The plunging of a large intruder sphere into a prefluidized granular bed with various constant velocities and various sphere diameters is investigated using a state-of-the-art hybrid discrete particle and immersed boundary method, in which both the gas-induced drag force and the contact force exerted on the intruder can be investigated separately. We investigate low velocities, where velocity dependent effects first begin to appear. The results show a concave-to-convex dependence of the plunging force as a function of intruder depth. In the concave regime the force fits to a power law with an exponent around 1.3, which is in good agreement with existing experimental observations. Our simulation results further show that the force exerted on the frontal hemisphere of the intruder is dominant. At larger intruder velocities, friction with the granular medium causes a velocity-dependent drag force. As long as the granular particles have not yet closed the gap behind the intruder, this drag force is independent of the actual intruder depth. In this regime, the drag force experienced by intruders of different diameter moving at different velocities all fall onto a single master curve if plotted against the Reynolds number, using a single value for the effective viscosity of the granular medium. This master curve corresponds well to the Schiller-Naumann correlation for the drag force between a sphere and a Newtonian fluid. After the gap behind the intruder has closed, the drag force increases not only with velocity but also with depth. We attribute this to the effect of increasing hydrostatic particle pressure in the granular medium, leading to an increase in effective viscosity.

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1. INTRODUCTION

The plunging of an intruder into a dense granular bed occurs in many different cases, such as meteor impacts and footprints on sand. Many studies have tried to find the macroscopic force law describing the force \( f \) exerted on the intruder. In general, these force laws have the form

\[
f(z,v) = f_z(z) + f_v(z,v),
\]

where \( f_z \) is a force which depends on penetration depth \( z \) from the surface, while \( f_v \) is a drag force which depends on the intruder velocity \( v \), being zero in the limit of \( v \to 0 \), and possibly also on the penetration depth. Different variations of the details of this force law have been found.

Experimental results by Hou et al. [1] of a vertically free falling object impacting onto a horizontal bed of granular particles with relatively large initial contact velocities showed that the macroscopic force law contains a term \( f_z = k_z \), where the parameter \( k \) characterizes the increase in force with increasing depth \( z \), and a drag force which scales linearly with the velocity \( v \). In [1], hollow cenospheres (diameter range \( d = 74 \) to 100 \( \mu \)m and density \( \rho_g = 0.693 \) g/cm\(^3\)) were used as granular medium. Particles are first poured into the bed and then loosened by slowly pulling a sieve (mesh size = 0.4 mm) which was initially buried at the bottom. The volume fraction of the bed produced by this procedure consistently gives a value of about 0.54.

Similar experiments were performed by Katsuragi and Durian [2], who used spherical glass beads (diameter range \( d = 250 \) to 350 \( \mu \)m and density \( \rho_g = 1.52 \) g cm\(^{-3}\)). The medium was fluidized, and gradually defluidized by a uniform upflow of nitrogen gas. The volume fraction occupied by the beads was \( 0.590 \pm 0.004 \) after fluidization. A steel sphere of diameter \( D = 2.54 \) cm was used as a projectile. Similar to the results of Hou et al., the macroscopic force law includes a term linear in depth \( z \). However, the drag force was now found to scale quadratically with the velocity \( v \).

Investigations done by Lohse et al. [3] confirmed that for freely falling projectiles in the limit of zero impact velocity \( (v \to 0) \), the macroscopic force law can be described simply by the term \( kv \).

Completely different results are obtained for intruders moving with constant but relatively small vertical velocities. Stone et al. pushed a flat plate vertically into a granular medium [4,5], and found that the penetration force increases with increasing depth nearly linearly in an initial regime, then followed by a depth-independent regime. When the plate was pushed near the bottom of the container, the penetration force showed an exponential increase. The authors deem that the initial linear regime is due to hydrostatics while the depth-independent regime is a Janssen-like regime, which is due to side wall support.

Hill et al. [6] measured the drag force of an intruder plunging into and withdrawing from a shallow granular bed (about \( 0.1 \) m), and found that both the plunging and the withdrawing forces possessed power-law dependence on the immersion depth with exponents exceeding unity, i.e., 1.3 for plunging and 1.8 for withdrawing.

