boundary conditions. W.l.o.g we only consider Dirichlet boundary conditions, but of course different boundary conditions are possible. For that let \(\Gamma_D\) denote the boundary condition \(s^D(x) \in H^{\frac{1}{2}}(\Gamma_D), p_c^D(x) \in H^{\frac{1}{2}}(\Gamma_D)\) and \(s^0 \in H^1(\Omega)\). By \(H^{\frac{1}{2}}(\Gamma_D)\) we denote the traces on \(\Gamma_D\) of \(H^1(\Omega)\)-functions:

\[
\text{For all } x \in \Omega \quad S_w(x,0) = s^0(x) \\
\text{For all } x \in \Gamma_D \text{ and } t \in [0,T] \quad S_w(x,t) = s^D(x), \quad p_n(x,t) = p_c^D(x) \quad (9) \]

Furthermore let \(p_c^D = \Pi_{H^{\frac{1}{2}}}(p_{c,eq}(s^D))\) denote the \(H^{\frac{1}{2}}\) projection of the Dirichlet boundary conditions for the capillary pressure onto \(\Gamma_D\). For simplicity we assume constant in time Dirichlet conditions for the saturation. However, it is possible to extend those conditions to time-dependent ones. This leads to an extra dynamic part in the boundary conditions for the capillary pressure, but does not change the forthcoming analysis.

By \(\| \cdot \|_{L^p(\Omega)}\) or \(\| \cdot \|_{W^{k,p}(\Omega)}\) we denote the standard norm on \(L^p(\Omega)\) or \(W^{k,p}(\Omega)\) for \(1 \leq p \leq \infty\). For simplicity denote the \(W^{k,2}(\Omega)\)-norm as \(\| \cdot \|_{\Omega,k}\) and use the notation \(H^k(\Omega)\) for \(W^{k,2}(\Omega)\). Furthermore we use \(H^0(\Omega)\) for the subspace of \(H^1(\Omega)\) with 0 Dirichlet boundary conditions. Furthermore by \(L^p([0,T],W^{k,p}(\Omega))\) we will denote the space of Bochner-integrable function on \([0,T] \times \Omega\). With \(H^1([0,T],L^2(\Omega))\) denote the space of functions in \(L^2(\Omega)\), with a weak time-derivative in \(L^2(\Omega)\).

The weak formulation of (7) reads:

**Problem 1 (Weak formulation).** Find \(s_w - s^D \in H^1([0,T],H^1_0(\Omega)), p_n - p_c^D \in L^2([0,T],H^1_0(\Omega)), p_c - p_c^D \in L^2([0,T],H^1_0(\Omega))\) and \(s_w = s^0\) in \(t=0\), such that for all \(\psi_p \in H^1_0(\Omega)\) and \(\psi_s \in H^1_0(\Omega)\) holds:

\[- \int_\Omega \partial_t S_w \phi \psi_p + \int_\Omega \lambda_n(S_w)K(\nabla p_n) \cdot \nabla \psi_p = \int_\Omega q_n \psi_p,\]
\[\int_\Omega \partial_t S_w \phi \psi_p + \int_\Omega \lambda_w(S_w)K(\nabla p_n - \nabla p_c) \cdot \nabla \psi_p = \int_\Omega q_w \psi_p,\]
\[\int_\Omega p_c \psi_s = \int_\Omega p_{c,eq}(S_w) \psi_s - \int_\Omega \tau \partial_t S_w \psi_s.\]  \(11\)

Existence is proved in [28], [21], [10] and [9].

**3 Numerical scheme**

**Preliminaries** We let \(\mathcal{T}\) be a decomposition into \(N\) non-degenerate elements \(T_i\), and assume it is admissible in the sense of Definition 2.1 in [13]. In this context we will use triangles or quadrilaterals respectively the 3-dimensional counterparts. Let \(\mathcal{F}\) denote the union of all interior and exterior faces \(F_i\) and \(h\) the maximal diameter of the elements. Given \(T_i \in \mathcal{T}\) and \(F_j \in \mathcal{F}\), let

\[F(T_i) := \bigcup_{F_j \subseteq F_i} F_j : F_j \subset T_i\]
and

\[ T(F_i) := \left\{ \bigcup_{T_j \in T} T_j : F_i \subset T_j \right\}. \]

The latter defines all the elements sharing one face. In the conforming case, \( T(F_i) \) consists of exactly two elements. With each face \( F \in \mathcal{F} \) connecting element \( T_i \) and \( T_j \) we associate a normal-vector \( \vec{n} \) directed from \( T_i \) to \( T_j \) \((j > i)\). Let \( \Pi^k(T) \) denote the space of polynomials on \( T \) with degree \( \leq k \). For the saturation we consider the broken Sobolev space with polynomials of order \( k_s \)

\[ V^s_h(\Omega) := \{ v \in L^2(\Omega) : v|_T \in \Pi^{k_s}(T) \text{ for all } T \in \mathcal{T} \} \tag{12} \]

and for the pressure \( p_n \) respectively \( p_c \) with polynomials of order \( k_p \)

\[ V^p_h(\Omega) := \{ v \in L^2(\Omega) : v|_T \in \Pi^{k_p}(T) \text{ for all } T \in \mathcal{T} \}. \tag{13} \]

For an interior face \( F \), connecting elements \( T_i \) and \( T_j \) with \( i < j \), \( \psi^i \in V_h(\Omega) \) with \( \psi^i = (\psi|_{T_i})|_F \) is the trace of \( F \) on the side \( T_j \) and similarly \( \psi^j \). Define the jump \([\cdot]\) and the average \(\{\cdot\}\) over the face as:

\[ [\psi] = (\psi^i - \psi^j) \quad \{\psi\} = \frac{1}{2} (\psi^i + \psi^j). \tag{14} \]

If \( F \) is a boundary face with \( T_i \) being the adjacent element, then

\[ [\psi] = \psi^i \quad \{\psi\} = \psi^i. \tag{15} \]

We use the following norm on the broken Sobolev-Space.

\[ \|v\|_{\Omega, DG}^2 := \sum_{T_i \in \mathcal{T}} \|\nabla v\|_{T_i,0}^2 + \sum_{F_i \in \mathcal{F}} \frac{1}{|F|} \|v\|_{F_i,0} \tag{16} \]

and utilize the following lemma, which can be found in [13]:

**Lemma 3.1.** For all \( q \) such that

\[
1 \leq q \leq \frac{2d}{d-2}, \text{ if } d \geq 3 \\
1 \leq q < \infty, \text{ if } d = 2
\]

there exists a constant \( \hat{C} \) depending on the polynomial degree, mesh-parameters and \( |\Omega| \) such that:

\[ \|v\|_{L^q(\Omega)} \leq \hat{C} \|v\|_{\Omega, DG} \tag{17} \]

Furthermore the following trace inequalities are used, which can be found in [39], [34] and [14]:

**Lemma 3.2.** Let \( \gamma_0 \) and denote the trace operator. There exists a constant \( C_t \) independent of the mesh size \( h \), such that for any \( T \in \mathcal{T} \) with \( F \subset \mathcal{F} \) and for all \( v \in H^k(T) \):

\[ \|\gamma_0 v\|_{0,F} \leq C_t \sqrt{\frac{1}{|F|}} (\|v\|_{0,T} + |F| \|\nabla v\|_{0,T}) \tag{18} \]

For \( v \in \Pi^k(T) \) and \( f(k) \) a function of the polynomial degree \( k \) holds:

\[ \|\gamma_0 v\|_{0,F} \leq C_t \sqrt{\frac{f(k)}{|F|}} \|v\|_{0,T} \tag{19} \]
We will also use the elementary lemma (see [18]):

**Lemma 3.3.** Let $\tilde{C}$ be the maximal number of elements sharing one face, and let $A : \mathcal{T} \to [0, \infty)$ be a function defined on the triangularization $\mathcal{T}$. Then one has:

$$\sum_{T_i \in \mathcal{T}} \sum_{T(F_i)} A(T) \leq \tilde{C} \sum_{T_i} A(T_i)$$

Finally the following well known (in-)equalities for $a, b \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^+$ are used throughout the paper:

$$(a - b) \cdot a = \frac{1}{2} (a - b)^2 + \frac{1}{2} (a^2 - b^2)$$  \hspace{1cm} (20)

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2$$  \hspace{1cm} (21)

**Discretization** Based on those definitions, we can now define the discretization in space:

**Problem 2 (Space discretization).** Given the penalty parameters $\sigma_n, \sigma_w \in \mathbb{R}^+$ and the parameter $\theta \in \{-1, 0, 1\}$, find $s_w \in V_h^s(\Omega)$, $p_n \in V_h^p(\Omega)$ and $p_c \in V_h^p(\Omega)$, such that for all $\psi_s \in V_h^s(\Omega)$, $\psi_n \in V_h^p(\Omega)$ and $\psi_w \in V_h^p(\Omega)$ holds:

$$\sum_{T_i \in \mathcal{T}} \int_{T_i} -\partial_t s_w \psi_n + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_n(s_w) K \nabla p_n \nabla \psi_n$$

$$- \sum_{F_i \in F} \int_{F_i} \{ \lambda_n(s_w) K \nabla p_n \cdot \tilde{n} \} [\psi_n] + \theta \sum_{F_i \in F} \int_{F_i} [p_n \{ \lambda_n(s_w) K \nabla \psi_n \}$$

$$+ \sigma_n \sum_{F_i \in F} \int_{F_i} \frac{f(k_p)}{|F_i|} [p_n] [\psi_n]$$

$$= \theta \sum_{F_i \in F_D} \int_{F_i} [p_n^D \{ \lambda_n(s^D) K \nabla \psi_n \} + \sigma_n \sum_{F_i \in F_D} \int_{F_i} \frac{f(k_p)}{|F_i|} [p_n^D] [\psi_n]$$

$$\sum_{T_i \in \mathcal{T}} \int_{T_i} \partial_t s_w \psi_w + \sum_{T_i \in \mathcal{T}} \int_{T_i} \lambda_w(s_w) K \nabla (p_n - p_c) \nabla \psi_w$$

$$- \sum_{F_i \in F} \int_{F_i} \{ \lambda_w(s_w) K \nabla (p_n - p_c) \cdot \tilde{n} \} [\psi_w]$$

$$+ \theta \sum_{F_i \in F} \int_{F_i} \{ \lambda_w(s_w) K \nabla \psi_w \cdot \tilde{n} \} [p_n - p_c] + \sigma_w \sum_{F_i \in F} \int_{F_i} \frac{f(k_p)}{|F_i|} [p_n - p_c] [\psi_w]$$

$$= \theta \sum_{F_i \in F_D} \int_{F_i} \{ \lambda_w(s^D) K \nabla \psi_w \cdot \tilde{n} \} [p_n^D - p_c^D] + \sigma_w \sum_{F_i \in F_D} \int_{F_i} \frac{f(k_p)}{|F_i|} [p_n^D - p_c^D] [\psi_w]$$

$$\sum_{T_i \in \mathcal{T}} \int_{T_i} p_c \psi_s = \sum_{T_i \in \mathcal{T}} \int_{T_i} p_{ceq}(s_w) \psi_s - \sum_{T_i \in \mathcal{T}} \int_{T_i} \tau \partial_t s_w \psi_s$$  \hspace{1cm} (23)

$$\sum_{T_i \in \mathcal{T}} \int_{T_i} \Delta t \psi_s = \sum_{T_i \in \mathcal{T}} \int_{T_i} \tau \partial_t s_w \psi_s$$  \hspace{1cm} (24)

The parameters $\sigma_n$ and $\sigma_w$ penalize jumps over the faces, i.e. discontinuities in the solutions. The choices $\theta = -1$ respectively $0$ or $1$, give the non-symmetric- (NIPG), incomplete- (IIP) or symmetric-interior-penalty (SIPG) scheme.

