Stochastic networked control systems with dynamic protocols

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STOCHASTIC NETWORKED CONTROL SYSTEMS WITH DYNAMIC PROTOCOLS

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ABSTRACT

We consider networked control systems in which sensors, controllers, and actuators communicate through a shared network that introduces stochastic intervals between transmissions, delays, and packet drops. Access to the communication medium is mediated by a protocol that determines which node (one of the sensors, one of the actuators, or the controller) is allowed to transmit a message at each sampling/actuator-update time. We provide conditions for mean exponential stability of the networked closed loop in terms of matrix inequalities, both for investigating the stability of given protocols, such as static round-robin protocols and dynamic maximum error first-try once discard protocols, and conditions to design new dynamic protocols. The main result entailed by these conditions is that, if the networked closed loop is stable for a static protocol, then we can provide a dynamic protocol for which the networked closed loop is also stable. The stability conditions also allow for obtaining an observer-protocol pair that reconstructs the state of a linear time invariant plant in a mean exponential sense and for less conservative stability results than other conditions previously appearing in the literature.

Key Words: Networked control systems, dynamic protocols, scheduling, stochastic systems.

I. INTRODUCTION

The proliferation of network communication systems in recent years has paved the way for important research in the area of networked control systems. This research area addresses control loops closed via a shared network that provides the medium for sensor, actuator, and controller nodes to communicate.

Walsh and co-authors [1] made strides in the analysis of control systems closed via a local area network, such as a controller-area network, an ethernet, and wireless 802.11 networks. The key assumptions in [1] are that there exists a bound on the interval between transmissions denoted by maximum allowable transfer interval (MATI), and that transmission delays and packet drops are negligible. In [1], an emulation set-up is considered in the sense that the controller

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Recently, [7] addressed a model of networked control systems with i.i.d. intervals between transmissions and stochastic delays for a class of quadratic protocols that is more general than MEF-TOD. Through a convex over-approximation approach, sufficient conditions were given for mean exponential stability. In [8], a method was proposed to design an observer-protocol pair to asymptotically reconstruct the states of a linear time invariant (LTI) plant, where the plant outputs are sent through a network with constant intervals between transmissions. The protocol to be designed can be viewed as a weighted version of the MEF-TOD.

In the present paper, we follow this line of research considering that the network imposes i.i.d. intervals between transmissions. We also take into account stochastic delays modeled as in [7] and packet drops. We consider that access to the network is mediated by a dynamic protocol specified as follows. Associated to each node, there is a set of quadratic state functions, which are evaluated at a given transmission time. The node allotted permission to transmit is the one corresponding to the least value of these quadratic state functions. These protocols are more general than the quadratic protocols considered in [7]; thus they are more general than the MEF-TOD protocol.

We establish two stability results for the networked control system. Both of these provide conditions in terms of linear matrix inequalities (LMIs) for investigating the stability in a mean exponential sense of given protocols and provide conditions in terms of BMIs to design quadratic state functions, specifying the dynamic protocol, that yield the networked closed loop stable. The first stability result allows one to prove that, if the networked closed loop is stable for a static protocol, then we can provide a dynamic protocol for which the networked closed loop is also stable. This is the main contribution of this paper and gives an analytical justification on why one should utilize dynamic protocols rather than static, while, e.g., in [1], this conclusion is only illustrated through simulation. The second stability result allows us to extend the work [8] to the case where transmission intervals are stochastic. We also address the relation of this stability result with the necessity of existence of a quadratic stochastic Lyapunov function that assures stability for the networked control system.

We illustrate through benchmark examples that the conditions in this paper are significantly less conservative than other conditions previously appearing in the literature.

A preliminary version of the results presented here appeared in a conference paper [9]. Besides including all of the formal proofs of the results, here we establish the connection between the second of our two main stability results and the existence of a quadratic stochastic Lyapunov function that assures stability for the networked control system.

The remainder of the paper is organized as follows. The networked control problem setup is given in Section II. The main results are stated in Section III. In Section IV, we compare our results with previous works. Concluding remarks are given in Section V. The proofs of the results are provided in the Appendix.

**Notation.** We denote by $I_n$ and $O_n$ the $n \times n$ identity and zero matrices, respectively, and by $\text{diag}(\{A_1, \ldots, A_n\})$ a block diagonal matrix with blocks $A_i$. For dimensionally compatible matrices $A$ and $B$, we define $(A, B) := [A^\top B^\top]^\top$.

**II. PROBLEM FORMULATION**

We start by introducing the networked control stability problem before showing that it can be cast into analyzing the stability of an impulsive system.

2.1 Networked control setup

We consider a networked control system for which sensors, actuators, and a controller are connected through a communication network, possibly shared with other users. The plant and controller are described by the following state-space model.

**Plant:**

\[
\dot{x}_p = A_p x_p + B_p \hat{u}, \quad y = C_p x_p
\]

(1)

**Controller:**

\[
\dot{x}_c = A_c x_c + B_c \hat{y}, \quad u = C_c x_c + D_c \hat{y}
\]

(2)

Following an emulation approach, we assume that the controller has been designed to stabilize the closed loop when the process and the controller are directly connected, i.e., $\hat{u}(t) = u(t)$, $\hat{y}(t) = y(t)$, $t \geq 0$, and we are interested in analyzing the effects of the network on the stability of the closed loop. Note that this assumption implies that $(A_p, C_p)$ is detectable and that $(A_c, B_c)$ is stabilizable. We denote the times at which a node transmits a message by \{ $t_k$, $k \in \mathbb{N}$ \}, and we assume that $\hat{u}$ and $\hat{y}$ are held constant between transmission times, i.e.,

\[
\hat{u}(t) = \hat{u}(t_k), \quad \hat{y}(t) = \hat{y}(t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{Z}_{\geq 0}
\]

(3)

We denote by $e$ the error signal between the process output and controller input ($\hat{y} - y$) and between the controller output and process input ($\hat{u} - u$). In particular,

\[
e := (\hat{y} - y, \hat{u} - u).
\]

(4)

