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Lubrication analysis of interacting rigid cylindrical particles in confined shear flow

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Lubrication analysis is used to determine analytical expressions for the elements of the resistance matrix describing the interaction of two rigid cylindrical particles in two-dimensional shear flow in a symmetrically confined channel geometry. The developed model is valid for non-Brownian particles in a low-Reynolds-number flow between two sliding plates with thin gaps between the two particles and also between the particles and the walls. Using this analytical model, a comprehensive overview of the dynamics of interacting cylindrical particles in shear flow is presented. With only hydrodynamic interactions, rigid particles undergo a reversible interaction with no cross-streamline migration, irrespective of the confinement value. However, the interaction time of the particle pair substantially increases with confinement, and at the same time, the minimum distance between the particle surfaces during the interaction substantially decreases with confinement. By combining our purely hydrodynamic model with a simple on/off non-hydrodynamic attractive particle interaction force, the effects of confinement on particle aggregation are qualitatively mapped out in an aggregation diagram. The latter shows that the range of initial relative particle positions for which aggregation occurs is increased substantially due to geometrical confinement. The interacting particle pair exhibits tangential and normal lubrication forces on the sliding plates, which will contribute to the rheology of confined suspensions in shear flow. Due to the combined effects of the confining walls and the particle interaction, the particle velocities and resulting forces both tangential and perpendicular to the walls exhibit a non-monotonic evolution as a function of the orientation angle of the particle pair. However, by incorporating appropriate scalings of the forces, velocities, and doublet orientation angle with the minimum free fraction of the gap height and the plate speed, master curves for the forces versus orientation angle can be constructed. © 2015 AIP Publishing LLC.

I. INTRODUCTION

Many consumer and industrial products are suspensions or composites consisting of particles or fibers with either microscopic or nanoscopic dimensions dispersed in a continuous phase. In addition to being a commonly used rheology modifier, particles also allow tailoring material properties such as electrical conductivity, mechanical strength, or barrier resistance.\textsuperscript{1} Even in the latter case, when the final product can be a solid material, a liquid particle suspension is often encountered during processing in the molten state. When present in a sufficiently high concentration, particles...
will interact both hydrodynamically and via direct interparticle forces, the most common of which are van der Waals attraction, depletion attraction due to dissolved polymers or nanometer-sized particles in the continuous phase, steric repulsion, and electrostatic interactions.\textsuperscript{2} In addition, the flow conditions during processing can be exploited to steer particle assembly resulting in a range of particle structures.\textsuperscript{3} Moreover, in recent years, there is a trend towards miniaturization, resulting in mixing and processing equipment for which at least one dimension of the channels is of the same order of magnitude as the particle size.\textsuperscript{4} In this case, in addition to particle-particle interactions also particle-wall interactions become important. It has been shown that such interactions can lead to peculiar particle or droplet structures such as clusters, strings, and pearl necklaces.\textsuperscript{5–7} Effects similar to geometrical confinement by means of solid walls can also be produced by surrounding particles in concentrated suspensions or by normal stresses in suspensions with viscoelastic fluids. For example, the formation of strings\textsuperscript{8} or two-dimensional crystals\textsuperscript{9} has also been observed in unconfined suspensions with viscoelastic fluids. To optimize the synergistic combination of particle and fluid properties by tailoring particle assembly, it is of primary importance to fully understand the relations between flow and particle assembly on the one hand and between particle interactions and rheology on the other hand.

A large amount of research has been devoted to developing a fundamental understanding of the hydrodynamics of particle-particle and particle-wall interactions by studying these processes on the level of single particles or particle pairs rather than considering concentrated suspensions. The dynamics of non-Brownian spherical or cylindrical particles in low-Reynolds-number shear flow of Newtonian fluids has been experimentally explored by Mason and coworkers already 45 yr ago.\textsuperscript{10,11} Upon hydrodynamic interaction, the particles pass over each other on a symmetrical path, with the distance between their centers gradually decreasing as long as they are in the compressional quadrant of the flow and subsequently increasing when they transit into the extensional quadrant of the flow. For particles with a sufficiently small center-to-center distance in the velocity gradient direction, closed trajectories can occur in which the particles keep rotating around each other or continuously tumble as a rigid doublet. Both the linear and angular velocities of the particles are also affected by their interaction. Similarly, for a spherical or cylindrical particle in shear flow, both its motion parallel to the wall and its rotation are affected by the nearby presence of a plane wall.\textsuperscript{10,11}

The geometry of these particle-particle or particle-wall problems allows analytical solutions of the Stokes equations by using tangent sphere or bipolar coordinates for, respectively, touching (e.g., Refs. 12–14) and non-touching (e.g., Refs. 10, 11, 15, and 16) cases. In addition, lubrication theory, perturbation methods, the multipole expansion, the method of reflections, and the boundary collocation technique have been used to study the dynamics of interacting spheres/cylinders in shear flow (e.g., Refs. 17–22) or a sheared sphere/cylinder in close proximity of a plane wall (e.g., Refs. 22–26). These techniques, providing approximate analytical solutions, have also been employed to investigate the dynamics of a sphere/cylinder in shear flow confined between two parallel walls.\textsuperscript{27–29} Due to the linearity of the Stokes equations, there is a linear relation between, on the one hand, the force, torque, and stresslet on a particle and, on the other hand, its linear and angular velocities as well as the velocity gradient of the ambient flow.\textsuperscript{30} The set of linear relations between these parameters is defined by means of either the resistance or the mobility matrix.\textsuperscript{31} Several authors have calculated (part of) the elements of these matrices for interacting spheres in shear flow (e.g., Refs. 12, 18, 19, and 21). This approach makes the results readily available for subsequent problems related to arbitrary motions of spherical particles in linear flow fields.

As compared to interacting spheres/cylinders or a sphere/cylinder confined between two walls, the dynamics of a pair of interacting spheres/cylinders confined between two parallel walls has received much less attention up to now. Zurita-Gotor et al.\textsuperscript{32} and Yan et al.\textsuperscript{33} showed that two spherical particles in confined shear flow can exhibit swapping trajectories, in which the particles exchange their transverse position and subsequently reverse their direction of motion upon interaction. Yoon et al.\textsuperscript{34} provide a phase diagram mapping out the conditions for swapping trajectories of confined spheres and cylinders in a viscoelastic Oldroyd-B fluid.

In addition to these swapping trajectories for particles with a small center-to-center distance in the velocity gradient direction, geometrical confinement may also affect the dynamics of particles passing over each other upon interaction. Contrary to two spheres/cylinders or a sphere/cylinder
next to a plane wall, the geometry of two interacting particles confined between two parallel plates does not allow a convenient representation of the boundary conditions in any coordinate system, thereby hampering (semi-)analytical approaches. Nevertheless, explicit expressions for the particle dynamics would allow a comprehensive overview of the effects of confinement. Hence, in the present work, the interaction of two-dimensional rigid cylindrical particles in confined shear flow is studied by means of a lubrication analysis. This type of analysis has been applied before, as indicated above, and for cases of confinement has been used to study a single rotating and translating cylinder between two parallel walls with the aim of optimizing a viscous micropump (e.g., Ref. 35) or elucidating the flagellar propulsion of microorganisms.36 The basic linear structure of the problem, accounting for both particle-particle and particle-wall interactions, is presented in Sec. II. The lubrication analysis is detailed in Sec. III and summarized for force-free and torque-free particles. A variety of results are explored in Sec. IV, including the trajectories of the two particles, the thinning of the gap between the particle interfaces, which governs particle aggregation, and finally the forces on the sliding plates, which give insight into the suspension rheology. Finally, we provide concluding remarks in Sec. V.