For the discretization in time, a backward Euler scheme is used. For that the interval $[0, T]$ is subdivided into $N$ intervals of the size $\Delta t > 0$ with $T = N \cdot \Delta t$. Write $t_i = i \cdot \Delta t$ for the $i$-th discrete time-step. Given a function $g(t, x)$ depending on $t$ and $x$ its time derivative is approximated by:

$$\partial^- g(t^{n+1}, x) = \frac{g(t^{n+1}, x) - g(t^n, x)}{\Delta t}$$  \hspace{1cm} (25)
With this the fully-discrete scheme is written as:

**Problem 3** (P\textsubscript{h}\textsuperscript{n+1}). Let P\textsubscript{n} \in V\textsubscript{h}\textsuperscript{n}(Ω), P\textsubscript{c} \in V\textsubscript{h}\textsuperscript{p}(Ω) and S\textsubscript{w} \in V\textsubscript{h}\textsuperscript{s}(Ω), find P\textsubscript{n+1} \in V\textsubscript{h}\textsuperscript{p}(Ω), P\textsubscript{c+1} \in V\textsubscript{h}\textsuperscript{p}(Ω) and S\textsubscript{w+1} \in V\textsubscript{h}\textsuperscript{s}(Ω), such that for all ψ\textsubscript{a} \in V\textsubscript{h}\textsuperscript{s}(Ω), ψ\textsubscript{b} \in V\textsubscript{h}\textsuperscript{p}(Ω) and ψ\textsubscript{w} \in V\textsubscript{h}\textsuperscript{p}(Ω) holds:

\[
\sum_{T_1 \in T_1} \int_{T_1} -\partial^+ S\textsubscript{w+1} \phi_1 + \sum_{T_1 \in T_1} \int_{T_1} \lambda_n(S\textsubscript{w+1}) K \nabla P\textsubscript{n+1} \nabla \psi_1 \\
- \sum_{F_1 \in F_1} \int_{F_1} \{\lambda_n(S\textsubscript{w+1}) K \nabla P\textsubscript{n+1} \cdot \bar{n}\} \psi_1 \\
+ \theta \sum_{F_1 \in F_1} \int_{F_1} \left[ p\textsubscript{n+1} \{\lambda_n(S\textsubscript{w+1}) K \nabla \psi_1 \} + \sigma_n \sum_{F_1 \in F_1} \int_{F_1} \frac{f(k_p)}{|F_1|} [p\textsubscript{n+1}][\psi_1] \right] \\
= \theta \sum_{F_1 \in F_1} \int_{F_1} \{p\textsubscript{n+1} \{\lambda_n(S\textsubscript{w+1}) K \nabla \psi_1 \} + \sigma_n \sum_{F_1 \in F_1} \int_{F_1} \frac{f(k_p)}{|F_1|} [p\textsubscript{n+1}][\psi_1] \right] \tag{26}
\]

\[
\sum_{T_1 \in T_1} \int_{T_1} -\partial^+ S\textsubscript{w+1} \phi_2 + \sum_{T_1 \in T_1} \int_{T_1} \lambda_n(S\textsubscript{w+1}) K \nabla (P\textsubscript{n+1} - P\textsubscript{c+1}) \nabla \psi_2 \\
- \sum_{F_1 \in F_1} \int_{F_1} \{\lambda_n(S\textsubscript{w+1}) K \nabla (P\textsubscript{n+1} - P\textsubscript{c+1}) \cdot \bar{n}\} \psi_2 \\
+ \theta \sum_{F_1 \in F_1} \int_{F_1} \{\lambda_n(S\textsubscript{w+1}) K \nabla \psi_2 \cdot \bar{n}\} \left[ p\textsubscript{n+1} - P\textsubscript{c+1} \right] \\
+ \sigma_n \sum_{F_1 \in F_1} \int_{F_1} \frac{f(k_p)}{|F_1|} [p\textsubscript{n+1} - P\textsubscript{c+1}][\psi_2] \right] \tag{27}
\]

\[
\sum_{T_1 \in T_1} \int_{T_1} P\textsubscript{n+1} \psi_s = \sum_{T_1 \in T_1} \int_{T_1} p_{c.eq}(S\textsubscript{w+1}) \psi_s - \sum_{T_1 \in T_1} \int_{T_1} \tau \nabla S\textsubscript{w+1} \psi_s \tag{28}
\]

**Assumptions** The following assumptions are made to prove existence and convergence of the scheme.

A1 Assume that the boundary and initial conditions are sufficiently smooth in the sense of (8) and (9). Furthermore the initial condition shall be compatible with the boundary condition.

A2 The permeability matrix $K \in \mathbb{R}^{d \times d}$ is symmetric and positive definite, i.e. there exist two constants $\overline{\pi}$ and $\underline{\pi}$, such that for any vector $x \in \mathbb{R}^d$ it holds:

$$\underline{\pi} \|x\|^2 \leq x^T K x \leq \overline{\pi} \|x\|^2$$

A3 The equilibrium capillary pressure function $p_{c.eq}(\cdot)$ is in $C^2(\mathbb{R})$. Furthermore it is assumed to be positive, bounded and decreasing. Let $P_{c.eq}(\cdot)$ define the primitive, i.e.:

$$P_{c.eq}(S) := \begin{cases} 
\int_1^S p_{c.eq}(\xi) d\xi = \int_0^S p_{c.eq}(\xi) d\xi - \int_0^1 p_{c.eq}(\xi) d\xi & \text{for } S \leq 1 \\
0 & \text{otherwise.} \tag{29}
\end{cases}$$

It is easy to see that $P_{c.eq}(S)$ is concave and negative.

A4 The functions $\lambda_w(\cdot)$ and $\lambda_n(\cdot)$ are Lipschitz-continuous, and bounded from above and below by the constants $0 < \overline{\lambda}_w < \underline{\lambda}_w < \infty$. 

7
For the error analysis, let \( s_w(t, x), p_n(t, x) \) and \( p_c(t, x) \) be the exact solutions of the problem. For simplicity we will use \( s'_w = s_w(t, x), p'_n = p_n(t, x) \) and \( p'_c = p_c(t, x) \). Denote by \( \tilde{p}_n(t), \tilde{p}_c(t) \) and \( \tilde{s}_w(t) \) the approximations of \( p_n(t), p_c(t) \) and \( s_w(t) \) in \( V_h^p(\Omega) \) or \( V_h^n(\Omega) \) for all \( t \in [0, T] \). Furthermore we assume that \( \tilde{p}_n(t) \in W^{1, \infty}(\Omega), \tilde{p}_c(t) \in W^{1, \infty}(\Omega) \) and \( \tilde{s}_w(t) \in W^{1, \infty}(\Omega) \) for all \( t \in [0, T] \) holds. We assume that the solutions possesses enough regularity, such that the following approximation properties are fulfilled:

For all \( t \in [0, T] \) and \( T \in \mathcal{T} \), for \( \tilde{p}_n(t) \in W^{1, \infty}, \tilde{p}_c(t) \in W^{1, \infty} \) and \( \tilde{s}_w(t) \in W^{1, \infty} \) there exists a constant \( C \) independent of \( h, k_s, k_{p_n}, k_{p_c} \) and \( \Delta t \) such that:

\[
\text{for all: } 0 < q \leq l_{p_n} \quad \|p_n(t) - \tilde{p}_n(t)\|_{T,q} \leq C \frac{h^{\min(k_{p_n}+1, l_{p_n})-q}}{k_{p_n}^{l_{p_n}-q}} \|p_n(t)\|_{T,l_{p_n}} \tag{30}
\]

\[
\text{for all: } 0 < q \leq l_{p_c} \quad \|p_c(t) - \tilde{p}_c(t)\|_{T,q} \leq C \frac{h^{\min(k_{p_c}+1, l_{p_c})-q}}{k_{p_c}^{l_{p_c}-q}} \|p_c(t)\|_{T,l_{p_c}} \tag{31}
\]

\[
\text{for all: } 0 < q \leq l_s \quad \|s_n(t) - \tilde{s}_n(t)\|_{T,q} \leq C \frac{h^{\min(k_s+1, l_s)-q}}{k_s^{l_s}-q} \|s_n(t)\|_{T,l_s} \tag{32}
\]

The proof for those results can be found in [1]. Further the numerical errors for \( i = 1, \ldots, N \) are written as:

\[
e^i_{s,h} = S^i - \tilde{S}_w^i, \quad e^i_s = \tilde{S}_w^i - S^i, \quad e^i_{p_n,h} = P^i_n - \tilde{P}_n^i, \quad e^i_{p_c} = \tilde{P}_c^i - P_c^i.
\]

### 4 Numerical Analysis

After having defined the basic notations and lemmas, we prove in this section that the scheme is well-posed and convergent. For the existence we use a fix-point argument and show energy estimates for the discrete solutions. Afterwards we show convergence of the scheme by proving error estimates.

#### 4.1 Existence of a discrete solution

We prove the existence of a discrete solution of Problem 3. For given \( P_{n,i} \in \mathbb{R}, P_{c,i} \in \mathbb{R} \) and \( S_{w,k} \in \mathbb{R} \) we define \( \tilde{P}_n, \tilde{P}_c \in V_h^p(\Omega) \) and \( \tilde{S}_w \in V_h^n(\Omega) \) by

\[
\tilde{P}_n = \sum_{i=0}^{d_p} P_{n,i} \varphi_i^p, \quad \tilde{P}_c = \sum_{i=0}^{d_p} P_{c,i} \varphi_i^p, \quad \tilde{S}_w = \sum_{k=0}^{d_s} S_{w,k} \varphi_k^w.
\]

(33)

Here \( \varphi_i^p \) and \( \varphi_k^w \) are elements of a basis for \( V_h^p(\Omega) \) and \( V_h^n(\Omega) \). Furthermore for a given real numbers \( S_{w,k} \in \mathbb{R} \) define \( P_{w,i} \in \mathbb{R} \) for \( i = 0, \ldots, d_p \) and \( dS_{w,k} \in \mathbb{R} \) for \( k = 0, \ldots, d_s \) with

\[
P_{w,i} := P_{n,i} - P_{c,i}, \quad dS_{w,k} := \frac{1}{\Delta t} (S_{w,k} - S_{w,k}^n).
\]

(34)

This gives us \( \tilde{P}_w \in V_h^p(\Omega), S_{w}^n \in V_h^n(\Omega) \) and \( dS_{w}^n \in V_h^n(\Omega) \) such that:

\[
\tilde{P}_w := \tilde{P}_n - \tilde{P}_c = \sum_{i=0}^{d_p} P_{w,i} \varphi_i^p, \quad dS_{w} := \frac{1}{\Delta t} (\tilde{S}_w - S_{w}^n) = \sum_{k=0}^{d_s} dS_{w,k} \varphi_k^w
\]

(35)

