Direct experimental visualization of the global Hamiltonian progression of two-dimensional Lagrangian flow topologies from integrable to chaotic state

Citation for published version (APA):

Baskan, O., Speetjens, M. F. M., Metcalfe, G., & Clercx, H. J. H. (2015). Direct experimental visualization of the global Hamiltonian progression of two-dimensional Lagrangian flow topologies from integrable to chaotic state. *Chaos*, 25, 103106-1/9. [103106]. <https://doi.org/10.1063/1.4930837>

DOI:

[10.1063/1.4930837](https://doi.org/10.1063/1.4930837)

Document status and date:

Published: 01/01/2015

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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Citation: *Chaos* **25**, 103106 (2015); doi: 10.1063/1.4930837

View online: <http://dx.doi.org/10.1063/1.4930837>

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Direct experimental visualization of the global Hamiltonian progression of two-dimensional Lagrangian flow topologies from integrable to chaotic state

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(Received 12 January 2015; accepted 31 August 2015; published online 14 September 2015)

Countless theoretical/numerical studies on transport and mixing in two-dimensional (2D) unsteady flows lean on the assumption that Hamiltonian mechanisms govern the Lagrangian dynamics of passive tracers. However, experimental studies specifically investigating said mechanisms are rare. Moreover, they typically concern *local* behavior in specific states (usually far away from the integrable state) and generally expose this *indirectly* by dye visualization. Laboratory experiments explicitly addressing the *global* Hamiltonian progression of the Lagrangian flow topology *entirely* from integrable to chaotic state, i.e., the fundamental route to efficient transport by chaotic advection, appear non-existent. This motivates our study on experimental visualization of this progression by *direct* measurement of Poincaré sections of passive tracer particles in a representative 2D time-periodic flow. This admits (i) accurate replication of the experimental initial conditions, facilitating true one-to-one comparison of simulated and measured behavior, and (ii) direct experimental investigation of the ensuing Lagrangian dynamics. The analysis reveals a close agreement between computations and observations and thus experimentally validates the full global Hamiltonian progression at a great level of detail. © 2015 AIP Publishing LLC.

[<http://dx.doi.org/10.1063/1.4930837>]

The Lagrangian equations of motion of passive tracers advected by two-dimensional (2D) unsteady flows define a Hamiltonian system. This Hamiltonian structure has fundamental consequences for the transport and mixing properties of such flows and forms the backbone of many studies on this matter. However, despite its relevance, experimental studies seeking to validate this Hamiltonian structure are few and far between. Moreover, they typically investigate the Hamiltonian nature of Lagrangian transport indirectly and locally by monitoring the evolution of patches of dye. The present experimental study, on the other hand, directly and globally measures the Lagrangian transport characteristics of the flow by tracking small tracer particles distributed over the entire flow domain. This explicitly visualizes essentially Hamiltonian features of tracer dynamics that are in close agreement with theory and computations. Thus, the present study experimentally validates the Hamiltonian structure of 2D unsteady flows in a more direct sense and at a greater level of detail compared to existing studies.

I. INTRODUCTION

Advection of passive tracers in a two-dimensional (2D) incompressible steady flow defines an autonomous Hamiltonian system with one degree of freedom, where the stream function acts as the Hamiltonian. Here, passive

tracers are restricted to individual streamlines and, in consequence, always perform non-chaotic motion. Introducing unsteadiness to the flow field causes breakdown of this situation and thus enables (yet not guarantees) chaotic tracer dynamics. This has first been demonstrated for the blinking-vortex flow in the seminal paper by Aref¹ and has since been investigated in numerous studies on a great variety of 2D fluid systems.^{2–14} Essentially, similar dynamics occurs in the continuum regime of 2D unsteady granular flows^{15–19} and cross-sections of certain 3D steady flows.^{20–24}

Unsteadiness is (due to its simplicity) commonly introduced by time-periodic variation of the flow field with a certain period time T , where $T=0$ corresponds with the steady (and thus non-chaotic) state.^{2,25} Increasing T from zero generically causes the characteristic Hamiltonian disintegration of the global streamline pattern at $T=0$ (integrable state) into regular and chaotic regions in the Poincaré section of the flow following the famous Kolmogorov-Arnold-Moser (KAM) and Poincaré-Birkhoff theorems.²⁵ Here, regular regions comprise (arrangements of) island-like structures known as “KAM tori.”