II. MODEL FOR TWO INTERACTING PARTICLES IN CONFINED SHEAR FLOW

In this work, two non-Brownian cylindrical particles interacting in shear flow between two parallel walls are considered. Sedimentation is assumed to be negligible. When the distance in the velocity gradient direction between two approaching cylindrical particles in shear flow is sufficiently small, the particles will “collide” and form a particle doublet, which will rotate in the shear flow. The aim of the present work is to elucidate the effects of geometrical confinement on the dynamics of such a doublet of equal-sized cylindrical particles in shear flow. Figure 1 provides a schematic representation of the doublet of particles, each with radius \( a \), at a random time instant during the particle interaction. The top and bottom walls of the channel move with equal but opposite velocities \( U \) and the movement of each particle during the interaction is described by means of velocities \( v_{ix} \) and \( v_{iy} \) in the \( x \)- and \( y \)-directions, respectively, for \( i = 1,2 \) as labels of the two particles. In addition, the particles can rotate around their own axis with a rotational speed \( \omega_i \). Thus, the particle positions and orientations change continuously during their interaction.

The orientation of the particle doublet within the channel is indicated by means of the orientation angle \( \theta(t) \), which represents the angle between the velocity direction and the line connecting the particle centers. Additional geometrical parameters that are needed to describe the hydrodynamics are the dimensionless gaps \( \epsilon_i(\theta) \), \( \epsilon_b(\theta) \), and \( \epsilon_g(\theta) \) between, respectively, the top particle and the wall, the bottom particle and the wall, and between the two particles, as shown in Fig. 1 (all gaps are scaled by the particle radius \( a \)). We consider geometries where each of \( \epsilon_i(\theta) \), \( \epsilon_b(\theta) \), and \( \epsilon_g(\theta) \) are \( \ll 1 \), which allows a lubrication analysis (Sec. III). The overall confinement of the particles within the

![FIG. 1. Schematics of two interacting cylindrical particles with radius \( a \) in confined shear flow indicating the terminology for (a) velocities and (b) forces.](image-url)
The channel is quantified by the dimensionless parameter $\delta_{\text{min}}$, which is defined as

$$\delta_{\text{min}} = \epsilon_t(\pi/2) + \epsilon_b(\pi/2) + 2\epsilon_g(\pi/2).$$  \hspace{1cm} (1)$$

$\delta_{\text{min}}$ provides the minimum dimensionless channel height available for fluid flow, which occurs when the doublet orientation angle $\theta = \pi/2$. Hence, $\delta_{\text{min}}$ is a measure for geometrical confinement, with a smaller value corresponding to a larger confinement.

The wall and particle velocities induce forces and torques on both the particles and the walls that are indicated in Fig. 1(b). Many applications of suspensions involve low-Reynolds-number flows. Hence, the Stokes equations are adequate to describe the hydrodynamics of the flow around the interacting particles. Due to the linearity of the Stokes equations, the relations between, on the one hand, the forces and torques on the particles and, on the other hand, their velocities and rotational speeds are linear. Here, the motion is driven by the speed of the moving walls $U$. Hence, the dynamics of the particles can be described by means of the following matrix equation:

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ L_1 \\ F_{2x} \\ F_{2y} \\ L_2 \end{bmatrix} = \eta A \begin{bmatrix} v_{1x} \\ v_{1y} \\ \omega_1 \\ v_{2x} \\ v_{2y} \\ \omega_2 \end{bmatrix} + \eta U B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$  \hspace{1cm} (2)$$

for which the velocities $v_i$, rotational speeds $\omega_i$, forces $F_i$, and torques $L_i$ are defined in Fig. 1. $A$ is a $6 \times 6$ matrix and $B$ is a $6 \times 1$ matrix and the matrix elements of $A$ and $B$ are functions of the particle radius $a$, the doublet orientation angle $\theta$, and the dimensionless gaps $\epsilon_t(\theta)$, $\epsilon_b(\theta)$, and $\epsilon_g(\theta)$. The derivation of the matrix elements will be described in Sec. III and the detailed results are listed in Appendix. Taking into account the fact that the interacting particles are force- and torque-free, Eq. (2) can be used to calculate the particle velocities for each doublet orientation angle $\theta(t)$.

The goal of the present work is to track the particle trajectories and the forces during the interaction and rotation of a particle pair. Therefore, the analysis is started when the dimensionless distance between the particle surfaces $\epsilon_t(\theta(0)) = 0.1$. In this work, only particle pairs that are positioned symmetrically in the gap ($\epsilon_t(\theta) = \epsilon_b(\theta)$) will be considered. For each value of the minimum free fraction of the channel height $\delta_{\text{min}}$, the dimensionless gap height $b/a$ is fixed,

$$b/a = (4 + \delta_{\text{min}}),$$  \hspace{1cm} (3)$$

The remaining parameter that is expected to affect the particle dynamics is the initial doublet orientation angle $\theta$ (at $\epsilon_g(\theta(0)) = 0.1$), which is governed by the distance between the particle centers in the velocity gradient direction. From the initial values of $\epsilon_g$ and $\theta$, the initial particle-wall gaps $\epsilon_t$ and $\epsilon_b$ can be determined,

$$b/a = 2 + \epsilon_t(\theta) + \epsilon_b(\theta) + \left(2 + 2\epsilon_g(\theta)\right) \sin \theta.$$  \hspace{1cm} (4)$$

The dynamics of the interacting particles can then be traced by updating the particle positions at each time instant by using the velocities obtained from Eq. (2) in an Euler step, $\Delta t = \nu\Delta t$, with the positions of the two particles, which vary as a function of time during the interaction, denoted by $(x_1, y_1)$ and $(x_2, y_2)$. These coordinates can be obtained from the dimensionless gaps,

$$2(1 + \epsilon_g(\theta)) \cos \theta = \frac{x_2 - x_1}{a},$$  \hspace{1cm} (5a)$$

$$\frac{b}{2a} = \frac{y_1}{a} + 1 + \epsilon_t(\theta),$$  \hspace{1cm} (5b)$$

$$-\frac{b}{2a} = \frac{y_2}{a} - 1 - \epsilon_b(\theta).$$  \hspace{1cm} (5c)$$
Since distances between particles and walls rather than particle coordinates determine the interaction forces, in every time step, Eqs. (5a)-(5c) are used to make conversions between particle coordinates and dimensionless gaps, which all depend on time. A suitable time step $\Delta t$ was determined by verifying convergence of the calculated gap between the particle surfaces $\epsilon_g(\theta)$ with subsequent reductions of the time step. Time was made dimensionless with the shear rate $\dot{\gamma} = 2U/b$. For the largest confinement value used, which corresponds to a minimum free fraction of the gap height $\delta_{min} = 10^{-3}$, a time step $\dot{\gamma}\Delta t = 5 \times 10^{-6}$ was needed.

III. LUBRICATION ANALYSIS OF THE DIFFERENT GAP REGIONS

The motions of the particles cause hydrodynamic forces due to interactions with the walls and interactions between the particles. When the dimensionless gaps $\epsilon_t(\theta)$, $\epsilon_b(\theta)$, and $\epsilon_g(\theta)$ are sufficiently small, the forces and torques will be primarily located in the gap regions. Hence, the forces and torques in the gap regions will be determined first. For the geometry shown in Fig. 1, two different gap types should be considered, namely, the gap between a cylinder and a wall and the gap between two cylinders. Due to the linearity of the Stokes equations, individual solutions of the equations will not affect each other and solutions for problems with different boundary conditions can be added to provide the solution for a problem with more complex boundary conditions. Hence, for each of the gap types, two cases will be considered, namely, that of a tangential movement and that of a perpendicular movement. This approach leads in total to four cases that are depicted in Figs. 2 and 3 for, respectively, tangential and perpendicular motions and will be discussed in Secs. III A and III B, respectively.