Define \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) as the \( \ell^2 \)-scalar product respectively the induced \( \ell^2 \)-norm on \( \mathbb{R}^d \) and note that for \( X \in V_h^p(\Omega) \) or \( X \in V_h^n(\Omega) \), \( \|X\|^2 = \langle X, X \rangle \) and \( \|X\|_{0,\Omega} \) are equivalent norms.
Having this, we define as above for \( i = 0, 1, \ldots, d_p \) and \( k = 0, 1, \ldots, d_s \):

\[
F_{i}^{P_n} := \sum_{T_i \in T} \int_{T_i} \frac{1}{\Delta t} (\tilde{S}_w - S^n_w) \phi_i \varphi_i^p + \sum_{T_i \in T} \int_{T_i} \lambda_n (\tilde{S}_w) K \nabla \tilde{P}_n \nabla \varphi_i^p \\
- \sum_{F_i \in F} \int_{F_i} \{ \lambda_n (\tilde{S}_w) K \nabla \tilde{P}_n \cdot \tilde{n} \} [\varphi_i^p]
\]

\[
+ \theta \sum_{F_i \in F} \int_{F_i} \{ \lambda_n (\tilde{S}_w) K \nabla \varphi_i^p \} + \sigma_n \sum_{F_i \in F} \int_{F_i} \frac{f(k_p)}{|F_i|} [\tilde{P}_n][\varphi_i^p]
\]

\[
- \theta \sum_{F_i \in \Gamma_D} \int_{F_i} [p_n^D]\{ \lambda_n (s^D) K \nabla \psi_n \} - \sigma_n \sum_{F_i \in \Gamma_D} \int_{F_i} \frac{f(k_p)}{|F_i|} [p_n^D][\psi_n],
\]

(36)

\[
F_{i}^{P_c} := \sum_{T_i \in T} \int_{T_i} \frac{1}{\Delta t} (\tilde{S}_w - S^n_w) \phi_i \varphi_i^p + \sum_{T_i \in T} \int_{T_i} \lambda_n (\tilde{S}_w) K \nabla (\tilde{P}_n - \tilde{P}_c) \nabla \varphi_i^p \\
- \sum_{F_i \in F} \int_{F_i} \{ \lambda_n (\tilde{S}_w) K \nabla (\tilde{P}_n - \tilde{P}_c) \cdot \tilde{n} \} [\varphi_i^p]
\]

\[
+ \theta \sum_{F_i \in F} \int_{F_i} \{ \lambda_n (\tilde{S}_w) K \nabla \varphi_i^p \} [\tilde{P}_n - \tilde{P}_c] + \sigma_w \sum_{F_i \in F} \int_{F_i} \frac{f(k_p)}{|F_i|} [\tilde{P}_n - \tilde{P}_c][\varphi_i^p]
\]

\[
- \theta \sum_{F_i \in \Gamma_D} \int_{F_i} \{ \lambda_n (s^D) K \nabla \psi_n \cdot \tilde{n} \} [p_n^D - p_c^D] - \sigma_w \sum_{F_i \in \Gamma_D} \int_{F_i} \frac{f(k_p)}{|F_i|} [p_n^D - p_c^D][\psi_n],
\]

(37)

\[
F_{k}^{S} := \sum_{T_i \in T} \int_{T_i} \phi \tilde{P}_c \varphi_k^s - \sum_{T_i \in T} \int_{T_i} \phi p_{c,eq}(\tilde{S}_w) \varphi_k^s - \sum_{T_i \in T} \int_{T_i} \phi \frac{1}{\Delta t} (\tilde{S}_w - S^n_w) \varphi_k^s.
\]

(38)

Observe that if \( F_{i}^{P_n} = F_{i}^{P_c} = F_{k}^{S} = 0 \) for all \( i = 0, 1, \ldots, d_p \) and \( k = 0, 1, \ldots, d_s \), \( \tilde{P}_n, \tilde{P}_c \) and \( \tilde{S}_w \) are a solution Problem 3. We proof the following Lemma:

**Lemma 4.1.** For sufficiently large \( \sigma_n, \sigma_w \) Problem 3 has a solution.

**Proof.** We use Lemma 1.4 in [36], p. 164. Specifically we show that a \( R \in \mathbb{R} \) with \( R > 0 \) exists, such that for

\[
\langle \tilde{P}_n, \tilde{P}_n \rangle + \langle \tilde{P}_w, \tilde{P}_w \rangle + \langle \tilde{d}S_w, \tilde{d}S_w \rangle = R
\]

holds

\[
\langle F_{i}^{P_n}, \tilde{P}_n \rangle + \langle F_{i}^{P_c}, \tilde{P}_w \rangle + \langle F_{i}^{S}, \tilde{d}S_w \rangle = \sum_{i=0}^{d_p} P_{n,i} F_{i}^{P_n} + \sum_{i=0}^{d_p} (P_{n,i} - P_{c,i}) F_{i}^{P_c} + \sum_{k=0}^{d_s} dS_{w,k} F_{k}^{S} > 0.
\]

We estimate

\[
\sum_{i=0}^{d_p} P_{n,i} F_{i}^{P_n} + \sum_{i=0}^{d_p} P_{w,i} F_{i}^{P_c} + \sum_{k=0}^{d_s} dS_{w,k} F_{k}^{S} = (I) + (II) + (III)
\]

separately.
Estimates for (I)

For the first part, we start with:

$$(I) = \sum_{T_i \in T} \lambda_n(\tilde{S}_w) \left| K^{\frac{1}{2}} \nabla \tilde{P}_n \right|^2 + \sigma_n \sum_{F_i \in F} \int_{F_i} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2$

$$- (1 - \theta) \sum_{F_i \in F} \int_{F_i} \{\lambda_n(\tilde{S}_w) K^{\frac{1}{2}} \tilde{P}_n \cdot \tilde{n}\} \left\| \tilde{P}_n \right\| - \sum_{T_i \in T} \int_{T_i} \frac{1}{\Delta_T}(\tilde{S}_w - S^w_n) \phi \tilde{P}_n$$

$$- \theta \sum_{F_i \in \Gamma_D} \int_{F_i} \{\lambda_n(s^D) K^{\frac{1}{2}} \tilde{P}_n\} - \sigma_n \sum_{F_i \in \Gamma_D} \int_{F_i} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2$$

$$= P_1 + P_2 - P_3 - P_4 - P_5 - P_6.$$  

By A4 for $P_1 + P_2$ one has:

$$P_1 + P_2 \geq \lambda_n \sum_{T_i \in T} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_i,0} + \sigma_n \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2_{F_i,0}. \quad (39)$$

Using Cauchy-Schwarz inequality and A4 again one gets:

$$P_3 \leq \overline{x}_n(1 - \theta) \sum_{F_i \in F} \left\| \{K^{\frac{1}{2}} \nabla \tilde{P}_n\} \right\|_{F_i,0} \left\| \tilde{P}_n \right\|_{F_i,0}.$$

For a fixed face $F_i$ let $T_{\pm}$ be the adjacent elements. By the trace inequality holds:

$$\overline{x}_n(1 - \theta) \sum_{F_i \in F} \left\| \{K^{\frac{1}{2}} \nabla \tilde{P}_n\} \right\|_{F_i,0} \left\| \tilde{P}_n \right\|_{F_i,0}$$

$$\leq \overline{x}_n(1 - \theta) C_1 \left( \sum_{F_i \in F} \left( \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|_{T_{\pm},0} + \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|_{T_{0},0} \right) \right) \left\| \tilde{P}_n \right\|_{F_i,0}.$$  

Further with Lemma 3.3 and Cauchy-Schwarz inequality we end up with:

$$\overline{x}_n(1 - \theta) C_1 \left( \sum_{F_i \in F} \left( \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|_{T_{\pm},0} + \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|_{T_{0},0} \right) \right) \left\| \tilde{P}_n \right\|_{F_i,0}$$

$$\leq \left( \sum_{T_i \in T} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_{\pm},0} \right)^{\frac{1}{2}} \left( \overline{x}_n^2 (1 - \theta)^2 C^2_i \tilde{C}^2 \frac{f(k_p)}{|F_i|} \sum_{F_i \in F} \left\| \tilde{P}_n \right\|^2_{F_i,0} \right)^{\frac{1}{2}}.$$

Using the scaled Young’s inequality the above leads to:

$$P_3 \leq \frac{\overline{C}_1}{2} \sum_{T_i \in T} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_{0},0} + \frac{1}{2\epsilon_1} \sum_{F_i \in F} \lambda_n^2 (1 - \theta)^2 C^2_i \tilde{C}^2 \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2_{F_i,0}. \quad (40)$$

For $P_5$ and $P_6$ we estimate directly:

$$P_5 \leq \frac{\epsilon_2}{2} \sum_{T_i \in T} \left\| K^{\frac{1}{2}} \nabla \tilde{P}_n \right\|^2_{T_{0},0} + \frac{1}{2\epsilon_2} \sum_{F_i \in \Gamma_D} \lambda_n^2 \theta^2 C^2_i \tilde{C}^2 \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2_{F_i,0} \quad (41)$$

and

$$P_6 \leq \frac{\epsilon_3}{2} \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2_{F_i,0} + \frac{\sigma_n}{2\epsilon_3} \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \left\| \tilde{P}_n \right\|^2_{F_i,0} \quad (42)$$
Choosing $\epsilon_1 = \epsilon_2 = \frac{\lambda_n}{2}$, and $\epsilon_3 = \sigma_n$, (39), (40), (41) and (42) give:

\begin{equation}
(\text{I}) \geq \sum_{T_i \in T} \int_{T_i} - \frac{1}{\Delta t}(\tilde{S}_w - S^n_w)\phi \tilde{P}_n + \sum_{T_i \in T} \frac{\lambda_n}{2} \|K^{\frac{1}{2}} \nabla \tilde{P}_n\|_{T_i,0}^2
+ \left( \frac{\sigma_n}{2} - \frac{(1 - \theta)^2 \lambda_n^2 C^2 T^2}{2\lambda_n} \right) \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} \|\tilde{P}_n\|_{F_i,0}^2
- \left( \frac{\sigma_n}{2} + \frac{\lambda_n^2 \theta^2 C^2 T^2}{\lambda_n} \right) \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \|p^n_w\|_{F_i,0}^2.
\end{equation}

**Estimates for (II)** As for (I) A2, Lemma 3.3, Cauchy-Schwarz and scaled Young’s inequality are applied. After choosing $\epsilon_4 = \epsilon_5 = \frac{\lambda_n}{2}$ and $\epsilon_6 = \sigma_w$ we arrive at:

\begin{equation}
(\text{II}) \geq \sum_{T_i \in T} \int_{T_i} \frac{1}{\Delta t}(\tilde{S}_w - S^n_w)\phi \tilde{P}_w + \sum_{T_i \in T} \frac{\lambda_n}{2} \|K^{\frac{1}{2}} \nabla \tilde{P}_w\|_{T_i,0}^2
+ \left( \frac{\sigma_w}{2} - \frac{(1 - \theta)^2 \lambda_n^2 C^2 T^2}{2\lambda_n} \right) \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} \|\tilde{P}_w\|_{F_i,0}^2
- \left( \frac{\sigma_w}{2} + \frac{\lambda_n^2 \theta^2 C^2 T^2}{\lambda_n} \right) \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \|p^n_w\|_{F_i,0}^2.
\end{equation}

**Estimates for (III)** For the last part we start with:

\begin{equation}
(\text{III}) = \sum_{T_i \in T} \int_{T_i} \phi \tilde{P}_c \frac{1}{\Delta t}(\tilde{S}_w - S^n_w) - \sum_{T_i \in T} \int_{T_i} \phi \rho c,eq(\tilde{S}_w) \frac{1}{\Delta t}(\tilde{S}_w - S^n_w)
+ \sum_{T_i \in T} \int_{T_i} \phi_T \frac{1}{\Delta t} \tilde{S}_w - S^n_w)^2.
\end{equation}