Investigations on this Hamiltonian progression in 2D incompressible flows to date almost exclusively concern numerical studies. A substantial body of work does exist on experimental analysis of 2D chaotic advection and its impact on transport processes. However, such studies focus predominantly on ramifications and signatures of chaotic advection as, e.g., (exponential) stretching and folding of material

elements,^{26–28} transport enhancement and anomalous diffusion,^{29–33} crossing of transport barriers,³⁴ and the formation of persistent patterns.³⁵

Laboratory experiments dedicated specifically to the Hamiltonian dynamics and kinematic mechanisms that underly the above phenomena (e.g., emergence of periodic points, formation and breakdown of KAM tori, and manifold dynamics) are rare, on the other hand. Moreover, they typically concern local behavior in specific states (usually far away from the integrable state) instead of the entire global progression from integrable to chaotic state and generally expose this via dye visualizations.^{10–14,24} Particularly, detailed dye visualizations of KAM tori are those in the 3D steady counterparts to 2D time-periodic flow examined in various studies.^{20–23} Similar visualization experiments, i.e., on correlations between KAM tori and segregation patterns, have been performed for 2D time-periodic granular flows.^{15–18} However, *direct* experimental visualization of the *global* Hamiltonian progression of 2D Lagrangian flow topologies *entirely* from integrable to chaotic state is, to the best of our knowledge, non-existent. This motivates our study on visualization of this progression in a representative flow: the 2D time-periodic Rotated Arc Mixer (RAM).

Laboratory experiments will be reconciled with theory through comparison of the measured flow and Lagrangian dynamics with simulated predictions. To this end, computations will be performed using an analytical solution to the formal 2D RAM flow and a data-fitted 2D approximation to the measured flow field (so as to account for experimental and modeling imperfections). This enables detailed comparative investigations.

This study, besides to fluid mechanics, contributes to the broader field of experimental state-space visualization and analysis of dynamical systems in two ways. First, existing studies in this context generally concern visualization of (chaotic) attractors in non-Hamiltonian non-fluid systems, e.g., magneto-elastic^{36–38} and micro-electromechanical⁴³ oscillators, gravity-driven motion of objects,³⁹ electrical circuits,^{40,41} and nonlinear pendulums.⁴² Our study is dedicated to visualization of essentially Hamiltonian dynamics. Second, said non-fluid systems are finite-dimensional, i.e., their state is described by a finite (and typically small) set of variables (e.g., the tip position of an oscillator³⁷). Fluid systems, on the other hand, unite characteristics of both finite-dimensional and infinite-dimensional systems and thus also in this sense belong to a different class. Their state is infinite-dimensional by consisting of an infinite union of fluid-parcel positions; these positions, in turn, are each governed by a finite-dimensional (Hamiltonian) system. Hence, the evolution of a single initial state of fluid systems is analogous to the *simultaneous* evolution of *all* initial states of said non-fluid systems. Moreover, the physical and state spaces are the same for fluid systems. These properties facilitate direct and full visualization of Hamiltonian dynamics with individual experiments in 2D fluid systems.

II. PROBLEM DEFINITION

The RAM is given schematically in Fig. 1(a) and consists of a circular domain of radius R enclosed by a wall

FIG. 1. 2D rotated arc mixer: (a) circular flow domain bounded by stationary (black) and 4 moving (grey) arcs (opening angle $\Delta = \pi/4$); (b) streamline patterns induced by individual motion of arcs 1–4 in clockwise direction (black: arc 1; grey: arcs 2–4).

composed of stationary (black) and moving (grey) arcs.⁴⁴ The four moving arcs, offset by an angle $\Theta = \pi/2$ and each spanning an angle $\Delta = \pi/4$, drive the internal flow by viscous drag as, e.g., in various lid-driven cavity flows investigated in literature.^{3,6,9,11} Clockwise steady motion only of arc 1 at an angular velocity Ω gives the dark streamline pattern in Fig. 1(b); individual activation of arcs 2–4 gives shown reorientations of this pattern. 2D time-periodic flow is accomplished by successive activation of arcs 1–4 each for a duration τ , resulting in $T = 4\tau$ as total period time. Dimensional analysis yields the Reynolds number $Re = \Omega R^2/\nu$, with ν the kinematic viscosity, and dimensionless period time $\mathcal{T} = \Omega T$ as system parameters. Our study is restricted to Stokes flow ($Re = 0$), which can be done without loss of generality, leaving $\mathcal{T}$ as sole parameter.

The dynamics of passive tracers, described by their current position $\mathbf{x}(t)$ and released at initial position $\mathbf{x}_0$, is governed by the Hamiltonian equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad (1)$$

with $H(x, y, t) = H(x, y, t + T)$ the corresponding (time-periodic) Hamiltonian.^{1,25} Here, H can (using polar coordinates (r, θ)) be arc-wise constructed from the stream function $\psi(x, y)$ corresponding with arc 1, i.e., $H(r, \theta)|_{\text{arc}n} = \psi(r, \theta - (n-1)\Theta)$ for $1 \leq n \leq 4$, where ψ is available in closed form through Hwu *et al.*⁴⁵ This analytical solution will be employed in two ways: (i) description of the Stokes flow in the 2D RAM; (ii) determination of a Stokes-flow approximation to the experimental flow so as to reconcile observed and predicted behavior (Sec. IV).

