Uniqueness of weak solutions for a pseudo-parabolic equation modeling two phase flow in porous media

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Uniqueness of weak solutions for a pseudo-parabolic equation modeling two phase flow in porous media

by

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Uniqueness of weak solutions for a pseudo-parabolic equation modeling two-phase flow in porous media

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Abstract. In this paper, we prove the uniqueness of weak solutions for a pseudo-parabolic equation modeling two-phase flow in a porous medium, where dynamic effects are included in the capillary pressure.

Keywords: Pseudo-parabolic equation, dynamic capillary pressure, two-phase flow, weak solution, uniqueness.

1 Introduction

We consider a two-phase flow model in a porous medium which includes dynamic effects in the capillary pressure. If determined under equilibrium condition, the difference of the pressures in the two phases (wetting and non-wetting) is a decreasing function in terms of the (say) wetting phase saturation $u$,

$$p_n - p_w = p_c(u).$$

(1)

Such models are analyzed in [4, 13]. However, the equilibrium assumption does not hold in several real life applications, e.g. in paper drying and also some recent works provides the experimental evidence for non-equilibrium assumption (see [2, 5, 12]). In this case, dynamic effects have to be included. An example in this sense is the model proposed in [10],

$$p_n - p_w = p_c(u) - \tau \partial_t u.$$

(2)

Here $\tau > 0$ is a parameter accounting for the dynamic effects.

Inspired by this, we consider here a simplified model obtained from mass conservation and the Darcy law for each phase (see [1, 11]), assuming that the medium is fully saturated by the two phases. Generally, this leads to a system of two equations, a parabolic one for the wetting phase saturation, and an elliptic one for the total flux.
Here we assume the latter known, and focus on the mass balance for the wetting phase. With the phase pressure difference in (2), the model reads (see [3, 15])

\[ \partial_t u + \nabla \cdot F(u) + \nabla \cdot (H(u)\nabla(p_c(u) - \tau \partial_t u)) = 0. \] (3)

It is defined in a bounded, connected domain \( \Omega \) in \( \mathbb{R}^d \) \((d = 1, 2, 3)\) and for a given time interval \((0, T]\). Further, by \( \overline{\Omega} \) we mean the closure of \( \Omega \), and by \( \partial \Omega \) its boundary.

In the above, \( F \) and \( H \) denote the water fractional flow function and the capillary induced diffusion function, and \( p_c \) is the equilibrium capillary pressure (see (1) and (2)). These are non-linear functions defined for the physically relevant interval \( u \in [0, 1] \). For the mathematical analysis, we extend \( H, p_c \) and \( F \) continuously to the entire \( \mathbb{R} \).

Throughout this paper, we assume

- **A1**: \( H: \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous, and a \( h_0 \) exists such that
  \[ 0 < h_0 \leq H(u) \text{ if } 0 < u < 1, \text{ and } H(u) = h_0 \text{ otherwise.} \] (4)

- **A2**: \( p_c \in C^1(\mathbb{R}) \) is a decreasing function, and \( m_p, M_p \) exist such that \( 0 < m_p \leq |p'_c(u)| \leq M_p < +\infty \), for all \( u \in \mathbb{R} \).

- **A3**: \( F: \mathbb{R} \rightarrow \mathbb{R}^d \) is Lipschitz continuous.

- **A4**: The functions \( F \) and \( H \) are bounded, \( |F(u)| + |H(u)| \leq M, \) for all \( u \in \mathbb{R} \).

To complete the model, the initial and boundary conditions are given

\[ u(\cdot, 0) = u^0, \text{ and } u = 0, \text{ at } \partial \Omega. \] (5)

For the initial data, we assume

- **A5**: \( u^0 \in C^{0,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1] \), \( u^0 = 0 \) at \( \partial \Omega \).

Further, the domain \( \Omega \) has a smooth boundary:

- **A6**: \( \Omega \) is a \( C^{1,\alpha} \) domain.

Finally we mention that throughout this paper, \( C \) denotes a generic constant.