Using the primitive (29) one gets:

\begin{equation}
(\text{III}) \geq \sum_{T_i \in T} \int_{T_i} \phi \tilde{P}_c \frac{1}{\Delta t}(\tilde{S}_w - S^n_w) + \sum_{T_i \in T} \int_{T_i} \phi \frac{1}{\Delta t} \left( |P_{c,eq}(\tilde{S}_w)| + |P_{c,eq}(S^n_w)| \right)
+ \sum_{T_i \in T} \int_{T_i} \phi_T \frac{1}{\Delta t} \tilde{S}_w - S^n_w)^2.
\end{equation}

For sufficiently large $\sigma_n$ and $\sigma_w$ and using (17) with $q = 2$, summing (43), (44) and (45) gives:

\begin{equation}
\sum_{i=0}^{d_{p_n}} P_{n,i} F^n_{i} + \sum_{j=0}^{d_{p}} P_{w,j} F^P_{j} + \sum_{k=0}^{d_s} dS_{w,k} F^S_{k} \geq
C\|\tilde{P}_n\|^2_{0,0} + C\|\tilde{P}_w\|^2_{0,0} + C|\tilde{S}_w|^2_{0,0} + \sum_{T_i \in T} \int_{T_i} \phi \frac{1}{\Delta t} |P_{c,eq}(\tilde{S}_w)| - \sum_{T_i \in T} \int_{T_i} \phi \frac{1}{\Delta t} |P_{c,eq}(S^n_w)|
- \left( \frac{\sigma_n}{2} + \frac{\lambda_n^2 \theta^2 C^2 T^2}{\lambda_n} \right) \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} \|p^n_w\|_{F_i,0}^2.
\end{equation}

To finalize the proof, (35), (36), (37) and (38) define a continuous mapping

\[ \mathcal{P} : (V_{p_n}^p(\Omega), V_{p}^p(\Omega), V_{s}^p(\Omega)) \rightarrow (V_{p_n}^p(\Omega), V_{p}^p(\Omega), V_{s}^p(\Omega)) \]
Proof. Starting with the discrete equations, we test (26) with \( k \) then (28) with Lemma 4.2.

\[
\langle \tilde{P}_n, \tilde{P}_n \rangle + \langle \tilde{P}_w, \tilde{P}_w \rangle + \langle \tilde{d}S_w, \tilde{d}S_w \rangle = R + \epsilon, \tag{46}
\]

with \( \epsilon \in \mathbb{R}, \epsilon > 0 \) and

\[
R := \sum_{T_i \in T} \int_{T_i} \frac{1}{\Delta t} |P_{c,eq}(S_w^n)| + \left( \frac{\sigma_n}{2} + \frac{\lambda_n^2 \theta^2 C_t^2 \tilde{C}^2}{\lambda_n} \right) \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} ||P_n^{n+1}||_{F_i,0}^2 \\
+ \left( \frac{\sigma_w}{2} + \frac{\lambda_w^2 \theta^2 C_t^2 \tilde{C}^2}{\lambda_w} \right) \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} ||P_w^{n+1}||_{F_i,0}^2 \tag{47}
\]

then

\[
\langle \tilde{P}_n, \tilde{P}_n \rangle + \langle \tilde{P}_w, \tilde{P}_w \rangle + \langle \tilde{d}S_w, \tilde{d}S_w \rangle > 0.
\]

Furthermore we just showed, that

\[
\langle F_{\gamma n}, \tilde{P}_n \rangle + \langle F_{\gamma w}, \tilde{P}_w \rangle + \langle F, \tilde{d}S_w \rangle > 0
\]

The existence of a zero for \( \mathcal{P} \) and the solution to the system follows now directly by Lemma 1.4 in page 164 from [36].

\[
\square
\]

4.2 Discrete energy estimate

Lemma 4.2. For \( \sigma_n \) and \( \sigma_w \) sufficiently large there exists a constant \( C \) independent of \( \Delta t, h \) and the polynomial degrees \( k_p \) and \( k_s \) such that the following energy estimate holds:

\[
\Delta t \sum_{n=0}^{N} \sum_{T_i \in T} ||\partial^2 S_w^{n+1}||_{T_i,0}^2 + \Delta t \sum_{n=0}^{N} \sum_{T_i \in T} ||K_t^2 \nabla P_n^{n+1}||_{T_i,0}^2 + \Delta t \sum_{n=0}^{N} \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} ||P_n^{n+1}||_{F_i,0}^2 \\
+ \Delta t \sum_{n=0}^{N} \sum_{T_i \in T} ||K_t^2 \nabla P_w^{n+1}||_{T_i,0}^2 + \Delta t \sum_{n=0}^{N} \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} ||P_w^{n+1}||_{F_i,0}^2 + \Delta t \sum_{n=0}^{N} \int_{T_i} |P_{c,eq}(S_w^n)| \\
\leq C \sum_{n=0}^{N} \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} ||P_n^{n}||_{F_i,0}^2 + C \Delta t \sum_{n=0}^{N} \sum_{F_i \in \Gamma_D} \frac{f(k_p)}{|F_i|} ||P_w^{n}||_{F_i,0}^2
\]

Proof. Starting with the discrete equations, we test (26) with \( P_n^{n+1} \), (27) with \( P_w^{n+1} = P_n^{n+1} - P_c^{n+1} \) and (28) with \( \partial^2 S_w^{n+1} \). Note that \( C = C(\gamma, \sigma_n, \lambda_n, \lambda_w, \theta, C_t, \tilde{C}) \) for \( \alpha = w, n \) and proceed as in the proof of Lemma 4.1 to obtain:

\[
\sum_{T_i \in T} \phi_T ||\partial^2 S_w^{n+1}||_{T_i,0}^2 + \frac{\lambda_n}{2} \sum_{T_i \in T} ||K_t^2 \nabla P_n^{n+1}||_{T_i,0}^2 \\
+ (\sigma_n - \frac{1}{2\lambda_n} \lambda_n^2 (1 - \theta)^2 C_t^2 \tilde{C}^2) \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} ||P_n^{n+1}||_{F_i,0}^2 \\
+ \frac{\lambda_w}{2} \sum_{T_i \in T} ||K_t^2 \nabla P_w^{n+1}||_{T_i,0}^2 + (\sigma_w - \frac{1}{2\lambda_w} \lambda_w^2 (1 - \theta)^2 C_t^2 \tilde{C}^2) \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} ||P_w^{n+1}||_{F_i,0}^2
\]
\[
\leq \sum_{T_i \in T} \int_{T_i} \frac{1}{\Delta t} \left( P_{c,eq}(S_w^{n+1}) - P_{c,eq}(S_w^n) \right) + \left( \sigma_n + \frac{\sum_{t = 1}^2 C_i c_i}{\lambda_n} \right) \sum_{F_i \in F_D} f(k_p) \| p_{D} \|_{F,D}^2 \\
+ \left( \sigma_w + \frac{\sum_{t = 1}^2 C_i c_i}{\lambda_w} \right) \sum_{F_i \in F_D} f(k_p) \| p_{D} \|_{F,D}^2 
\]

Multiplying the inequality by \( \Delta t \) and summing over all \( n = 0 \ldots N \) one gets:

\[
\Delta t \sum_{n=0}^{N} \sum_{T_i \in T} \phi_T \| \partial^2 S_w^{n+1} \|_{T_i,0}^2 + \frac{\lambda_n}{2} \Delta t \sum_{n=0}^{N} \sum_{T_i \in T} \left\| K^{\frac{1}{2}} \nabla P_n^{n+1} \right\|_{T_i,0}^2 \\
+ (\sigma_n - \frac{1}{2\lambda_n} \lambda_n^2 (1 - \theta)^2 C_i c_i^2) \Delta t \sum_{n=0}^{N} \sum_{F_i \in F} f(k_p) \| [P_n^{n+1}] \|_{F_i,0}^2 \\
+ \frac{\lambda_w}{2} \Delta t \sum_{n=0}^{N} \sum_{T_i \in T} \left\| K^{\frac{1}{2}} \nabla P_n^{n+1} \right\|_{T_i,0}^2 + (\sigma - \frac{1}{2\lambda_w} \lambda_n^2 (1 - \theta)^2 C_i c_i^2) \Delta t \sum_{n=0}^{N} \sum_{F_i \in F} f(k_p) \| [P_n^{n+1}] \|_{F_i,0}^2 \\
\leq \sum_{T_i \in T} \int_{T_i} \left( P_{c,eq}(S_w^{n+1}) - P_{c,eq}(S_w^n) \right) + \left( \sigma_n + \frac{\sum_{t = 1}^2 C_i c_i}{\lambda_n} \right) \Delta t \sum_{n=0}^{N} \sum_{F_i \in F_D} f(k_p) \| p_{D} \|_{F,D}^2 \\
+ \left( \sigma_w + \frac{\sum_{t = 1}^2 C_i c_i}{\lambda_w} \right) \Delta t \sum_{n=0}^{N} \sum_{F_i \in F_D} f(k_p) \| p_{D} \|_{F,D}^2 
\]

Which leads to the desired estimate. 

\[\square\]

### 4.3 Error Estimates

After general energy estimates and the existence of a discrete solution were derived, we want to show a convergence result for the scheme.

#### Estimates for the non-wetting phase

**Lemma 4.3.** For a sufficiently large \( \sigma_n \) there exists a constant \( C \) independent of \( h, \Delta t, k_p \) and \( k_s \) such that the following estimate holds:

\[
\sum_{T_i \in T} \int_{T_i} \left[ -\partial^2 S_w^{n+1} + \partial_t S_w \right] \phi_{e_n^{n+1}} \\
+ \sum_{T_i \in T} \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,n,h} \|_{T_i,0}^2 + \sum_{F_i \in F} f(k_p) \| [e_{n+1}^{n+1}] \|_{F_i,0}^2 \\
\leq C \left( \frac{5}{2\lambda_n} + \frac{3f(k_s)}{2\sigma_n f(k_p)} \right) \sum_{T_i \in T} \| K^{\frac{1}{2}} \nabla p_{n+1}^{n+1} \|_{H^\infty} \| e_{n+1}^{n+1} \|_{T_i,0}^2 \\
+ C \left( \frac{5}{2\lambda_n} \| K^{\frac{1}{2}} \nabla p_{n+1}^{n+1} \|_{H^\infty} \| e_{n+1}^{n+1} \|_{T_i,0}^2 \\
+ C \left( \frac{3\sigma_n C_i c_i^2}{2} + \frac{50\lambda_n^2 C_i c_i^2}{\lambda_n} \right) \| K^{\frac{1}{2}} \nabla e_{n+1}^{n+1} \|_{H^\infty} \| e_{n+1}^{n+1} \|_{T_i,0}^2 \\
+ C \left( \frac{3\lambda_n^2 C_i c_i^2}{2\sigma_n} \left( \| K^{\frac{1}{2}} \nabla e_{n+1}^{n+1} \|_{H^\infty}^2 + h^2 \| K^{\frac{1}{2}} \nabla e_{n+1}^{n+1} \|_{H^\infty}^2 \right) \right) \\
+ C \left( \frac{3\lambda_n^2 C_i c_i^2}{2\sigma_n} \right) \left( \| K^{\frac{1}{2}} \nabla e_{n+1}^{n+1} \|_{H^\infty}^2 + h^2 \| K^{\frac{1}{2}} \nabla e_{n+1}^{n+1} \|_{H^\infty}^2 \right) 
\]
Proof. As for the energy estimate, we use equations (22) and (23) and test with $e_{p_n,h}^{n+1}$ to get:

$$
\sum_{T_i \in T} \int_{T_i} \left[ -\partial^T S^{n+1}_w + \partial_t S_w \right] \phi e_{p_n,h}^{n+1} + \sum_{T_i \in T} \int_{T_i} \left[ \lambda_n(S_w^{n+1}) K \nabla P_{n+1}^n - \lambda_n(S_w) K \nabla p_n \right] \nabla e_{p_n,h}^{n+1} + \sigma_n \sum_{T_i \in T} \int_{F_i} \frac{f(k_p)}{|F_i|} \left[ \|P_{n+1}^n\| - |p_n| \right] \left[ e_{p_n,h}^{n+1} \right]
$$

$$
= \sum_{F_i \in F} \int_{F_i} \left[ \{\lambda_n(S_w^{n+1}) K \nabla P_{n+1}^n \cdot \vec{n}\} - \{\lambda_n(S_w) K \nabla p_n \cdot \vec{n}\} \right] \left[ e_{p_n,h}^{n+1} \right]
$$

$$
- \theta \sum_{F_i \in F} \int_{F_i} \left[ \|P_{n+1}^n\| \{\lambda_n(S_w^{n+1}) K \nabla e_{p_n,h}^{n+1}\} - \|p_n\| \{\lambda_n(S_w) K \nabla e_{p_n,h}^{n+1}\} \right]
$$