A. Motion tangential to a wall or the surface of another particle

First, the forces and torques for particle motions parallel to a wall or along the surface of another particle will be determined. As the derivation is similar for both cases, only case 2 for the shearing motion of two cylinders next to each other will be detailed here. A lubrication analysis for the case of a cylinder moving tangentially along a plane wall can be found in the works of Jeffrey and Onishi and Yang et al.

The relevant forces and torques are located in the narrow gap region between the cylinders. In this region, the cylinder surface can be approximated by

$$h(x) = h_0 \left(1 + \frac{x^2}{2ah_0}\right), \tag{6}$$

![FIG. 2. Schematics of a cylindrical particle translating (a) parallel to a wall or (b) parallel with the surface of another particle. A cross indicates the center of the $xy$ coordinate system. The first subscript of the translational velocities indicates the particle number, whereas the second subscript indicates whether the motion is with respect to the wall or to the other particle and the third subscript indicates tangential or perpendicular motion.](image-url)
with the $x$ direction as defined in Fig. 2. The minimum gap height is then denoted as $h_0 = \epsilon a$. Eq. (6) demonstrates that the relevant length scale along the $x$-direction is $\sqrt{\epsilon a}$. In addition, the $y$-coordinate will be scaled with the gap height $\epsilon a$, and for the characteristic velocity, the net velocity difference was found to be convenient,

$$V_S = a\omega_1 + a\omega_2 + v_{1p\parallel} - v_{2p\parallel},$$

with the translational velocities $v_{1p\parallel}$ and $v_{2p\parallel}$ as defined in Fig. 2. It should be noted here that the characteristic velocity $V_S$ varies as a function of time, which does not pose a problem due to the quasi-steady form of the Stokes equations. These scales lead to rescaled variables,

$$X = \frac{x}{\epsilon a}, \quad Y = \frac{y}{\epsilon a}, \quad V_X = \frac{v_x}{V_S}, \quad V_Y = \frac{v_y}{V_S\sqrt{\epsilon}}, \quad P = \frac{ae^{3/2}p}{\eta V_S},$$

in which the latter two are obtained from scaling of the continuity equation and the $x$ component of the Stokes equations. With these scalings, the dimensionless expression for the gap region becomes

$$H(X) = 1 + \frac{X^2}{2},$$

and the lubrication approximation of the Stokes equations in the scaled variables is

$$\frac{\partial^2 V_X}{\partial Y^2} = \frac{\partial P}{\partial X},$$

$$\frac{\partial P}{\partial Y} = 0,$$

$$\frac{\partial V_X}{\partial X} + \frac{\partial V_Y}{\partial Y} = 0.$$ 

Eq. (10b) illustrates that the pressure only depends on $x$. The boundary conditions for the particle motions shown in Fig. 2(b), in terms of the scaled variables, become

$$Y = H : \quad V_X = \frac{a\omega_1 + v_{1p\parallel}}{V_S}, \quad V_Y = \frac{a\omega_1 X}{V_S},$$

$$Y = -H : \quad V_X = \frac{-a\omega_2 + v_{2p\parallel}}{V_S}, \quad V_Y = \frac{a\omega_2 X}{V_S},$$

in which terms proportional to $\epsilon$ have been omitted.
The solution of these equations follows standard steps. Integrating Eq. (10a) twice and using boundary conditions (11a) and (11b), $V_X$ becomes

$$V_X(X,Y) = \frac{1}{2} \frac{dP}{dx}(y^2 - H^2) + \frac{1}{2H} Y + \frac{a_1 - a_2 + v_{1p} + v_{2p}}{2S}.$$  

(12)

By substituting Eq. (12) in Eq. (10c), integrating in $Y$, and using boundary conditions (11a) and (11b), both $V_Y(X,Y)$ and a differential equation for $P(X)$ are found,

$$\frac{d^2P}{dx^2} + \frac{dP}{dx} \frac{3X}{H} = \frac{3}{2H^3} \frac{(\omega_1 - \omega_2)AX}{V_S}.$$  

(13)

By integrating, the derivative $\frac{dP}{dx}$ can be obtained,

$$\frac{dP}{dx} = \frac{3/4(\omega_1 - \omega_2)AX^2 + AV_S}{(1 + X^2/2)^2V_S},$$  

(14)

where $A$ is an integration constant. By integrating Eq. (14) in $X$, the pressure profile $P(X)$ is obtained,

$$P(X) = \frac{AV_S X - 3/2a_1 - \omega_2)X}{V_S(2 + X^2)^2} + \frac{3/4AXV_S + 3/8a_1 - \omega_2)X}{V_S(2 + X^2)} + \left(\frac{3}{8}A\sqrt{2} + \frac{3}{16} \frac{\sqrt{2}a_1 - \omega_2)}{V_S}\right) \arctan\left(\frac{\sqrt{2}}{2} X\right) + B,$$

(15)

where $B$ is an integration constant. For the case of interacting particles in shear flow in a symmetrical configuration, the pressure drop over the gap region is zero. Hence, the constant $A$ can be determined from the difference between the pressure at $\infty$ and $-\infty$, where $\infty$ should be interpreted as a distance $\gg \sqrt{\eta} a$. In addition, the pressure at infinity can be arbitrarily taken to be zero, which leads to $B = 0$. These steps result in

$$P(X) = -\frac{2a_1 - \omega_2)X}{V_S(2 + X^2)^2}. $$  

(16)

Once the velocity and pressure profiles in the gap are known, the force (per unit length) from the fluid on the top cylinder in the $x$ direction can be obtained from the integral,

$$F_{1p} = e_x \cdot \int_{-x^*}^{x^*} n \cdot T \, dx,$$  

(17)

where $T$ is the stress tensor, $n$ is the unit normal on the cylinder surface directed into the fluid, $e_x$ is the unit vector in the $x$ direction, and $x^*$ indicates a distance over which the lubrication approximation is valid ($\sqrt{\eta} a \ll x^* \ll a$). In the gap region, for which $n$ has an orientation that is not too different from the $y$ direction, $n \approx -h e_x - e_y$. Therefore, consistent with the lubrication approximation, $F_{1p}$ becomes

$$F_{1p} = \int_{-x^*}^{x^*} (h' \tau_{xx} - \tau_{yx}) \, dx,$$  

(18)

where $\tau_{xx} = -p$ and $\tau_{yx} = \eta \frac{\partial v_x}{\partial y}$. After substituting with the scaled variables, the force has the form

$$F_{1p} = -\frac{\eta V_S}{\sqrt{\eta}} \int_{-\infty}^{\infty} \left( XP + \frac{\partial V_x}{\partial Y} \right) \, dX.$$  

(19)

Using Eqs. (12) and (16), a linear relation between the tangential force and the tangential velocities for two cylinders in a shearing motion is obtained,

$$F_{1p} = -\frac{\sqrt{2} \pi \eta}{2 \sqrt{\eta}} (a_1 + a_2 + v_{1p} + v_{2p}).$$  

(20)

When the force along the line of centers of the cylinders is calculated in a similar way, it is found to be zero, which means that a tangential motion of two rigid cylinders only causes a tangential force.
For the case of a cylinder moving tangential to a wall, a similar procedure can be used, resulting in an expression for the force parallel to the wall,

\[ F_{1w\parallel} = \frac{2\sqrt{2} \pi \eta}{\sqrt{\epsilon}} (-v_{1w\parallel} + U). \]  

(21)

Similar to the tangential motion of two cylinders, during the tangential motion of a cylinder next to a wall, the normal force on the wall is zero. From Eq. (21), we observe that the force on a rotating cylinder next to a plane wall is zero. This fact was already pointed out by Jeffrey and Onishi, and indicates that the pressure and viscous stress contributions exactly cancel each other in that case.