Also Peng et al. [7] found that the plunging force curves of fully immersed intruders exhibit a concave-to-convex transition where the depth dependence of the force turns from supralinear to sublinear. They found that the plunging force at the inflection point is proportional to the intruder’s volume and the inflection point occurs when the intruder is fully buried to a level of around twice its diameter. Within the shallow regime, i.e., before reaching the inflection point, the plunging force
exhibits a power-law dependence for all spherical intruders, with an exponent of $1.3 \pm 0.02$.

We have summarized some of the phenomenological models for the macroscopic force $f$ in Table I.

Clearly, the form of the force experienced by an impacting intruder is still not unified, and understanding of this complex system remains limited. During the plunging process, the complex behavior of the force on the intruder is mainly caused by the inhomogeneity of the force propagation and the requirement for grain reorganization. To investigate the plunging force, simulation techniques have become an effective alternative.

In our former work, a state-of-the-art discrete particle method (DPM) combined with an immersed boundary method (IBM) was introduced to investigate the kinematics of an intruder freely dropping on a prefluidized granular bed. We showed that our simulations results are in good agreement with experimental observations, thus offering a powerful tool to investigate the gas-induced forces and contact forces directly and independently. The direct gas-induced force experienced by an intruder is small, but not negligible, for relatively large granular particles of 0.5-mm diameter and an intruder of 1-cm diameter. We also showed that for smaller granular particles (0.25-mm diameter), the relative effect of the gas becomes significant [8].

According to the experimental results discussed above, the granular plunging process is influenced by many parameters, such as the intruder diameter $D$, plunging depth $z$, and intruder velocity $v$. Thus, a straightforward idea would be to vary one parameter while keeping the other parameters constant. In this paper, we use our simulation method to further investigate the macroscopic force exerted on an intruder moving with a relatively small but constant velocity through a prefluidized bed. The relation between the macroscopic force $f$ and the intruder penetration depth $z$, intruder diameter $D$, and intruder velocity $v$ are investigated, while keeping the granular medium the same. We will show that these relations can be interpreted physically using dimensionless quantities, such as Reynolds and Froude numbers, in an effort to collectively describe all results of simulations using different intruder diameters and different velocities. We substantiate our physical interpretation by investigation of the solids pressure of the granular particles and the rearrangement of the granular particles by using a layer colored bed.

This paper is organized as follows. In Sec. II we introduce the numerical model. In Sec. III we give the simulation settings, In Sec. IV we show and discuss our results. Finally, we give our conclusions in Sec. V.

### II. MODEL DESCRIPTION

The DP-IB method used in this work has been detailed in our previous paper [8] and the applicability of the model has been verified by its good agreement with existing experimental data. Our model consists of two major submodels, the discrete particle (DP) and immersed boundary (IB) models. The DP method deals with the motion of suspended small (granular) particles, taking into account the action of gravity, gas-solid drag forces, as well as particle-particle and particle-wall collisions. In this model, the gas phase is solved on a computational mesh with a length scale exceeding the size of the small particles and the gas-particle coupling is treated by empirical drag relations [10]. The IB method deals with the motion of the large intruder through the continuous phase, enforcing no-slip boundary conditions with the surrounding gas.

We now give a brief technical description of the DP method and the IB method. A schematic representation of the DP model and IB method is shown in Fig. 1.

#### A. Equations of motion for the small particles

The motion of a spherical granular particle $a$ with mass $m_a$, moment of inertia $I_a$, and coordinate $r_a$ is described by

\[
\frac{d^2 r_a}{dt^2} = \sum_i f_{ia} + g_a + \tau_a
\]

where $f_{ia}$ is the contact force between the particles at positions $r_i$ and $r_a$ and $\tau_a$ is the torque due to the interaction of the gas with the particle. The gravity force $g_a$ is given by

\[
g_a = m_a g
\]

where $g$ is the acceleration due to gravity.