We assume that, while $m := n_s + n_a$ nodes compete for the network, where $n_s$ and $n_a$ denote the number of sensor and actuator nodes, respectively, only one of them is allowed to transmit at each given transmission time. Hence, in our terminology, a single transmitting node can be associated with several entries of the process output $y$ or with several entries of the controller output $u$. We partition the process output vector as $y = (y_1, \ldots, y_m)$, the controller output vector as
\[ u = (u_1, \ldots, u_n), \] and the error vector as \[ e = (e_1, \ldots, e_n), \] where each \( e_i(t) \in \mathbb{R}^n \) is the error associated with node \( i \in M := \{1, \ldots, m\} \), i.e., \( e_i \) corresponds to a sensor node \( y_i \) if \( i \in \{1, \ldots, n_1\} \) and to an actuator node \( u_{i+n_1} \) if \( i \in \{n_1+1, \ldots, n+n_1\} \). The setup is shown in Figure 1. The state of the networked control system is defined by the vector \( x := (x_r, x_c, e) \), where \( x_r \in \mathbb{R}^{mP} \), \( x_c \in \mathbb{R}^n \), \( e \in \mathbb{R}^n \), and \( x \in \mathbb{R}^n \). We are interested in scenarios for which the following assumptions hold:

(i) The time intervals \( \{h_k := t_{k+1} - t_k\} \) are i.i.d. described by a probability measure \( \mu \) with support on \([0, \gamma]\), \( \gamma \in \mathbb{R} \cup \{+\infty\} \), i.e., \( \text{Prob}(a \leq h_k \leq b) = \int_a^b \mu(dr) \) for \( a, b \in [0, \gamma] \).

(ii) Corresponding to a transmission at time \( t_k \) there is a transmission delay \( d_k \) no greater than \( h_k = t_{k+1} - t_k \); A joint stationary probability density \( \chi \) describes \( (h_k, d_k) \), in the sense that

\[
\text{Prob}(a \leq h_k \leq b, c \leq d_k \leq d) = \int_a^b \chi(dr, ds)
\]

where \( a, b \in [0, \gamma] \) and \( \text{Prob}(a \leq h_k \leq b, c \leq d_k \leq d) = 0 \) if \( c > b \). In view of (i) and (ii), we see that \( \mu([a, b]) = \chi([a, b]), [0, b] \).

(iii) At each transmission time, there is a probability \( p_{\text{drop}} \) that a packet may not arrive at its destination or that it may arrive corrupted (packet drop).

(iv) The nodes implement one of the two protocols:

**Dynamic protocol (DP):** This protocol is specified by \( m_D \) symmetric matrices \( \{R_i, i \in M_D\}, M_D := \{1, \ldots, m_D\}, m_D \geq m \). A subset of these matrices \( \{R, i \in I\} \) is associated with node \( j \in M \) where \( I_j := \{i_1, \ldots, i'_j\} \) is an index subset of \( M_D \). These subsets are assumed to be nonempty, i.e., \( n_j \geq 1 \), disjoint, and the \( r_j \) are such that \( \sum_{j=1}^{n_j} r_j = m_D \). The node \( j \) allotted to transmit at \( t_k \) is determined by the map \( d : \mathbb{R}^n \mapsto \mathcal{M} \),

\[
d(x(t_k)) = d_1 \circ d_2 (x(t_k)),
\]

where \( d_2 : \mathbb{R}^n \mapsto M_D \) is given by

\[
d_2(x(t_k)) := \text{argmin}_{i \in M_D} x(t_k)' R_i x(t_k),
\]

and \( d_1 : M_D \mapsto \mathcal{M} \) is given by

\[
d_1(i) := \{j : i \in I_j\}.
\]

In case the minimum in (7) is achieved simultaneously for several values of the index \( i \), stability of the networked control system should be guaranteed regardless of the specific choice for the argmin. In view of (6), the error \( e \) is updated at time \( t_k \) according to

\[
e(t_k) = (I_m - \Lambda_{d_1(t_k)} e(t_k)),
\]

where \( \Lambda_j := \text{diag}(0_{s_j \times s_j}, 1_{(s_j)^\prime}), j \in \mathcal{M} \). That is, only the components of \( \tilde{y} \) or \( \tilde{u} \) associated with the node that transmits are updated by the corresponding components of \( y(t_k) \) or \( u(t_k) \). We call a dynamic protocol regular if for every \( j \in M_D \) there exists a state \( x \) such that \( j \) is the unique index such that \( j = \text{argmin}_{i \in M_D} x_i R_i \). An irregular dynamic protocol can always be made regular by discarding unnecessary matrices \( R_i \).

**Static Protocol (SP):** The nodes transmit in a \( m_s \)-periodic sequence determined by a periodic function

\[
s : \mathbb{N} \mapsto \mathcal{M}
\]

with period \( m_s \). In this case, the error \( e \) is updated at time \( t_k \) according to

\[
e(t_k) = (I_m - \Lambda_{s(t_k)} e(t_k)),
\]

We assume that \( s \) is onto, i.e., each node transmits at least once in a period. When \( m_s = m \), each node transmits exactly once in a period.

As mentioned in Section I, Assumptions (i)–(iii) are appropriate for networked control systems in which feedback loops are closed via local area networks (cf. [1,10]). In particular, Assumption (i) holds for scenarios in which nodes attempt to do periodic transmissions of data, but these regular transmissions may be perturbed by the medium access. It is typically the case in carrier sense multiple access (CSMA) protocols that nodes may be forced to back off for a typically random amount of time until the network becomes available. Note that networks with CSMA protocols protocol, such as the ethernet of wireless 802.11, are prevalent in modern communication systems. The probability distribution of the time interval between transmissions, which can be estimated experimentally or by running Monte Carlo simulations of the
is determined by two factors: the congestion of the network and the delay introduced by the medium access protocol.

The class of dynamic protocols that we describe in (iv) allows a node to transmit if the state of the networked control system lies in a given region of the state space, partitioned according to quadratic restrictions. This class of protocols boils down to the quadratic protocols introduced in [7] when \( m_D = m \). Thus, our definition allows for ampler partitions of the state-space than quadratic protocols, and, as we shall see, it also allows us to see that dynamic protocols are in a sense better than static ones. If we make \( m_D = m \) and choose \( P > 0 \) such that \( R = P - \text{diag}(0_{n_{r+p+c}}) \Lambda_i) > 0 \), then (6) becomes the usual MEF-TOD protocol, where the node that transmits is the one with the maximum norm of the error \( e(t) \) between its current value and its last transmitted value.