The torque on the particle taken about the particle center, which is only different from zero along the \( z \) axis, can be found from the integral,

\[ L_{1p} = \mathbf{e}_z \cdot \int_{-x^*}^{x^*} a \mathbf{n} \times (\mathbf{n} \cdot \mathbf{T}) \, dx, \]  

(22)

which results in

\[ L_{1p} = \mathbf{e}_z \cdot \int_{-x^*}^{x^*} a (h^2 \tau_{xy} - \tau_{yx}) \, dx. \]  

(23)

After substituting the scaled variables, it can be seen that the first term can be neglected, which results in an expression for the torque,

\[ L_{1p} = \frac{a \eta V_s}{\sqrt{\epsilon}} \int_{-\infty}^{\infty} -\frac{\partial V_X}{\partial Y} \, dX. \]  

(24)

Evaluating the integral leads to

\[ L_{1p} = -\frac{\sqrt{2} \pi a}{2\sqrt{\epsilon}} (2a \omega_1 + v_{1p\parallel} - v_{2p\parallel}). \]  

(25)

With a similar procedure, the torque on a cylindrical particle moving tangential to a wall is found to be

\[ L_{1w} = -\frac{2\sqrt{2} \pi \eta a}{\sqrt{\epsilon} \epsilon} (a \omega_1). \]  

(26)

The latter equation shows that the torque on a translating cylinder next to a plane wall is zero. This result is consistent with the force on a rotating cylinder as the resistance matrix is symmetric so the force on a rotating cylinder equals the torque on a translating cylinder and here both equal zero.

### B. Motion perpendicular to a wall or the surface of another particle

In addition to shearing motions, the particle interaction also causes squeezing motions between the particles and between the particles and the walls. Similar to Sec. III A, only case 4 (see Fig. 3) for the movement of two cylindrical particles with respect to each other will be described here. A lubrication analysis for the motion of a cylinder perpendicular to a wall can be found in the work of Jeffrey and Onishi.

The steps in the lubrication analysis are standard so we only give the main ideas here. The expressions for the gap region remain unaltered and are provided in Eqs. (6) and (9). In case 4 shown in Fig. 3, the characteristic velocity is given by the difference between the particle velocities along the line of centers of the cylinders. The scaling procedure now starts with the scaling of \( v_y \). Substituting the result in the continuity equation and the \( x \) component of the Stokes equations results in the following set of scaled variables:

\[ X = \frac{x}{\sqrt{\epsilon} a}, Y = \frac{y}{\epsilon a}, V_X = \frac{\sqrt{\epsilon} v_x}{(v_{1p\perp} - v_{2p\perp})}, V_Y = \frac{v_y}{(v_{1p\perp} - v_{2p\perp})}, P = \frac{a \epsilon^2 p}{\eta (v_{1p\perp} - v_{2p\perp})}. \]  

(27)
In scaled variables, the boundary conditions for $V_X$ and $V_Y$ become

$$Y = H : \quad V_X = 0, \quad V_Y = \frac{v_{1p\perp}}{(v_{1p\perp} - v_{2p\perp})}, \quad (28a)$$

$$Y = -H : \quad V_X = 0, \quad V_Y = \frac{v_{2p\perp}}{(v_{1p\perp} - v_{2p\perp})}. \quad (28b)$$

The scaled Stokes equations were already given in Eq. (10), and integrating Eq. (10a) twice and consequently using the boundary conditions (28a) and (28b), $V_X$ becomes

$$V_X(X,Y) = \frac{1}{2}\frac{dP}{dX} (Y^2 - H^2). \quad (29)$$

By substituting expression (29) in Eq. (10c), integrating in $Y$, and using boundary conditions (28a) and (28b), $V_Y(X,Y)$ and the following differential equation for $P(X)$ are obtained:

$$\frac{d^2P}{dX^2} + \frac{dP}{dX} \frac{3X}{H} = \frac{3}{2} \frac{1}{H}. \quad (30)$$

Integrating Eq. (30) leads to

$$\frac{dP}{dX} = \frac{3}{2} \frac{X + A}{(1 + X^2/2)^{3/2}}, \quad (31)$$

which can be integrated further to obtain $P(X)$,

$$P(X) = \frac{(AX - 3)}{(2 + X^2)} + \frac{3}{4} \frac{AX}{(2 + X^2)} + \frac{3}{8} A \sqrt{2} \arctan \left( \frac{\sqrt{2}}{X} \right) + B, \quad (32)$$

where $A$ and $B$ are constants. Taking into account the fact that the pressure difference between $X = -\infty$ and $X = \infty$ should be zero due to symmetry and that the pressure at infinity can be arbitrarily set to zero, we have $A = B = 0$ and the pressure profile obeys

$$P(X) = -\frac{3}{(2 + X^2)}. \quad (33)$$

Once the velocity and pressure profiles in the gap are known, the lift force $F_{1p\perp}$ on the top cylinder can be obtained from the integral,

$$F_{1p\perp} = e_y \cdot \int_{x^*}^{x^*} \mathbf{n} \cdot \mathbf{T} \, dx. \quad (34)$$

Taking into account the fact that in the lubrication limit $\mathbf{n}$ can be approximated as $\mathbf{n} = h' \mathbf{e}_x - \mathbf{e}_y$, this integral becomes

$$F_{1p\perp} = \int_{x^*}^{x^*} (h'\tau_{xy} - \tau_{yy}) \, dx. \quad (35)$$

After substituting $\tau_{xy}$ and $\tau_{yy}$ and introducing the scaled variables, it becomes clear that the first term can be neglected, which provides the expression

$$F_{1p\perp} = \frac{\eta (v_{1p\perp} - v_{2p\perp})}{\epsilon^{3/2}} \int_{-\infty}^{\infty} P \, dX. \quad (36)$$

When using Eq. (33) for $P(X)$, a linear relation between the lift force and the particle velocities along the line through the particle centers is obtained,

$$F_{1p\perp} = -\frac{3\sqrt{2}\pi \eta}{8\epsilon^{3/2}} (v_{1p\perp} - v_{2p\perp}). \quad (37)$$

A projection of the force on the particle in the tangential direction shows that in the case of a perpendicular movement, only the lift force is different from zero. Due to symmetry, also the torque on the particle is zero for this type of motion. For a particle moving in a direction perpendicular to a
wall, a similar approach leads to the lift force:  
\[ F_{1w\perp} = -\frac{3\sqrt{2}\pi \eta}{\epsilon^{3/2}} v_{1w\perp}, \]  
with \( v_{1w\perp} \) as defined in Fig. 3(b).

C. Combination of the lubrication forces

Eqs. (20), (21), (25), (26), (37), and (38) form a set of expressions providing the forces and torques in the gap regions due to particle-particle and particle-wall interactions caused by shearing and squeezing motions. In addition to this, the shear flow and particle motions cause a flow in the outer region, which generates viscous stresses on the cylinder surfaces. The contribution of these stresses in the outer region can be estimated from a simple scaling analysis. Inside the gap region, the distance over which the lubrication forces and torques act is proportional to \( \sqrt{2}\epsilon a \) and the viscous stress in this region corresponds to \( \eta U/\epsilon a \). In the outer region on the other hand, the forces and torques act over a distance proportional to the cylinder radius \( a \) and the stress corresponds to \( \eta U/b \approx \eta U/2a \). Hence, the ratio of the force or torque in the outer region to that in the inner region is proportional to \( \sqrt{\epsilon} \). This feature means that for sufficiently small gaps, the whole problem is dominated by the forces and torques in the inner region. In order to incorporate the outer contributions into the results, an outer solution should be derived and matched to the inner solution. Due to the complexity of the geometrical setup and the fact that such a geometry cannot be described easily in any curvilinear coordinate system, deriving the solution for the outer problem will not be attempted here. Rather, the results obtained from the forces and torques in the inner region will be used to provide a qualitative understanding of the dynamics of interacting cylindrical particles in confined shear flow.