### TABLE I. Comparison of phenomenological models for the macroscopic force $f$ acting on an intruder moving through a granular bed.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Phenomenological model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hou et al. [1]</td>
<td>$f = k z + \gamma v + f_0$</td>
</tr>
<tr>
<td>Katsuragi and Durian [2]</td>
<td>$f = a z + b v^2$</td>
</tr>
<tr>
<td>Lohse et al. [3]</td>
<td>$f = k z$</td>
</tr>
<tr>
<td>Hill et al. [6]</td>
<td>$f = k z^{1.3}$</td>
</tr>
<tr>
<td>Peng et al. [7]</td>
<td>$f = k z^{1.3}$</td>
</tr>
</tbody>
</table>

Figure 1. (Color online) Schematic representation of the DP and IB methods. In the DP method, the motion of the small granular particles is solved, taking into account detailed contact forces, as well as drag forces caused by motion relative to the interstitial gas. The particles are smaller than the grid on which the gas phase equations are solved (left), requiring empirical drag relations. The intruder is much larger than the gas grid and coupled to the gas phase through the IB method, which enforces no-slip boundary conditions by a distribution of force points across the surface of the intruder (right). A schematic representation of half the bed geometry is also shown (center); particles are color coded according to their initial position in the $z$ direction. Visualization was carried out using OVITO [9].
Newton’s equations for rigid body motion:

\[ m_a \frac{d^2 \mathbf{r}_a}{dt^2} = \mathbf{F}_{g,a} + \mathbf{F}_{d,a} + \mathbf{F}_{p,a} + \mathbf{F}_{c,a}, \]  

(2)

\[ I_a \frac{d\omega_a}{dt} = \mathbf{T}_a. \]  

(3)

The four terms on the right-hand side of Eq. (2) account for the gravitational force, the gas drag force, the force due to pressure gradients in the gas phase, and the sum of the individual contact forces exerted by all other particles in contact with particle \( a \), respectively. In Eq. (3), \( \omega_a \) is the angular velocity and \( \mathbf{T}_a \) is the torque around the center-of-mass of particle \( a \).

Regarding the contact model, two types of collision models are widely used, namely, the hard sphere model and the soft sphere model. In our simulation, the soft sphere model is used since the hard sphere model is not suited for systems where quasistatic particle configurations exist. More detailed information can be found in [10,11]. For the calculation of \( \mathbf{F}_{c,a} \), a three-dimensional linear spring and dashpot type soft sphere collision model along the lines of Cundall and Strack is used [12–14]. In this model, the total contact force on particle \( a \) of radius \( R_a \) is given by a sum of normal and tangential pair forces with neighboring particles,

\[ \mathbf{F}_{c,a} = \sum_{b \in \text{contactlist}} (\mathbf{F}_{n,ab} + \mathbf{F}_{t,ab}). \]  

(4)

where the normal forces \( \mathbf{F}_{n,ab} \) depend linearly on the overlap \( \delta = R_a + R_b - |\mathbf{r}_a - \mathbf{r}_b| \) and the relative normal velocity \( \mathbf{v}_{n,ab} = (\mathbf{v}_a - \mathbf{v}_b) \cdot \mathbf{n}_{ab} \mathbf{n}_{ab} \), with \( \mathbf{n}_{ab} \) the unit vector pointing from the center of \( a \) to the center of \( b \). Similarly, the tangential forces \( \mathbf{F}_{t,ab} \) depend linearly on the tangential overlap \( \delta_t \), defined as the time integral of the relative tangential velocity from the time of first contact, and on the relative tangential velocity \( \mathbf{v}_{t,ab} = (\mathbf{v}_a - \mathbf{v}_b) - \mathbf{v}_{n,ab} \), itself. These tangential forces also cause a torque on the particles, given by

\[ \mathbf{T}_a = \sum_{b \in \text{contactlist}} (\mathbf{R}_a \mathbf{n}_{ab} \times \mathbf{F}_{t,ab}). \]  

(5)

B. Governing equations for the gas phase

The gas flow is governed by the conservation equations for mass and momentum:

\[ \frac{\partial (\rho g)}{\partial t} + \nabla \cdot \rho g \mathbf{u} = 0, \]  

(6)

\[ \frac{\partial (\rho g \mathbf{u})}{\partial t} + \nabla \cdot \rho g \mathbf{u} \mathbf{u} = -\nabla p - \nabla \cdot (\mu \nabla \mathbf{u}) + \rho g \mathbf{g} + \mathbf{s}_p + \mathbf{s}_{\text{ibm}}, \]  

(7)

where \( \rho g \) is the gas density, \( \mathbf{u} \) the gas velocity, \( p \) the gas pressure, \( \mu \) the dynamic gas viscosity, \( \mathbf{g} \) the gravitational acceleration, and \( \mathbf{s}_p \) and \( \mathbf{s}_{\text{ibm}} \) are source terms that describe the momentum exchange with the small solid particles:

\[ \mathbf{s}_p = \sum_{a=1}^{N_{\text{part}}} \mathbf{F}_{d,a} \delta(\mathbf{r} - \mathbf{r}_a). \]  