### 2.2 Impulsive systems

Suppose that there are no delays, i.e., \( d_i = 0 \), and no packet drops, i.e., \( p_{\text{drop}} = 0 \). Then, we can write the networked control system (1), (2), (3), (4), in the form of the following impulsive system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t), \quad t \in \mathbb{R}_{\geq 0}, t \neq t_k \\
x(t_k) &= J_{p(t_k)} x(t_k) \quad k \in \mathbb{N}_0
\end{align*}
\]

where \( x \in \mathbb{R}^{n_x}, x(0^-) := x_0, \) and \( t_{k+1} - t_k \) are i.i.d. random variables characterized by the probability density \( \mu \) and where the map \( p \) takes the following form for dynamic and static protocols

\[
\begin{align*}
\text{DP:} & \quad p(x(t_k), k) = \text{d}(x(t_k)) \\
\text{SP:} & \quad p(x(t_k), k) = s(k).
\end{align*}
\]

For example, the following expressions for \( A \) and \( \{J_i, i \in \mathcal{M}\} \) correspond to the case in which the controller and plant are directly connected and only the outputs are transmitted through the network, i.e., \( \bar{u}(t) = u(t), x = (x_p, x_c, \bar{y}, y) \).

\[
\begin{align*}
A &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_2 \end{bmatrix} \\
A_{11} &= \begin{bmatrix} A_p + B_pD_pC_p & B_pC_p \\ B_cC_p & A_c \end{bmatrix} \\
A_{12} &= \begin{bmatrix} B_pD_p \\ B_c \end{bmatrix} \\
A_{21} &= -[C_p 0]A_1 \\
A_{22} &= -[C_p 0]A_2 \\
J_i &= \text{diag}(I_{n_p+n_c} I_{n_c} - \Lambda_i), \quad i \in \mathcal{M}.
\end{align*}
\]

This case will be considered in Section IV. Expressions for the general case considered in Section II can be obtained (see, e.g., [11, p. 5]). To take into account delays and packet drops modeled as described in Section II, we consider the following impulsive system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t), \quad t \in \mathbb{R}_{\geq 0}, t \neq t_k \\
x(t_k) &= K_1^{n_{r+p+c}} x(t_k), \quad k \in \mathbb{N}_0 \\
x(s_k) &= L x(s_k), \quad t_k \leq s_k \leq t_{k+1},
\end{align*}
\]

where \( p(x_k, k) \) is defined as in (13) for dynamic protocols and as in (14) for static protocols and where the initial condition is set to \( x(0^-) := x_0 \). The random variables \( t_k \) and \( s_k \) are defined completely by the inter-sampling times \( h_k := t_{k+1} - t_k \) and by the delays \( d_k := s_k - t_k \). The \( (h_k, d_k) \) are i.i.d., and are as described by (5). The \( q_k \in \{1, \ldots, n_q\} \) are i.i.d., and such that \( \text{Prob}(q_k = j) = w_j \), \( \forall j \in \{1, \ldots, n_q\}, k \geq 0 \). Below we provide expressions for \( A, L, w_0, \) and \( K_j, i \in \mathcal{M}, j \in \{1, \ldots, n_q\} \), which model the case where the controller and the plant are directly connected and only the plant outputs are transmitted through the network, i.e., \( \bar{u}(t) = u(t) \). The state is now considered to be \( x = (x_p, x_c, \bar{y}, y) \in \mathbb{R}^{n_x} \) where \( v \in \mathbb{R}^{n_v} \) is an auxiliary vector \( \{v_1, \ldots, v_{n_v}\} \) that is updated with the sampled value \( v_j = y_j(t_k) \) at each sampling time \( t_k \) at which node \( j \) is allowed to transmit. Nevertheless, the update only takes place if a packet sent at \( t_k \) is not dropped and the sampled value \( v_j \) is used only to update the value of \( \bar{y} \), after a transmission delay \( d_k \) at time \( s_k = t_k + d_k \).

\[
\begin{align*}
A &= \begin{bmatrix} A_1 & A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
A_{11} &= \begin{bmatrix} A_p & B_pC_p \\ 0 & A_c \end{bmatrix} \\
A_{12} &= \begin{bmatrix} B_pD_p \\ B_c \end{bmatrix} \\
n_q = 2, w_1 = 1 - p_{\text{drop}}, w_2 = p_{\text{drop}} \\
K_1 &= \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ A_cC_p & 0 & I_{n_c} - \Lambda_i \end{bmatrix} \\
K_2 &= I_{n_p+n_c+2n_v}, i \in \mathcal{M} \\
L &= \begin{bmatrix} 0 & 0 & I_{n_v} \\ 0 & 0 & I_{n_v} \end{bmatrix}.
\end{align*}
\]

Again, the expressions for the general case considered in Section II can be obtained. It is also important to mention that there are other ways to model the setup with delays and packets.
packet drops described in Section II. For example, one can find a similar model to (16) but introduce the dependency on the variable \( q_k \) modeling the packet drops in the matrix \( L \).

### 2.3 Stability notion

We define stability for System (12) in terms of the following auxiliary system obtained by considering the state of (12) only at times \( t_k \):

\[
z_{k+1} = e^{A \Delta t} J_{p(t_k)} z_k, \quad k \in \mathbb{N}_{\mu},
\]

where \( z_k := x(t_k) \), and \( z_0 = x_0 \). We say that (12) is mean exponentially stable (MES) if there exist constants \( c > 0 \) and \( 0 < \alpha < 1 \) such that, for any initial condition \( x_0 \), we have:

\[
E[z_k^T z_k] \leq c \alpha^k z_0^T z_0, \quad \forall k \geq 0.
\]

The same definition of MES is used for System (16). We assume that the following condition holds:

\[
e^{\tilde{A} \Delta t} r(t) < ce^{-\alpha t} \quad \text{for some } c > 0, \alpha > 0.
\]

where \( \tilde{A}(A) \) is the real part of the eigenvalues of \( A \) with largest real part and \( r(t) := \mu(t, \gamma) \) denotes the survivor function. Assuming (20), we were able to prove in [6], considering only static protocols, that (19) is equivalent to the more common notion of mean exponential stability in continuous-time where one requires \( E[x(t)^T x(t)] \) to decrease exponentially. In the present paper, we make no such assertion, although assuming (20) is still useful (e.g., (20) guarantees that (22) is bounded).