As the forces for relative motions between the particles and the walls on the one hand and between the particles on the other hand are derived in different coordinate systems, the different forces are projected on the general \( xy \) coordinate system shown in Fig. 1. For the case of particle 1, the following relations are obtained:

\begin{align*}
F_{1x} &= -F_{1w\parallel} + F_{1p\parallel} \sin \theta - F_{1p\perp} \cos \theta, \\
F_{1y} &= -F_{1w\perp} + F_{1p\parallel} \cos \theta + F_{1p\perp} \sin \theta, \\
L_1 &= L_{1w\parallel} + L_{1p\parallel}. 
\end{align*}

For particle 2, similar equations can be derived. As all forces and torques in Eqs. (39a)-(39c) are provided as a function of the velocities tangential and perpendicular to the walls and the surface of the other particle, they have to be written explicitly in terms of the velocities \( v_x \) and \( v_y \) in the general coordinate system of Fig. 1,

\begin{align*}
v_{1w\parallel} &= -v_{1x}, \\
v_{1w\perp} &= -v_{1y}, \\
v_{1p\perp} &= -v_{1x} \cos \theta + v_{1y} \sin \theta, \\
v_{1p\parallel} &= v_{1x} \sin \theta + v_{1y} \cos \theta. 
\end{align*}

After substituting Eqs. (20), (21), (25), (26), (37), and (38) and subsequently Eqs. (40a)-(40d) into Eqs. (39a)-(39c), the different matrix elements of the matrices \( A \) and \( B \), as defined in Eq. (2), can be obtained. These different elements are listed in Appendix. The thus obtained model consisting of Eq. (2) together with Eqs. (A1a)–(A1pp) in Appendix will be used to study the dynamics of interacting force-free and torque-free rigid cylindrical particles in confined shear flow. More in particular, three aspects are of primary interest. First, the effects of geometrical confinement on the particle interaction time and the thinning of the gap between the particle surfaces will be discussed in Sec. IV A. When the gaps between either the particles or between the particles and the wall become sufficiently small, non-hydrodynamic forces such as van der Waals attraction or electrostatic repulsion can come into play. Taking into account van der Waals attraction allows to predict effects of geometrical confinement on particle aggregation, an aspect that is discussed in
Fig. 4. (a) Schematic of interacting particles in shear flow and (b) particle trajectories for different minimum free fractions of the channel height $\delta_{\text{min}}$. The initial contact angle $\theta$ at which $\epsilon_g = 0.1$ is 45°. The double line in (a) indicates the movement of the particle around its own axis.

Sec. IV B. Finally, the lubrication contributions to the shear and normal force in confined suspensions in shear flow will determined in Sec. IV C. This allows to assess the relative importance of these contributions in the rheology of suspensions in confined shear flow.

IV. DYNAMICS OF INTERACTING PARTICLES IN CONFINED SHEAR FLOW

A. Particle dynamics

The dynamics of two interacting particles in shear flow is schematically depicted in Fig. 4(a) and the corresponding trajectories are provided in Fig. 4(b) for three different values of confinement $\delta_{\text{min}}$. As a reference, the trajectory of a doublet of touching cylindrical particles ($\epsilon_g = 0$) is also included in Fig. 4(b). The calculations are started for a particle pair with an orientation angle $\theta$ of 45° (Fig. 4(a) panel 1) and a dimensionless gap $\epsilon_g = 0.1$. Upon interaction, the particles rotate over each other (Fig. 4(a) panel 2). At the same time, they are being pushed together and their trajectory converges towards that of a pair of touching cylindrical particles. Once the orientation angle of the pair exceeds 90°, the particles are being pulled apart by the moving plates and separate again (Fig. 4(a) panel 3). It can be seen from Fig. 4(b) that for all confinement values, the distance between the particle centers in the velocity gradient direction after the interaction is exactly the same as that before interaction. Hence, the particle collision does not cause any cross-streamline migration of the particles. Such symmetric particle trajectories have been reported before for non-confined particles in shear flow\textsuperscript{10,11} and are expected based on the reversibility of the Stokes equations.\textsuperscript{31} The double lines on the particle interface in Fig. 4(a) illustrate the rotation of the particles around their own axis. The total rotation angle of the particles varies between 60° and 55° when $\delta_{\text{min}}$ is decreased from 0.1 to 0.001, showing that the total particle rotation is rather insensitive to the geometrical confinement.

It can be concluded from Fig. 4(b) that the overall trajectory of highly confined interacting particles in shear flow is not affected by the confinement value. Thus, for a range of confinement values, corresponding to small free fractions of the channel height, the macroscopic behavior of interacting particles in shear flow is nearly universal. Hence, in the absence of any particle interactions other than hydrodynamic interactions, no effect of geometrical confinement on structure formation or particle assembly in shear flow can be expected. However, when additional non-hydrodynamic interactions come into play, the gap between the particle surfaces is an essential parameter for particle assembly. Since this gap can be orders of magnitude smaller than the particle diameter, it cannot be sufficiently resolved from Fig. 4(b). Hence, the dimensionless gap between the particle surfaces $\epsilon_g$ is plotted in Fig. 5 as a function of the orientation angle of the particle pair for three values of confinement. In agreement with the reversibility of the particle trajectories shown in Fig. 4(b), the dimensionless gap $\epsilon_g(\theta)$ for each case starts at 0.1 when $\theta = 45°$ and returns to this value at an orientation angle of 135°. During the interaction, the minimal distance between the
FIG. 5. Thinning of the gap between the particles as a function of doublet orientation angle for different minimum free fractions of the channel height $\delta_{\text{min}}$. The initial contact angle at which $\epsilon_g = 0.1$ is 45°. The inset shows the linear relation between the minimum dimensionless gap $\epsilon_{g,\text{min}}$ between the particles and the minimum free fraction of the channel height $\delta_{\text{min}}$.

particle surfaces occurs at an orientation angle of 90°, which corresponds to the case depicted in panel 2 of Fig. 4(a). This minimum gap decreases if the confinement is increased, with the additional thinning mainly occurring for orientation angles around 90°. This feature can be attributed to the decreased distance between the particle surfaces and the walls at these orientation angles, thus leading to more pronounced hydrodynamic interacting forces, which will be discussed in more detail in Sec. IV C. However, the minimum distance between the particle surfaces $\epsilon_{g,\text{min}}$ becomes much smaller as compared to the minimum free fraction of the channel height $\delta_{\text{min}}$. The inset in Fig. 5 shows that the relation between the minimum gap between the particle surfaces $\epsilon_{g,\text{min}}$ and the minimum free fraction of the channel $\delta_{\text{min}}$ is fairly linear with the gap between the particle surfaces accounting for only approximately 4% of the minimum free fraction of the channel. This latter percentage obviously depends on the initial orientation angle of the particle pair, as will be illustrated in Sec. IV B.

Figure 5 demonstrates an increased thinning of the gap between the particle surfaces with decreasing values of the minimum free fraction of the channel height $\delta_{\text{min}}$. The lower value of the minimum gap between the particle surfaces could be caused by either a faster gap thinning or an extended interaction time. In order to further elucidate the mechanism of gap thinning, the evolution of the gap as a function of dimensionless time is provided in Fig. 6. The time is made dimensionless with the shear rate $\dot{\gamma} = 2U/b$. This graph clearly shows that the kinetics of gap thinning is not

FIG. 6. Thinning of the gap between the particles as a function of dimensionless time for different minimum free fractions of the channel height $\delta_{\text{min}}$. The initial contact angle at which $\epsilon_g = 0.1$ is 45°. The time is made dimensionless with the shear rate $\dot{\gamma} = 2U/b$. 
affected by confinement. However, when the confinement is increased (or the free fraction of the channel is decreased), thinning occurs for a longer time. Since the minimum gap occurs when the particle pair reaches an orientation angle of 90°, irrespective of the confinement value, this means that the rotation of the particle pair is slowed down due to the presence of the walls and more time becomes available for gap thinning.