(8)

Here the summation is performed over all particles and the drag force \( \mathbf{F}_{d,a} \) is identical to what is used in the equation of motion of the particles. For the momentum exchange between the gas and small solid particles which are smaller than the Eulerian grid, it is necessary to introduce an empirical drag correlation:

\[ \mathbf{F}_{d,a} = 6 \pi \eta_d R_a (\mathbf{u} - \mathbf{v}_a) F(Re, \varepsilon), \]  

(9)

where \( \eta_d \) is the dynamic gas viscosity. \( F(Re, \varepsilon) \) represents a correction factor for finite particle Reynolds number \( Re = 2 R_a \rho_g \varepsilon \mathbf{u} - \mathbf{v}_a / \eta_d \) and higher solids volume fractions. Here we use the [15] and [16] correlations:

\[ F(Re, \varepsilon) = \begin{cases} \varepsilon^{-2.65} & \text{for } \varepsilon > 0.8, \\ \frac{150}{Re^{0.5}} + \frac{175 \varepsilon}{Re} & \text{for } \varepsilon < 0.8. \end{cases} \]  

(10)

C. Immersed boundary method

The interaction of the gas phase with an intruder larger than the size of the Computational Fluid Dynamics (CFD) cells is modeled with the IB method where Lagrangian marker points are situated on the boundary of the intruder. Each marker exerts a force on the gas phase such that the local velocity of the gas is equal to the velocity of that marker. The IBM has been widely used to study fluid-structure interaction and was pioneered by Peskin to investigate cardiac flow problems [17]. Subsequently, the method has been extended to flow around rigid bodies. The implementation that we adopt is along the lines of [18]. The IBM source term \( \mathbf{s}_{\text{ibm}} \) at the grid cell faces is calculated by summing the contribution of all Lagrangian force points,

\[ \mathbf{s}_{\text{ibm}} = \sum_{m=1}^{N_{\text{source}}} \mathbf{F}_m \delta(\mathbf{r} - \mathbf{r}_m), \]  

(11)

where in our discretized simulations \( \mathbf{F}_m \) is constructed such that the forcing results in a zero slip velocity at the surface of the sphere and \( \delta(\mathbf{r} - \mathbf{r}_m) \) is a volume weighing \( \delta \) function which distributes the forces to the surrounding grid cell faces. For detailed implementation of this method we refer to [19–21].

III. SIMULATION SETTINGS

In the simulation, we use a container of dimensions 0.06 × 0.06 × 0.12 m³ (width, depth, height). It contains one large intruder and 2,000,000 granular particles. The intruder has a diameter \( D = 4, 6, 8, \) or 10 mm, while the granular particles are of average diameter \( d = 0.5 \) mm, with a Gaussian size distribution (standard deviation \( \sigma_d = 0.02 \) mm) to avoid excessive ordering of the bed. According to the experimental results shown in [22] and [23], the ratio between the width \( L_{\text{box}} \) of the container and the largest intruder in our simulation (\( L_{\text{box}} / D = 6 \) for \( D = 10 \) mm) is large enough so that the surrounding walls have negligible effect on the force on the
The influence of varying the spring stiffness between the intruder and the walls is shown in Fig. 1. A summary of the simulation parameters is given in Table II.

In all simulations the coefficient of restitution is set to 0.97 for the normal direction, and to 0.33 for the tangential direction. For the particle-wall interaction the same collision parameters are used as for the particle-particle interaction. The friction coefficient is set to 0.1. All these values are typical for glass spheres and walls. We note that the normal spring stiffness \( k_n \) is in principle related to the Young’s modulus and the Poisson ratio of the solid material; however, in practice its value must be chosen much smaller to prevent the use of impractically small integration time steps. Therefore the spring stiffness is chosen as low as possible while ensuring that the lowered spring stiffness does not have a significant influence on the macroscopic system behavior. We investigated the influence of varying the spring stiffness between \( k_n = 100, 200, 400, \) and 800 N/m with an intruder initially located at a height of 0.11 m and initial velocity equal to 2 m/s. The results showed that the lower values for the normal stiffness do not significantly influence the collision kinematics and the solid fraction of the bed (not shown). Thus, to enhance the computational efficiency, \( k_n = 100 \) N/m was used in all subsequent simulations. Under these settings, the static angle of repose is \((17 \pm 1)^\circ\), determined by an independent simulation.