The equation is estimated componentwise by splitting it up in the following way:

$$
P_1 + P_2 + P_3 = P_4
$$

For each component the approximations $\tilde{p}_n$ is inserted. For $P_2$ one gets:

$$
P_2 = \sum_{T_i \in T} \int_{T_i} \left[ \lambda_n(S_w^{n+1}) K \nabla e_{p_n,h}^{n+1} + (\lambda_n(S_w^{n+1}) - \lambda_n(S_w)) K \nabla \tilde{p}_n^{n+1} + \lambda_n(S_w) K \nabla e_{p_n,h}^{n+1} \right] \nabla e_{p_n,h}^{n+1}
$$

$$
= P_{2,1} + P_{2,2} + P_{2,3}
$$

Which can be estimated as:

$$
P_{2,1} \geq \sum_{T_i \in T} \lambda_n \|K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 (48)
$$

$$
P_{2,2} \leq \sum_{T_i \in T} \int_{T_i} \lambda_n(S_w^{n+1} - S_w) K \nabla \tilde{p}_n^{n+1} \cdot \nabla e_{p_n,h}^{n+1} \leq \sum_{T_i \in T} \int_{T_i} \lambda_n(e_{s,n,h}^{n+1} + e_s^{n+1}) K \nabla \tilde{p}_n^{n+1} \cdot \nabla e_{p_n,h}^{n+1}
$$

$$
\leq \frac{\epsilon_{2,2}}{2} \sum_{T_i \in T} \|K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 + \frac{\lambda_n^2}{2 \epsilon_{2,2}} \|K^{\frac{1}{2}} \nabla \tilde{p}_n^{n+1}\|_{T_i,0}^2 \sum_{T_i \in T} (\|e_{s,n,h}^{n+1}\|_{T_i,0}^2 + \|e_s^{n+1}\|_{T_i,0}^2) (49)
$$

$$
P_{2,3} \leq \frac{\epsilon_{2,3}}{2} \sum_{T_i \in T} \|K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 + \frac{\lambda_n^2}{2 \epsilon_{2,3}} \sum_{T_i \in T} \|K^{\frac{1}{2}} \nabla e_{p_n,h}^{n+1}\|_{T_i,0}^2 (50)
$$

Continuing with $P_3$ we derive:

$$
P_3 = \sigma_n \sum_{F_i \in F} \int_{F_i} \frac{f(k_p)}{|F_i|} \left[ \|e_{p_n,h}^{n+1}\| + \|e_{p_n,h}^{n+1}\| \right] = P_{3,1} + P_{3,2}
$$

$$
P_{3,1} = \sigma_n \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} \|e_{p_n,h}^{n+1}\|^2_{F_i,0}
$$

$$
P_{3,2} \leq \frac{\epsilon_{3,2}}{2} \sum_{F_i \in F} \frac{f(k_p)}{|F_i|} \|e_{p_n,h}^{n+1}\|^2_{F_i,0} + \frac{1}{2 \epsilon_{3,2}} \sigma_n^2 C^2 \tilde{C}^2 (h^{-2} \|e_{p_n,h}^{n+1}\|_{\Omega,0}^2 + \|v e_{p_n,h}^{n+1}\|_{\Omega,0}^2) (52)
$$

For $P_4$ we get the following estimates:

$$
P_4 = (1 - \theta) \sum_{F_i \in F} \int_{F_i} \left[ \{\lambda_n(S_w^{n+1}) K \nabla e_{p_n,h}^{n+1} \cdot \vec{n}\} \right] e_{p_n,h}^{n+1}
$$

$$
+ \sum_{F_i \in F} \int_{F_i} \left[ \{\lambda_n(S_w^{n+1}) - \lambda_n(S_w)\} K \nabla \tilde{p}_n^{n+1} \cdot \vec{n} + \{\lambda_n(S_w) K \nabla e_{p_n,h}^{n+1} \cdot \vec{n}\} \right] e_{p_n,h}^{n+1}
$$

$$

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\[- \theta \sum_{F_i \in \mathcal{F}} \int_{F_i} \left[ [e_{p_n}^{n+1}] \{ \lambda_n(s_w) K \nabla e_{p_n,h}^{n+1} \} + [\bar{p}_n^{n+1}] \{ (\lambda_n(S_{\nu}^{n+1}) - \lambda_n(s_w)) K \nabla e_{p_n,h}^{n+1} \} \right] = P_{4.1} + \cdots + P_{4.5} \]

From which we get the following estimates:

\begin{align*}
P_{4.1} & \leq \frac{\varepsilon_{4.1}}{2} \sum_{T_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 + (1 - \theta) \frac{2}{\varepsilon_{4.1}} \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \left\| e_{p_n,h}^{n+1} \right\|_{F_i,0}^2 \right) + \frac{1}{\varepsilon_{4.1}} \sum_{T_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 \right) \tag{53} \\
P_{4.2} & \leq \lambda_n^2 \tilde{C} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{\Omega, \infty}^2 \sum_{F_i \in \mathcal{F}} \int_{F_i} \left\{ (e_{s_n}^{n+1} + e_s^{n+1}) \right\} e_{p_n,h}^{n+1} \right) \leq \frac{\varepsilon_{4.2}}{2} \sum_{F_i \in \mathcal{F}} \frac{f(k_s)}{|F_i|} \left\| e_{p_n,h}^{n+1} \right\|_{F_i,0}^2 + \frac{1}{\varepsilon_{4.2}} \sum_{T_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 \right) \tag{54} \\
P_{4.3} & \leq \left( \lambda_n^2 \tilde{C}^2 \sum_{T_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 + \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 \right) \right) \cdot \left( \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \left\| e_{p_n,h}^{n+1} \right\|_{F_i,0}^2 \right) + \frac{1}{\varepsilon_{4.3}} \left( \lambda_n^2 \tilde{C}^2 \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \left\| e_{p_n,h}^{n+1} \right\|_{F_i,0}^2 \right) \tag{55} \\
P_{4.4} & \leq \lambda_n^2 \tilde{C}^2 \sum_{T_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 \right) + \frac{1}{\varepsilon_{4.4}} \sum_{t_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 \right) + \frac{1}{\varepsilon_{4.4}} \left( \lambda_n^2 \tilde{C}^2 \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \left\| e_{p_n,h}^{n+1} \right\|_{F_i,0}^2 \right) \tag{56} \\
P_{4.5} & \leq \frac{\varepsilon_{4.5}}{2} \sum_{T_i \in \mathcal{T}} \left( \left\| K^{1/2} \nabla e_{p_n,h}^{n+1} \right\|_{T_i,0}^2 + \frac{1}{\varepsilon_{4.5}} \lambda_n^2 \tilde{C}^2 \left( h^{-2} \left\| e_{p_n,h}^{n+1} \right\|_{\Omega, 0}^2 + \left\| \nabla e_{p_n,h}^{n+1} \right\|_{\Omega, 0}^2 \right) \right) \tag{57} \\
\end{align*}

If \( \bar{p}_n^{n+1} \) is continuous, the jump term in \( P_{4.5} \) is vanishing and we are done. Otherwise we proceed as in \( P_{4.4} \). We use the continuity of \( p_n \) to replace \( \bar{p}_n^{n+1} \) by \( e_{p_n,h}^{n+1} \).
\[
+ \left( \frac{1}{2e_{4,4}} + \frac{1}{2e_{4,5}} \right) \theta^2 \lambda_n^2 C^2_i \tilde{C}^2 (h^{-2} \| e_{p_n}^{n+1} \|_{\Omega,0}^2 + \| \nabla e_{p_n}^{n+1} \|_{\Omega,0}^2)
\]

Choosing
\[
\epsilon_{2,2} = \epsilon_{2,3} = \epsilon_{4,1} = \epsilon_{4,4} = \epsilon_{4,5} = \frac{\lambda_n}{5}
\]
and
\[
\epsilon_{3,2} = \frac{f(k_s)}{f(k_p)} \epsilon_{4,2} = \epsilon_{4,3} = \frac{\sigma_n}{3}
\]
we arrive at the desired estimate.

**Estimates for the wetting phase**

**Lemma 4.4.** For a sufficiently large \( \sigma_w \) there exists a constant \( C \) independent of \( h, \Delta t, k_p \) and \( k_s \) such that the following estimate holds:

\[
\sum_{T_i \in \mathcal{T}} \int_{T_i} \left[ -\partial^t S_{w}^{n+1} + \partial_t s_w \right] \phi e_{p_{n+1}}^{n+1}
+ \sum_{T_i \in \mathcal{T}} \| K^2 \nabla e_{p_{n+1}}^{n+1} \|_{T_i,0}^2 + \sum_{F_i \in \mathcal{F}} \frac{f(k_p)}{F_i} \| e_{p_{n+1}}^{n+1} \|_{F_i,0}^2
\]

\[
\leq C \left( \frac{5}{2\lambda_w} + \frac{3f(k_s)}{2\sigma_w f(k_p)} \right) \lambda_{\infty}^2 \| K^2 \nabla p_{\tilde{w}}^{n+1} \|_{\Omega,\infty}^2 \sum_{T_i \in \mathcal{T}} \| e_{s_{n+1}}^{n+1} \|_{T_i,0}^2
\]

\[
+ C \lambda_{\infty}^2 \| K^2 \nabla e_{s_{n+1}}^{n+1} \|_{\Omega,\infty}^2 (\| e_{s_{n+1}}^{n+1} \|_{\Omega,0} + \| \nabla e_{s_{n+1}}^{n+1} \|_{\Omega,0})
\]

\[
+ \frac{3\sigma_w C^2 \tilde{C}^2}{2} \left( \| K^2 \nabla e_{p_{n+1}}^{n+1} \|_{\Omega,0}^2 + h^2 \| \nabla e_{p_{n+1}}^{n+1} \|_{\Omega,0}^2 \right)
\]

**Proof.** The proof is the same as in the case of the non-wetting phase and is therefore left out.

For the final error estimate Lemma 4.3 and 4.4 are combined into:

**Lemma 4.5.** For sufficiently large \( \sigma_n \) and \( \sigma_w \) there exists a constant \( C \) independent of \( h, \Delta t, k_p \) and \( k_s \) such that the following estimate holds:

\[
\sum_{T_i \in \mathcal{T}} \| e_{p_{n+1}}^{n+1} \|_{T_i,0}^2 + \sum_{F_i \in \mathcal{F}} \| e_{p_{n+1}}^{n+1} \|_{F_i,0}^2
\]

\[
+ \frac{L^2 \phi}{|p'_{\text{eq}}|} \| e_{s_{n+1}}^{n+1} \|_{\Omega,0}^2 + \frac{3\phi}{2\tau} \| e_{p_{\text{eq}}^{n+1}} \|_{\Omega,0}^2 + \frac{5}{4} \| e_{p_{\text{eq}}^{n+1}} \|_{\Omega,0}^2 + \left( \frac{16 - \phi}{2} \right) \| e_{s_{n+1}}^{n+1} \|_{\Omega,0}^2 + \left( \frac{16 - \phi}{2} \right) \| e_{s_{\alpha}}^{n+1} \|_{\Omega,0}^2 + \frac{3f(k_s)\lambda_{w}}{2\sigma_w f(k_p)} \| K^2 \nabla p_{\tilde{w}}^{n+1} \|_{\Omega,\infty} (\| e_{s_{n+1}}^{n+1} \|_{\Omega,0}^2 + h^2 \| \nabla e_{s_{n+1}}^{n+1} \|_{\Omega,0}^2)
\]

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\[ + C \frac{5\lambda_\alpha}{2\lambda_\alpha} \| K \frac{1}{2} \nabla e_{p,w}^{n+1} \|^2_{\Omega,0} + C \left( \frac{3\sigma_\alpha C_2^2 \tilde{C}^2}{2} + C \frac{5\theta_\alpha^2 \lambda_\alpha^2}{\lambda_\alpha} \right) \left( h^{-2} \| e_{p,w}^{n+1} \|^2_{\Omega,0} + \| \nabla e_{p,w}^{n+1} \|^2_{\Omega,0} \right) \]
\[ + C \frac{\lambda_\alpha^2}{2\sigma_\alpha} \left( \left\| K \frac{1}{2} \nabla e_{p,w}^{n+1} \right\|^2_{\Omega,0} + h^2 \left\| K \frac{1}{2} \nabla^2 e_{p,w}^{n+1} \right\|^2_{\Omega,0} \right) \]

(59)

Proof. The estimate is obtained by adding up the results of Lemma 4.4 and 4.3:

\[ \sum_{T_i \in T} \int_{T_i} \left[ -\partial^* S_{w}^{n+1} + \partial_s S_{w} \right] \phi_{p,w,h}^{n+1} + \sum_{T_i \in T} \int_{T_i} \left[ \partial^* S_{w}^{n+1} - \partial_s S_{w} \right] \phi_{p,w,h}^{n+1} \]
\[ + \sum_{T_i \in T} \left( \left\| K \frac{1}{2} \nabla e_{p,w}^{n+1} \right\|^2_{\Omega,0} + h^2 \left\| K \frac{1}{2} \nabla^2 e_{p,w}^{n+1} \right\|^2_{\Omega,0} \right) + \sum_{F_i \in F} f(k_p) \left( \| e_{p,w}^{n+1} \|^2_{F_i,0} + \| e_{p,w}^{n+1} \|^2_{F_i,0} \right) \]
\[ \leq \sum_{\alpha=w,n} \left[ C \left( \frac{5}{2\lambda_\alpha} + \frac{3f(k_p)}{2\sigma_\alpha f(k_p)} \right) \lambda_\alpha \| K \frac{1}{2} \nabla\tilde{p}_{a}^{n+1} \|^2_{\Omega,\infty} \right] \sum_{T_i \in T} \| e_{s,h}^{n+1} \|^2_{T_i,0} \]
\[ + \sum_{\alpha=w,n} \left[ C \frac{5\lambda_\alpha}{2\lambda_\alpha} \sum_{T_i \in T} \left\| K \frac{1}{2} \nabla e_{p,w}^{n+1} \right\|^2_{T_i,0} + C \left( \frac{3\sigma_\alpha C_2^2 \tilde{C}^2}{2} + C \frac{5\theta_\alpha^2 \lambda_\alpha^2}{\lambda_\alpha} \right) \right] \left( h^{-2} \| e_{p,w}^{n+1} \|^2_{\Omega,0} + \| \nabla e_{p,w}^{n+1} \|^2_{\Omega,0} \right) \]
\[ + C \frac{3\lambda_\alpha^2}{2\sigma_\alpha} \left( \left\| K \frac{1}{2} \nabla e_{p,w}^{n+1} \right\|^2_{\Omega,0} + h^2 \left\| K \frac{1}{2} \nabla^2 e_{p,w}^{n+1} \right\|^2_{\Omega,0} \right) \]

(60)

To estimate the first term, we consider (28) and write for the error:

\[ \sum_{T_i \in T} \int_{T_i} (P_{c}^{n+1} - p_c) \psi_{p_c} = \sum_{T_i \in T} \int_{T_i} (p_{c,eq}(S_{w}^{n+1}) - p_{c,eq}(S_{w})) \psi_{p_c} - \sum_{T_i \in T} \int_{T_i} \tau (\partial^* S_{w}^{n+1} - \partial_s S_{w}) \psi_{p_c} \]

Extending the equation and using \( \psi_{p_c} = \partial e_{s,h}^{n+1} \) one gets:

\[ \sum_{T_i \in T} \int_{T_i} (e_{p,c}^{n+1} + e_{p,c}^{n+1}) \partial^* e_{s,h}^{n+1} = \sum_{T_i \in T} \int_{T_i} (p_{c,eq}(S_{w}^{n+1}) - p_{c,eq}(S_{w}) + p_{c,eq}(S_{w}) - p_{c,eq}(S_{w})) \partial^* e_{s,h}^{n+1} \]
\[ - \sum_{T_i \in T} \int_{T_i} \tau (\partial^* e_{s,h}^{n+1} + (\partial^* - \partial_s) S_{w}^{n+1} + \partial_s e_{s}^{n+1}) \partial^* e_{s,h}^{n+1} \]

(61)

Continuing with (60), rearranging the first two sums leads to:

\[ S_1 + P_1 = \sum_{T_i \in T} \int_{T_i} \phi \left[ \partial^* e_{s,h}^{n+1} + (\partial^* - \partial_s) S_{w}^{n+1} + \partial_s e_{s}^{n+1} \right] (e_{p,c}^{n+1} - e_{p,c}^{n+1}) = \]
\[ \sum_{T_i \in T} \int_{T_i} \phi \left[ -\partial^* e_{s,h}^{n+1} e_{p,c}^{n+1} + (\partial^* - \partial_s) S_{w}^{n+1} e_{p,c}^{n+1} + \partial_s e_{s}^{n+1} e_{p,c}^{n+1} \right] = \]
\[ = PS_{1,1} + PS_{1,2} + PS_{1,3} \]

Using (61) in \( PS_{1,1} \) one gets:

\[ PS_{1,1} = \phi \sum_{T_i \in T} \int_{T_i} e_{p,c}^{n+1} \partial^* e_{s,h}^{n+1} - \phi \sum_{T_i \in T} \int_{T_i} (p_{c,eq}(S_{w}^{n+1}) - p_{c,eq}(S_{w}^{n+1}) + p_{c,eq}(S_{w}) - p_{c,eq}(S_{w})) \partial^* e_{s,h}^{n+1} \]
\[ + \phi \sum_{T_i \in T} \int_{T_i} \tau (\partial^* e_{s,h}^{n+1} + (\partial^* - \partial_s) S_{w}^{n+1} + \partial_s e_{s}^{n+1}) \partial^* e_{s,h}^{n+1} \]
\[ = PS_{1,11} + PS_{1,12} + PS_{1,13} \]

We estimate the first term by Hölder's and Young's inequality:

\[ PS_{1,1,1} \leq \frac{\phi^2}{2\epsilon_{1,1,1}} \sum_{T_i \in T} \int_{T_i} \| e_{p,e}^{n+1} \|_T^2 + \frac{\epsilon_{1,1,1}}{2} \sum_{T_i \in T} \| \partial^- e_{s,h}^{n+1} \|_{T_i,0}^2 \]

For the second term we write:

\[ PS_{1,1,2} = -\phi \sum_{T_i \in T} \int_{T_i} (p_{c,eq}(s_{w}^{n+1}) - p_{c,eq}(s_{w}^{n})) \partial^- e_{s,h}^{n+1} - \phi \sum_{T_i \in T} \int_{T_i} (p_{c,eq}(s_{w}^{n+1}) - p_{c,eq}(s_{w})) \partial^- e_{s,h}^{n+1} \]

\[ = PS_{1,1,2,1} + PS_{1,1,2,2} \]

which is estimated as:

\[ PS_{1,1,2,1} = \frac{\phi}{2} \sum_{T_i \in T} \int_{T_i} \| p_{c,eq}'(\xi) \|_{T_i,0} \| e_{s,h}^{n+1} - e_{s,h}^{n} \|_{T_i,0} \]

\[ PS_{1,1,2,2} \leq \frac{\phi}{2} \sum_{T_i \in T} \int_{T_i} \| p_{c,eq}'(\xi) \|_{T_i,0} \| e_{s,h}^{n+1} - e_{s,h}^{n} \|_{T_i,0} \]

Using Young's Inequality and Lipschitz continuity gives:

\[ PS_{1,1,2,2} \leq \frac{\epsilon_{1,1,2}}{2} \sum_{T_i \in T} \| e_{s,h}^{n+1} \|_{T_i,0}^2 + \frac{L_{p,c,eq}^2 \phi^2}{2\epsilon_{1,1,2}} \sum_{T_i \in T} \| e_{s,h}^{n+1} \|_{T_i,0}^2 \]

For \( PS_{1,1,3} \) one continues with:

\[ PS_{1,1,3} = \phi \| \partial^- e_{s,h}^{n+1} \|_{T_i,0}^2 + \phi \sum_{T_i \in T} \int_{T_i} (\partial^- - \partial_t) s_{w}^{n+1} \partial^- e_{s,h}^{n+1} + \phi \sum_{T_i \in T} \int_{T_i} \partial_t e_{s,h}^{n+1} \partial^- e_{s,h}^{n+1} \]

\[ = PS_{1,1,3,1} + PS_{1,1,3,2} + PS_{1,1,3,3} \]

With the help of a Taylor expansion, the consistency error in \( PS_{1,1,3,2} \) can be approximated as:

\[ \frac{1}{\Delta t} (s_{w}^{n+1} - s_{w}^{n}) - \partial_t s_{w}^{n+1} = \Delta t \partial_t s_{w}^{n+1} + \mathcal{O}(\Delta t^2) \]

which leads to:

\[ PS_{1,1,3,2} \leq \frac{\tau^2 \phi^2}{2\epsilon_{1,1,3,2}} \sum_{T_i \in T} \| (\partial^- - \partial_t) s_{w}^{n+1} \|_{T_i,0}^2 + \frac{\epsilon_{1,1,3,2}}{2} \sum_{T_i \in T} \| \partial^- e_{s,h}^{n+1} \|_{T_i,0}^2 \]

\[ \leq \frac{\epsilon_{1,1,3,2}}{2} \sum_{T_i \in T} \| \partial^- e_{s,h}^{n+1} \|_{T_i,0}^2 + \frac{\tau^2 \phi^2}{2\epsilon_{1,1,3,2}} \sum_{T_i \in T} \| \partial_t e_{s,h}^{n+1} \|_{T_i,0}^2 \]