Based on the solution curves in Fig. 6, the interaction time of the particles can be derived. This time is defined as the time needed for the particles to separate again up to their initial distance \( \epsilon_g = 0.1 \), as indicated by the arrows in Fig. 6. The obtained dimensionless interaction times are provided as a function of \( \delta_{\text{min}} \) in Fig. 7, where it can be seen that the interaction time gradually increases when the free fraction of the gap \( \delta_{\text{min}} \) decreases and converges towards a plateau value at very high confinement values. The values presented in Fig. 7 can be compared with the predictions of Jeffery \(^{37}\) for a rigid prolate spheroid, which can be used to represent a pair of touching cylinders.\(^{10}\) The Jeffery equations for a spheroid in an unbounded shear flow provide a dimensionless interaction time of 2.4 for a rotation from \( \theta = 45^\circ \) to 135°. Geometrical confinement thus results in an increase of the interaction time of around 50% with respect to the bulk value.

B. A simplified characterization of aggregate formation

Lubrication forces prevent rigid particles with smooth surfaces to come into physical contact due to the divergence of the drainage force when the gap between the particles tends to zero, as shown in Eq. (37). However, the presence of non-hydrodynamic attractive forces between the particles can lead to particle aggregation. These forces only become effective if the interparticle distance is sufficiently small. It is beyond the scope of the present work to extend the model presented in Secs. II and III with non-hydrodynamic forces. However, using simple arguments, an estimate can be made of the effects of confinement on the capture cross section and corresponding capture efficiency of interacting attractive particles in shear flow. The capture efficiency provides the percentage of colliding particles that coagulate into particle aggregates and is thus an essential parameter for the development of aggregation models. For bulk shear flow, the capture efficiency has been determined both experimentally and theoretically, for example, for systems with repulsive electrostatic interactions and attractive van der Waals attraction.\(^{38,39}\)

To estimate the effect of geometrical confinement on the capture cross section, a simplified profile of the attraction force is assumed, in which the attraction is zero when the gap between the particle surfaces is larger than a certain critical value and becomes dominant with respect to the hydrodynamic lubrication force when the gap between the particles becomes smaller than the critical value. This critical value is system-dependent, and for the purpose of illustrating the qualitative effect, a value \( \epsilon_{g,\text{crit}} = 2 \times 10^{-3} \) is assumed. Then, different particle collisions were considered,
FIG. 8. Diagram indicating conditions for aggregation versus no aggregation for interacting rigid cylindrical particles in confined shear flow. The critical minimum dimensionless gap spacing $\epsilon_{g,\text{crit}} = 2 \times 10^{-3}$. The upper region of the diagram corresponds to orientation angles for which a particle encounter with $\epsilon_g = 0.1$ is not possible due to space limitations in the channel.

each starting with a gap between the particle surfaces $\epsilon_g(\theta(0)) = 0.1$ but with different values of the initial orientation angle $\theta(0)$. This initial condition corresponds to particles that are on streamlines that have different separations in the velocity gradient direction. For each case, the evolution of the gap between the particle surfaces is traced, and when this value drops below the critical value, aggregation is assumed to occur due to the attractive force. Hence, aggregation will only occur when the orientation angle at the start of the interaction is sufficiently low, providing sufficient interaction time for gap thinning. The initial orientation angle was changed in steps of $10^\circ$.

In Fig. 8, the highest orientation angle leading to particle aggregation and the lowest orientation angle leading to particle interaction without aggregation are indicated with, respectively, filled and open symbols. The figure clearly shows that, with increasing confinement, or decreasing $\delta_{\text{min}}$, the range of orientation angles over which aggregation occurs substantially increases for a limited change in confinement value. Although only a two-dimensional particle configuration is used in this work, the fact that two spheres in the same velocity-velocity gradient plane exhibit qualitatively similar trajectories to cylinders, both in bulk and confined shear flow, supports the hypothesis that a similar substantial effect of geometrical confinement on the aggregation of rigid spherical particles in shear flow can be expected. The cylinders represented by the two-dimensional calculations presented here could also be used as a model for fibers. However, it should be kept in mind that in the case of fiber flocculation, other aspects such as friction forces between the fibers, elastic bending and twisting torques, and non-straight fiber shapes play dominant roles.

To our best knowledge, the predicted effects of geometrical confinement on particle aggregation, as shown in Fig. 8, have not yet been reported in the literature. However, the effect of geometrical confinement on the coalescence efficiency of droplet pairs in shear flow has been experimentally investigated by De Bruyn et al. These authors report an increase of the highest initial offset in the velocity gradient direction between the droplets leading to coalescence and hence an increase in the coalescence efficiency with confinement. Their results are thus in qualitative agreement with the aggregation diagram presented in Fig. 8. However, as compared to rigid particles, the deformability of droplets introduces extra complexities as it affects both the thinning of the continuous phase fluid in between the droplet interfaces and the interaction of the droplets with the moving plates. For example, droplet deformability will lead to forces and hence migration perpendicular to a sliding plate.

C. Lubrication forces on the sliding plates

A last aspect that is of interest is the fact that the interacting pair of particles exerts lubrication forces on the sliding plates. This way, the particle pair directly contributes to the rheology of a
particle suspension in shear flow by causing shear and normal stresses in addition to the bulk stresses of the matrix fluid. In semi-dilute suspensions, the hydrodynamic contribution of particle interactions to the bulk rheology consists of a mean-field contribution, resulting from long-range hydrodynamic interactions between particles that are well separated as well as contributions from a small number of particle pairs with particles that are in close proximity at any given time.\(^2\) In bulk suspensions, the contribution of each particle is obtained from the particle stresslet exerted on the matrix fluid, which involves an integral of the local stresses over the particle surface and incorporates effects of the complete outer region.\(^4\) However, in confined shear flow, particles can also exert a direct shear and normal force on the sliding plates. From the model described in Secs. II and III, the shear and normal force contribution of particle pairs as a function of particle separation and doublet orientation can be obtained for confined shear flow. The overall contribution to the suspension rheology should then be derived from an ensemble average, accounting for all possible relative separations of a representative pair of particles and their occurrence,\(^2\) which is however beyond the scope of the present work. It should be noted that in this case also the stresses from particle pairs that are not positioned symmetrically in the gap should be taken into account. This possible asymmetric positioning of the particle pair is included in the model presented in Secs. II and III.

The shear force (per unit length) on the upper plate caused by the interacting particle pair is equal to the parallel wall force from the upper plate on particle 1, given in Eq. (21), but with opposite sign. The dimensionless shear force on the upper plate is provided in Fig. 9 as a function of the orientation angle of the particle pair. At the start and end of the particle interaction, the shear force tends towards a plateau, corresponding to the shear force resulting from a single cylinder translating with a constant velocity next to a plane wall. The value of the shear force increases during the rotation of the particle pair from 45° to 90°. The small dip in the curves for orientation angles around 90° is caused by the fact that the relative velocity of the particle with respect to the moving plate \(U - v_{1w}\) substantially decreases when the particle pair reaches an orientation angle of 90°, which counteracts the increase of the shear force due to the reduced gap spacing between the particle and the wall. With increasing confinement, the shear force on the upper plate increases, as expected from Eq. (21) since the gap \(\epsilon_g\) decreases proportionally with \(\delta_{\text{min}}\). The shear stress due to the bulk fluid is \(O(\eta \dot{\gamma})\) and it acts over a length \(O(a)\), which results in a bulk shear force that is \(O(\eta U)\). Hence, Fig. 9 shows that for the presented confinement values, the contribution of the lubrication force originating from particle pairs is substantially higher than the shear force due to the bulk fluid. In semi-dilute and concentrated confined suspensions, particle interactions are thus expected to substantially increase the shear stress on the sliding plates.