In these simulations, the intruder is moved at constant velocity through the granular bed. Because the time step needs to be small to successfully capture the details of the collision processes, the simulation time for extremely small intruder velocities would become prohibitively long. Thus, in the current simulations the lowest intruder velocity was limited to 0.01 m/s. According to Albert et al. [24], when the velocity of the intruder \( v < \sqrt{2gd} / 10 \approx 0.01 \) m/s, the system is in the so-called low velocity regime where the force exerted on the intruder is independent of the plunging velocity. On the other hand, due to the limited size of the system, boundary effects are expected to become important at very large velocities. Also by checking the animations, we found that for larger intruder velocities, a long channel following the intruder will form and will take more time to be refilled by granular particles. We therefore limited our intruder velocity to a maximum of 0.40 m/s.

The granular particles are prefluidized by a uniform upflowing gas, which was gradually turned off with the goal to make a homogeneous particle bed with a flat surface. The depth of the resulting static bed is about 0.0584 m with a solids volume fraction of about \( \varepsilon_s = 0.62 \). The solid fraction is larger than the ones measured in [2,25]. We think this is due to the higher friction between particles in the experiments, possibly aggravated by slight nonsphericities in particle shape. The influence of the friction coefficient on the solid fraction can be found in [26].

We let the intruder move constantly downwards from a height of 0.11 m, which is about 0.05 m above the bed, and stop near the bottom. Here and in the following \( z \) is the depth of the intruder inside the granular bed, where \( z = 0 \) is defined as the position of initial contact between the intruder and the topmost particles of the granular bed (see Fig. 1).

### IV. RESULTS AND DISCUSSION

#### A. Qualitative features of granular bed cratering

Figure 2 shows snapshots of the granular bed configuration at the time when a \( D = 10 \) mm spherical intruder has moved one diameter since initial contact \( (z = D) \), at impact velocities of \( v = 0.1, 0.2, 0.3, \) and 0.4 m/s, respectively. The granular particles are color coded according to their velocities. Note how the granular particles are perturbed much more at higher velocities, leading to formation of a granular crater. These differences in qualitative behavior of the granular medium...
lead to differences in the force $f$ experienced by the intruder. In the following we will focus on this force.

### B. Total force on intruder

Figure 3 shows the force experienced by a 10 mm spherical intruder under different impact velocities, as a function of penetration depth $z$. At the lowest intruder velocity of 0.01 m/s, we find a good agreement with a total force scaling as a power law with an exponent 1.3, up to a depth of slightly more than twice the intruder diameter $D$. For higher intruder velocities, the force is somewhat larger, caused by additional drag effects. We will discuss these additional drag forces in detail later.

A concave-to-convex transition is observed at all intruder velocities. These results are very similar to the former experimental results by [6,7]. It was found that the inflection point occurs when the intruder is fully buried to a level slightly more than twice its diameter. Note that for high intruder velocities the impact is relatively violent and the forces exerted on the intruder are quite noisy. Thus for velocities larger than 0.10 m/s, forces have been averaged over different runs. Further smoothing using an averaging window of 2.5 mm was applied for all velocities.

Figure 4 shows the force experienced by spherical intruders of different sizes, ranging from $D = 4$ mm to 10 mm, at an impact velocity of 0.01 m/s. When we scale the penetration depth $z$ by the intruder diameter $D$, and scale the force $f$ by $D^3$, as conducted in [7], all data collapses for depths beyond approximately $2D$. A slightly different offset is observed for different intruder diameters, which is related to different drag forces experienced by different sized intruders at this finite velocity, as will be discussed in detail later.