## III. MAIN RESULTS

For simplicity, we assume in Subsections 3.1 and 3.2, that there are no delays, i.e., \( d_i = 0, \forall i \), and no packet drops, i.e., \( p_{\text{drop}} = 0 \), and we consider the general case in Subsection 3.3.

### 3.1 Stability result I and dynamic vs. static protocols

The following is our first stability result for (12), providing a test for sufficient conditions for (12) to be mean exponentially stable.

**Theorem 1.** The system (12) with dynamic protocol (13) is MES if there exist scalars \( \{0 \leq p_{ji} \leq 1, \forall i, j \in \mathcal{M}_D\} \) with \( \sum_{p_{ji}} p_{ji} = 1, \forall i \in \mathcal{M}_D \) and \( n_i \times n_i \) symmetric matrices \( \{R_i > 0, \forall i \in \mathcal{M}_D\} \) such that:

\[
J_{d(i)}^T \left( \sum_{p_{ji}} p_{ji} E(R_i) \right) J_{d(i)} - R_i < 0, \forall i \in \mathcal{M}_D,
\]

where

\[
E(R_i) := \int_0^T e^{-\lambda t} R_i e^{\lambda t} \mu(dh).
\]

This result can be used to analyze if a given protocol yields the networked control system stable or to synthesize a protocol that achieves this.

**Analysis.** Note first that a given dynamic protocol specified by \( R_i > 0, \forall i \in \mathcal{M}_D \) is equivalent to a dynamic protocol specified by

\[
\tilde{R}_i = P + R_i > 0, \forall i \in \mathcal{M}_D,
\]

where \( P \) can be any symmetric matrix such that \( P + R_i > 0 \). If we replace in (21) the matrices \( R_i \) by \( \tilde{R}_i \), given by (23), we obtain that (21), (23) are LMIs in the variables \( P \) and \( p_{ji} \) (using the fact that \( \sum_{p_{ji}} p_{ji} = 1, \forall i \in \mathcal{M}_D \)).

**Synthesis.** If we allow \( R_i \) to be variables in (21), then (21) are basically BMIs. In fact, if we chose a basis \( B_i \) for the linear space of symmetric matrices, we have \( R_i = \sum_{j} b_{ji} B_j \) and (21) depends on the products \( p_{ji} b_{ji} \). In this case, the dynamic protocol, determined by the matrices \( R_i \), comes out from the solution to (21).

**Remark 2.** Stability condition (21) resembles a stability condition for Markov jump linear systems [12]. Indeed, suppose that we choose protocol \( p \) for (12) according to the following stochastic rule:

\[
\text{Prob}(\omega_{k+1} = j | \omega_k = i) = \mu_{ji}, \quad i, j \in \mathcal{M}_D
\]

\[
p(x(t_i^k), k) = d_i(\omega_k), \quad k \geq 0,
\]

where \( \omega_k \) is a Markov chain with \( m_D \geq m \) states, \( \sum_{\mu_{ji}} \mu_{ji} = 1, \forall i \in \mathcal{M}_D \), and \( d_i \) is described by (8). Then, it is known (cf. [12, ch. 4]) that (18) with scheduling (24) is mean square stable (\( \lim_{t \to \infty} E[x(t_i^k)^T x(t_i^k)] = 0, \forall i \in \mathcal{M}_D \)) if and only if there exist \( n_i \times n_i \) symmetric matrices \( \{R_i > 0, \forall i \in \mathcal{M}_D\} \) such that (21) holds for \( p_{ji} = \mu_{ji}, \forall i, j \in \mathcal{M}_D \). Note, however, that contrary to Protocol (24), Protocol (6), (8) is a deterministic state-dependent protocol and Theorem 1 assures stability in the deterministic sense (19).

To state the next theorem, we need the following result, which provides necessary and sufficient stability conditions under which (12) is mean exponentially stable for a static protocol (14). The proof can be found in [6] and is omitted.
here. Let $[i+1]=i+1$ if $i \in \{1, \ldots, m_5-1\}$ and $[i+1]=1$ if $i=m_5$. Let $\mathcal{M}_b := \{1, \ldots, m_b\}$.

**Theorem 3.** The system (12), with static protocol (14) is MES if and only if there exist $n_i \times n_i$ symmetric matrices $\{R_i > 0, i \in \mathcal{M}_b\}$ such that:

$$J_{\dot{d}(i)}E(R_{\dot{d}(i)})J_{\dot{d}(i)} - R_i < 0, \forall ic_{i}\mathcal{M}_b$$

(25)

where $E(R_{\dot{d}(i)})$ is given as in (22).

The following result, which builds upon Theorems 1 and 3, establishes that, if the networked closed loop is stable for a static protocol, then we can provide a dynamic protocol for which the networked closed loop is also stable. This is one of the main contributions of the paper.

**Theorem 4.** If the networked control system is MES for a static protocol with period $m_5$ then there exists a dynamic protocol taking the form (6), with $m_D = m_S$, that yields the networked control system MES.

**Proof.** Since the stability conditions of Theorem 3 are necessary and sufficient, there exists a static protocol with period $m_5$ that yields the networked control system MES if and only if there exist $\{R_i, i \in \mathcal{M}_b\}$ such that (25) holds for (12) with matrices defined by (15). This implies that, if we consider a dynamic protocol with $m_D = m_S$, $I_j = \{k \in \mathcal{M}_b : s(k) = j\}, j \in \mathcal{M}$, then $d_i(i) = s(i)$, for $i \in \mathcal{M}_b$, and (21) holds with

$$p_j = \begin{cases} 1, & \text{if } i < m_D \text{ and } j = i+1, \\ 1, & \text{if } i = m_D \text{ and } j = 1, \\ 0 & \text{otherwise} \end{cases}$$

and with $\{R_i, i \in \mathcal{M}_b = \mathcal{M}_D\}$ taken to be the solution to (25).

From the proof of Theorem 4, we see that the matrices $\{R_i, i \in \mathcal{M}_D\}$ that characterize the dynamic protocol mentioned in its statement can be taken to be the solution to (25). Note that, in the special case where $m_D = m = m_S$, Theorem 4 states that, if there exists a round-robin protocol with period $m_S = m$, i.e., each node only transmits exactly once in a period, which yields the networked control system MES, then one can find a quadratic protocols, as introduced in [7], that also yields the networked control system MES.