For the last term, we use Young's inequality:

\[ PS_{1,1,3,3} \leq \frac{\epsilon_{1,1,3,3}}{2} \sum_{T_i \in T} \| \partial^- e_{s,h}^{n+1} \|_{T_i,0}^2 + \frac{\tau^2 \phi^2}{2\epsilon_{1,1,3,3}} \sum_{T_i \in T} \| \partial_t e_{s,h}^{n+1} \|_{T_i,0}^2 \]

The remaining terms are estimated similar to the ones shown before, leading to:

\[ PS_{1,2} \leq \frac{\epsilon_{1,2}}{2} \sum_{T_i \in T} \| e_{p,c,h}^{n+1} \|_{T_i,0}^2 + \frac{\phi^2}{2\epsilon_{1,2}} \Delta t^2 \sum_{T_i \in T} \| \partial_t s_{w}^{n+1} \|_{T_i,0}^2 \]
Putting estimates (62), (63), (64), (65), (66), (68), (69) and (70) together gives:

\[
\frac{|p_{c, eq}|}{2} \sum_{T_i \in T} \partial^- \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{|p_{c, eq}|}{2} \sum_{T_i \in T} \frac{1}{\Delta t} \|e_{s,h}^{n+1} - e_{s,h}^n\|^2_{T_i,0} + \frac{\phi_T}{2} \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \left( \phi_T - \frac{\epsilon_{1,1,3,2}}{2} - \frac{\epsilon_{1,1,3,3}}{2} - \frac{\epsilon_{1,1,1}}{2} \right) \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} \\
\leq \left( \frac{\epsilon_{1,2}}{2} + \frac{\epsilon_{1,3}}{2} \right) \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{\epsilon_{1,1,2}}{2} \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{L_p^2 \phi_T^2}{2 \epsilon_{1,1,2}} \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \left( \frac{3 \tau \phi^2}{2} + \frac{\phi_T^2}{2} \right) \Delta t^2 \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} + \left( \frac{3 \tau \phi}{2} + \frac{\phi_T}{2} \right) \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \frac{3 \phi_T}{2} \sum_{T_i \in T} \int_{T_i} \|e_{s,h}^{n+1}\|^2_{T_i,0} 
\]

Choosing

\[
\epsilon_{1,1,3,2} = \epsilon_{1,1,3,3} = \epsilon_{1,1,1} = \frac{\phi_T}{3}
\]

this leads to:

\[
\frac{|p_{c, eq}|}{2} \sum_{T_i \in T} \partial^- \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{|p_{c, eq}|}{2} \sum_{T_i \in T} \frac{1}{\Delta t} \|e_{s,h}^{n+1} - e_{s,h}^n\|^2_{T_i,0} + \frac{\phi_T}{2} \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} \\
\leq \left( \frac{\epsilon_{1,2}}{2} + \frac{\epsilon_{1,3}}{2} \right) \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{\epsilon_{1,1,2}}{2} \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{L_p^2 \phi_T^2}{2 \epsilon_{1,1,2}} \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \left( \frac{3 \tau \phi}{2} + \frac{\phi_T}{2} \right) \Delta t^2 \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} + \left( \frac{3 \tau \phi}{2} + \frac{\phi_T}{2} \right) \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \frac{3 \phi_T}{2} \sum_{T_i \in T} \int_{T_i} \|e_{s,h}^{n+1}\|^2_{T_i,0} 
\]

Which can be used to further estimate (60):

\[
\frac{|p_{c, eq}|}{2} \sum_{T_i \in T} \partial^- \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{|p_{c, eq}|}{2} \sum_{T_i \in T} \frac{1}{\Delta t} \|e_{s,h}^{n+1} - e_{s,h}^n\|^2_{T_i,0} + \frac{\phi_T}{2} \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \sum_{T_i \in T} \left( \left| K \frac{\nabla e_{p,w,h}}{2 \lambda_a} \right| + \left| K \frac{\nabla e_{p,w,h}}{2 \lambda_a} \right| \right) \sum_{F_i \in F} \frac{f(k_p)}{|F|} \left( \|e_{s,h}^{n+1}\|^2_{T_i,0} + \|e_{s,h}^{n+1}\|^2_{T_i,0} \right) \\
\leq \sum_{\alpha = w, h} \left[ C \left( \frac{5}{2 \lambda_a} + \frac{3 f(k_p)}{2 \sigma_a f(k_p)} \right) \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \epsilon_{1,1,2} \right] \\
+ \left( \frac{\epsilon_{1,2}}{2} + \frac{\epsilon_{1,3}}{2} \right) \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{L_p^2 \phi_T^2}{2 \epsilon_{1,1,2}} \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} + \frac{3 \phi_T}{2} \sum_{T_i \in T} \|e_{s,h}^{n+1}\|^2_{T_i,0} \\
+ \left( \frac{3 \tau \phi}{2} + \frac{\phi_T^2}{2} \right) \Delta t^2 \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} + \left( \frac{3 \tau \phi}{2} + \frac{\phi_T^2}{2} \right) \sum_{T_i \in T} \|\partial^- e_{s,h}^{n+1}\|^2_{T_i,0} 
\]
\[
+ \sum_{\alpha=w,n} \left[ C \frac{5\lambda_\alpha}{2\lambda_\alpha} \left| K^2 \nabla e_{p,\alpha}^{n+1} \right|_{T_i,0}^2 + \frac{3f(k_\alpha)}{2\sigma_\alpha f(k_p)} \left| K^2 \nabla e_{p,\alpha}^{n+1} \right|_{T_i,0}^2 + h^2 \left| \nabla e_{p,\alpha}^{n+1} \right|_{\Omega,0}^2 \right]
\]

\[
+ C \frac{5\lambda_\alpha}{2\lambda_\alpha} \sum_{T \in \mathcal{T}} \left| K^2 \nabla e_{p,\alpha}^{n+1} \right|_{T_i,0}^2 + C \left( \frac{3\sigma_\alpha C^2 2}{2} + \frac{5^2 \lambda_\alpha^2 C^2 2}{\lambda_\alpha} \right) \left( h^{-2} \left| e_{p,\alpha}^{n+1} \right|_{\Omega,0}^2 + \left| \nabla e_{p,\alpha}^{n+1} \right|_{\Omega,0}^2 \right)
\]


\[
\sum_{T \in \mathcal{T}} \left| e_{p,\alpha}^{n+1} \right|_{T_i,0}^2 \leq \sum_{T \in \mathcal{T}} \left| \nabla e_{p,\alpha}^{n+1} \right|_{T_i,0}^2 + \sum_{F \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \left( \left| e_{p,\alpha}^{n+1} \right|_{F_i,0}^2 + \left| e_{p,\alpha}^{n+1} \right|_{T_i,0}^2 \right)
\]

Choosing

\[
\epsilon_{1,2} = \epsilon_{1,3} = \frac{1}{2}
\]

and

\[
\epsilon_{1,1,2} = \frac{\|p_{c,eq}\|_\phi}{2}
\]

leads to the final result.

Using Lemma 4.5 we can now deduce the following theorem about the convergence of the scheme:

**Theorem 4.1.** For sufficiently large \( \sigma_n \) and \( \sigma_w \) there exists an constant \( C \) independent of \( h \) and \( \Delta t \), such that the following estimate holds:

\[
\sum_{T \in \mathcal{T}} \left| e_{s,h}^{N+1} \right|_{T_i,0}^2 + \sum_{n=0}^{N} \sum_{T \in \mathcal{T}} \left| e_{s,h}^{n+1} - e_{s,h}^n \right|_{T_i,0}^2 + \Delta t \sum_{n=0}^{N} \sum_{T \in \mathcal{T}} \left| \partial^+_e e_{s,h}^{n+1} \right|_{T_i,0}^2
\]

\[
+ \Delta t \sum_{n=0}^{N} \sum_{T \in \mathcal{T}} \left( \left| \nabla e_{p,w,h}^{n+1} \right|_{T_i,0}^2 + \left| \nabla e_{p,w,h}^{n+1} \right|_{T_i,0}^2 \right) + \Delta t \sum_{n=0}^{N} \sum_{F \in \mathcal{F}} \frac{f(k_p)}{|F_i|} \left( \left| e_{p,w,h}^{n+1} \right|_{F_i,0}^2 + \left| e_{p,w,h}^{n+1} \right|_{T_i,0}^2 \right)
\]

\[
\leq C \sum_{T \in \mathcal{T}} \left| e_{s,h}^0 \right|_{T_i,0}^2 + C \Delta t \frac{h^2 \min(k_1, l_1, s, 1)}{k^2_{s,2}} \left( 1 + \frac{1}{k^2_2} \right) \sum_{n=0}^{N} \left| s_n(t) \right|_{\Omega,0}^2 + C \Delta t^3
\]

\[
+ C \Delta t \frac{h^2 \min(k_1, l_1, s, 1)}{k^2_{s,2}} \sum_{n=0}^{N} \left| \partial_s s_n(t) \right|_{\Omega,0}^2 + C \Delta t^2 \frac{h^2 \min(k_1, l_1, p_c, 1)}{k^2_{p_c,2}} \sum_{n=0}^{N} \left| p_n(t) \right|_{\Omega,0}^2 + C \Delta t \frac{h^2 \min(k_1, l_1, p_c, 2)}{k^2_{p_c,2}} \sum_{n=0}^{N} \left| p_c(t) \right|_{\Omega,0}^2
\]

**Proof.** Starting from (59) one has:

\[
\frac{|p_{c,eq}|_\phi}{2} \sum_{T \in \mathcal{T}} \partial^+ \left| e_{s,h}^{n+1} \right|_{T_i,0}^2 + \frac{|p_{c,eq}|_\phi}{2} \sum_{T \in \mathcal{T}} \frac{1}{\Delta t} \left| e_{s,h}^{n+1} - e_{s,h}^n \right|_{T_i,0}^2 + \phi \sum_{T \in \mathcal{T}} \left| \partial^- e_{s,h}^{n+1} \right|_{T_i,0}^2
\]
\[ + \frac{1}{2} \sum_{T_i \in T} \left( \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,w,h} \|^2_{T_i,0} + \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,w} \|^2_{T_i,0} \right) + \frac{1}{2} \sum_{F_i \in F} f(k_p) \left( \| e^{n+1}_{p,w,h} \|^2_{F_i} + \| e^{n+1}_{p,w,h} \|^2_{F_i} \right) \]
\[ \leq \left( C + \frac{|p_{e,eq}| \phi}{4} \right) \sum_{T_i \in T} C \| e^{n+1}_{s,h} \|^2_{T_i,0} + C \| e^{n+1}_{s} \|^2_{\Omega,0} + \frac{C \Delta t}{2} \sum_{F_i \in F} f(k_p) \left( \| e^{n+1}_{p,w,h} \|^2_{F_i} + \| e^{n+1}_{p,w,h} \|^2_{F_i} \right) \]
\[ + C \| \partial e^{n+1}_{s} \|^2_{\Omega,0} + C h^2 \| \nabla e^{n+1}_{s} \|^2_{\Omega,0} + \sum_{\alpha=w,n} \left[ C \| \nabla e^{n+1}_{p,w} \|^2_{\Omega,0} + C h^2 \| e^{n+1}_{p,w} \|^2_{\Omega,0} + C h^2 \| \nabla^2 e^{n+1}_{p,w} \|^2_{\Omega,0} \right] \]