### C. Forces on the upper and lower half of the intruder

Peng et al. [7] stated in their paper: “The emergence of the inflection point is a signal that the filled-in granules on top of the intruder have been reorganized in bulk, and this fill-in effect shall be taken into account in the plunging force.” A direct visualization of the bed structure by coloring the bed into different layers, as shown in Fig. 1, shows how the granular particles have reorganized above the intruder. Clearly granular particles originating from the bed surface fall into the gap created by the intruder.
Because the granular particles fall into the gap behind the intruder, one may naively expect that the contact force on the upper half of the intruder is substantial. However, in our simulations we find that the contact force exerted on the upper hemisphere is relatively small for all cases studied, always amounting to less than 3% of the contact force exerted on the lower hemisphere. Figure 5 shows the contact forces on the upper and lower hemispheres of a $D = 10$ mm intruder. Although the contact force exerted on the upper hemisphere increases as the velocity decreases, the change is small. Thus, from our results we can conclude that the concave-to-convex behavior is caused predominantly by the lower part of the intruder. Some former works have shown similar behavior. For example, numerical simulations by Ding et al. [27] showed that the force on the leading surface of horizontally dragged intruders in granular media is dominant. Also for higher impact velocities Albert et al. [28] found that there was almost no difference between the force on a cylinder and the force on half a cylinder (with the cutting plane perpendicular to the direction of motion) when they are vertically moving within a granular medium, which also implies that the force on the frontal part of the intruder is crucial. In addition, both Brzinski et al. [29] and Clark et al. [30] tested the geometry dependence of the force explicitly and their results support the same argument that forces at the leading face dominate the dynamics.

For the 0.5 mm diameter granular particles used in this study, there is a small effect of the interstitial gas on the dynamics of the granular particles [8]. The gas also has a more direct effect on the intruder through the no-slip boundary conditions enforced on its surface. Figure 6 shows this direct gas-induced drag force on the upper and lower hemispheres, respectively. The magnitude of the direct gas-induced drag force on the lower hemisphere is of the order of 1% of the contact force exerted on the lower hemisphere. In contrast, the direct gas-induced drag force on the upper hemisphere is a large fraction of the contact force on the upper hemisphere, but both are small compared to the contact force on the lower hemisphere. Oscillations occur in the drag force because the granular particles, which are hydrodynamically coupled to the gas, experience stick-slip behavior caused by the reorganization of the force chains underneath the intruder.

**D. Granular drag force on the intruder**

We investigate the effective drag force exerted by the granular medium on the intruder by subtracting the force when the intruder velocity is small enough, in this case $v_{\text{small}} = 0.01$ m/s, from the force at larger intruder velocity:

$$f_c(z,v) \approx f(z,v) - f(z,v_{\text{small}}).$$

The results are shown in Fig. 7 for all intruder diameters and all intruder velocities studied. From this figure it can be seen that the granular drag force can be divided into two regimes: first a regime where the drag force is approximately independent of penetration depth $z$, and then a regime where the drag force.
increases with increasing \( z \):

\[
f_z(z,v) = \begin{cases} 
  f_z^*(v) & (z < z^*) \\
  f_z(v; z - z^*) & (z > z^*) 
\end{cases}
\]  

(13)

In the following, we will separately investigate the transition depth \( z^* \), the plateau drag force \( f_z^* \) for \( z < z^* \), and the increasing drag force \( f_z \) for \( z > z^* \).

1. Transition depth for granular drag force

We found that for all intruder diameters and velocities studied, the transition between the two drag force regions coincides with the depth \( z^* \) where the intruder is being engulfed by the granular particles. At this depth and time, the contact force exerted on the upper hemisphere of the intruder suddenly increases, as shown in Fig. 8.

It is apparent from Fig. 7 that for different intruder velocities, the transition depth \( z^* \) is different. The same applies for different intruder diameters. Let us now try to collectively describe all our observations of \( z^* \). Suppose the granular particles will start to freely fall towards the intruder when its penetration depth \( z \) is a factor \( \alpha \) of its diameter. On geometrical grounds, we expect \( \alpha \) to be larger than 1/2 (and equal to 1/2 in the limit of a very small angle of repose). Furthermore, because the granular medium immediately surrounding the intruder is already dragged downwards, we assume that the initial vertical velocity of the granular particle is typically a factor \( \beta \) of the intruder velocity, with \( \beta \) between 0 and 1. So we start with a situation in which a granular particle is in contact with the intruder and has a velocity \( \beta v \). The intruder continues to move down with a larger constant velocity \( v \), but the granular particle is accelerated by gravity, so after some time \( t \) the granular particle will again be in contact with the intruder.