In this paper we prove the uniqueness of a weak solution to (3) with the given initial and boundary conditions. This solution solves

**Problem \( P \)**: Find \( u \in W^{1,2}(0, T; W_0^{1,2}(\Omega)) \), such that \( u(\cdot, 0) = u^0, \nabla \partial_t u \in L^2(0, T; L^2(\Omega)^d) \) and

\[
\int_0^T \int_\Omega \partial_t u \phi dxdt - \int_0^T \int_\Omega F(u) \cdot \nabla \phi dxdt \\
- \int_0^T \int_\Omega H(u) \nabla p_c(u) \cdot \nabla \phi dxdt + \tau \int_0^T \int_\Omega H(u) \nabla \partial_t u \cdot \nabla \phi dxdt = 0, \] (6)
for any $\phi \in L^2(0,T;W^{1,2}_0(\Omega))$.

Note the nonlinearity appearing in the highest order term, $\nabla \cdot (H(u)\nabla \partial_t u)$. For the linear case, when only $\Delta \partial_t u$ is appearing, existence and uniqueness results are obtained in [8]. Also we refer to [3, 15, 16] for the existence of weak solutions to the nonlinear Problem P. However, the uniqueness in this case is still an open issue. To overcome the difficulty related to the nonlinearity appearing in the highest order term, we use an additional unknown $p$ to represent the phase pressure difference. To avoid the confusion between $u$ given by Problem P and the solution pair of the following system, we denote the saturation latter by $s$. With this, (3) can be rewritten formally as the system

$$\partial_t s + \nabla \cdot F(s) + \nabla \cdot (H(s)\nabla p) = 0,$$

$$p = p_c(s) - \tau \partial_t s. \tag{8}$$

For the sake of presentation we finally assume

- **A7**: $p_c(0) = 0$.

**Remark:** A7 can be avoided: if $p_c(0) \neq 0$, one defines $p = p_c(s) - p_c(0) - \tau \partial_t s$ in (8).

For this system, a weak solution is a pair $(s,p)$ solving

**Problem P**: Find $s \in W^{1,2}(0,T;L^2(\Omega))$ and $p \in L^2(0,T;W^{1,2}_0(\Omega))$, such that $s(\cdot,0) = u^0$ and

$$\langle \partial_t s, \phi \rangle - \langle F(s), \nabla \phi \rangle - \langle H(s)\nabla p, \nabla \phi \rangle = 0, \tag{9}$$

$$\langle p, \psi \rangle = \langle p_c(s), \psi \rangle - \tau \langle \partial_t s, \psi \rangle, \tag{10}$$

for any $\phi \in L^2(0,T;W^{1,2}_0(\Omega))$, $\psi \in L^2(0,T;L^2(\Omega))$.

Following [7], the Problem P and Problem $P_s$ are equivalent. Specifically, under the non-degeneracy assumptions A1-A4, if $(s,p)$ is a solution to Problem $P_s$, then $u = s$ solves Problem P. Conversely, if $u$ solves Problem P, then $(s,p)$ with $s = u$ and $p$ defined in (8) solves Problem $P_s$. This equivalence will be exploited below.

## 2 Uniqueness

In this section we prove the uniqueness of weak solutions for Problem $P_s$. The equivalence result also provides uniqueness for Problem P. The first step is to obtain the essential boundedness of $\nabla p$:

**Lemma 2.1.** The solution component $p$ of Problem $P_s$ satisfies $\nabla p \in L^\infty((0,T] \times \Omega)$. 

Proof. As shown in [3], if $u$ solves Problem P, one has $u \in L^\infty(0,T;W_0^{1,2}(\Omega))$, $\partial_t u \in L^\infty(0,T;W_0^{1,2}(\Omega))$. In view of the equivalence result, from (10) one has $p \in L^\infty(0,T;W_0^{1,2}(\Omega))$.