**Remark 5.** The fact that the stability conditions of Theorem 3 are necessary and sufficient the key to obtaining Theorem 4. In [4], a similar reasoning to Theorem 4 can be used to prove that, if the stability conditions provided there for quadratic protocols (cf. [4, Theorem 3]) hold, then so do the stability conditions for a static protocol in the special case where each node transmits only once in a period (cf. [7, Theorem 3]). Nevertheless, since the conditions provided in [4] are only sufficient for the RR protocol, it does not allow one to conclude that, if a stabilizing static protocol exists, then so does a dynamic protocol, as stated in Theorem 4. Although [7] does not explicitly present stability conditions for a static protocol, the same remarks should apply, since convex over-approximations introduce conservativeness.

### 3.2 Stability result II and observer-protocol design

The following is our second stability result for (12). Similar to Theorem 1, this theorem provides testable sufficient conditions for (12) to be mean exponentially stable. Nevertheless, it will allow us to obtain different results than Theorem 1.

**Theorem 6.** The system (12) with dynamic protocol (13) is MES if there exist an $n_i \times n_i$ symmetric matrix $W > 0$, scalars $\{c_{ij} \geq 0, i, j \in \mathcal{M}_D, i \neq j\}$, and $n_i \times n_i$ matrices $R_i, i \in \mathcal{M}_D$, such that:

$$J_{\dot{d}(i)}E(W)J_{\dot{d}(i)} + \sum_{j=1, j \neq i}^{m_D} c_{ij}(R_j - R_i) - W < 0, \forall ic_{i}\mathcal{M}_D$$

(26)

where $E(W) := \sum_{i=1}^{m_D} (e^{Ai})^T W e^{Ai} \mu (dh)$.

Given a quadratic protocol, i.e., specific values for the matrices $R_i$, testing if (26) holds is an LMI feasibility problem. To design a protocol for which mean exponential stability of the networked control system is guaranteed, we can take the $\{R_i, i \in \mathcal{M}_D\}$ as additional unknowns and (26) should now be viewed as a BMI feasibility problem.

The proof of Theorem 6 builds upon establishing that, if there exists a positive definite matrix $W$, positive constants $c_{ij}$ and matrices $R_i$ such that (26) holds, then the quadratic function

$$V(x) := x^T W x$$

(27)

is a stochastic Lyapunov function for System (18) (which models (12) at sampling times) in the sense that the following condition holds for (18)

$$\mathbb{E}[V(z_{i+1}) - V(z_i)] \leq -z_i^T Z z_i, \forall \mathbb{E} \in \mathbb{R}^n,$$

(28)

for some $Z > 0$. The next result shows that, under certain conditions, which include the case $m_D = m = 2$, i.e., only two nodes pertaining to the closed loop access the network, the converse holds. Let $\text{co}(\mathcal{A})$ denote the convex hull of a set $\mathcal{A}$, and, for each $i \in \mathcal{M}_D$, define the function...
\[ g_i(y) := y^T (W - J_{d(i)}^T E(W) J_{d(i)}) y \]

and the \( m_D - 1 \) functions

\[ g'_i(y) = y^T (R_i - R_i y), \quad j \in \mathcal{M}_D - \{i\}. \]

Define also the following sets in \( \mathbb{R}^{m_n} \)

\[ \mathcal{K}' := \{ (g_0(y), g'_1(y), \ldots, g'_i(y), g'_i(y), \ldots, g_{m_D}(y)) : y \in \mathbb{R}^{m_n}, \quad i \in \mathcal{M}_D, \} \]

and

\[ \mathcal{N}' := \{ (\eta_0, \eta_1, \ldots, \eta_{m_D}) : \eta_0 < 0, \quad \eta_k > 0, 1 \leq k \leq m_D \}. \]

**Theorem 7.** Suppose that the dynamic protocol (13) is regular. Then, if

\[ \mathcal{K}' \cap \mathcal{N}' = \emptyset \Rightarrow c_{\mathcal{K}'}(\mathcal{N}') = \emptyset, \quad \forall \mathcal{M}_D, \]

there exists an \( n_i \times n_i \) symmetric matrix \( W > 0 \), scalars \( \{ c_y \geq 0, i, j \in \mathcal{M}_D, i \neq j \} \), and \( n_i \times n_i \) matrices \( R_i, i \in \mathcal{M}_D \) such that (26) holds if and only if there exists a quadratic stochastic Lyapunov function taking the form (27) such that (28) holds. In particular, (29) holds if \( m_D = 2 \).

The proof of Theorem (7) relies on the S-Procedure [13] which is a relaxation technique that can be used to provide stability conditions for linear systems with quadratic constraints. In particular, Condition (29) is a condition for the S-Procedure to be lossless, and it is always satisfied in the case in which there is only one quadratic constraint (\( m_D = 2 \)).

We show next that Theorem 6 allows one to extend the observer-protocol design proposed in [8].

### 3.2 Observer design

Suppose that we wish to estimate the state of the following plant

\[ \dot{x}_p(t) = A_p x_p(t), \quad y(t) = C_p x_p(t), \quad x_p(0) = x_{p0} \]

where the \( m \) outputs \( y(t) = (y_1, \ldots, y_m) \), \( y_i \in \mathbb{R}^{n_i} \) are sent through a network that imposes i.i.d. intervals between transmissions to a remote observer. As in Section II, we denote by \( \mu \) the measure that defines the inter-transmission times \( h_i = t_{i+1} - t_i \) and we let \( \mathcal{M} = \{1, \ldots, m\} \). Also let \( \Psi_j := \text{diag}([0_{n_j}, \ldots, 0_{n_j}, 0_{n_j}]) \), for \( j \in \mathcal{M} \) A natural model based linear remote observer for this system is defined by

\[ \dot{x}(t) = A_p \dot{x}(t) + L_p \Psi_{\text{diag}}(C_p \dot{x}(t) - y(t)), \]

where the observer gains \( L_p \) to be designed are allowed to depend on the index \( k \) and the map

\[ c(x_i(t_i)) := \arg \min_{c_{ij} \in S_i} x_i(t_i)^T C_p^T S_i C_p x_i(t_i) \]

determines which node transmits at \( t_i \), based on the estimation error \( x_i(t_i) = \hat{x}(t_i) - x_p(t_i) \), where \( \{ S_i, j \in \mathcal{M}_i \} \) is a set of \( m \) matrices. As argued in [8], the sensors should run a replica of the remote observer to access \( \hat{x}(t) \), which allows each node to encode in the message arbitration field \( x_i(t_i)^T C_p^T S_i C_p x_i(t_i) \), where \( C_p x_i(t_i) = C_p \dot{x}(t_i) - y_j(t_i), j \in \mathcal{M} \).