This is the transition time, and it can be found by equating the distance traveled by the granular particle, \( \frac{1}{2}gt^2 + \beta vt \), to the distance traveled by the intruder, \( vt \), yielding a transition time \( t = \frac{2(1 - \beta) v}{g} \). At this time the depth of the intruder is equal to \( z^* = \alpha D + vt \), so the normalized transition depth is

\[
\frac{z^*}{D} = \alpha + 2(1 - \beta)Fr,
\]  

(14)

where \( Fr \) is the dimensionless Froude number of the intruder, defined as

\[
Fr = \frac{v^2}{gD}.
\]  

(15)

Figure 9 shows that indeed it is possible to collapse all transition depth measurements to a single curve by plotting \( z^*/D \) against the Froude number. For all Froude numbers, except possibly the smallest, a linear relation between \( z^*/D \) and \( Fr \) is found, in agreement with Eq. (14). From a least squares fit we find \( \alpha = 1.02 \) and \( \beta = 0.57 \), both of which are within the expected range.

2. Granular drag force before transition depth (plateau region)

For \( z < z^* \) the effective drag force is independent of intruder depth, but highly dependent on both intruder velocity and intruder diameter. Again we wish to collectively describe all our observations.

If we interpret the granular medium as a homogeneous complex fluid, then the correlation of Schiller and Naumann
moving through a Newtonian fluid of (effective) viscosity \( \mu_f \) and (effective) density \( \rho_f \). This correlation leads to a drag force that can be written as

\[
f_v^*(v) = 3\pi \mu_f D \nu \left[ 1 + 0.15 \left( \frac{\rho_f D \nu}{\mu_f} \right)^{0.687} \right]. \tag{16}
\]

where the last factor in round brackets is the intruder Reynolds number:

\[
\text{Re} = \frac{\rho_f D \nu}{\mu_f}. \tag{17}
\]

Note that the above analogy is not entirely accurate, because in our case the intruder is not yet engulfed by the granular fluid, and the drag force therefore predominantly arises from the front (lower hemisphere) of the intruder. However, also for a sphere which is fully embedded in a Newtonian fluid, the drag force for a large part arises from the front of the sphere. Also note that we assume that we can treat the granular medium as having a constant viscosity \( \mu_f \), while in case of an impacting intruder the granular medium close to the intruder is being perturbed much more strongly than the granular medium further away from the intruder.

Nevertheless, inspired by Eq. (16), we try to collapse all data by plotting the measurements of \( f_v^* \) against the product \( D \nu \). Figure 10 shows that such a collapse is possible. Moreover, we find a good fit between the measured drag force and the Schiller-Naumann correlation, Eq. (16), if we use the effective density \( \rho_f = \rho_s \epsilon_s \), and an effective viscosity of \( \mu_f = 0.20 \) Pa s. Note that the effective viscosity was the only free fit parameter in this exercise. Its value is somewhat lower than the viscosity of a quasistatic particle bed. Schügerl et al. [32], using a Couette-type viscometer, found that the granular viscosity for a quasistatic bed of 500 \( \mu \)m glass particles varies between 1.2 and 5 Pa s. King et al. [33] found a granular viscosity of 2.2 Pa s for 475 \( \mu \)m glass particles. However, these are the granular viscosities of quasistatic beds which are slowly sheared in a Couette cell. In our case the granular fluid is actively pushed aside by an impacting intruder. We hypothesize that this leads to greater inhomogeneities in the number of direct neighbor contacts in the granular fluid. Because the viscosity is determined by the weakest link in the granular fluid, such inhomogeneities will generally lead to a lower viscosity. We are strengthened in our belief in this picture by the results of Grace [34], who estimated the granular viscosity using an empirical approach based on the shape of spherical-cap bubbles rising in conventional liquids, and also found a lower viscosity (compared to Couette cell measurements) of 0.95 Pa s for 530 \( \mu \)m ballotini.

3. Granular drag force after transition depth

Figure 7 shows that the drag force \( \tilde{f}_v \) experienced after the transition depth, \( z > z^* \), increases with increasing \( (z - z^*) \). Recent publications show that the increase is caused by the hydrostatic loading; the role of hydrostatic particle pressure on stopping force has been carefully characterized by [29,35,36].