The tracer dynamics is examined by Poincaré sections $X(\mathbf{x}_0) = \{\mathbf{x}_0, \mathbf{x}_1, \dots\}$, with $\mathbf{x}_p = \mathbf{x}(pT)$ the position after p periods, versus dimensionless period time $\mathcal{T}$:

- Limit $\mathcal{T} \rightarrow 0$ yields an autonomous Hamiltonian consisting of the average of the arc-wise stream functions: $H(r, \theta) = \frac{1}{4} \sum_{n=1}^4 \psi(r, \theta - (n-1)\Theta)$. This corresponds with simultaneous activation of all arcs and defines the integrable limit (Fig. 2(a)).
- Non-zero $\mathcal{T} > 0$ introduces unsteadiness and breaks the integrable state. This manifests itself in the characteristic

FIG. 2. Hamiltonian progression of the Lagrangian flow topology of the 2D time-periodic RAM from integrable to (nearly) chaotic state: simulated Poincaré sections using the analytical 2D Stokes flow (top) versus experiments (center) and simulations using the Stokes-flow fit to the experimental surface flow (bottom). The experimental analysis is discussed in Secs. III–V; the Stokes-flow fit and adjustment in period times are discussed in Sec. IV.

Hamiltonian breakdown of the global island of the integrable state into a progressively smaller central island surrounded by emerging island chains and a chaotic sea (Figs. 2(b)–2(d)).

The Poincaré sections in Figs. 2(b)–2(d) are simulated by numerical integration of (1) using the analytical stream function ψ following Hwu *et al.*⁴⁵ and 20 initial positions $\mathbf{x}_0$ consisting of two equidistant distributions of 10 tracers each on the x - and y -axis (origin at domain center). The key objective of our study is to experimentally visualize and validate this progression by *direct* measurement of the Poincaré sections via tracking of tracer particles.

The tracking approach has key advantages over dye visualization for the present kind of experiments. Particles, namely, mark *individual* fluid parcels and, inherently, visualize Lagrangian entities in the Poincaré section from the first period on. Dye patterns, on the other hand, converge on such entities strictly only in the limit of infinite time, meaning

that finite-time dye traces can basically only approximate Lagrangian entities (Compare, e.g., with the evolution of concentration patterns in distributive mixing.⁴⁶). An alternative dye-visualization method exists in time-averaging of successive dye patterns.²³ However, this strictly also requires an infinite sequence. Moreover, employment of tracer particles enables accurate replication of the experimental initial conditions in numerical simulations, which facilitates true one-to-one comparison and validation of features and behavior. A further argument in favor of tracer particles here is that molecular diffusion is far less relevant than for dye. This is a crucial practical factor for the long-term visualization experiments in our study (Sec. III).

III. EXPERIMENTAL PROCEDURE

The experimental apparatus is shown schematically in Fig. 3. It consists of a shallow circular tank of depth $h = 10$ mm and radius $R = 250$ mm with apertures at the

FIG. 3. Schematic of the experimental facility: test section (dark grey) enclosed by external annular region (light grey) holding 4 computer-controlled motor-belt systems (black) that create the moving arcs. Poincaré sections are measured by tracking of tracer particles floating on the fluid interface (dots indicate initial positions) by an overhead CCD camera.

location of the four arcs. The moving arcs are realized by elastic belts gently pressed against the apertures from the outside (so as to seal-off the flow domain) and driven by electrical motors. Note that the belts are somewhat larger than the apertures so as to ensure that they follow the curvature of the circular boundary upon being pressed against it from the outside. The belt-motor systems (Maxon Motor, Germany) are placed in an external annular region enclosing the flow domain. Two layers of fluid are used to as closely as possible create a 2D flow in the fluid surface:⁴⁷ glycerol-water solution (66% by volume) at the bottom layer to dampen bottom-wall friction effects; silicon oil (type AK10000 by Wacker, Germany) at the top layer serving as the fluid of interest. These liquids are immiscible so that the layers remain separated throughout the experiment. Time-periodic flow is achieved by sequential actuation of the belts by a computerized motion-control system at a constant belt velocity $U = 5 \text{ mm/s}$ with a maximum variation of 0.07%. This gives, together with the kinematic viscosity $\nu = 0.01 \text{ m}^2/\text{s}$ of the silicon oil, to good approximation Stokes flow ($Re = 0.1$).