Further, for almost every $t$, one can use (8) to eliminate $\partial_t s$ in (7),

$$\nabla \cdot (H(s)\nabla p) + \frac{1}{\tau} p = \frac{1}{\tau} p_c(s) - \nabla \cdot F(s).$$  (11)

Note that given $s \in L^\infty(0,T;W_1^{1,2}(\Omega))$, (11) is a linear elliptic equation in $p$ with the right hand side $\frac{1}{\tau} p_c(s) - \nabla \cdot F(s) \in L^2(\Omega)$. Then as in [4], one obtains

$$\|p\|_{L^\infty(0,T;C^0(\bar{\Omega}))} \leq C.$$  (12)

Note: The inequality can also be obtained by Theorem 14.1 in [14].

Then for almost every $t \in (0,T]$ and for almost every $x, y \in \Omega$, $x \neq y$, by using (8), we obtain

$$\frac{p(t, x) - p(t, y)}{|x - y|^\alpha} = \frac{p_c(u)(x) - p_c(u)(y)}{|x - y|^\alpha} - \tau \frac{\partial_t u(t, x) - u(t, y)}{|x - y|^\alpha}.$$  (13)

Since $p \in C^{0,\alpha}(\bar{\Omega})$, this implies

$$\sup_{x \neq y \in \Omega} \frac{|p(t, x) - p(t, y)|}{|x - y|^\alpha} \leq C.$$  (14)

Defining $w : (0,T] \times \Omega \to \mathbb{R}$:

$$w = \frac{s(t, x) - s(t, y)}{|x - y|^\alpha}.$$  (15)

Clearly, a $\xi$ exists such that

$$\tau \partial_t w - p_c'(\xi) w = \frac{p(t, x) - p(t, y)}{|x - y|^\alpha}.$$  (16)

Multiplying $w$ in the two sides of (16), and integrating from 0 to $\tilde{t}$ ($\tilde{t}$ is arbitrary in $(0,T]$) gives us

$$\frac{\tau}{2} \tilde{w}^2 - \int_0^{\tilde{t}} p_c'(\xi) w^2 dt = \int_0^{\tilde{t}} \frac{p(t, x) - p(t, y)}{|x - y|^\alpha} w dt + \tau \left( w(0, \cdot) \right)^2.$$  (17)

According to $A2$ and $A5$, by using Cauchy Schwarz’s inequality, one has

$$\tau w^2 \leq \tau \left( \frac{u^0(x) - u^0(y)}{|x - y|^\alpha} \right)^2 + \int_0^{\tilde{t}} \left( \frac{p(t, x) - p(t, y)}{|x - y|^\alpha} \right)^2 + \int_0^{\tilde{t}} w^2 dt$$

$$\leq C + \int_0^{\tilde{t}} w^2 dt.$$  (18)
By Gronwall’s inequality we obtain
\[ w^2 \leq C, \] (19)
for any time \( \tilde{t} \in (0, T] \).
This implies
\[ \left| \frac{s(t, x) - s(t, y)}{|x - y|\alpha} \right| \leq C, \quad \text{for almost every } x, y \in \Omega, \text{ for every } t. \] (20)
Then we consider \( \Omega = \Omega_c \cup \Omega_0 \), here \( m(\Omega_0) = 0 \). For any \( x, y \in \Omega_c \), one has
\[ |s(t, x) - s(t, y)| \leq C|x - y|\alpha, \quad \text{for every } t. \] (21)
Further, for any \( x \in \Omega_0 \), we define
\[ s(t, x) = \lim_{y \to x} s(t, y). \] (22)
This yields \( s \in C^{0,\alpha}(\tilde{\Omega}) \) ([6]).
Finally, by Theorem 8.33 and Corollary 8.35 in Chapter 8 in [9], we get
\[ |p|_{1,\alpha} \leq C(|p|_0 + |p_c(s)|_0 + |F(s)|_{0,\alpha}), \] (23)
implying \( \nabla p \in L^\infty((0, T] \times \tilde{\Omega}) \).