In addition to a shear force, the interacting particle pair also exerts a normal force on the moving plates. The normal force on the top plate is equal in value and opposite to the force of the plate on the top particle, given in Eq. (38). This force (per unit length), non-dimensionalized with \(\eta U\) is

![Figure 9](image-url)
shown in Fig. 10. Similar to the shear force, the evolution of the normal force during the particle interaction is governed by two competing factors, namely, the gap between the top plate and the particle surface on one hand and the velocity $v_{1w\perp}$ of the particle perpendicular to the wall on the other hand. The former decreases during the rotation of the particle pair up to an orientation angle of 90°, thereby leading to an increase of the normal force up to this angle. However, while the gap between the particle and the wall decreases, $v_{1w\perp}$ shows a more complex behaviour. Before particle interaction, $v_{1w\perp} = 0$ resulting in the absence of a normal force on the plates for a single cylinder in shear flow. Due to the particle interaction, $v_{1w\perp}$ increases, which results in an increase of $F_{normal}$. However, due to the fact that the particles approach the wall, $v_{1w\perp}$ decreases again and becomes zero when the particle pair has an orientation angle of 90°. Afterwards, $v_{1w\perp}$ reverses direction and the particles separate. The combination of both effects leads to a maximum of the normal force for an orientation angle close to 90° followed by a reversal of the normal force from an upwards push to a downwards pull. Obviously, when many particle pairs are present in a random manner, the normal force of the different particle pairs should be averaged out leading to a lubrication normal force of zero. However, it has been shown that in concentrated particle suspensions, there can be an excess of particle pairs in the compressional quadrant of the shear flow. Hence, when considering applications in which only one or a few particle pairs are present in a confined gap or the particle pairs are not present in a random manner, Fig. 10 shows that the normal force of an interacting particle pair steeply increases with confinement and moreover is an order of magnitude larger than the shear force.

For a single cylinder translating parallel to a plane wall with a constant velocity $v_{1w\parallel}$, the shear force is proportional to the tangential velocity difference $U - v_{1w\parallel}$ and inversely proportional to $\epsilon_t^{1/2}$, as shown in Eq. (21). For two interacting particles in confined shear flow, the only externally applied velocity is that of the sliding plate. Rather than being externally applied, the velocity $v_{1w\parallel}$ is now determined by the interactions of the particle with both the sliding plate and the second particle. The dependence of the tangential velocity difference $U - v_{1w\parallel}$ on the confinement value for a particle doublet with an orientation angle of 90° is explored in Fig. 11(a). This figure shows that the dimensionless relative velocity of each particle with respect to the sliding plate is independent of the confinement value. For doublet orientation angles around 90°, Fig. 5 shows a linear scaling between $\epsilon_t$ and hence $\epsilon$ and the minimum free fraction of the channel height $\delta_{min}$. Hence, Eqs. (20) and (21) show that the forces due to, on the one hand, the wall and, on the other hand, the second particle show the same dependence on confinement, thus resulting in a constant relative velocity between the particle and the wall. As a consequence, the shear force on the sliding plates from a particle doublet with an orientation angle of 90° is proportional to $\epsilon_t^{1/2}$, as shown in Fig. 11(a).

As mentioned before, the normal force on the sliding plates is zero when the doublet orientation angle is 90°. However, the maximum normal force is reached for orientation angles that approach 90° when the minimum free fraction of the channel height goes to zero. Hence, the scaling of the
normal force with the minimum free fraction of the channel height is explored for orientation angles around 90° in Fig. 11(b). This figure shows that the normal force is inversely proportional to the minimum free fraction of the channel height. Similar to the shear force, the scaling can be rationalized starting with Eq. (38) for a single cylinder translating with a constant velocity perpendicular to a plane wall. It should be noted here that, for a single cylinder in simple shear flow, there is no driving force for translation perpendicular to the walls. However, in the presence of a second particle, the interaction results in an evolution of $v_{1\perp}$ as a function of the doublet orientation angle. Since this movement is driven by the external plate velocity $U$, it can be expected to increase proportional to $\delta_{\text{min}}^{1/2}$ (Eq. (27)), which was confirmed by the data (not shown). Inserting this in Eq. (38) leads to a predicted scaling of $F_{\text{normal}}$ with $\delta_{\text{min}}^{-1}$, which is confirmed in Fig. 11(b). By subsequently taking into account the fact that in the gap regions the distance over which the velocities and pressure evolve to zero is proportional to $\epsilon^{-1/2}$, a master curve can be constructed for the normal force as a function of the doublet orientation angle $\theta$, as shown in Fig. 11(b). It should be noted however that this scaling is only valid for orientation angles of the particle pair around 90°, with the validity range steeply decreasing when the confinement is increased. A similar master curve could be constructed for the drainage force between the particle surfaces (not shown). For the shear force, the range of orientation angles of the particle pair for which the scaling of Fig. 11(a) is valid is only about half that of the normal force.

In conclusion, the combined effects of particle interaction and the presence of confining walls lead to a complex evolution of the shear and normal forces on the sliding plates as a function of the doublet orientation angle. However, since all particle movements are driven by the externally applied plate speed and the geometrical setup, appropriate scalings with the minimum free fraction of the channel height $\delta_{\text{min}}$ and the plate speed $U$ lead to master curves of velocities and forces as a function of doublet orientation angle. Hence, Figs. 9–11 allow determination of the contribution of the lubrication shear and normal forces due to interacting particle pairs on the rheology of

FIG. 11. (a) Dimensionless tangential velocity difference and dimensionless shear force (per unit length) on the upper plate for $\theta = 90°$ as a function of the minimum free fraction of the channel height. (b) Rescaled dimensionless normal force (per unit length) on the upper plate versus rescaled orientation angle for different minimum free fractions of the channel height $\delta_{\text{min}}$. The initial contact angle at which $\epsilon_g = 0.1$ is 45°.
confined suspensions in shear flow for a range of confinement values. As mentioned before, also normal stresses in suspensions with viscoelastic fluids and surrounding particles in concentrated suspensions cause some form of confinement. Hence, using a cell model as in the work of Choi and Schowalter, with the boundary conditions being determined by the surrounding suspension rather than the sliding walls, the approach presented here could allow to provide insight into the dynamics and rheology. However, it should be noted that due to the 2D nature of the calculations, the absolute values of the obtained stresses may not quantitatively match those in real 3D systems.

V. CONCLUSIONS

The effect of geometrical confinement on the interaction of two rigid cylindrical particles in shear flow was investigated in the limit of low Reynolds numbers and thin gaps between the particles and the walls and between the particle surfaces. For these conditions, the forces and torques on the particles in the particle-particle and particle-wall gaps dominate over the contributions of the outer flow. A lubrication analysis of the flow in the different gap regions was performed for both squeezing and shearing motions. Combining the obtained forces and torques allowed the determination of analytical expressions for the elements of the resistance matrix, linking forces and torques on the particles to their linear and angular velocities. Using this model, consisting of Eq. (2) together with expressions (A1a)–(A1pp) in Appendix, the dynamics of interacting cylinders in shear flow in a symmetrically confined channel geometry was studied. A priori it is known from the Stokes equations that the particles undergo a kinematically reversible interaction with no cross-streamline migration, irrespective of the confinement ratio. When the geometrical confinement is increased, the minimum distance between the particle surfaces during the interaction decreases due to the increased interaction time of the particle pair. In agreement with the smaller gaps, geometrical confinement causes a substantial increase of both the normal and tangential lubrication forces on the particles and the sliding plates. The combined effects of geometrical confinement and particle interaction result in a complex evolution of the particle velocities and the resulting lubrication forces as a function of the doublet orientation angle. However, the curves of particle velocities and forces versus orientation angle of the doublet pair for different confinement values can be superposed onto a master curve by using appropriate scalings with the minimum free fraction of the channel height and the plate speed. Finally, by assuming a simple on/off non-hydrodynamic attractive particle interaction force, an aggregation diagram could be constructed, mapping out the initial relative particle positions for which aggregation occurs. Based on this qualitative diagram, geometrical confinement is expected to enhance particle aggregation, a finding that warrants further exploration of confined suspensions by means of experimental and numerical studies.