Direct measurement of the Poincaré sections is achieved by combining the successive positions of tracer particles in exactly the same way as the numerical Poincaré sections are attained. To this end, polystyrene foam particles (diameter $d_p = 1.5 \text{ mm}$ and density $\rho_p = 500 \text{ kg/m}^3$) are released on the free surface of the top layer ($\rho_p < \rho_{\text{siliconoil}} = 970 \text{ kg/m}^3$ ensures they remain floating throughout the experiment). The typical response time of particles to changes in velocity is estimated at $\tau_p = d_p^2 \rho_p / 18 \rho \nu = 6.4 \mu\text{s}$, which is negligible compared to the typical flow time scale $\tau_f = R/U = \Omega^{-1} = 50 \text{ s}$, meaning they are indeed passively advected by the flow.⁴⁸ Tracer particles are released at the same 20 positions as in the numerical simulations, and their subsequent positions after each period are recorded by a CCD camera synchronized with the motion-control system (MegaPlus ES2020, Princeton Instruments, USA) placed above the fluid surface (Fig. 3). The particle positions are determined from the imagery in sub-pixel accuracy by a dedicated

particle-detection code implemented in the high-level programming language MATLAB and combined into experimental Poincaré sections. Note that detection of particles suffices to construct Poincaré sections; actual tracking of individual particles is unnecessary. One pixel corresponds for given camera resolution of $1600 \times 1200 \text{ pixels}^2$ with approximately $0.3 \times 0.4 \text{ mm}^2$, meaning that an individual particle covers about $5 \times 4 \text{ pixels}^2$, which ensures reliable detection of the particle location. Experiments are run for 250 periods in all cases. The actual period time is $T = \Omega^{-1} T = 50T \text{ s}$, amounting for a typical dimensionless period time $T = 5$ to $T = 250 \text{ s}$ and a total duration of about 17 h. It is important to note that dye visualization over such extensive time spans is extremely difficult if not impossible due to molecular diffusion. This basically leaves tracer particles as the sole option for this kind of experiment. The results below demonstrate that this indeed enables successful visualizations.

Furthermore, velocity measurements by way of Particle-Image Velocimetry (PIV) are performed in the current laboratory setup using the approach following Baskan *et al.*⁴⁹ so as to support the analyses. This involves employment of the same optical setup and tracer particles as described above. Particle imagery has been processed with the commercial PIV package PIVview 3C Version 2.4 using interrogation windows of size $24 \times 24 \text{ pixels}^2$ with an overlap factor of 50%. Consult Baskan *et al.*⁴⁹ for further details.

IV. EXPERIMENTAL FLOW FIELD

One premise of the current analysis is that the experimental surface flow adequately represents the analytic 2D Stokes flow introduced in Sec. II. Examinations of the surface flow in Baskan *et al.*⁴⁹ revealed a close agreement with 2D Stokes flow. However, for the current study, compliance with these conditions is far more critical, since (experimental) visualization of the Lagrangian flow topology by passive tracers is a long-term process that is very sensitive to minute deviations. This necessitates further analysis.

Said premise holds true wherever the experimental surface flow admits expression as a 2D Stokes flow driven by azimuthal motion of the circular boundary. This is, expanding on Baskan *et al.*,⁴⁹ investigated below for the base flow corresponding with the first window. To this end, the azimuthal boundary condition is expressed in the generic form

$$u_\theta(1, \theta) = f(\theta) = \sum_{n=1}^{\infty} \alpha_n f_n(\theta), \quad (2)$$

with $f_k(\theta) = \delta(\theta - \theta_k)$, $0 \leq \theta_k < 2\pi$ a single angular position on the circular boundary and $\delta(\cdot)$ the Kronecker delta function ($f_n(\theta) = 1$ for $\theta = \theta_n$ and zero elsewhere) (Note, $u_r(1, \theta) = 0$ for all θ). This structure, by virtue of linearity of Stokes flows, carries over to the internal flow, yielding

$$\mathbf{u}(\mathbf{x}) = \sum_{n=1}^{\infty} \alpha_n \mathbf{u}_n(\mathbf{x}), \quad (3)$$

with $\mathbf{u}_n$ the elementary flow field given by the analytical solution of Hwu *et al.*⁴⁵ for boundary condition $u_r(1, \theta) = 0$ and $u_\theta(1, \theta) = f_n(\theta)$. Discrete approximation of (2) as

$$f(\theta) \approx \sum_{n=1}^N \alpha_n f_n^*(\theta), \quad (4)$$

with $f_n^*(\theta)$ the top-hat function ($f_n^*(\theta) = 1$ for $2\pi(n-1)/N \leq \theta < 2\pi n/N$ and zero elsewhere) enables determination of expansion coefficients α_n from the experimental flow field obtained through PIV via the least-squares method (see the Appendix). Fig. 4 gives the boundary profile $f(\theta)$ according to (4) thus attained (black solid) for $N=90$ in comparison with that of the true 2D configuration (grey dashed) (Refer to the Appendix for the particular setting of the order of approximation N). This reveals an overall good correspondence, giving a first indication that the surface flow indeed (largely) behaves as a 2D Stokes flow (Discrepancies with full 2D Stokes flow are discussed below.). It must be stressed that $f(\theta)$ does not represent the true experimental boundary condition, but the boundary condition of the *Stokes fit* to the surface flow. Hence, departures of $f(\theta)$ from the imposed boundary condition do not signify experimental imperfections. Moreover, $f(\theta)$ must not be interpreted in terms of physical characteristics of the flow. Its profile, instead of reflecting flow physics, most and for all is a consequence of fitting a 2D Stokes flow to an experimental surface flow that *not* everywhere behaves as such.