To show the uniqueness, let \( g \in L^2(\Omega) \) and define \( G_g \) as the solution of the elliptic problem (see [7]):
\[ -\tau \nabla (H(u)\nabla G_g) + G_g = g, \quad \text{in } \Omega, \] (24)
\[ G_g = 0, \quad \text{at } \partial \Omega, \] (25)
here \( u \) is the solution of Problem P. It is easy to show
\[ G_g \in W_0^{1,2}(\Omega) \text{ and } \|G_g\|_{W^{1,2}(\Omega)} \leq C\|g\|_{L^2(\Omega)}. \] (26)
We have the following result:

**Theorem 2.1.** Under the assumptions A1-A7, Problem P has a unique solution \( u \).

**Proof.** As discussed above, the uniqueness of a solution to Problem P follows directly if uniqueness is proved for Problem \( P_s \). Assume now Problem \( P_s \) has two solutions, \((u, p_u), (v, p_v)\), then one has
\[ (\partial_t(u - v), \phi) - (H(u)\nabla p_u - H(v)\nabla p_v, \nabla \phi) - (F(u) - F(v), \nabla \phi) = 0, \] (27)
(p_u - p_v, \psi) - (p_c(u) - p_c(v), \psi) + \tau(\partial_t(u - v), \psi) = 0, \quad (28)

for any \( \phi \in L^2(0, T; W_0^{1,2}(\Omega)) \), \( \psi \in L^2(0, T; L^{1,2}(\Omega)) \).

Further, we rewrite the above system as follows:

\[
(\partial_t(u - v), \phi) - (F(u) - F(v), \nabla \phi) - (H(u)(\nabla p_u - \nabla p_v), \nabla \phi) - (\nabla (H(u) - H(v))\nabla p_v, \nabla \phi) = 0, \quad (29)
\]

(\( F \approx -u \)).

Further, substituting (36) into (35) gives

\[
(p_u - p_v, \psi) - (p_c(u) - p_c(v), \psi) + \tau(\partial_t(u - v), \psi) = 0. \quad (30)
\]

Taking \( g = u - v \) into (24), one gets \( G_{u-v} \in W_0^{1,2}(\Omega) \) and

\[
(G_{u-v}, \lambda) + \tau(H(u)\nabla G_{u-v}, \nabla \lambda) = (u - v, \lambda), \quad (31)
\]

for any \( \lambda \in W_0^{1,2}(\Omega) \).

Further, we have

\[
\|G_{u-v}\|_{W^{1,2}(\Omega)} \leq C\|u - v\|_{L^2(\Omega)}. \quad (32)
\]

Setting \( \phi = G_{u-v} \) in (29) gives

\[
(\partial_t(u - v), G_{u-v}) - (H(u)(\nabla p_u - \nabla p_v), \nabla G_{u-v}) - ((H(u) - H(v))\nabla p_v, \nabla G_{u-v}) = 0, \quad (33)
\]

and choose \( \lambda = p_u - p_v \) in (31), we find that

\[
-H(u)\nabla (p_u - p_v), \nabla G_{u-v}) = -\frac{1}{\tau}(u - v, p_u - p_v) + \frac{1}{\tau}(G_{u-v}, p_u - p_v). \quad (34)
\]

Substituting (34) into (33) leads to

\[
(\partial_t(u - v), G_{u-v}) - \frac{1}{\tau}(u - v, p_u - p_v) + \frac{1}{\tau}(G_{u-v}, p_u - p_v)
- ((H(u) - H(v))\nabla p_v, \nabla G_{u-v}) - (F(u) - F(v), \nabla G_{u-v}) = 0. \quad (35)
\]

Taking the test function \( \psi = G_{u-v} \) in (30), we obtain

\[
(\partial_t(u - v), G_{u-v}) + \frac{1}{\tau}(G_{u-v}, p_u - p_v) = \frac{1}{\tau}(p_c(u) - p_c(v), G_{u-v}). \quad (36)
\]