Multiplying the inequality by \( \Delta t \) summing over \( n = 0, \ldots, N \) and absorbing \| e^{n+1}_{s,h} \|^2_{\Omega,0} one gets:
\[ \left( \frac{|p_{e,eq}| \phi}{2} - C \Delta t \right) \sum_{T_i \in T} \| e^{n+1}_{s,h} \|^2_{T_i,0} + \frac{|p_{e,eq}| \phi}{2} \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| e^{n+1}_{s,h} - e^{n}_{s,h} \|^2_{T_i,0} + \frac{\phi_{T}}{2} \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s,h} \|^2_{T_i,0} \]
\[ + \frac{\Delta t}{2} \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \left( \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,w,h} \|^2_{T_i,0} + \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,w} \|^2_{T_i,0} \right) + \frac{\Delta t}{2} \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \sum_{F_i \in F} f(k_p) \left( \| e^{n+1}_{p,w,h} \|^2_{F_i} + \| e^{n+1}_{p,w,h} \|^2_{F_i} \right) \]
\[ \leq \left( \frac{|p_{e,eq}| \phi}{2} - C \Delta t \right) \sum_{T_i \in T} \| e^{n+1}_{s,h} \|^2_{T_i,0} + \left( C + \frac{\phi_{T}}{4} \right) \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} C \| e^{n+1}_{s,h} \|^2_{T_i,0} + C \| e^{n+1}_{s} \|^2_{\Omega,0} + C \Delta t \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s,h} \|^2_{T_i,0} \]
\[ + C \Delta t \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s} \|^2_{\Omega,0} + C \Delta t \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \nabla e^{n+1}_{s} \|^2_{\Omega,0} + C \Delta t \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \nabla^2 e^{n+1}_{s} \|^2_{\Omega,0} \]

Using Grönwall’s inequality for \( \Delta t \) small enough, there exists a constant independent of \( \Delta t, h, k_p \) or \( k_s \) such that:
\[ \left( \frac{|p_{e,eq}| \phi}{2} - C \Delta t \right) \sum_{T_i \in T} \| e^{n+1}_{s,h} \|^2_{T_i,0} + \left( C + \frac{\phi_{T}}{4} \right) \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} C \| e^{n+1}_{s,h} \|^2_{T_i,0} + C \| e^{n+1}_{s} \|^2_{\Omega,0} + C \Delta t \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s,h} \|^2_{T_i,0} \]
\[ + \frac{\Delta t}{2} \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \left( \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,w,h} \|^2_{T_i,0} + \| K^{\frac{1}{2}} \nabla e^{n+1}_{p,w} \|^2_{T_i,0} \right) + \frac{\Delta t}{2} \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \sum_{F_i \in F} f(k_p) \left( \| e^{n+1}_{p,w,h} \|^2_{F_i} + \| e^{n+1}_{p,w,h} \|^2_{F_i} \right) \]
\[ \leq \left( \frac{|p_{e,eq}| \phi}{2} - C \Delta t \right) \sum_{T_i \in T} \| e^{n+1}_{s,h} \|^2_{T_i,0} + \left( C + \frac{\phi_{T}}{4} \right) \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} C \| e^{n+1}_{s,h} \|^2_{T_i,0} + C \| e^{n+1}_{s} \|^2_{\Omega,0} + C \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s,h} \|^2_{T_i,0} \]
\[ + C \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s} \|^2_{\Omega,0} + C \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \nabla e^{n+1}_{s} \|^2_{\Omega,0} + C \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \nabla^2 e^{n+1}_{s} \|^2_{\Omega,0} \]

Using (30), (31) and (32) and the triangle inequality for the error terms in \( p_w = p_n - p_c \) we can write:
\[ \left( \frac{|p_{e,eq}| \phi}{2} - C \Delta t \right) \sum_{T_i \in T} \| e^{n+1}_{s,h} \|^2_{T_i,0} + \left( C + \frac{\phi_{T}}{4} \right) \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| e^{n+1}_{s,h} - e^{n}_s \|^2_{T_i,0} + \frac{\phi_{T}}{2} \Delta t \sum_{n=0}^{N} T_i \in T \sum_{n=0}^{\frac{N}{T_i}} \| \partial e^{n+1}_{s,h} \|^2_{T_i,0} \]
Table 1: Properties for Test problem 1

<table>
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<tr>
<th>Phase Properties</th>
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<tbody>
<tr>
<td>dyn. viscosity water</td>
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<tr>
<td>dyn. viscosity oil</td>
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<tr>
<td>density water</td>
</tr>
<tr>
<td>density oil</td>
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<tr>
<td>res. water saturation</td>
</tr>
<tr>
<td>res. oil saturation</td>
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<tr>
<td>porosity</td>
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<tr>
<td>retardation coefficient</td>
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<table>
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<tr>
<th>Brooks-Correy Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>entry pressure</td>
</tr>
<tr>
<td>pore size distr. index</td>
</tr>
</tbody>
</table>

\[
+ \frac{\Delta t}{2} \sum_{n=0}^{N} \sum_{T_i \in T} \left( \| K^2 \nabla e_{p,w,h}^{n+1} \|^2_{T_i,0} + \| K^2 \nabla e_{p,o,h}^{n+1} \|^2_{T_i,0} \right) + \frac{\Delta t}{2} \sum_{n=0}^{N} \sum_{T_i \in T} f(k_p) \left( \| e_{p,w,h}^{n+1} \|^2_{F_i,0} + \| e_{p,o,h}^{n+1} \|^2_{F_i,0} \right)
\leq C \sum_{T_i \in T} \| e_{s,h}^0 \|^2_{T_i,0} + C \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{s+1,l_s})}}{k_{s2}^2} \| s_n(t) \|^2_{\Omega,ls} + C \Delta t^3
\]

\[
+ C \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{s+1,l_s})}}{k_{s2}^2} \| \partial_t s_n(t) \|^2_{\Omega,ls} + C h^2 \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{s+1,l_s})-2}}{k_{s4}^{2}} \| s_n(t) \|^2_{\Omega,ls}
\]

\[
+ C \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{p_n+1,l_{p_n}})-2}}{k_{p_n4}^{2}} \| p_n(t) \|^2_{\Omega,l_{p_n}} + C h^{-2} \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{p_n+1,l_{p_n}})}}{k_{p_n2}^{2}} \| p_n(t) \|^2_{\Omega,l_{p_n}}
\]

\[
+ C \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{p_n+1,l_{p_n}})-4}}{k_{p_n4}^{2}} \| p_n(t) \|^2_{\Omega,l_{p_n}} + C \Delta t \sum_{n=0}^{N} \frac{h^{2 \min(k_{p_n+1,l_{p_n}})-2}}{k_{p_n2}^{2}} \| p_n(t) \|^2_{\Omega,l_{p_n}}
\]

from where follows the stated estimate.

5 Numerical Experiments

**Test problem** In this problem, we verify the convergence rates in a numerical experiment. For that we consider a analytical solution to compute the \( L^2 \)- and \( H^1 \)-errors and do a sequential refinement of the grid.

We consider the domain \( \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2 \) and \( t \in [0, 1] \). The properties are chosen as follows: The right hand side in the equations are chosen such, that the exact solution for \( t \geq 0 \) equals:

\[
p_n(t, x, y) = \frac{1}{4} \cos((x + y)\pi - t) + \frac{1}{2}
\]

\[
S_w(t, x, y) = \frac{1}{4} \sin((x + y)\pi - t) + \frac{1}{2}
\]

\[
p_c(t, x, y) = p_{c,eq}(S_w(t, x, y)) - \tau \partial_t S_w(t, x, y)
\]
We chose \( \theta = 1 \), which gives a NIPG scheme, and the penalty parameters as \( \sigma_w = \sigma_n = 10 \). For the implementation we used DUNE-PDELab [3, 4]. We start with a refinement of two elements in each direction, and double the number of elements after each simulation. At \( t = 1 \) with polynomial order of 2 for all unknowns and a refinement of 1024 elements this results in Figures 1a, 1b and 1c. The plot over the diagonal is depicted at Figure 1d.

In Tables 2 and 3 and Figures 2c and 2d we show the calculated error for piecewise linear polynomials for the non-wetting pressure, capillary pressure and wetting saturation. We are able to obtain the stated convergence rates of Theorem 4.1. Afterwards we repeat the same simulation we piecewise quadratic polynomials. In Tables 4 and 5 and Figures 2c and 2d one can see the calculated errors and convergence rates. Once again it is possible to obtain the stated convergence rates.

### 6 Conclusions

We proved the convergence of a fully implicit interior penalty discontinuous Galerkin scheme. For that we showed existence using a fixed point argument, and proved convergence in dependency of the polynomial degree the mesh-size and the time-discretization.
Table 4: $H^1$ error for piecewise quadratic polynomials at $t = 1$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$H^1$ error $p_n$</th>
<th>rate</th>
<th>$H^1$ error $p_c$</th>
<th>rate</th>
<th>$H^1$ error $S_w$</th>
<th>rate</th>
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<tr>
<td>0.5</td>
<td>0.101565</td>
<td></td>
<td>0.216777</td>
<td></td>
<td>0.119231</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.0255655</td>
<td>1.986</td>
<td>0.0539399</td>
<td>2.009</td>
<td>0.0304559</td>
<td>1.957</td>
</tr>
<tr>
<td>0.125</td>
<td>0.00636018</td>
<td>2.01</td>
<td>0.0139612</td>
<td>1.932</td>
<td>0.00762787</td>
<td>1.996</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.00158656</td>
<td>2.004</td>
<td>0.00349532</td>
<td>1.997</td>
<td>0.00190618</td>
<td>2.001</td>
</tr>
<tr>
<td>0.03125</td>
<td>0.000396073</td>
<td>2.003</td>
<td>0.000873346</td>
<td>2.001</td>
<td>0.000476439</td>
<td>2.000</td>
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</tbody>
</table>

Table 5: $L^2$ error for piecewise quadratic polynomials at $t = 1$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$L^2$-error $p_n$</th>
<th>rate</th>
<th>$L^2$-error $p_c$</th>
<th>rate</th>
<th>$L^2$-error $S_w$</th>
<th>rate</th>
</tr>
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<tr>
<td>0.5</td>
<td>0.0112288</td>
<td></td>
<td>0.0161979</td>
<td></td>
<td>0.0234577</td>
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</tr>
<tr>
<td>0.25</td>
<td>0.00226715</td>
<td>2.476</td>
<td>0.00252684</td>
<td>3.205</td>
<td>0.00611598</td>
<td>1.918</td>
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<tr>
<td>0.125</td>
<td>0.000530639</td>
<td>2.136</td>
<td>0.000519114</td>
<td>2.434</td>
<td>0.00155186</td>
<td>1.970</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.000130055</td>
<td>2.040</td>
<td>0.000119633</td>
<td>2.17</td>
<td>0.000389647</td>
<td>1.991</td>
</tr>
<tr>
<td>0.03125</td>
<td>0.000322681</td>
<td>2.015</td>
<td>0.0000291316</td>
<td>2.053</td>
<td>0.0000975277</td>
<td>1.997</td>
</tr>
</tbody>
</table>

Figure 2: Convergence rates for Test problem 1
In upcoming work we plan to analyze the linearization techniques and investigate the convergence of nonlinear solver. Furthermore we will consider different extensions to the model as e.g. hysteretical effects, heterogenities with entry pressure and space-time adaptivity for better convergence.

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