The Stokes fit is shown in Fig. 5 (top) and overall captures the experimental surface flow to a high degree of accuracy; appreciable deviations $\Delta \mathbf{u}_{\text{exp-fit}} = \mathbf{u}_{\text{exp}} - \mathbf{u}_{\text{fit}}$ inside the flow domain occur only very locally in the direct proximity of the driving window (Fig. 5, center). Deviations in the radial component u_r are concentrated in peaks at the window edges (indicated by arrows) and in a small patch just above the lower window edge; deviations in the azimuthal component u_θ are confined to a thin layer directly at the window. Hence, save these localized areas, the experimental surface flow to a high degree of approximation behaves as a 2D Stokes flow. This validates the aforementioned premise of the present study. However, important to note is that the Stokes fit differs essentially from the full 2D Stokes flow for

FIG. 4. Boundary profile (2) of the azimuthal velocity: Stokes-flow fit to the experimental surface flow versus full 2D Stokes flow.

FIG. 5. Experimental surface flow $\mathbf{u}_{\text{exp}}$ versus 2D Stokes-flow representations: Stokes fit $\mathbf{u}_{\text{fit}}$ following (3) (top); departure Stokes fit from experimental flow $\Delta \mathbf{u}_{\text{exp-fit}} = \mathbf{u}_{\text{exp}} - \mathbf{u}_{\text{fit}}$ (center); departure full 2D Stokes flow from experimental flow $\Delta \mathbf{u}_{\text{exp-2D}} = \mathbf{u}_{\text{exp}} - \mathbf{u}_{2D}$ (bottom). Arrows indicate window edges.

the physical boundary conditions (dashed profile in Fig. 4). Comparison of the latter with the surface flow, namely, reveals, in contrast with the Stokes fit, a significant departure in substantial parts of the domain (Fig. 5, bottom). This implies that, despite indeed largely behaving as a 2D Stokes flow, the experimental surface flow exhibits different flow characteristics in the direct proximity of the window.

Direct comparison of Stokes fit and full 2D Stokes flow in Fig. 6 (top) exposes two important features that may offer an explanation for the above observations: (i) the Stokes-fit velocity is relatively lower; (ii) the deviations closely correlate with the window edges (indicated by arrows). The overall slowing down is in part the result of viscous friction with the bottom wall; this effect is significantly reduced by the two-fluid layer yet can never be fully eliminated (Sec. III). The impact of viscous friction increases near the window edges due to the strong velocity gradients that occur here. This is likely to be aggravated by the formation of an additional vertical boundary layer on the stationary part of the

FIG. 6. Comparison full 2D Stokes flow $\mathbf{u}_{2D}$ versus Stokes fit $\mathbf{u}_{fit}$ to experimental surface flow: departure $\Delta\mathbf{u}_{2D-fit} = \mathbf{u}_{2D} - \mathbf{u}_{fit}$ (top); streamlines base flow/integrable state of $\mathbf{u}_{2D}$ (red) versus $\mathbf{u}_{fit}$ (blue) (center); corresponding orbit times of streamlines base flow (bottom). Arrows indicate window edges.

side wall in this region. Fluid inertia is a probable secondary factor by suppressing the actual fluid acceleration relative to its Stokes limit and thus tending to smooth the local velocity gradients near the window edges. This may somewhat mitigate said viscous friction yet at the same time reduce the viscous drag—and thus the effective driving velocity—at the window (which is proportional to the surfacial velocity gradients) that sets up the surface flow. Moreover, such gradient smoothing—and resulting deviation in velocity—will be most pronounced near the window edges, which may explain why the deviations “radiate away” from these regions.⁵² Conclusive establishment of the exact causes for the discrepancies requires more detailed analysis. This is beyond the present scope, however. Relevant here is mainly the demonstration of the 2D Stokes nature of the experimental surface flow everywhere outside the direct vicinity of the window.