The resulting estimation error \( x := \hat{x} - x_p \) evolves according to

\[ \dot{x}(t) = A_p x(t) + L_p \Psi_{\text{diag}}(C_p \dot{x}(t) - y(t)), \]

where the observer gains \( L_p \) that yield the networked control system MES. To state the result we need the following assumption:

\[ H(s) := \int_0^s e^{\gamma r} dr \text{ is invertible for every } s \in [0, \gamma]. \]

While this assumption holds for a large class of matrices \( A_p \), it is possible to construct examples where it does not, as in the case where \( \gamma > s = 2\pi \) and \( A_p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) in which case \( H(s) = 0 \). Let \( n_p := \sum_{i=1}^m n_i \).

**Theorem 8.** Suppose that (34) holds. If there exist an \( n_p \times n_p \) symmetric matrix \( P > 0 \), an \( n_i \times n_i \) matrix \( Y \), an \( n_p \times n_p \) matrix \( M \), \( n_i \times n_i \) matrices \( \{ S_i, i \in \mathcal{M} \} \), and scalars \( \{ c_y \geq 0, i, j \in \mathcal{M} \} \) such that

\[ F(P) + DM^T Y P + (DM^T Y P) + C_p^T Y C_p \]

\[ + \sum_{j=1}^m c_y (C_p^T S_i C_p - C_p^T S_j C_p) - P < 0, \quad \forall \mathcal{M} \]

\[ \begin{bmatrix} P & M \end{bmatrix} > 0, \quad \begin{bmatrix} x - x \end{bmatrix} \]

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where \( F(P) := \int_0^T e^{P r} P e^{P r} d\mu(dr) \) and \( D := \int_0^T e^{P r} d\mu(dr) \), then the observer gain \( L_k = H(h_k)^{-1} P^{-1} M \) yields (12) with matrices (33) MES.

Note that our proposed observer gain \( L_k \) depends on the length \( h_k \) of the time interval \( \{ t_{k+1} - t_k \} \), which is not known at time \( t_k \leq t < t_{k+1} \) (30). In practice, this results in a delay in constructing the state estimate that never needs to exceed \( h_k \) since the state of the remote observer (30) can only be updated with measurement \( y(t_k) \) at time \( t_{k+1} \), at which \( h_k \) can be computed.

Similar to Theorem 6, the conditions of Theorem 8 can be used to investigate the stability of a given protocol determined by matrices \( R_k \), in which case the problem reduces to an LMI feasibility problem, or they can be used to design a protocol, where one needs to solve a BMI feasibility problem.

**Remark 9.** When the intervals between transmission are constant, one can show that the stability conditions (35) and (36) are equivalent to the ones given in [8], where such an assumption is made. In this case, the matrices \( L_k \) do not depend on \( k \), and can be computed off-line.

### 3.3 Extensions to handle delays and packet drops

Theorems 6 and 1 can be extended to the case where the network introduces packet drops and delays modeled by (16) with Matrices (17). We state these extensions next.

**Theorem 10.** System (16) with dynamic protocol (13) is MES if there exist scalars \( 0 \leq p_{ij} \leq 1, \forall_{i \in M_P} \), with \( \sum_{j \neq i} n_i \leq 1, \forall_{i \in M_P} \), and \( n_i \times n_i \) symmetric matrices \( R > 0, \forall_i \in M_P \) such that:

\[
\sum_{i=1}^{n_k} W_i \left( K^{T}_{d(i)} \left( \sum_{j \neq i}^{n_i} p_{ij} E(R_i) \right) K_{d(i)} \right) - R_i < 0, \forall_i \in M_P,
\]

where

\[
E(R_i) := \int_0^T \int_0^T (e^{Pr} - e^{Pr})^T R_i e^{Pr} - e^{Pr} \chi(dh, ds).
\]  

**Theorem 11.** System (16) with dynamic protocol (13) is MES if there exist an \( n_i \times n_i \) symmetric matrix \( W > 0 \), scalars \( \{q_{ij} \geq 0, i, j \in M_P, i \neq j \} \), and \( n_i \times n_i \) matrices \( K, i \in M_P \) such that:

\[
\sum_{i=1}^{n_k} W_i (K^{T}_{d(i)} E(W) K_{d(i)} + \sum_{j \neq i}^{n_i} c_{ij} (R_j - R_i) - W) < 0, \forall_i \in M_P,
\]

where \( E(W) \) is defined as in (37).

### IV. NETWORKED CONTROL RESULTS

In this section, we show that Theorems 6 and 11 reduce the conservativeness of the results in [3], [5], and [7]. These three works use the same benchmark problem for the control of a batch reactor, where the plant (1) and controller (2) matrices are given by:

\[
A_r = \begin{bmatrix}
1.38 & -0.2077 & 6.715 & -5.676 \\
-0.5814 & -4.29 & 0 & 0.675 \\
1.067 & 4.273 & -6.654 & 5.893 \\
0.048 & 4.273 & 1.343 & -2.104
\end{bmatrix},
\]

\[
B_r = \begin{bmatrix}
5.679 & 0 \\
1.136 & -3.146 \\
1.136 & 0
\end{bmatrix},
\]

\[
C_r = \begin{bmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

Only the two outputs are sent through the network, i.e., \( u(t) = \hat{u}(t) \). The network imposes i.i.d. intervals between transmissions, possibly contains packet drops, and has no delays. The networked control closed loop can be written as in (12), (15) in the absence of drops and as in (16)–(17) when drops occur. Thus, the stability of the networked control system can be tested by Theorems 1, 6, and 10, 11. The results are shown in Table I, considering two distributions \( \mu \) for the inter-transmissions intervals \( h_i \): uniform in the interval \([0, \gamma]\), and exponential with expected value \(1/\lambda_{exp}\).