We will show that in our regime of low velocity impact it is possible to describe the drag force beyond \( z^* \) again by the Schiller-Naumann correlation, but using an effective granular viscosity which is increasing linearly with depth beyond the transition depth. In other words, our results can be explained if we speculate that the effective granular viscosity will increase linearly with depth beyond the transition depth:

\[
\mu_f(z) = \begin{cases} 
\mu_f^* & (z < z^*) \\
\mu_f^* + C(z - z^*) & (z > z^*) 
\end{cases}
\tag{18}
\]

where for our granular system \( \mu_f^* = 0.20 \) Pa s is the granular viscosity found in the previous subsection and \( C \) remains to be determined. It is essential to realize that \( C \), representing the increase in viscosity with depth, is a property of the granular bed, and therefore does not depend on the velocity or diameter of the intruder.

Inserting the viscosity Eq. (18) into the Schiller-Naumann correlation Eq. (16), we can describe the measured drag force for all intruder diameters and all velocities by choosing a single value \( C = 60 \) Pa m s\(^{-1}\). Figure 7 shows that the theoretically predicted drag forces (dashed lines) are in very good agreement with the measurements from our simulations (symbols), even correctly predicting the exchange in order of drag forces with increasing velocity near the transition depth (notice the intersection of the blue and black curves).

The predictions for the smallest intruder of \( D = 6 \) mm are slightly higher than observed in the simulations. We attribute this to the fact that for the smallest intruder the ratio \( D/d \) is smallest, leading to the largest error in approximating the granular medium as a continuum fluid.

We also observe for the relatively higher intruding velocities that the drag force in the plateau region decreases slightly right before the transition depth. At this moment we cannot explain the detailed reason behind this phenomenon, but suspect it is related to the more dynamic granular behavior at higher impact velocities.

Our results show that for shallow to intermediate depths, apparently conflicting experimental observations of different scaling laws for the drag force as a function of intruder velocity \( v \) can be unified using the effective Reynolds number. We find
that there is no single power-law scaling for the drag force. Depending on experimental details, the drag force can scale as $v$ ($\text{Re} \ll 1$), as $v^2$ ($\text{Re} \gg 1$), or in between these two extremes ($\text{Re} \approx 1$).

E. Particle pressure

The evolution of the macroscopic force exerted on the intruder at intermediate to larger depths can also be related to the evolution of the granular particle pressure beneath the intruder.

In our simulations, we investigate the spatial distribution of the average overlap between particles in the bed. Because our soft sphere model is linear, the average overlap is directly proportional to the particle pressure. The result is shown in Fig. 11 for an unperturbed bed and for a $D = 10$ mm intruder at different depths.

In an unperturbed bed, the particle pressure is determined by its force balance with gravity, i.e., the granular particles at each depth must carry the weight of all particles above. Because the solids volume fraction does not vary a lot, this leads to a particle pressure which increases linearly with depth. Indeed, the isobaric lines in Fig. 11 (No contact) are more or less equally spaced.

When an intruder impacts on the bed, extra forces are generated predominantly below the intruder, leading to a larger particle pressure. Figure 11 shows how from the first moment of impact the particle pressure increases beneath the intruder, and grows in extent as the intruder penetrates deeper into the bed. Note that in our simulations, the disturbed pressure field comes in contact with the bottom wall when the gap width between the intruder and bottom wall is of the order of one intruder diameter. In this paper we focused on the forces for shallow to intermediate intruder depths, for which these wall effects are not yet dominant.

\section{V. CONCLUSION}

A DP-IB method, validated in our former work \cite{Stone2004}, has been extended to study the vertical plunging force on a spherical intruder moving with relatively low but constant velocity into a granular bed. The results from the present study can be summarized as follows:

(i) Our simulation results show a concave-to-convex transition in the plunging force at intermediate depth (2 to 3 particle diameters). Prior to this transition, the force fits to a power law with an exponent around 1.3, which is in good agreement with existing experimental observations.

(ii) The force exerted on the front part of the immersed intruder dominates, even at these relatively small intrusion velocities where the granular particles have enough time to close the gap behind the intruder.

(iii) The effective drag force is depth independent at shallow depths, and increases with depth for intermediate depths. The transition depth between these two regimes can be characterized in terms of a Froude number.

(iv) The effective drag force at shallow and intermediate depths can be described by the Schiller-Naumann correlation, using an effective Reynolds number based on a granular viscosity which is constant before the transition depth and linearly increasing beyond the transition depth.

In our future work, we will investigate different intruder shapes and validate our simulation results by a one-to-one comparison with force-transducer experiments in which an intruder is moved at constant velocity through a granular bed.

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