Further examination reveals that the discrepancy of the experimental flow with the analytic flow primarily concerns

the magnitude of the flow yet not its direction, as evidenced in Fig. 6 (center) by the close resemblance of the streamline patterns (emanating from identical initial conditions on the x -axis) of both the base flow and integrable states of Stokes fit (blue) and full 2D Stokes flow (red). This strongly suggests that the Lagrangian dynamics will predominantly differ quantitatively in terms of distance traveled along a trajectory for a given time and thus closely relate via an offset in time. Fig. 6(e) gives the time T to complete one loop on each closed streamline of the base flow for the full 2D Stokes flow (black) versus the Stokes fit (grey) parameterized by the initial position on the x -axis (Note that only one side of the stagnation point need be considered.). This exposes a structurally shorter orbit time for the full 2D Stokes flow, or equivalently, a delay of the Stokes-fit case, by an approximately constant factor $T_{2D}/T_{fit} \approx 0.86$ (Fig. 6(f)). Accounting for this shift will be important for proper interpretation and comparison of the results on Lagrangian dynamics.

V. EXPERIMENTAL POINCARÉ SECTIONS

The above revealed that the Lagrangian dynamics of the base flows of the Stokes fit and the full 2D Stokes case closely correspond up to a temporal scaling factor: $t_{2D}/t_{Stokesfit} \approx T_{2D}/T_{Stokesfit} \approx 0.86$. This implies, given the full periodic flow being a composition of reoriented base flows, that the progressions of the Poincaré sections versus period time for the Stokes fit of the surface flow—and thus the experimental Poincaré sections—will closely follow that of the full 2D Stokes flow upon tuning the period times as

$$\frac{T_{fit}}{T_{exp}} = 1, \quad \frac{T_{2D}}{T_{exp}} = \frac{T_{2D}}{T_{fit}} = 0.86, \quad (5)$$

so as to account for said scaling factor.

The experimental Poincaré sections are shown for increasing T_{exp} in Fig. 2 (center) versus their simulated counterparts using the Stokes fit at identical period time $T_{fit} = T_{exp}$ (bottom) and the full 2D Stokes flow with T_{2D} rescaled following (5) (top). Comparison of the measured and predicted progressions reveals an excellent agreement (Note that time spans for the simulated Poincaré sections are chosen to ensure optimal visualization of features; the number of periods may thus vary and differ from the fixed 250 periods employed in the experiments (Sec. III).). The central island of the experimental progression clearly undergoes the same Hamiltonian breakdown from its original integrable state ($T_{exp} = 0$) to its strongly diminished state just before the onset of global chaos ($T_{exp} = 10$). Moreover, both these states as well as the intermediate states at $T_{exp} = 4$ and $T_{exp} = 8$ are in close agreement with their corresponding simulated states during this progression. This is strong evidence of the fact that simulated and measured dynamics result from the same fundamental (Hamiltonian) mechanisms. Furthermore, this substantiates the Stokes-flow nature of the experimental surface flow established in Sec. IV as well as its translation to the full 2D Stokes flow via a scaling factor.

The behavior near the integrable limit can be further examined via the rotation number

$$\mathcal{R}(x_0) = \lim_{P \rightarrow \infty} \frac{1}{P} \frac{\sum_{p=0}^{P-1} \Delta\theta_p}{2\pi}, \quad (6)$$

with $\Delta\theta_p = \theta_p - \theta_{p+1}$ and p the step number, describing the average step-wise rotation of a tracer about the origin (The employed definition of $\Delta\theta_p$ yields $\mathcal{R} > 0$ and $\mathcal{R} < 0$ in case of clock-wise and counter-clock-wise rotation, respectively.). Tracer motion diminishes with decreasing period time $\mathcal{T}$, implying $\lim_{\mathcal{T} \rightarrow 0} \mathcal{R}(x_0) = 0$ for all x_0 . This asymptotic behavior is demonstrated in Fig. 7 for experiments at $\mathcal{T}_{\text{exp}}$ as indicated (panel (a)) and simulations using the 2D Stokes field with corresponding $\mathcal{T}_{2D}$ according to (5) (panel (b)) for tracers released at the indicated positions x_0 on the x -axis. This reveals the distribution of $\mathcal{R}$ over the concentric streamlines that occur near the integrable limit (Fig. 2(a)) (Note that isolation of individual trajectories from experimental Poincaré sections for the evaluation of $\mathcal{R}$ is straightforward in this $\mathcal{T}$ -range.). Rotation numbers $\mathcal{R}_{\text{exp}}$ and $\mathcal{R}_{2D}$ both meet $\mathcal{R} > 0$, signifying counter-clockwise tracer rotation along with the rotor (Fig. 1(a)). Moreover, they closely agree with respect to magnitude and (in particular) qualitative dependence on initial position x_0 and approach the limit $\mathcal{R} = 0$ with decreasing $\mathcal{T}$ at a comparable rate. Minor quantitative differences exist in that the experiments asymptote somewhat faster towards the integrable limit.