Further, substituting (36) into (35) gives

\[
\frac{1}{\tau}(p_c(u) - p_c(v), G_{u-v}) - \frac{1}{\tau}(u - v, p_u - p_v)
- ((H(u) - H(v))\nabla p_v, \nabla G_{u-v}) - (F(u) - F(v), \nabla G_{u-v}) = 0. \quad (37)
\]

Setting \( \psi = u - v \) in (30) gives

\[
-\frac{1}{\tau}(u - v, p_u - p_v) = (\partial_t(u - v), u - v) - \frac{1}{\tau}(p_c(u) - p_c(v), u - v). \quad (38)
\]
Combining (37) and (38) to eliminate $-\frac{1}{\tau}(u - v, p_u - p_v)$ and integrating the resulting equation in time over $(0, t)$, with $t \in (0, T]$ arbitrary lead to
\[
\int_0^t (\partial_t (u - v), u - v) dz - \frac{1}{\tau} \int_0^t (u - v, p_c(u) - p_c(v)) dz \\
- \int_0^t (F(u) - F(v), \nabla G_{u-v}) dz \\
= - \frac{1}{\tau} \int_0^t (G_{u-v}, p_c(u) - p_c(v)) dz + \int_0^t ((H(u) - H(v))\nabla p_v, \nabla G_{u-v}) dz.
\] (39)

We proceed by estimating each term called $T_1, T_2, T_3, T_4, T_5$. For $T_1$, since $u(0) = v(0) = u^0$, one has
\[
\int_0^t \int_\Omega \partial_t (u - v)(u - v) dxdz = \frac{1}{2} \|(u - v)(t)\|_{L^2(\Omega)}^2.
\] (40)

For $T_2$, by A2, one obtains
\[
- \frac{1}{\tau} \int_0^t \int_\Omega (u - v)(p_c(u) - p_c(v)) dxdz \geq 0.
\] (41)

Similarly, since $F$ is Lipschitz continuous, and according to (32), we get the estimates for $T_3$ and $T_4$:
\[
\left| \frac{1}{\tau} \int_0^t \int_\Omega (F(u) - F(v))\nabla G_{u-v} dxdz \right| \leq \frac{C}{\tau} \int_0^t \|u - v\|_{L^2(\Omega)}^2 dz,
\] (42)

\[
\left| \frac{1}{\tau} \int_0^t \int_\Omega G_{u-v}(p_c(u) - p_c(v)) dxdz \right| \leq \frac{C}{\tau} \int_0^t \|u - v\|_{L^2(\Omega)}^2 dz.
\] (43)

Finally, for the last term, we use Cauchy Schwarz inequality and $\nabla p \in L^\infty((0, T] \times \bar{\Omega})$, then we have
\[
\int_0^t \int_\Omega (H(u) - H(v))\nabla p_v \cdot \nabla G_{u-v} dxdz \leq \int_0^t \int_\Omega |(H(u) - H(v))\cdot \nabla G_{u-v}| |\nabla p_v| dxdz \\
\leq C \int_0^t \int_\Omega |(H(u) - H(v))\cdot \nabla G_{u-v}| dxdz \\
\leq C \int_0^t \|u - v\|_{L^2(\Omega)}^2 dz.
\] (44)

Summarizing the above leads to
\[
\|(u - v)(\cdot, t)\|_{L^2(\Omega)}^2 \leq C \int_0^t \|u - v\|_{L^2(\Omega)}^2 dz.
\] (45)

Since $t$ is arbitrary, the Gronwall Lemma shows that
\[
\|(u - v)(t)\|_{L^2(\Omega)}^2 = 0, \text{ for all } t,
\] (46)

implying the uniqueness for Problem $P_s$ and therefore for Problem $P$ as well. □
3 Conclusion

In this paper, we prove the uniqueness of a weak solution to a pseudo-parabolic equation modeling the two-phase flow in porous media, and including dynamic effects in the capillary pressure. The major difficulty is in the nonlinearity of the third order derivative term. The proof uses the equivalence of the original problem with a mixed form of it. By this, the third order derivative term is avoided.

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References


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