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APPENDIX: MATRIX ELEMENTS FOR PARTICLE DYNAMICS

Here, we list the elements of the 6 × 6 matrix $A$ and the 6 × 1 matrix $B$. It is worthwhile to note that $A$ is symmetric. In addition, for the elements above the diagonal, it can be noted that $A_{12} = -A_{13} = -A_{34} = A_{45}$, $A_{13} = A_{16} = -A_{34} = -A_{46}$, and $A_{23} = A_{26} = -A_{35} = A_{56}$ whereas similar relations hold for the elements below the diagonal. These equalities result from the symmetry of the problem combined with the fact that the force on a rotating cylinder near a plane wall and the torque on a translating cylinder near a plane wall are zero. Based on this symmetry of the resistance matrix, it can be concluded that for interacting cylinders symmetrically placed in a confined channel, not only the torque on a translating cylinder equals the force on a rotating cylinder but also the torque on
one cylinder due to the translation of the other equals the force on the second one due to the rotation of the first one,

\[ A_{11} = -\sqrt{2\pi} \left( \frac{2}{\sqrt{\epsilon_i}} + \frac{\sin^2\theta}{2\sqrt{\epsilon_g}} + \frac{3\cos^2\theta}{8\epsilon_g^{3/2}} \right), \]  
(A1a)

\[ A_{12} = \sqrt{2\pi} \sin \theta \cos \theta \left( -\frac{1}{2\sqrt{\epsilon_g}} + \frac{3}{8\epsilon_g^{3/2}} \right), \]  
(A1b)

\[ A_{13} = -\frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \sin \theta, \]  
(A1c)

\[ A_{14} = \sqrt{2\pi} \left( \frac{\sin^2\theta}{2\sqrt{\epsilon_g}} + \frac{3\cos^2\theta}{8\epsilon_g^{3/2}} \right), \]  
(A1d)

\[ A_{15} = \sqrt{2\pi} \sin \theta \cos \theta \left( \frac{1}{2\sqrt{\epsilon_g}} - \frac{3}{8\epsilon_g^{3/2}} \right), \]  
(A1e)

\[ A_{16} = -\frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \sin \theta, \]  
(A1f)

\[ A_{21} = \sqrt{2\pi} \sin \theta \cos \theta \left( -\frac{1}{2\sqrt{\epsilon_g}} + \frac{3}{8\epsilon_g^{3/2}} \right), \]  
(A1g)

\[ A_{22} = -\sqrt{2\pi} \left( \frac{3}{\epsilon_i^{3/2}} + \frac{\cos^2\theta}{2\sqrt{\epsilon_g}} + \frac{3\sin^2\theta}{8\epsilon_g^{3/2}} \right), \]  
(A1h)

\[ A_{23} = -\frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \cos \theta, \]  
(A1i)

\[ A_{24} = \sqrt{2\pi} \sin \theta \cos \theta \left( \frac{1}{2\sqrt{\epsilon_g}} - \frac{3}{8\epsilon_g^{3/2}} \right), \]  
(A1j)

\[ A_{25} = \sqrt{2\pi} \left( \frac{\cos^2\theta}{2\sqrt{\epsilon_g}} + \frac{3\sin^2\theta}{8\epsilon_g^{3/2}} \right), \]  
(A1k)

\[ A_{26} = -\frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \cos \theta, \]  
(A1l)

\[ A_{31} = -\frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \sin \theta, \]  
(A1m)

\[ A_{32} = -\frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \cos \theta, \]  
(A1n)

\[ A_{33} = -\sqrt{2\pi} a^2 \left( \frac{2}{\sqrt{\epsilon_i}} + \frac{1}{\sqrt{\epsilon_g}} \right), \]  
(A1o)

\[ A_{34} = \frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \sin \theta, \]  
(A1p)

\[ A_{35} = \frac{\sqrt{2\pi} a}{2\sqrt{\epsilon_g}} \cos \theta, \]  
(A1q)

\[ A_{36} = 0, \]  
(A1r)

\[ A_{41} = \sqrt{2\pi} \left( \frac{\sin^2\theta}{2\sqrt{\epsilon_g}} + \frac{3\cos^2\theta}{8\epsilon_g^{3/2}} \right), \]  
(A1s)

\[ A_{42} = \sqrt{2\pi} \sin \theta \cos \theta \left( \frac{1}{2\sqrt{\epsilon_g}} - \frac{3}{8\epsilon_g^{3/2}} \right), \]  
(A1t)
\[ A_{43} = \frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \sin \theta, \quad (A1u) \]

\[ A_{44} = -\frac{\sqrt{2\pi}}{\sqrt{\epsilon_b}} \left( \frac{2}{\sqrt{\epsilon_g}} \frac{\sin^2 \theta}{2\sqrt{\epsilon_g}} + \frac{3\cos^2 \theta}{8\epsilon_g^{3/2}} \right), \quad (A1v) \]

\[ A_{45} = \sqrt{2\pi} \sin \theta \cos \theta \left( -\frac{1}{2\sqrt{\epsilon_g}} + \frac{3}{8\epsilon_g^{3/2}} \right), \quad (A1w) \]

\[ A_{46} = \frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \sin \theta, \quad (A1x) \]

\[ A_{51} = \sqrt{2\pi} \sin \theta \cos \theta \left( \frac{1}{2\sqrt{\epsilon_g}} - \frac{3}{8\epsilon_g^{3/2}} \right), \quad (A1y) \]

\[ A_{52} = \sqrt{2\pi} \left(\frac{\cos^2 \theta}{2\sqrt{\epsilon_g}} + \frac{3\sin^2 \theta}{8\epsilon_g^{3/2}} \right), \quad (A1z) \]

\[ A_{53} = \frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \cos \theta, \quad (A1aa) \]

\[ A_{54} = \sqrt{2\pi} \sin \theta \cos \theta \left( \frac{1}{2\sqrt{\epsilon_g}} - \frac{3}{8\epsilon_g^{3/2}} \right), \quad (A1bb) \]

\[ A_{55} = -\sqrt{2\pi} \left( \frac{3}{8\epsilon_g^{3/2}} + \frac{3\sin^2 \theta}{2\sqrt{\epsilon_g}} + \frac{3\cos^2 \theta}{8\epsilon_g^{3/2}} \right), \quad (A1cc) \]

\[ A_{56} = \frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \cos \theta, \quad (A1dd) \]

\[ A_{61} = -\frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \sin \theta, \quad (A1ee) \]

\[ A_{62} = -\frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \cos \theta, \quad (A1ff) \]

\[ A_{63} = 0, \quad (A1gg) \]

\[ A_{64} = \frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \sin \theta, \quad (A1hh) \]

\[ A_{65} = \frac{\sqrt{2\pi}a}{2\sqrt{\epsilon_g}} \cos \theta, \quad (A1ii) \]

\[ A_{66} = -\sqrt{2\pi}a^2 \left( \frac{2}{\sqrt{\epsilon_b}} + \frac{1}{\sqrt{\epsilon_g}} \right), \quad (A1jj) \]

\[ B_{11} = \frac{2\sqrt{2\pi}}{\sqrt{\epsilon_l}}, \quad (A1kk) \]

\[ B_{21} = 0, \quad (A1ll) \]

\[ B_{31} = 0, \quad (A1mm) \]

\[ B_{41} = -\frac{2\sqrt{2\pi}}{\sqrt{\epsilon_b}}, \quad (A1nn) \]

\[ B_{51} = 0, \quad (A1oo) \]

\[ B_{61} = 0. \quad (A1pp) \]