The particular distribution of $\mathcal{R}$ over the concentric streamlines reveals that the tracer motion basically consists of two regimes separated by a “minimum-rotation” streamline at $r \approx 0.5$ (The streamline pattern can, for the purpose of this discussion, be treated as being axisymmetric, allowing substitution of x_0 by r). Development of a plateau in $\mathcal{R}$ towards the center signifies solid-body-like behavior ($d\theta/dt \sim \omega \leftrightarrow R \sim \omega T_{\text{step}}$); linear growth towards the boundary signifies shear-like behavior ($d\theta/dt \sim \omega r \leftrightarrow R \sim \omega T_{\text{step}} r \propto r$). Thus, the departure from integrability sets in via emergence of these coexisting fluid motions. Shear and solid-body flow are of comparable strength in the simulations for all T_{step} (maximum of $\mathcal{R} \approx 0.005$ revolutions per step in both

FIG. 7. Tracer dynamics near the integrable limit $\mathcal{T} \rightarrow 0$ investigated by rotation number $\mathcal{R}$ versus step time $\mathcal{T}_{\text{step}} = \mathcal{T}/4$: experiments versus simulations by full 2D Stokes flow with rescaling (5). Symbols differentiate $\mathcal{T}_{\text{step}}$: $\mathcal{T}_{\text{step}} = 0.05$ ($\diamond$); $\mathcal{T}_{\text{step}} = 0.1$ ($\triangleleft$); $\mathcal{T}_{\text{step}} = 0.15$ ($\circ$); $\mathcal{T}_{\text{step}} = 0.2$ ($\times$); $\mathcal{T}_{\text{step}} = 0.25$ ($*$); x_0 indicates initial tracer position on the x -axis.

regimes). The experiments, on the other hand, exhibit a relative intensification of shear versus solid-body flow with growing $\mathcal{T}_{\text{step}}$: $\mathcal{R}_{\text{shear}}/\mathcal{R}_{\text{solid-body}} \approx 1$ at $\mathcal{T}_{\text{step}} = 0.05$ to $\mathcal{R}_{\text{shear}}/\mathcal{R}_{\text{solid-body}} \approx 5/3$ at $\mathcal{T}_{\text{step}} = 0.25$. However, overall, the flow remains very weak; a tracer typically takes at least $R^{-1} \approx 0.005^{-1} = 200$ steps for one revolution. The close agreement between experiments and simulations near the integrable limit, notwithstanding minor quantitative differences, further substantiates the earlier finding that they are subject to the same (Hamiltonian) mechanisms.

The experiments also provide (circumstantial) evidence of island chains. The region surrounding the central island at $\mathcal{T}_{\text{fit}} = 8$ ($\mathcal{T}_{2D} = 6.9$) is, e.g., dominated by the two period-7 island chains highlighted in Figs. 8(a) and 8(c). The simulated outer island chain (blue in Figs. 8(a) and 8(c)) coincides well with the “blank zones” in the corresponding experimental Poincaré section at $\mathcal{T}_{\text{exp}} = 8$. This is demonstrated in Figs. 8(b) and 8(d) by inserting the simulated period-7 island chains of Figs. 8(a) and 8(c), respectively, in the experimental Poincaré section (black), revealing that both parts indeed fit like pieces of a puzzle. Moreover, an inner period-7 chain and accompanying chaotic band can be identified in the experimental results that coincide with a period-7 island chain in the simulations (red in Figs. 8(a) and 8(c)). This coincidence is demonstrated in Figs. 8(b) and 8(d) by overlaying these entities with the numerical island chains of Figs. 8(a) and 8(c), respectively. The mismatch in dynamical state

FIG. 8. Circumstantial experimental evidence for island chains: coincidence of the simulated outer (blue) and inner (red) period-7 chains using the Stokes fit to the surface flow (panel (a)) and the adjusted full 2D Stokes flow (panel (c)) with the “blank zones” of the experimental Poincaré section at $\mathcal{T}_{\text{exp}} = 8$ (right; simulated island chains of panels (a) and (c) inserted in, respectively, panels (b) and (d)).

(i.e., intact simulated islands versus partially disintegrated experimental islands), rather than signifying a fundamental difference, must be attributed to the high sensitivity—and intrinsic unpredictability—of such island chains (typically increasing with smaller size) to parametric variations and weak (experimental) disturbances (e.g., finite-size effects of tracer particles). They, namely, emanate from instability of resonant orbits of the original island and are therefore far less robust than the latter. The experimental period-7 chain may thus already be in a relatively higher state of disintegration, which is consistent with the fact that this chain is embedded in a chaotic band that coincides well with the simulated inner period-7 island chain. Hence, despite lack of one-to-one correspondence between all individual features, also for island chains, a close agreement between simulations and experiments is observed. Important to note is that this element of unpredictability in the actual state of the island chains is inherent in the nature of the system and not a consequence of experimental imperfections *per se*.