From Table I, we can conclude that our results allow a significantly reduction the conservativeness of the conditions in [5] and [3] for the same benchmark examples. The results in [7] are very close to the ones obtained with Theorem 1, and both outperform the results obtained with Theorem 6.

In Table II, we show the results obtained by allowing \( R_i \) in Theorem 1 to be additional variables, i.e., the protocol is to be designed. Note that Theorem 4 assures that the values obtained with Theorem 1 for the maximum support of a uniform distribution that preserves stability when a dynamic protocol (obtained from solving (21)) is utilized are larger than the ones obtained with the necessary and sufficient conditions provided by Theorem 3 for the static protocol, which matches well with the results in Table II.

### V. CONCLUSIONS AND FUTURE WORK

We provided stability results for networked control systems with stochastic intervals between transmissions,
Table I. Stability results for the batch reactor example-MEF-TOD and Round Robin protocol. NA stands for Not Available.

<table>
<thead>
<tr>
<th></th>
<th>Dynamic Protocol</th>
<th>Static Protocol</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>no drops</td>
<td>$p = 0.5$</td>
</tr>
<tr>
<td>Max. $\gamma$:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_i \sim \text{Uni.}(\gamma)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results from [3]</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Results from [5]</td>
<td>0.0372</td>
<td>0.0170</td>
</tr>
<tr>
<td>Results from [7]</td>
<td>0.11</td>
<td>NA</td>
</tr>
<tr>
<td>Ths. 6 and 11</td>
<td>0.0550</td>
<td>0.024</td>
</tr>
<tr>
<td>Ths. 1</td>
<td>0.111</td>
<td>NA</td>
</tr>
<tr>
<td>Th. 3</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Max. $1/\lambda_{exp}$:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_i \sim \text{Exp}(\lambda_{exp})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results from [3]</td>
<td>0.0095</td>
<td>0.0046</td>
</tr>
<tr>
<td>Results from [5]</td>
<td>0.0158</td>
<td>0.00795</td>
</tr>
<tr>
<td>Results from [7]</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Ths. 6 and 11</td>
<td>0.0226</td>
<td>0.01124</td>
</tr>
<tr>
<td>Ths. 1</td>
<td>0.0357</td>
<td>NA</td>
</tr>
<tr>
<td>Th. 3</td>
<td>NA</td>
<td>NA</td>
</tr>
</tbody>
</table>

Table II. Stability results for the batch reactor example-Protocol design, no packet drops.

<table>
<thead>
<tr>
<th></th>
<th>Dyn. Prot. Design (Th. 1)</th>
<th>Static Prot. (Th. 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. $\gamma$:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h_i \sim \text{Uni.}(\gamma)$</td>
<td>0.140</td>
<td>0.112</td>
</tr>
</tbody>
</table>

Additional possible line for future research is to consider related optimal control problems in the scenarios addressed in the present paper.

VI. APPENDIX A. PROOFS

Proof of Theorem 1. The discrete-time process $z_i$, described by (18), can be shown to be a Markov process due to the i.i.d. assumption on $h_i$. In particular

\[ E_{z_k}[E_{z_{k+1}}[V(z_{k+1})]] = E_{z_k}[V(z_{k+1})] \]  \hspace{1cm} (38)

for any bounded measurable function $V$, where $E_{z_k}$ denotes expectation given $z_k$. i.e., $E_{z_k}[] := E[|z_k|]$. If one can find a function $V$ and positive constants $c_1, c_2, c_3$, such that

\[ c_1\|z\|^2 \leq V(z) \leq c_2\|z\|^2, \forall z \in \mathbb{R}^n \]

and

\[ E_{z_k}[V(z_{k+1})] - V(z_k) \leq -c_3\|z_{k}\|^2, \forall z_k \in \mathbb{R}^n \]

then we can prove that

\[ \mathbb{E}[z_1^2] \leq c\alpha^2z_0, \forall \epsilon > 0 \text{ for some } 0 < \alpha < 1, c > 0. \]  \hspace{1cm} (41)

which implies MES for (12) according to Definition (19) since $x(t_i) = Jz_i$ for some $i \in \mathcal{M}$. In fact, if (38) and (40) hold, then

\[ E_{z_k}[V(z_{k+1})] \leq \alpha V(z_k) \]  \hspace{1cm} (42)

where $0 < \alpha = 1 - \frac{c_3}{c_1} < 1$ must be greater than zero since $V$ is positive. From (38) and (42), we can conclude that

\[ E_{z_k}[V(z_k)] \leq \alpha^k V(z_0). \]  \hspace{1cm} (43)

From (39) and (43), we obtain:

\[ E_{z_k}[\|z_k\|^2] \leq \frac{c_3}{c_1} \alpha^k \|z_0\|^2, k \geq 0. \]

Take $V(z_i) := \min_{i \in \mathcal{M}} z_i^2R_i$, which satisfies (39) since $R_i > 0 \forall i \in i - \emptyset$. Suppose that $z_{\hat{i}}$ is such that $i = d(z_{\hat{i}}) = \min_{i \in \mathcal{M}} z_i^2R_iz_i$, i.e., $V(z_\hat{i}) = z_{\hat{i}}^2R_{\hat{i}}z_{\hat{i}}$. Note that, in the case where the minimum is achieved simultaneously for several values of the index $i$, any of these indices can be chosen without affecting the present proof. Then, for any $p_j \geq 0$, $\sum_{j=1}^{\infty} p_j = 1$, we have:
\[
\mathbb{E}_{z_i}[V(z_{i+1})] = \mathbb{E}_{z_i}\left[ \min_{y_{d(i)}} z_i^{T} J_{d(i)}^{T} e^{\epsilon h} R_e e^{\epsilon h} J_{d(i)} z_i \right] \\
\leq \mathbb{E}_{z_i} \left[ \sum_{j=1}^{m_0} p_j z_i^{T} J_{d(i)}^{T} e^{\epsilon h} R_e e^{\epsilon h} J_{d(i)} z_i \right] \\
= z_i^{T} J_{d(i)}^{T} \sum_{j=1}^{m_0} p_j E(R_j) J_{d(i)} z_i \\
= z_i^{T} J_{d(i)}^{T} (\sum_{j=1}^{m_0} p_j E(R_j)) J_{d(i)} z_i
\] 

(44)

Suppose we choose \( p_j \) such that (21) holds, i.e.,

\[
J_{d(i)}^{T} \sum_{j=1}^{m_0} p_j E(R_j) J_{d(i)} = -Q,\]

for some \( Q > 0 \). Then, from (44), we conclude that \( \mathbb{E}_{z_i}[V(z_{i+1})] - V(z_i) \leq -z_i^{T} Q z_i, i = d(z_i) \), which implies (40) and concludes the proof.