VI. CONCLUSIONS

This study provides (to the best of our knowledge) the first experimental investigation of the *global* Hamiltonian progression of the Lagrangian flow topology of 2D time-periodic flows *entirely* from integrable to chaotic state by *direct* measurements of Poincaré sections. To this end, the 2D time-periodic Rotated Arc Mixer has been adopted as representative flow. The analysis reveals a close agreement between simulated and measured dynamics and thus experimentally validates the Hamiltonian mechanisms that are assumed to govern the Lagrangian dynamics in the considered flow class.

The first analyses by the rotation number $\mathcal{R}$ lay the groundwork for quantitative experimental studies on the onset of chaos. Key to the latter are symmetry breaking and resonance of trajectories, which are inextricably linked to symmetry and mode locking. These “locking phenomena” admit quantification by (generalized definitions of) $\mathcal{R}$ and have thus been investigated theoretically and numerically in parametric studies by Lester *et al.*⁵¹ Corresponding laboratory experiments with the current setup are underway.

ACKNOWLEDGMENTS

This work has been supported by the Dutch Technology Foundation STW (Project No. 11054), which is part of The Netherlands Organisation for Scientific Research (NWO). We thank Freek van Uittert, Gerald Oerlemans, and Ad Holten for their invaluable technical assistance. The authors are indebted to the anonymous reviewers for their constructive and encouraging feedback.

APPENDIX: DETERMINING STOKES-FLOW FIT TO EXPERIMENTAL SURFACE FLOW

Objective is determination of expansion coefficients α_n in expansion (3) of the surface flow. This is achieved by the standard least-squares method, which in matrix notation yields the linear set of equations⁵⁰

$$A\alpha = b, \quad (A1)$$

with $\alpha = [\alpha_1, \dots, \alpha_N]$

$$A = \begin{bmatrix} u_{1,x}(x_1) & \cdots & u_{N,x}(x_1) \\ \vdots & & \vdots \\ u_{1,x}(x_M) & \cdots & u_{N,x}(x_M) \\ u_{1,y}(x_1) & \cdots & u_{N,y}(x_1) \\ \vdots & & \vdots \\ u_{1,y}(x_M) & \cdots & u_{N,y}(x_M) \end{bmatrix} \quad (A2)$$

and

$$b^T = [u_{x,\text{exp}}(x_1) \cdots u_{x,\text{exp}}(x_M), \\ u_{y,\text{exp}}(x_1) \cdots u_{y,\text{exp}}(x_M)], \quad (A3)$$

where x_m are the M data positions. The coefficients subsequently follow the from

$$\alpha = (A^T A)^{-1} A^T b \quad (A4)$$

and accomplish an orthogonal projection of the experimental field on the 2D Stokes flow.

The order of approximation N of expansion (4) is determined via the resolution of PIV. The employed settings according to Sec. III yield interrogation windows with relative size $(\Delta x/R, \Delta y/R) = (0.03, 0.04)$, signifying a relative spatial resolution of $\Delta = \max(\Delta x/R, \Delta y/R) = 0.04$. One $\Delta \times \Delta$ cell within the spatial grid thus defined can hold a circular boundary segment with maximum arc length of approximately $\Delta s = \sqrt{2}\Delta$. This determines the corresponding relative spatial resolution on the circular boundary and translates into a grid of $2\pi/\Delta s = \sqrt{2}\pi/\Delta \approx 110$ boundary segments. The latter sets the upper bound for N , since it slightly overestimates the true boundary resolution. The arc length of boundary segments is, namely, estimated by the diagonal of a $\Delta \times \Delta$ cell, while the actual segments are curved and thus slightly longer. Hence, N must be set somewhat below this upper bound for (4) to be consistent with the PIV resolution.

Important to note is that N cannot be determined through an unambiguous convergence criterion. The quality of the Stokes fit is, namely, determined by the degree to which it adequately captures that part of the experimental surface flow that behaves as a 2D Stokes flow. However, strict separation between surface regions with Stokes and non-Stokes behavior—and thus definition of said criterion—is impossible. Instead, convergence and quality of the Stokes fit has been examined by its sensitivity to variation of N in the range $80 \leq N \leq 110$. This revealed appreciable variations only in the direct window proximity, that is, the area identified as the non-Stokes-flow region in Sec. IV. Sensitivity to changes in N outside these areas proved marginal, on the other hand, signifying (sufficient) convergence of the Stokes fit to that part of the experimental surface flow that exhibits 2D Stokes behavior. Thus, N can basically be chosen arbitrarily in the examined range; here, $N = 90$ has been adopted.

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