**Proof of Theorem 6.** As in the proof of the Theorem 1, it is sufficient to find a function \( V \) such that (39), (40) hold. Take \( V(z) = z^{T} W z, \) where \( W \) is the solution to (26), and suppose that \( i = \arg\min_{j \in \mathcal{M}} z_i^{T} R_i z_i \). Then

\[
\mathbb{E}_{z_i}[V(z_{i+1})] - V(z_i) = z_i^{T} [J_{d(i)}^{T} E(W) J_{d(i)} - W] z_i, \\
= -z_i^{T} \left[ \sum_{j=1}^{m_0} c_j(R_j - R_i) + Q \right] z_i
\] 

(45)

where \( Q > 0 \). Since \( i = \arg\min_{j \in \mathcal{M}} z_i^{T} R_j z_i \), we have \( z_i^{T} \left[ \sum_{j=1}^{m_0} c_j(R_j - R_i) \right] z_i \geq 0 \) for every \( z_i \in \mathbb{R}^n \). Therefore from (45), we conclude that \( V \) satisfies (40). It is also clear that \( V \) satisfies (39), which concludes the proof.

**Proof of Theorem 7.** If, for a given \( z_i \neq 0 \), \( i \in \mathcal{M}_D \) is such that \( i = \arg\min_{j \in \mathcal{M}_D} z_i^{T} R_j z_i \), which is equivalent to

\[
g_j'(z_i) \geq 0, \quad \forall j \in \mathcal{M}_D - \{i\},
\] 

(46)

then we have that \( z_{i+1} = e^{\epsilon h} J_{d(i)} z_i \), in which case (28) boils down to

\[
g_0'(z_i) > 0.
\] 

(47)

The fact that the dynamic protocol is regular implies that there exists at least one \( y \) such that (46) holds with strict inequality. Then, a straightforward adaptation of the lossless S-Procedure theorem provided in [13, Theorem 2] to handle strict inequalities in the objective function \( g_0(y) \) assures that, under Condition (29), (47) holds if and only if there exist \( c_j \geq 0, j \in \mathcal{M}_D - \{i\} \) such that:

\[
g_0(y) - \sum_{j \in \mathcal{M}_D - \{i\}} c_j g_j'(y) > 0, \quad \forall y \in \mathbb{R}^n - \{0\}.
\] 

(48)

The result then follows by noticing that (48) is equivalent to (26). The fact that (29) holds if \( m_D = 2 \) (in which case there is only one quadratic constraint) follows from [13, Theorem 3].

**Proof of Theorem 8.** We first prove that, if there exists \( P > 0 \) such that

\[
\int_0^T (e^{2h} + H(h)L_\Psi C_P)^{T} P \ldots (e^{2h} + H(h)L_\Psi C_P) \mu(dh) \\
+ \sum_{j=1,j \neq i} c_j (C_j S_j C_P - C_j S_j C_P) - P < 0
\]

(49)

then (26) holds for (12) with Matrices (33). Note that we can assume that \( d(i) = i, \forall i \in \mathcal{M} \), since, if this is not the case, we can relabel the sensor nodes in such a way that this holds. For \( A, J \) given by (33) we have:

\[
e^{4h} J_i = \begin{bmatrix} e^{4h} + H(h)L_\Psi C_P & 0 \\
L_\Psi C & 0 \end{bmatrix}.
\]

Using this expression and considering \( W = \text{diag}(Pe_{I_{n_p}}) \) in (26), where \( P \) satisfies (49) and \( \epsilon \) is a given positive constant, we have:

\[
\int_0^T \begin{bmatrix} B(h) & 0 \\
0 & 0 \end{bmatrix} \mu(dh) - \begin{bmatrix} P & 0 \\
0 & \epsilon I_{n_p} \end{bmatrix} \\
+ \sum_{j=1,j \neq i} c_j \begin{bmatrix} C_j S_j C_P - C_j S_j C_P & 0 \\
0 & 0 \end{bmatrix} < 0, \quad \forall i \in \mathcal{M},
\]

(50)

where

\[
B(h) = e(L_\Psi C_P)^{T} (L_\Psi C_P) \\
+ (e^{2h} + H(h)L_\Psi C_P)^{T} P(e^{2h} + H(h)L_\Psi C_P).
\]

From this expression we conclude that, if (49) holds, then (50) holds for sufficiently small \( \epsilon \). Set \( L_\Psi = H(h_{n_p})^{-1} P^{-1} M \) for an \( n_p \times m \) matrix. Then, (49) can be written as:

\[
F(P) + DMW \Psi C_P + (DMW \Psi C_P)^{T} + C_j MP^{-1} MC_P \\
+ \sum_{j=1,j \neq i} c_j (C_j S_j C_P - C_j S_j C_P) - P < 0, \forall i \in \mathcal{M}_M.
\]

(51)

If we let \( Y > 0 \) be such that \( M^T P^{-1} M < Y \), which, applying the Shur complement, can be seen to be equivalent to (36), we see that, if (35) holds, then (51) holds, which concludes the proof.

**Proof of Theorem 10.** The proof is obtained by following the same steps of Theorem 1 and by noticing that, using a similar reasoning to (44), one obtains:

\[
\mathbb{E}_{n_i}[V(z_{i+1})] \leq \sum_{i=1}^{m_0} w_i \left[ K_i^{d(i)} \left( \sum_{j=1}^{m_0} p_j E(R_j) \right) K_i^{d(i)} \right].
\]

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Proof of Theorem 11. The proof is obtained by following the same steps of Theorem 6 and by noticing that, in (45)

$$E_{z_{k}}[V(z_{k+1})] = \sum_{i=1}^{n} w_{i}(K_{d_{i}k})^{T} E(W)K_{d_{i}k}$$

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